CONSTRUCTIONS OF COMPLEX HADAMARD MATRICES VIA TILING ABELIAN GROUPS

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Abstract. Applications in quantum information theory and quantum tomography have raised current interest in *complex Hadamard matrices*. In this note we investigate the connection between tiling of Abelian groups and constructions of complex Hadamard matrices. First, we recover a recent very general construction of complex Hadamard matrices due to Dita [2] via a natural tiling construction. Then we find some necessary conditions for any given complex Hadamard matrix to be *equivalent* to a Dita-type matrix. Finally, using another tiling construction, due to Szabó [8], we arrive at *new parametric families* of complex Hadamard matrices of order 8, 12 and 16, and we use our necessary conditions to prove that these families do not arise with Dita's construction. These new families *complement the recent catalogue* [10] of complex Hadamard matrices of small order.

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1. Introduction

Hadamard matrices, real or complex, appear in various branches of mathematics such as combinatorics, Fourier analysis and quantum information theory. Various applications in quantum information theory have raised recent interest in *complex* Hadamard matrices.

One example, taken from quantum tomography, is the problem of existence of *mutually unbiased bases*, which is known to be a question on the existence of certain complex Hadamard matrices. The existence of $d+1$ such bases is known for any prime power dimension d, but the problem remains open for all non prime power dimensions, even for $d = 6$ (for a more detailed exposition of this example see the Introduction of [10]).

Other important questions in quantum information theory, such as construction of teleportation and dense coding schemes, are also based on complex Hadamard matrices. Werner in [11] proved that the construction of *bases of maximally entangled states*, *orthonormal bases of unitary operators*, and *unitary depolarizers* are all equivalent in the sense that a solution to any of them leads to a solution to any other, as well as to a corresponding scheme of teleportation and dense coding. A general construction procedure for orthonormal bases of unitaries, involving complex Hadamard matrices, is also presented in [11].

On the one hand, it seems to be impossible to give any complete, or satisfactory *characterization* of complex Hadamard matrices of high order. On the other hand, we can hope to give fairly *general constructions* producing large families of Hadamard matrices, and we can also hope to characterize Hadamard matrices of small order (currently a full characterization is available only up to order 5; very recently the self-adjoint complex Hadamard matrices

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of order 6 have also been classified in [1]). A recent paper by Dita [2] describes a general construction which leads to parametric families of complex Hadamard matrices in composite dimensions. Another recent paper by Tadej and Życzkowski [10] gives an (admittedly incomplete) *catalogue* of complex Hadamard matrices of *small order* (up to order 16).

The aim of this note is to show how tiling constructions of Abelian groups can lead to constructions of complex Hadamard matrices, and in this way to complement the catalogue of [10] with new parametric families. In particular, we first show how Dita's construction can be arrived at via a natural tiling construction (this part does not lead to new results, but it is an instructive example of how tiling and Hadamard matrices are related). Second, we observe some regularities satisfied by all Dita-type matrices, and thus arrive at an effective method to decide whether a given complex Hadamard matrix is of Dita-type. Then we use a combinatorial tiling construction due to Szabó [8] to produce Hadamard matrices *not of Dita-type*, and complement the catalogue of [10] with *new parametric families* of order 8, 12 and 16.

2. Recovering Dita's construction via tiling

This section describes a beautiful example of how seemingly distant parts of mathematics are related to each other. A short history of the construction is as follows.

