Construction of Universal Modal World based on Hyperset Theory

Toru Tsujishita

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August 1, 1994

Introduction

When cognitive agents live in a world, the knowledge states of the agents are important components of the the global status of the world. Thus the global state S of the world consists of the state of the external environment $e \in E$ and the knowledge states α_i of the agents living in the world:

$$S = \langle \alpha_1, \cdots, \alpha_n, e \rangle.$$

Now what is the knowledge state α_i of the agent i? One possible interpretation is that it is the subset of the global state of the world which seem possible for the agent i. Such an object as S can not be formulated as a set in the usual set theory, since the validity of knowledge, i.e., $S \in \alpha_i$, implies $S \ni \cdots \ni S$, which is incompatible with the foundation axiom.

Fortunately, a new set theory, called hyperset theory, is recently established by Aczel [1], which does permit us to think such set S as above.

In this note, we construct a universal modal world carrying KT modal logic (cf. [4]) in the universe of hypersets. We construct it as the largest fixed point of a natural set continuous class operator. In this modal world, not only the knowledge formulas but also the those expressing common knowledge have simple and natural interpretation.

This modal world can be regarded also as a universal KT Kripke structure, since every KT Kripke structure has a unique equivalent model in it. This implies in particular that a knowledge formula is valid if and only if it is true in this universal modal world which preserve the interpretation of the knowledge formulas.

An illustrative example of this new model of modal logic, applied to the famous Muddy Boys Puzzle, can be found in [6].

We note that, in the usual set theory with foundation axiom, modal worlds can be constructed only cumulatively ([2, 3]). In such well-founded modal worlds, not all knowledge formulas can be interpreted simultaneously. Moreover it is not possible to interpret common knowledge formulas.

We show that the hierarchy of modal worlds constructed in the usual universe of sets ([3]) are the images of certain natural maps from this universal modal world.

Although we confined ourselves to the KT modal logic, our approach is effective for most of the variants of modal logic, such as S4 and S5.

1 Anti-foundation axiom

We shall work in the hyperset theory ZFA. This is an axiomatic set theory having the Zermero-Fraenkel axioms except for the foundation axiom, which is replaced with the Anti-Foundation Axiom of Aczel. (cf. [1] for details.) In ZFA, we can uniquely solve any system of set equations: Let X be a class of atoms and $\{ a_x \mid x \in X \}$ be a system of X-sets. Consider the system of set equations:

(1.1) $$x = a_x, \quad x \in X$$

Then there exist unique family of sets $\{ b_x \mid x \in X \}$ such that

$$b_y = a_y[\{ b_x/x \}] \quad (\forall y \in X),$$

where $a[\{ b_x/x \}]$ denotes the set obtained from an X-set a by substituting b_x to x.

We denote the class of all the hypersets by $\mathbf{V}$ and the class of all the subsets of a class $\mathbf{X}$ by $\mathbf{pow}\ \mathbf{X}$.

2 Knowledge operator $\mathrm{K}_{I,E}$

Let I and E be sets. An element i of I denotes an agent and an $e \in E$ stands for a state of external world. We assume $I = \{ 1, \ldots, n \}$ for simplicity of notation. For each class $\mathbf{X}$, we define

$$\overline{\mathbf{K}}_{I,E}\mathbf{X} := (\mathbf{pow}\ \mathbf{X})^I \times E.$$

Since this is obviously set continuous, i.e.,

$$\overline{\mathbf{K}}_{I,E}\mathbf{X} = \bigcup_{x \subset \mathbf{X},\ x \text{ is a set}} \overline{\mathbf{K}}_{I,E}\, x,$$

it has a unique largest fixed point $\overline{\mathbf{W}}_{I,E}$:

$$\overline{\mathbf{W}}_{I,E} = \overline{\mathbf{K}}_{I,E}\, \overline{\mathbf{W}}_{I,E}.$$

By ZFA, this fixed point is not empty, since it has S_e $(e \in E)$ as its elements, where S_e is defined as the solution of the set equation

(2.1) $$x = \langle \{ x \}, \cdots, \{ x \}, e \rangle.$$

Let $\mathcal{U}$ be the subclass of $\overline{\mathbf{W}}_{I,E}$ consisting of S's satisfying $S \in \pi_i S$ for all $i \in I$. The class $\mathcal{U}$ is not empty, since it includes S_e.

