CONSTRUCTION WORK IN SOLID GEOMETRY.[1]

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"To measure is to know." Thus spoke the great Kepler some three hundred years ago. I shall take his words as my text and shall endeavor in the brief time allotted me to discuss the effect those words have had upon my method of conducting construction work in my solid geometry classes.

In the eventful year of 1914, I joined the mathematics staff of the high school in New Castle, Pennsylvania, as commanding officer. My first campaign led me into the fields of Euclid, directing a company of so-called "flunkers" in second semester geometry. The morale of such a company, as might be expected, was at the breaking point. Regulation drill work as outlined by the textbook writers on geometry was out of the question. I must seek a new method of attack. For a while, I must confess, I was at a loss how to command the company. One day as I was reading a book on geometry I saw the words of Kepler, "To measure is to know." Surely there is truth in this statement I thought to myself and the words kept ringing in my ears, as they still do. It was not long before I was asking myself the question: "Does it mean that if I give my students more measuring work in geometry they will know more geometry?" I tried it out and found my new method of attack.

Every day I required my class to measure some geometrical magnitude. At one time they were sent to the blackboard and required to construct a square of 16.5 inches on a side, not simply to construct a square; at another, they were asked to determine the area of all the blackboards in the room to the nearest square inch. It was not long before I had some real enthusiasm in this "flunkers" class. Near the close of the course the spirit ran high and when the last recitation was over a goodly number of the class came to me personally and said that they had not only enjoyed the course but had really pursued it with understanding. More than twenty-five per cent of the class continued in their geometrical studies by taking the course in solid geometry, an elective subject, which I accepted as additional proof of the sincerity of their remarks. Measuring magnitudes, drawing to scale and to specifications in plane geometry convinced me that Kepler's words were true. No sooner proved in plane ge-

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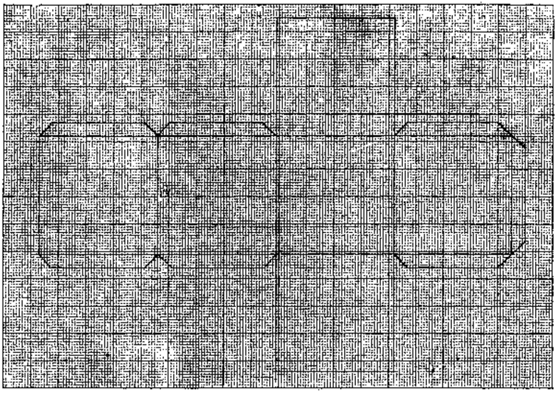
ometry, I applied the same methods in solid geometry with the same results, and thus you have the historic introduction of the construction work that is to follow.

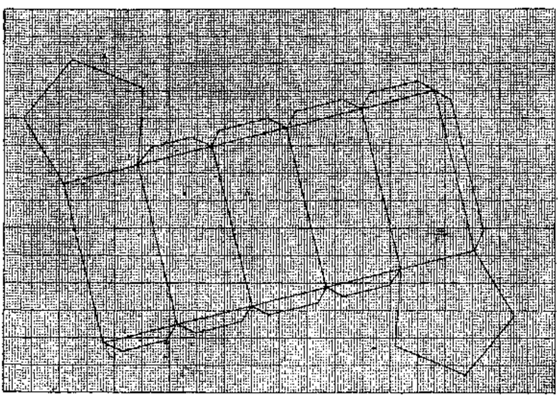
It has been my experience in the past that one of the benefits to be derived from a High School Conference of this nature is to hear someone relate definite classroom methods in a particular part or branch of a subject. Such will be my procedure.

The first part of the course in solid geometry, usually designated as Book VI, deals with lines and planes in space and offers little opportunity for interesting construction work. I require only one model in this division of the subject, namely a model to illustrate the theorem: If a line is perpendicular to each of two other lines at their point of intersection it is perpendicular to the plane of the two lines. This model is made according to specifications, insuring uniformity, and I make a freehand sketch on the board to show the class just how I want it made.