Fuglede's conjecture states that a set in a locally compact Abelian group (originally in $\mathbb{R}^d$) is spectral (a notion to be defined below) if and only if it tiles the group by translation. (We remark that this conjecture has been disproved in dimensions 3 and higher [3, 5] but remains open in dimensions 1 and 2.) While tiling is a 'natural' notion, spectrality is less so, and it is closely related to complex Hadamard matrices, as explained below. One approach to tackle the conjecture was to look for 'canonical' *constructions for tiling Abelian groups*, and see whether similar *constructions* work also *for spectral sets*. This, indeed, turned out to be the case for a very general construction (see Proposition 2.1 below), and then this general scheme of producing spectral sets *leads directly to Dita's construction of complex Hadamard matrices.*

First, let us recall the most general form of Dita's construction, formula (12) in [2] (his subsequent results on parametric families of complex Hadamard matrices with some free parameters follow easily from this formula, as described very well in Proposition 3 and Theorem 2 of [2]).

$$K := \begin{bmatrix} m_{11}N_1 & \cdot & \cdot & m_{1k}N_k \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m_{k1}N_1 & \cdot & \cdot & m_{kk}N_k \end{bmatrix} \tag{1}$$

In this formula Dita assumes m_{ij} to be the entries of any $k \times k$ complex Hadamard matrix M, while N_j are any $n \times n$ complex Hadamard matrices (possibly different from each other). Then he shows that K is a complex Hadamard matrix of order kn. While this construction seems fairly natural, it may be remarkable that it has only been discovered very recently [2] (we remark that an earlier, less general construction was given in [4]), and that it is so powerful that it leads to most of the parametric families included in [10].

Definition 2.1. *A complex Hadamard matrix K is called Dita-type if it is equivalent to a matrix arising with formula* (1) *(we use the standard notion of equivalence of Hadamard* *matrices (see e.g.* [10]*), i.e.* K_1 *and* K_2 *are equivalent if* $K_1 = D_1P_1K_2P_2D_2$ *with unitary diagonal matrices* D_1, D_2 *and permutation matrices* P_1, P_2*.)*

Let us now turn to the definition of spectral sets and tiles, and see how Dita's construction arises naturally via tiling of Abelian groups.

Definition 2.2. *Let* G *be a locally compact Abelian group, and* $\widehat{G}$ *its dual group (the group of characters). An open set* $T \subset G$ *is said to be a translational tile if there is a disjoint union of some translated copies of* T *covering the whole group* G *up to gaps of measure zero (w.r.t Haar measure).* $T \subset G$ *is spectral if it has a spectrum* $S \subset \widehat{G}$ *such that the characters* $\{\gamma | \gamma \in S\}$ *restricted to* T *form an orthogonal basis of* $L^2(T)$*. Then* (T, S) *is called a spectral pair.*

Remark 1. Let $\mathbb{Z}_N$ denote the cyclic group of N elements. If G is the Abelian group $\mathbb{Z}_N^d$, or $\mathbb{Z}^d$, or $\mathbb{T}^d$ then we identify elements of the group with column vectors $\mathbf{g}$ of length d (with entries $g_j = \frac{k}{N}$ $(0 \leq k \leq N-1)$, or $g_j \in \mathbb{Z}$, or $g_j \in \mathbb{T}$, respectively). Also, we identify characters with row vectors $\mathbf{h}$ of length d (with entries $h_j = m$ $(0 \leq m \leq N-1)$, or $h_j \in \mathbb{T}$, or $h_j \in \mathbb{Z}$, respectively; it is also convenient to identify $\mathbb{T}$ with the interval $[0, 1)$). The action of a character is then described conveniently as

$$\gamma_h(g) = e^{2\pi i \langle \mathbf{h}, \mathbf{g} \rangle} \qquad \mathbf{h} \in \widehat{G}, \mathbf{g} \in G. \tag{2}$$

These notations will be particularly useful to describe how spectral pairs lead to complex Hadamard matrices. Readers unfamiliar with this notation are advised to check the concrete numerical Example 1 in Section 3.