Define now

$$\mathrm{K}_{I,E}\mathbf{X} := \Big((\mathbf{pow}\ \mathbf{X})^I \times E \Big) \bigcap \mathcal{U},$$

which is obviously set continuous.

Define

$$\mathbf{W}_{I,E} := \bigcup_{x \subset \mathbf{K}_{I,E}x} x.$$

Note that $\mathbf{W}_{I,E}$ is not empty, since $S_e \in \mathbf{W}_{I,E}$.

By Theorem[6.5] of [1], we have

Proposition 2.1 (1)

$$\mathbf{W}_{I,E} = \left((\mathbf{pow}\ (\mathbf{W}_{I,E}))^I \times E\right) \bigcap \mathcal{U}.$$

(2) $\mathbf{X} \subset \mathbf{K}_{I,E}\mathbf{X}$ *implies* $\mathbf{X} \subset \mathbf{W}_{I,E}$.

Let $\pi_i : \mathbf{W}_{I,E} \to \mathbf{pow}\ \mathbf{W}_{I,E}$ and $\pi_E : \mathbf{W}_{I,E} \to E$ be natural projections.

We call a class $\mathbf{X}$ *coclosed* if $\mathbf{X} \subset \mathbf{K}_{I,E}\mathbf{X}$. Then $\mathbf{W}_{I,E}$ is by definition the union of coclosed set.

Proposition 2.2 *For each subset x of $\mathbf{W}_{I,E}$, there is a largest coclosed subset of x, which we call the coclosure of x and denote it by $\underline{x}$.*

Proof. The coclosure can be defined as the largest fixed point $\underline{x}$ of the set continuous operator $\Phi : \mathbf{X} \mapsto x \bigcap \mathbf{K}_{I,E}\mathbf{X}$. Then $\underline{x} = x \bigcap \mathbf{K}_{I,E}\underline{x}$, which imply that $\underline{x}$ is coclosed. Moreover, for any coclosed subset $z \subset x$, we have $z \subset \Phi z$, which implies $z \subset \underline{x}$. **q.e.d.**

We recall a fact which we need to show that $\overline{\mathbf{W}}_{I,E}$ and $\mathbf{W}_{I,E}$ include solutions of set equations of certain types. Let Φ be a set continuous class operator. An X-set a is called $\mathbf{K}_{I,E}$ *-local* if for every class $\mathbf{B}$ and every family of sets $\{ c_x \in \mathbf{B} \mid x \in X \}$, $a[c_x/x] \in \Phi\mathbf{B}$.

Proposition 2.3 *Suppose the X-sets in the RHS of the set equation* (1.1) *is Φ-local, then its solutions belong to the largest fixed point of Φ.*

3 KT Kripke structures

A tuple $\mathcal{K} = \langle \mathcal{W}, \mathcal{R}_1, \cdots, \mathcal{R}_n, \epsilon \rangle$ is called a *KT Kripke structure of type* (I, E) if $\mathcal{W}$ is a class, $\mathcal{R}_i$'s are reflexive binary relations on $\mathcal{W}$ and $\epsilon : \mathcal{W} \rightarrow E$ is a map.

A coclosed class $\mathbf{X}$ defines a KT Kripke structure$\mathcal{K}(\mathbf{X})$ defined by

$$\mathcal{K}(\mathbf{X}) := \langle \mathbf{X}, \mathcal{R}_1, \cdots, \mathcal{R}_n, \epsilon \rangle,$$

where

$$\mathcal{R}_i = \{ \langle S, T \rangle \mid \pi_i S \ni T \}.$$

The reflexivity is guaranteed by $S \in \mathcal{U}$. Especially the $\mathbf{W}_{I,E}$ defines a KT Kripke structure$\mathcal{K}_{I,E}$.