Though the remaining exercises deal entirely with polyhedrons I do not wait until I reach Book VII in the textbook before I begin operations in constructing some of the simpler polyhedrons. Two reasons may be given for so doing. (1) The work in polyhedrons offers excellent opportunity for the use of algebra and the sooner you can show the student where he can use his algebra as a tool the more interest he will take in the entire course. (2) I require each pupil to make ten models, one per week, and as ten weeks cannot be given to the study of polyhedrons if you would cover satisfactorily the subject of solid geometry as ordinarily outlined in the average textbook there must be an overlapping and I have found it better to do so in the beginning of the course than at the end.

Exercise No. 1 (as you have it before you)[2] reads: Required to construct a cube whose volume is ten cubic inches. To determine the length of the edge of the cube requires the solution of the equation $e^3=10$. Solving for e we obtain $e=\sqrt[3]{10}$ or 2.154, approximately. The approximate answer raises the question as to how the student is to extract the cube root of ten. Personally, if my class is not familiar with the use of logarithms I spend one or two recitations in explaining their use and you would be surprised how quickly this powerful computing tool is successfully operated by the average pupil. He does not need to know all about logarithms to be able to use logarithms. If he has access to a table of the powers and roots of numbers (which table ought to be in every high school textbook in mathematics, as well as a table of logarithms) he may use it to find the value of $\sqrt[3]{10}$. I require the approximate result to three places of decimals so that the nearest hundredth of an inch in the length of the edge is determined.

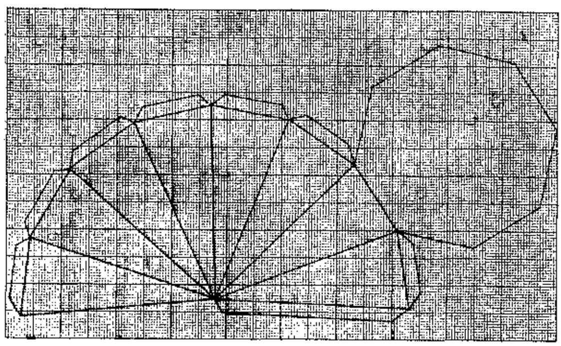


Next I impress upon the class the fact that they must draw to scale with greater accuracy than they have been accustomed to, the error allowed being one-hundredth of an inch. To insure such accuracy the diagrams are drawn upon a good quality of cross section paper (Eugene Dietzgen Co. No. 350). This is a tracing paper and therefore allows blue-prints to be made from the original diagram, a feature which I strongly recommend because of its duplicating powers. If an especially good diagram is submitted by a pupil the instructor can make several copies of the same for purposes of instruction or exhibition. Then, too, I find the easiest way to check up a diagram of some member of the class is to superimpose my own drawing upon that of the pupil

[2]See end of article.

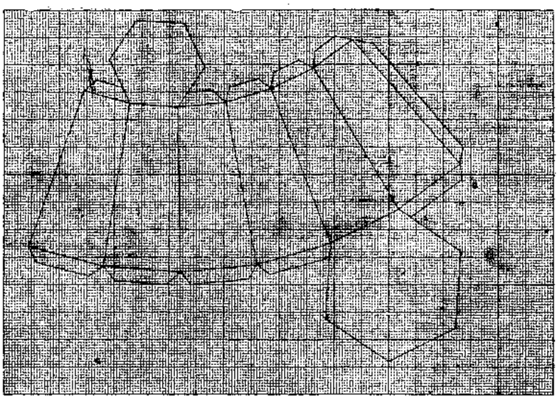
and if both coincide throughout I accept the pupil's drawing. This method is possible because the paper is semi-transparent.