In the case of $G = \mathbb{Z}_N^d$ or $\mathbb{Z}^d$, if a finite set $T = \{\mathbf{t}_1, \dots, \mathbf{t}_r\} \subset G$ has spectrum $S = \{\mathbf{s}_1, \dots, \mathbf{s}_r\} \subset \widehat{G}$ then by orthogonality

$$\sum_{k=1}^{r} e^{2\pi i \langle \mathbf{s}_i - \mathbf{s}_j, \mathbf{t}_k \rangle} = \sum_{k=1}^{r} \overline{\gamma}_{\mathbf{s}_j}(\mathbf{t}_k) \gamma_{\mathbf{s}_i}(\mathbf{t}_k) = r\delta_{ij}, \tag{3}$$

so the matrix $[H]_{i,k} := \left(e^{2\pi i \langle \mathbf{s}_i, \mathbf{t}_k \rangle}\right)$ is an $r \times r$ *complex Hadamard matrix* (i.e. a matrix with complex entries of absolute value 1, such that the rows (and hence the columns) are orthogonal). We call the matrix of exponents $[\log H]_{i,k} = \langle \mathbf{s}_i, \mathbf{t}_k \rangle$ a *log-Hadamard matrix* (note that there is a factor 2π difference between [6] and [10] as to the terminology 'log-Hadamard matrix'; here we adhere to the one used in [6]). Finally, we have arrived at the conclusion that S is a spectrum of T if and only if the matrix product ST is log-Hadamard. Accordingly, in the case of $G = \mathbb{T}^d$ we find it convenient to extend the definition of spectrality to finite sets, too (finite sets are not open and have measure zero in this case, so the original definition is meaningless). □

Definition 2.3. *We say that a finite set* $T \subset \mathbb{T}^d$ *is spectral if there exists a set* $S \subset \mathbb{Z}^d$ *(as row vectors) such that* ST *is log-Hadamard.*

We now recall Proposition 2.2 from [6] (the point is that the analogous construction is *natural* for tiles (see Proposition 2.1 in [6]), and that is how this construction was discovered for spectral sets).

Proposition 2.1. *Let* G *be a finite Abelian group, and* $H \leq G$ *a subgroup. Let* $T_1, T_2, \dots T_k \subset H$ *be subsets of* H *such that they share a common spectrum in* $\widehat{H}$*; i.e. there exists a set* $L \subset \widehat{H}$ *such that* L *is a spectrum of* T_m *for all* $1 \leq m \leq k$*. Consider any spectral pair* (Q, S) *in the* *factor group G/H, with $|Q| = k$, and take arbitrary representatives $\mathbf{q}_1, \mathbf{q}_2, \dots \mathbf{q}_k$ from the cosets of H corresponding to the set Q. Then the set $\Gamma := \cup_{m=1}^{k}(\mathbf{q}_m + T_m)$ is spectral in the group G.*

Proof. The proof is trivial, although the notations are somewhat cumbersome. We will simply construct a spectrum $\Sigma \subset \widehat{G}$ for Γ. Let n denote the number of elements in each T_m (they necessarily have the same number of elements as there exists a common spectrum), and $\mathbf{t}_r^m$ ($r = 1, \dots n$ and $m = 1, \dots k$) the rth element of T_m. By assumption, there exist characters $\mathbf{l}_j \in \widehat{H}$ ($j = 1, \dots n$) such that the matrices $[A_m]_{j,r} := [\mathbf{l}_j(\mathbf{t}_r^m)]$ are $n \times n$ complex Hadamard for each m. Let $\tilde{\mathbf{l}}_j$ denote any extension of $\mathbf{l}_j$ to a character of G (such extensions always exist, although not unique). Also, the elements $\mathbf{s}_1, \dots, \mathbf{s}_k$ of $S \subset \widehat{G/H}$ can be identified with characters $\tilde{\mathbf{s}}_i \in \widehat{G}$ which are constant on cosets of H. Then we consider the product characters $\tilde{\mathbf{s}}_i\tilde{\mathbf{l}}_j$ and let $\Sigma := \{\tilde{\mathbf{s}}_i\tilde{\mathbf{l}}_j\}_{i,j}$ where $i = 1, \dots, k$ and $j = 1, \dots, n$. We claim that Σ is a spectrum of Γ. For each $m = 1, \dots k$ let $D_{L\mathbf{q}_m}$ denote the $n \times n$ diagonal matrix with entries $[D_{L\mathbf{q}_m}]_{j,j} = \tilde{\mathbf{l}}_j(\mathbf{q}_m)$. Then, for fixed i and m the product characters $\tilde{\mathbf{s}}_i\tilde{\mathbf{l}}_j$ ($j = 1, \dots, n$) restricted to the set $\mathbf{q}_m + T_m = \{\mathbf{q}_m + \mathbf{t}_1^m, \dots, \mathbf{q}_m + \mathbf{t}_n^m\}$ simply give the $n \times n$ matrix