4 Interpretations of Knowledge Formulas

The set of knowledge formulas is defined, using the BNF, as

$$\varphi ::= \bot \mid p \mid \varphi \rightarrow \varphi \mid K_i \varphi,$$

where p and φ stands respectively for a subset of E and a knowledge formula.

A knowledge formula can be interpreted by a KT Kripke structureas follows: Let $\mathcal{K}$ be a KT Kripke structure. We define

$$\begin{aligned} \mathcal{K}, S &\not\models \bot \\ \mathcal{K}, S \models p &\overset{def}{\Longleftrightarrow} \epsilon(S) \in p \\ \mathcal{K}, S \models \varphi_1 \rightarrow \varphi_2 &\overset{def}{\Longleftrightarrow} \mathcal{K}, S \not\models \varphi_1 \text{ or } \mathcal{K}, S \models \varphi_2 \end{aligned}$$

$$\mathcal{K}, S \models K_i\varphi \overset{def}{\Longleftrightarrow} \forall T\,[(S,T) \in \mathcal{R}_i \Rightarrow \mathcal{K}, T \models \varphi].$$

We write

$$\begin{aligned} \mathcal{K} \models \varphi &\overset{def}{\Longleftrightarrow} \mathcal{K}, s \models \varphi \text{ for all } S \in \mathcal{W} \\ \models \varphi &\overset{def}{\Longleftrightarrow} \mathcal{K} \models \varphi \text{ for all KT Kripke structureKripke structure } \mathcal{K}. \end{aligned}$$

By the reflexivity of $\mathcal{R}_i$'s, we have

$$\models \ K_i\varphi \rightarrow \varphi,$$

for every formulat φ.

A knowledge formula φ efines a subclass of the modal space $\mathbf{W}_{I,E}$ by

$$[\varphi] := \{\ S \in \mathbf{W}_{I,E} \mid \mathcal{K}_{I,E}, S \models \varphi\ \}.$$

This can be described directly as follows:

$$\begin{aligned} [\bot] &:= \emptyset, \\ [p] &:= \pi_E^{-1} p, \\ [\varphi_1 \rightarrow \varphi_2] &:= [\varphi_1]^c \cup [\varphi_2], \\ [K_i\varphi] &:= \pi_i^{-1} \mathbf{pow}\,[\varphi]. \end{aligned}$$

In $\mathbf{W}_{I,E}$, we can also interpret the common knowledge formula. A common knowledge formula is defined by

$$\varphi ::= \bot \mid p \mid \varphi \rightarrow \varphi \mid K_i\varphi \mid C\varphi.$$

We define $[C\varphi]$ as the coclosure of $[\varphi]$. Then

$$\begin{aligned} [C\varphi] &= [\varphi] \bigcap \mathbf{K}_{I,E}[C\varphi] \\ &= [\varphi] \bigcap \pi_1^{-1}\mathbf{pow}\,[C\varphi] \bigcap \cdots \bigcap \pi_n^{-1}\mathbf{pow}\,[C\varphi]. \end{aligned}$$

Hence

$$S \models C\varphi \Longleftrightarrow S \models \varphi \wedge K_1 C\varphi \wedge \cdots \wedge K_n C\varphi,$$

which means exactly that φ is a common knowledge at S.

5 Universality of $\mathbf{W}_{I,E}$

Let $\mathcal{K}^i$ $(i = 1, 2)$ be KT Kripke structures of type (I, E). A map $f : \mathcal{W}_1 \to \mathcal{W}_2$ is called *a simulation* from $\mathcal{K}_1$ to $\mathcal{K}_2$ if $(f \times f)\mathcal{R}_i^1 \subset \mathcal{R}_i^2$ for all $i \in I$ and when $f(S) = S'$ and $(S', T') \in \mathcal{R}_i^2$ then there exists $T \in \mathcal{W}^1$ such that $f(T) = T'$ and $(S, T) \in \mathcal{R}_i^1$.