After the diagram has been presented and approved the pupil uses it as his pattern and the real construction of the cube is begun. It is made out of a good drawing paper (Eugene Dietzgen Co. No. 5 "Napoleon"). The pattern is placed upon the blank drawing paper and by means of a well sharpened pencil (4 or 6H) the necessary points are transferred by a slight downward pressure. The points are then joined by the required lines and thus the entire diagram is transferred. Perhaps you are wondering why I insisted upon a good diagram on the cross section paper, only to be transferred to the drawing paper. The reason is simple.



If the first model presented is not accepted, which is often the case with beginners, the time and effort required to make the original drawing is not lost. It is only a matter of using the pattern a second time. With the aid of a pair of scissors the diagram is cut out along the boundary lines and where folding along a line is necessary a knife is used to make a shallow cut. It has been my experience that the use of flaps in joining two edges together is better and makes a neater model than the use of gummed paper. Either glue or library paste may be employed as an adhesive but I find that the paste has a tendency to dry out in time and the model comes apart. The cube is now complete and if satisfactory is accepted and the pupil assigned a new construction exercise. Thus you have the "life-history" of a model cube.

Of the thirty exercises that I have designed for this construction work in solid geometry, you have fifteen before you and you may judge for yourselves the type of mental effort required for their solution. It is plain to be seen that some are more difficult than others and naturally I give the former to the better students. In a few instances the determination of the formula for solving the unknown edge is so complex that I give the pupil the formula, but he still solves the approximate answer himself. I have in mind Ex. 21: Required to construct a regular icosahedron whose volume is ten cubic inches. In this exercise I give the formula, $V=5e^3/12\ (\sqrt{5}+3)$ from which the approximate result for e is obtained when $V=10$. Occasionally I have in my class a pupil who has studied trigonometry or who is taking it along with his solid geometry and in that case I ask him to find his own formula, and it is indeed gratifying to see the light of success in the eye of the young computer when he exclaims, like Archimedes of old, "Eureka!" (I have found it!). I sometimes think that we make our courses in mathematics too cut and dried, not giving sufficient opportunity for the exceptional pupil to cut his own path into the beautiful field of number relations.



After each member of the class has constructed eight exercises out of the thirty he is asked to design and construct two original models, working out any ideas he may have or be interested in. The results obtained are, in my opinion, very profitable, for it allows the student to express himself mathematically, to use his own mathematical wings, as it were. It is amazing what wonderful concoctions are fabricated and what gifts are laid upon the altar of exact thinking.

In conclusion let me summarize some of the values to be realized from the course of construction exercises as outlined above. (1) In the first place the class really enjoys building mathematical models. It ministers to the building instinct found in every normal boy and girl. As the little child delights in making houses out of blocks so the larger child finds pleasure in giving tangible form to his ideas. When once you have your pupil liking his subject you have given him a decided impetus toward progress in it.

(2) The value of fixing the meaning of geometrical terms cannot be overestimated. I firmly believe that a student definitely knows what a truncated right triangular prism is after he has made one and that he will retain that knowledge for a greater length of time. Furthermore, he receives a training in exact measurement and constructs according to specifications, lessons he will not soon forget.

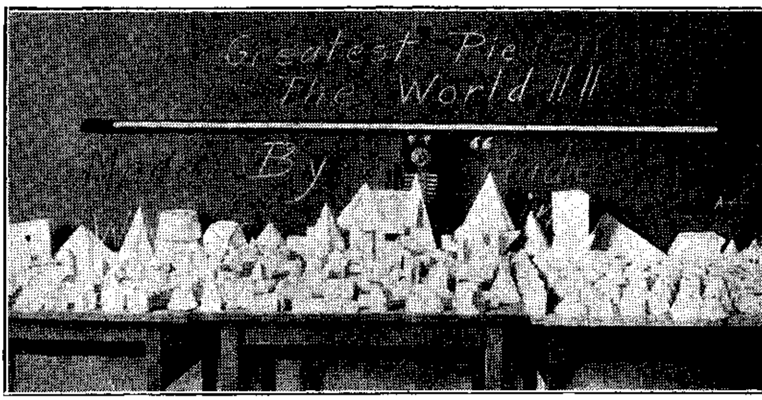


FIGURE 1. Three hundred models made by pupils at the New Castle High School. On the long strip of white paper in the background is written Shanks' value of π, to 707 decimal places. The strip is nearly seven feet long.