$$B^{i,m} := \tilde{\mathbf{s}}_i(\mathbf{q}_m) D_{Lq_m} A_m, \tag{4}$$

because the entries are given as $[B^{i,m}]_{j,r} = \tilde{\mathbf{s}}_i\tilde{\mathbf{l}}_j(\mathbf{q}_m + \mathbf{t}_r^m) = \tilde{\mathbf{s}}_i(\mathbf{q}_m)\tilde{\mathbf{l}}_j(\mathbf{q}_m)\tilde{\mathbf{l}}_j(\mathbf{t}_r^m)$. This means that the characters $\tilde{\mathbf{s}}_i\tilde{\mathbf{l}}_j \in \Sigma$ restricted to Γ will give the $nk \times nk$ block matrix

$$H := \begin{bmatrix} B^{1,1} & \cdot & \cdot & B^{1,k} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ B^{k,1} & \cdot & \cdot & B^{k,k} \end{bmatrix}. \tag{5}$$

Now, observe that each block $B^{i,m}$ is given as a product $\tilde{\mathbf{s}}_i(\mathbf{q}_m) D_{Lq_m} A_m$ where $N_m := D_{Lq_m} A_m$ is a complex Hadamard matrix (because A_m is such and D_{Lq_m} is a unitary diagonal matrix), and $\tilde{\mathbf{s}}_i(\mathbf{q}_m)$ is the entry of a $k \times k$ complex Hadamard matrix by the assumption that S is a spectrum of Q. Therefore H is seen to be a complex Hadamard matrix arising directly with formula (1), and hence Σ is indeed a spectrum of Γ. □

Remark 2. We see that *the constructed spectral pair* (Σ, Γ) *gives rise to a Dita-type matrix.* We remark, however, that the set Γ might well have many other spectra than the one constructed in the proof above (and other spectra might produce complex Hadamard matrices not of the Dita-type). There is no efficient algorithm known to list out all the spectra of a given set. □

Remark 3. The above Proposition was quoted verbatim from [6], and remains in the finite group setting. This has the disadvantage that the arising matrices are necessarily of the Butson-type (i.e. containing roots of unity only), and one cannot expect to obtain continuous parametric families of complex Hadamard matrices. However, the same construction works in the infinite setting $G = \mathbb{Z}^d$ or $G = \mathbb{T}^d$, too, and we now present how *every Dita-type matrix arises in this manner.*

Assume that matrices M and N_m ($m = 1, \dots, k$) are given, and K is constructed as in formula (1). We aim to recover K with the construction of Proposition 2.1.

Let $G = \mathbb{T}^d$ where $d = n + k$, and consider the subgroup $H_1 = \mathbb{T}^n$ (subgroup of vectors with last k coordinates 0), and $G/H_1 = H_2 = \mathbb{T}^k$ (vectors with first n coordinates 0). Then $G = H_1 \times H_2$.