A simulation map from a KT Kripke structure$\mathcal{K} = \langle \mathcal{W}, \mathcal{R}_1, \cdots, \mathcal{R}_n, \epsilon \rangle$ to $\mathcal{K}_{I,E}$ is exactly *a decoration*, i.e., a family of sets $\{ d(S) \mid S \in \mathcal{W} \}$ satisfying

$$d(S) = \langle \{ d(T) \mid (S,T) \in \mathcal{R}_1 \}, \cdots, \{ d(T) \mid (S,T) \in \mathcal{R}_n \}, \epsilon S \rangle.$$

Theorem 5.1 *Every KT Kripke structurehas a unique simulation map from itself to $\mathcal{K}_{I,E}$.*

Proof. This is essentially a special case of the special final coalgebra theorem of Aczel, although a KT Kripke structureis not a $\mathbf{K}_{I,E}$ -coalgebra because of the conditions of reflexivity and transitivity of $\mathcal{R}_i$'s.

Let $x_S = d(S)$ $(S \in \mathcal{W})$ be the solution of the set equation

$$x_S = \langle \{ x_T \mid (S,T) \in \mathcal{R}_1 \}, \cdots, \{ x_T \mid (S,T) \in \mathcal{R}_n \}, \epsilon S \rangle.$$

Put $\overline{\mathcal{W}} := \{ d(S) \mid S \in \mathcal{W} \}$. Then $\overline{\mathcal{W}}$ is coclosed and hence $\overline{\mathcal{W}} \subset \mathbf{W}_{I,E}$ by proposition(2.1). We define then $d : \mathcal{W} \to \overline{\mathcal{W}}$ by $S \mapsto d(S)$. The map d is obviously a decoration, whence a simulation. The uniqueness is obvious since any decoration is a solution of the above set equation. **q.e.d.**

We call a KT Kripke structure$\mathcal{K}$ is *minimal* if every simulation map from $\mathcal{K}$ to other KT Kripke structureis injective.

Corollary 5.2 *The image of the decoration of a KT Kripke structureis a minimal KT Kripke structure.*

Proof. Let $\varphi : \mathcal{K} \to \mathcal{K}'$ be an simulation. Let d' be the decoration of $\mathcal{K}'$. Then $d' \circ \varphi$ is a decoration of $\mathcal{K}$, whence it must be the identity, which is obviously a decoration. This implies that φ must be injective. q.e.d.

6 Invariance of Interpretation under Simulation Maps

The importance of Theorem5.1 is due to the fact that a simulation map preserves the interpretation of knowledge formula.

Proposition 6.1 *Let $\mathcal{K}^i$ $(i = 1, 2)$ be KT Kripke structures and f be a simulation from $\mathcal{K}^1$ to $\mathcal{K}^2$. Then*

$$\mathcal{K}^1, S \models \varphi \iff \mathcal{K}^2, f(S) \models \varphi, \tag{6.1}$$

for all knowledge formula φ.

Proof. It suffices to show that if (6.1) is valid for φ, then it is valid also for $K_i\varphi$. But

$$\mathcal{K}_2, f(S) \models K_i\varphi$$

means that for all T' satisfying $(f(S), T') \in \mathcal{R}_i^2$,

$$\mathcal{K}_2, T' \models \varphi. \tag{6.2}$$

But, by the simulation condition of f, we can find T for each such T'satisfying $f(T) = T'$ and $(S, T) \in \mathcal{R}_i^1$. Thus (6.2) is equivalent to

$$\mathcal{K}_2, f(T) \models \varphi \iff \mathcal{K}_1, T \models \varphi$$

for all T with $(S, T) \in \mathcal{R}_i^1$. This means exactly

$$\mathcal{K}_1, S \models K_i\varphi.$$

q.e.d.

We say that a knowledge formula is valid if it is true in all KT Kripke structure.

Corollary 6.2 *A knowledge formula is valid if and only if* $\mathbf{W}_{I,E} \models \varphi$.

7 Relation with cumulative modal worlds

In [2, 3], modal worlds are defined cumulatively. Let us briefly recall the definition. We will denote by E the set of all truth assignment to primitive propositions. Define a family of sets $\left\{ \overline{V}_n \mid n \in \mathrm{N} \right\}$ inductively by

$$\overline{V}_0 := E, \tag{1}$$
$$\overline{V}_{n+1} := \overline{V}_n \times \mathrm{pow}\,(\overline{V}_n)^I. \tag{2}$$

We denote the projections from $\overline{V}_{n+1}$ to the first and the second factor by π_R and π_L respectively.