(3) All models have a volume of ten cubic inches, making them equivalent solids. This offers an excellent opportunity to train the mind in space relations. I find that my students have a much better conception of size and volume, of three dimensions, if you please, which is a most important conception in the study of solid geometry.

(4) We are beginning to appreciate the problem of individual differences in classroom instruction. Here is ample room to apply this new idea in pedagogy. In assigning the various exercises you can give the good student the more difficult models and the slow plodder the easier ones. Both types are kept busy and up to their ability.

Lest in my enthusiasm I have caused you to believe that I have substituted for the logic of Euclid a course in model construction, let me emphatically state that such is not the case. A rough estimate of the time consumed in the work as outlined would place it at from fifteen to twenty per cent of the time of the entire course in solid geometry.

If I have given you food for thought my mission is fulfilled. But before I leave you I want to give tribute where tribute is due. How far that "candle throws his beams!" The candle is Kepler, and the beam, "to measure is to know."

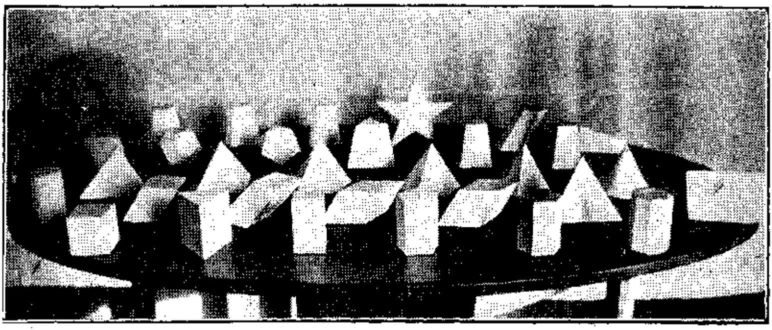


FIGURE 2. The thirty models made by pupils at the Proviso Township High School.

CONSTRUCTION EXERCISES IN SOLID GEOMETRY.

1. Required to construct a cube whose volume is ten cubic inches.

2. Required to construct a right prism whose base is an equilateral triangle, altitude three inches, and volume ten cubic inches.

4. Required to construct a right prism whose base is a regular pentagon, altitude three inches, and volume ten cubic inches.

7. Required to construct a right circular cylinder whose altitude is three inches and volume ten cubic inches.

11. Required to construct a parallelopiped oblique in three directions all of whose faces are congruent polygons, volume ten cubic inches, and all of whose face angles at a trihedral angle are sixty degrees.

12. Required to construct a regular tetrahedron whose volume is ten cubic inches.

14. Required to construct a regular quadrangular pyramid whose altitude is three inches and volume ten cubic inches.

17. Required to construct a regular octagonal pyramid whose altitude is three inches and volume ten cubic inches.

19. Required to construct a regular octahedron whose volume is ten cubic inches.

20. Required to construct a regular dodecahedron whose volume is ten cubic inches.

21. Required to construct a regular icosahedron whose volume is ten cubic inches.

25. Required to construct a frustum of a regular hexagonal pryamid whose volume is ten cubic inches, area of upper base two square inches, area of lower base five square inches.

27. Required to construct a frustum of a right circular cone whose volume is ten cubic inches, area of upper base two square inches, area of lower base, five square inches.

29. Required to construct an oblique prism whose base is a regular hexagon, altitude three inches, volume ten cubic inches, and one of whose dihedral angles between the base and a lateral face is sixty degrees.

30. Required to construct a truncated right triangular prism whose lateral edges are one, two, and three inches, respectively, whose base is an equilateral triangle, and whose volume is ten cubic inches.