Let $T_m = \log N_m$ denote the matrix of the exponents of the entries of N_m, i.e. $[N_m]_{i,j} = e^{2\pi i [T_m]_{i,j}}$ (each T_m is defined *mod* 1). Let $\tilde{T}_m \subset H_1$ denote the set of vectors consisting of the columns of the log-Hadamard matrix T_m extended by 0's in the last k coordinates. Then each $\tilde{T}_m$ is spectral in H_1 and a common spectrum of them is given by

$$E_1 := \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}. \tag{6}$$

(This is because $E_1 \tilde{T}_m = T_m$ is log-Hadamard for each m.) Also, let $Q := \log M$, and $\tilde{\mathbf{q}}_j \in H_2$ denote the jth column of the log-Hadamard matrix Q extended by 0's in the first n coordinates. Then the set $\tilde{Q} = \{\tilde{\mathbf{q}}_1, \dots \tilde{\mathbf{q}}_k\} \subset H_2$ is spectral in $H_2 = G/H_1$ with spectrum

$$E_2 := \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{7}$$

(This is because $E_2 \tilde{Q} = Q$ is log-Hadamard.)

As in Proposition 2.1 above we define

$$\Gamma := \cup_{m=1}^{k} (\mathbf{q}_m + T_m) = \left[\begin{array}{cccc|cccc|c|cccc} & T_1 & & & & T_2 & & & \dots & & T_k & & \\ \hline | & | & \dots & | & | & | & \dots & | & \dots & | & | & \dots & | \\ \mathbf{q}_1 & \mathbf{q}_1 & \dots & \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_2 & \dots & \mathbf{q}_2 & \dots & \mathbf{q}_k & \mathbf{q}_k & \dots & \mathbf{q}_k \\ | & | & \dots & | & | & | & \dots & | & \dots & | & | & \dots & | \end{array}\right] \tag{8}$$

Then, the spectrum Σ constructed in the proof of Proposition 2.1 takes the form '$\Sigma = E_1 + E_2$', i.e.

$$\Sigma := \left[\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \\ 0 & 0 & \dots & 1 & 1 & 0 & \dots & 0 \\ \hline 1 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \\ 0 & 0 & \dots & 1 & 0 & 1 & \dots & 0 \\ \hline \vdots & & & & \vdots & & & \\ \hline 1 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 1 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 1 \end{array}\right] \tag{9}$$

Finally, the $nk\times nk$ log-Hadamard matrix arising from the spectral pair (Γ,Σ) is the product $\Sigma\Gamma$, and it gives back exactly the log-Hadamard matrix $\log K$, as desired.

$$\Sigma\Gamma=\log K=\begin{bmatrix} q_{11}+T_1 & q_{12}+T_2 & \dots & q_{1k}+T_k\\ q_{21}+T_1 & q_{22}+T_2 & \dots & q_{2k}+T_k\\ \vdots & \vdots & & \vdots\\ q_{k1}+T_1 & q_{k2}+T_2 & \dots & q_{kk}+T_k\end{bmatrix} \tag{10}$$

□

3. Other tiling constructions yielding new families of complex Hadamard matrices

Once the connection between tiling and complex Hadamard matrices has been noticed, it is natural to look for tiling constructions other than that of Proposition 2.1 above, in the hope of producing new complex Hadamard matrices not of the Dita-type. Furthermore, when a new complex Hadamard matrix M is discovered, the 'linear variation of phases' method of [10] gives hope to find new parametric affine families of complex Hadamard matrices stemming from M. This is exactly the route we are going to follow in this section. First, we show how *a tiling method of Szabó* [8] *leads to complex Hadamard matrices not of the Dita-type.* Then, *stemming from these matrices*, we produce *new parametric families of order 8, 12, and 16* which have not been present in the literature so far and which complement the recent catalogue [10].

It turns out that the (tiling analogue of) the construction of Proposition 2.1 is so general that it is not trivial to produce tilings which do not arise in such manner. In fact, it was once asked by Sands [7] whether *every* tiling of finite Abelian groups is such that one of the factors is contained in a subgroup (note that such tilings correspond to the special case $Q=G/H$ in the tiling analogue of Proposition 2.1). This question was then answered in the negative by *a construction of Szabó* [8], which we now turn to.