Since

$$\overline{V}_{n+1} = \overline{V}_0 \times \mathrm{pow}\,(\overline{V}_0)^I \times \mathrm{pow}\,(\overline{V}_1)^I \times \cdots \times \mathrm{pow}\,(\overline{V}_n)^I,$$

an element of $\overline{V}_{n+1}$ can be expressed as

$$\langle f_0, \cdots, f_n \rangle$$

with $f_k \in \mathrm{pow}\,(\overline{V}_{k-1})^I$ for $k \geq 1$.

A subset $V_n \subset \overline{V}_n$ is defined by *the extensionality condition* : For all $0 < k < n$ and $i \in I$,

$$f_k(i) = \{ \langle g_0, \cdots, g_{k-1} \rangle \mid \langle g_0, \cdots, g_{n-1} \rangle \in f_n(i) \text{ for some } g_k, \cdots, g_{n-1} \}.$$

An element of V_n is called an *n-world* in [3].

We define now *KT modal worlds*. Define a subset $W_n \subset V_n$ by *the knowledge condition*: For all $0 \leq k < n$ and $i \in I$,

$$\langle f_0, \cdots, f_k \rangle \in f_{k+1}(i).$$

We construct, for each $m \in \mathrm{N}$, a map

$$\varpi_m : \overline{\mathbf{W}}_{I,E} \to V_m,$$

inductively by

$$\begin{aligned} \varpi_0 &:= \pi_E : \overline{\mathbf{W}}_{I,E} \to E, \\ \varpi_{m+1} &:= \varpi_m \times \mathbf{pow}\,(\varpi_m)^I, \end{aligned}$$

namely,

$$\varpi_{m+1} S = \langle \varpi_m S, g \rangle,$$

with

$$g(i) = \{\ \varpi_n T \mid T \in \pi_i S\ \} \quad \text{for } i \in I.$$

Theorem 7.1 $\varpi_m(\overline{\mathbf{W}}_{I,E}) = V_m$.

Proof. Consider the following system of set equations

$$x\langle f_0, \cdots, f_m \rangle = \langle \cdots, A\langle f_0, \cdots, f_m \rangle_i, \cdots, f_0 \rangle$$

for the family of unknown sets:

$$\{\ x\langle f_0, \cdots, f_m \rangle \mid \langle f_0, \cdots, f_m \rangle \in V_m,\ m \in \mathrm{N}\ \},$$

where

$$A\langle f_0, \cdots, f_m \rangle_i = \{\ x\langle g_0, \cdots, g_{n-1} \rangle \mid \langle g_0, \cdots, g_{n-1} \rangle \in f_n(i)\ \}$$

for $n \geq 1$ and $A\langle f_0 \rangle = \emptyset$. Let

$$x\langle f_0, \cdots, f_m \rangle = S\langle f_0, \cdots, f_m \rangle$$

be its unique solution. Then by Proposition 2.3 $S\langle f_0, \cdots, f_m\rangle \in \overline{\mathbf{W}}_{I,E}$ and

$$\pi_i S\langle f_0, \cdots, f_m\rangle = \{\ S\langle g_0, \cdots, g_{m-1}\rangle \mid \langle g_0, \cdots, g_{m-1}\rangle \in f_n(i)\ \}.$$

Now we show

$$\varpi_k S\langle f_0, \cdots, f_m\rangle = \langle f_0, \cdots, f_k\rangle \quad \text{for } k \leq m. \tag{7.1}$$

by double induction on k and m. For $m = 0$, (7.1) is obvious. Suppose, for some $M \geq 1$, (7.1) has been established for all k and m with $k \leq m < M$. Since

$$\varpi_0 S\langle f_0, \cdots, f_M\rangle = \pi_E S\langle f_0, \cdots, f_M\rangle = f_0,$$

(7.1) with $m = M$ is true for $k = 0$. Suppose (7.1) with $m = M$ holds for $k \leq \ell$. Then

$$\begin{aligned} \pi_R[\varpi_{\ell+1} S\langle f_0, \cdots, f_M\rangle](i) &= \varpi_{\ell *}\{\ S\langle g_0, \cdots, g_{M-1}\rangle \mid \langle g_0, \cdots, g_{M-1}\rangle \in f_M(i)\ \} \\ &= \{\ \langle g_0, \cdots, g_\ell\rangle \mid \langle g_0, \cdots, g_{M-1}\rangle \in f_M(i)\ \} \\ &= f_{\ell+1} \quad \text{(by the extensionality condition).} \end{aligned}$$

On the other hand

$$\begin{aligned} \pi_R[\varpi_{\ell+1} S\langle f_0, \cdots, f_M\rangle](i) &= \varpi_\ell S\langle f_0, \cdots, f_M\rangle \\ &= \langle f_0, \cdots, f_\ell\rangle \text{ (by the induction hypothesis).} \end{aligned}$$

Hence

$$\varpi_{\ell+1} S\langle f_0, \cdots, f_M\rangle = \langle f_0, \cdots, f_\ell, f_{\ell+1}\rangle.$$

This concludes the proof of (7.1) and also proves $\mathrm{Im}\varpi_m$ includes V_m.

We prove now

$$\mathrm{Im}\varpi_m \subset V_m \tag{7.2}$$

by induction on m. For $m = 0$, we have

$$\mathrm{Im}\varpi_0 = \mathrm{Im}\pi_E = E.$$

Suppose (7.2) is true for $m < M$. Let

$$\varpi_M S = \langle f_0, \cdots, f_{M-1}\rangle$$

for an $S \in \overline{\mathbf{W}}_{I,E}$. Since

$$\varpi_{M-1} S = \langle f_0, \cdots, f_{M-2} \rangle,$$

we have $\langle f_0, \cdots, f_{M-2} \rangle \in W_{M-1}$. It suffices to show that

$$f_{M-2}(i) = \{ \langle g_0, \cdots, g_{M-3} \rangle \mid \langle g_0, \cdots, g_{M-3}, \exists g_{M-2} \rangle \in f_{M-1}(i) \} \tag{7.3}$$

But

$$\begin{aligned} f_{M-2}(i) &= \{ \varpi_{M-2} T \mid T \in \pi_i S \} \\ &= \{ \pi_L \varpi_{M-1} T \mid T \in \pi_i S \} \\ &= \pi_{L*} \{ \varpi_{M-1} T \mid T \in \pi_i S \} \\ &= \pi_{L*} f_{M-1}(i), \end{aligned}$$

which means exactly (7.3). **q.e.d.**

We have a similar result for modal worlds with knowledge condition:

Theorem 7.2 $\varpi_m(\mathbf{W}_{I,E}) = W_m$.

Proof. The proof is similar to the previous one. Consider the following system of set equations

$$x\langle f_0, \cdots, f_m \rangle = \langle \cdots, B\langle f_0, \cdots, f_m \rangle_i, \cdots, f_0 \rangle$$

with unknown sets

$$\{ x\langle f_0, \cdots, f_m \rangle \mid \langle f_0, \cdots, f_m \rangle \in W_m,\ m \in \mathrm{N} \},$$

where

$$\begin{aligned} B\langle f_0, \cdots, f_m \rangle_i &= \{ x\langle f_0, \cdots, f_m \rangle \} \\ &\bigcup \{ x\langle g_0, \cdots, g_{m-1} \rangle \mid \langle g_0, \cdots, g_{m-1} \rangle \in f_m(i) \}. \end{aligned}$$

Let

$$x\langle f_0, \cdots, f_m \rangle = S\langle f_0, \cdots, f_m \rangle$$

be its unique solution. Then, by Proposition 2.3, the set $S\langle f_0, \cdots, f_m\rangle$ is in $\mathbf{W}_{I,E}$ and

$$\pi_i S\langle f_0, \cdots, f_m\rangle = \{ S\langle f_0, \cdots, f_m\rangle \} \bigcup \{ S\langle g_0, \cdots, g_{m-1}\rangle \mid \langle g_0, \cdots, g_{m-1}\rangle \in f_m(i) \}.$$