Assume $G=\mathbb{Z}_{p_1q_1}\times\mathbb{Z}_{p_2q_2}\times\mathbb{Z}_{p_3q_3}$ where $p_j,q_j\geq 2$. The idea of Szabó is to take the obvious tiling $G=A+B$ where

$$A=\{0,\frac{1}{p_1q_1},\dots\frac{p_1-1}{p_1q_1}\}\times\{0,\frac{1}{p_2q_2},\dots\frac{p_2-1}{p_2q_2}\}\times\{0,\frac{1}{p_3q_3},\dots\frac{p_3-1}{p_3q_3}\} \tag{11}$$

and $B=\{0,\frac{1}{q_1},\frac{2}{q_1},\dots\frac{q_1-1}{q_1}\}\times\{0,\frac{1}{q_2},\frac{2}{q_2},\dots\frac{q_2-1}{q_2}\}\times\{0,\frac{1}{q_3},\frac{2}{q_3},\dots\frac{q_3-1}{q_3}\}$ and then modify the grid B by pushing three grid-lines in different directions (see [8] for details; we do not describe the details here as we do not directly use this construction in this paper, it serves only as a guide to our spectral analogue below). Here we use the *analogous construction for spectral sets* which we now describe in detail (it may be easier to follow the general construction by looking at the specific Example 1 below).

Consider the set A above. By formula (3) a set $S\subset\widehat{G}$ is a spectrum of A if and only if $|S|=|A|$ and $S-S\subset Z_A\cup\{0\}:=\{\mathbf{r}\in\widehat{G}:\widehat{\chi}_A(\mathbf{r})=0\}\cup\{0\}$ (χ_A denotes the indicator function of A, and the Fourier transform $\widehat{\chi}_A$ is evaluated at some $\mathbf{r}\in\widehat{G}$ as $\widehat{\chi}_A(\mathbf{r})=\sum_{\mathbf{a}\in A}e^{2\pi i\langle\mathbf{r},\mathbf{a}\rangle}$). For a more detailed discussion of this fact see e.g. [6]. Recall that $\widehat{G}$ is identified with 3-dimensional row vectors. It is clear that if $\mathbf{r}=(r_1,r_2,r_3)\in\widehat{G}$ is such that q_1 divides r_1 and $r_1\neq 0$ then $\widehat{\chi}_A(\mathbf{r})=0$ (all sub-sums become 0 with fixing the second and third coordinate and letting the first one vary in A). Similarly, if $q_2|r_2 \neq 0$ or $q_3|r_3 \neq 0$ then $\widehat{\chi}_A(\mathbf{r}) = 0$. Therefore the grid

$$S = \{0, q_1, \dots (p_1-1)q_1\} \times \{0, q_2, \dots (p_2-1)q_2\} \times \{0, q_3, \dots (p_3-1)q_3\} \tag{12}$$

is a spectrum of A. Using an analogous idea to that of Szabó we now modify this grid.

Consider the grid-line $L_1 := \{\{0, q_1, \dots (p_1-1)q_1\} \times \{q_2\} \times \{0\}$ and change it to $L_1' := \{1, q_1+1, \dots (p_1-1)q_1+1\} \times \{q_2\} \times \{0\}$ (adding $+1$ to the first coordinates). Similarly, change $L_2 := \{0\} \times \{0, q_2, \dots (p_2-1)q_2\} \times \{q_3\}$ to $L_2' := \{0\} \times \{1, q_2+1, \dots (p_2-1)q_2+1\} \times \{q_3\}$, and change $L_3 := \{q_1\} \times \{0\} \times \{0, q_3, \dots (p_3-1)q_3\}$ to $L_3' := \{q_1\} \times \{0\} \times \{1, q_3+1, \dots (p_3-1)q_3+1\}$. It is easy to see that