As before we can show

$$\varpi_k S\langle f_0, \cdots, f_m\rangle = \langle f_0, \cdots, f_k\rangle \quad \text{for } k \leq m$$

using the knowledge condition. This proves that $\varpi\mathbf{W}_{I,E}$ includes W_m. Since

$$\varpi\mathbf{W}_{I,E} \subset \varpi\overline{\mathbf{W}}_{I,E} \subset V_m,$$

we have only to check that the knowledge condition is satisfied by

$$\langle f_0, \cdots, f_{m-1}\rangle := \varpi_m S$$

for $S \in \mathbf{W}_{I,E}$. But $S \in \pi_i S$ implies

$$\begin{aligned} \langle f_0, \cdots, f_{m-2}\rangle &= \varpi_{m-1} S \\ &\in \{ \varpi_{m-1} T \mid T \in \pi_i S \} \text{ (by } S \in \pi_i S) \\ &= f_{m-1}(i). \end{aligned}$$

q.e.d.

Finally we note that ϖ_m preserves the validity of knowledge formulas with depth less than or equal m. We recall that

$$\begin{aligned} \text{depth}(p) &= 0 \text{ if } p \in \text{pow}\,(E) \\ \text{depth}(\bot) &= 0 \\ \text{depth}(\varphi_1 \rightarrow \varphi_2) &= \max(\text{depth}(\varphi_1), \text{depth}(\varphi_2)) \\ \text{depth}(K_i \varphi) &= \text{depth}(\varphi) + 1. \end{aligned}$$

For $\langle f_0, \cdots, f_r\rangle \in V_{r+1}$ and a formula φ with depth$(\varphi) \geq r$, the satisfiability relation

$$\langle f_0, \cdots, f_r\rangle \models \varphi$$

is defined inductively by

$$\begin{array}{lcl}\langle f_0,\cdots,f_r\rangle & \models & p \text{ if } f_0\in p \quad (p\in \text{pow}\,(E)),\\ \langle f_0,\cdots,f_r\rangle & \not\models & \bot\\ \langle f_0,\cdots,f_r\rangle & \models & \varphi_1{\rightarrow}\varphi_2 \text{ if } \langle f_0,\cdots,f_r\rangle \not\models \varphi_1 \text{ or } \langle f_0,\cdots,f_r\rangle \models \varphi_2\\ \langle f_0,\cdots,f_r\rangle & \models & K_i\varphi \text{ if } \langle g_0,\cdots,g_{r-1}\rangle \models \varphi \text{ for each } \langle g_0,\cdots,g_{r-1}\rangle \in f_r(i).\end{array}$$

Theorem 7.3 *If* depth$(\varphi)\le r+1$ *then,*

$$S\models\varphi \iff \varpi_{r+1}S\models\varphi$$

for $S\in \mathrm{W}_{I,E}$.

Proof. By induction on r. The nontrivial part of the proof is to show

$$S\models K_i\varphi \iff \varpi_{r+1}S\models K_i\varphi.$$

Let $\varpi_{r+1}S=\langle f_0,\cdots,f_r\rangle$. Then

$$f_r(i)=\{\,\varpi_rT \mid T\in\pi_iS\,\}$$

be definition. Hence

$$\begin{array}{rcl}\varpi_{r+1}\models K_i\varphi & \iff & \varpi_rT\models\varphi \text{ for all } T\in\pi_iS\\ & \iff & T\models\varphi \text{ for all } T\in\pi_iS \text{ by induction hypothesis}\\ & \iff & S\models K_i\varphi.\end{array}$$

q.e.d.

We remark that we can also construct universal modal world for S5 modal logic. We simply replace the $\mathcal{U}$ by the subclass $\mathcal{V}$ of $\overline{\mathrm{W}}_{I,E}$ consisting of S's satisfying

- $S\in\pi_i$ for all $i\in I$,
- $\bigcup_{T\in\pi_iS}\pi_iT\subset\pi_iS$,
- If $T\in\pi_iS$ then $S\in\pi_iT$.

for all $i\in I$. This class is also non empty since $S_e\in\mathcal{V}$.

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