$$S' := S \cup (L_1' \cup L_2' \cup L_3') \setminus (L_1 \cup L_2 \cup L_3) \tag{13}$$

is still a spectrum of A. Indeed, for any $\mathbf{r} \in S' - S'$ it still holds that either the first coordinate is divisible by q_1 or the second by q_2 or the third by q_3. Then the spectral pair (A, S') gives rise to a complex Hadamard matrix of size $p_1p_2p_3$. Below we will apply this construction in the groups $G_1 = \mathbb{Z}_{2\cdot 2} \times \mathbb{Z}_{2\cdot 2} \times \mathbb{Z}_{2\cdot 2}$, $G_2 = \mathbb{Z}_{2\cdot 2} \times \mathbb{Z}_{2\cdot 2} \times \mathbb{Z}_{3\cdot 3}$ and $G_3 = \mathbb{Z}_{2\cdot 2} \times \mathbb{Z}_{4\cdot 2} \times \mathbb{Z}_{2\cdot 4}$ (it may be instructive to see the step-by-step numerical exposition of the construction in Example 1 in group G_1 below).

We will then prove that these matrices are *not of the Dita-type.* (It would be very interesting to see a proof of a general statement that all matrices arising with the above construction are non-Dita-type.) As a result we will conclude that these matrices have not been included in the catalogue [10].

Remark 4. We can see from the construction above that the *size* of the arising matrix is $p_1p_2p_3$, while the numbers q_1, q_2, q_3 are chosen arbitrarily to determine the group we are working in. It is not clear whether different choices of q_1, q_2, q_3 lead to non-equivalent Hadamard matrices. In this paper we only list the three examples for which the dimension is not greater than 16 (as in [10]) and for which we can *prove* that the arising matrices are new, i.e. non-equivalent to any matrix listed in [10]. □

Example 1. Let us follow the construction above, step by step, in $G_1 = \mathbb{Z}_{2\cdot 2} \times \mathbb{Z}_{2\cdot 2} \times \mathbb{Z}_{2\cdot 2} = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$.

By (11) we take $A = \{0, \frac{1}{4}\} \times \{0, \frac{1}{4}\} \times \{0, \frac{1}{4}\}$. This is a Cartesian product, each element of which is a 3-dimensional vector composed of 0's and $\frac{1}{4}$'s. We list out the elements in lexicographical order as

$$A = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \tag{14}$$

where the columns represent the elements of $A \subset G_1$, in accordance with our notation introduced earlier. (The order of the elements is up to our choice, but a permutation of the elements only corresponds to a permutation of the columns of the matrix S_8 below.)

Then, by equation (12) we have $S = \{0,2\} \times \{0,2\} \times \{0,2\}$, which we list out (also in lexicographical order) as

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} \tag{15}$$

Now, S is a spectrum of A, therefore the product SA already gives a log-Hadamard matrix but we do not take that matrix (which *is Dita-type*, as can be verified by the reader), but modify the set S first. The grid-line L_1 in S is given as $L_1 = \{0,2\} \times \{2\} \times \{0\} = \{(0,2,0);(2,2,0)\}$. This we replace by $L_1' = \{(1,2,0);(3,2,0)\}$. Similarly, the grid-line $L_2 = \{(0,0,2);(0,2,2)\}$ is replaced by $L_2' = \{(0,1,2);(0,3,2)\}$ and finally $L_3 = \{(2,0,0);(2,0,2)\}$ by $L_3' = \{(2,0,1);(2,0,3)\}$. Therefore, by (13) we get

$$S' = S \cup (L_1' \cup L_2' \cup L_3') \setminus (L_1 \cup L_2 \cup L_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 0 \end{bmatrix} \tag{16}$$

(Once again, the order of the elements of S' is arbitrary, and we take lexicographical order.) The point is, as explained above in the general description of this construction, that the set S' is still a spectrum of A. Therefore the matrix product $S'A$ is a log-Hadamard matrix (we reduce the entries *mod* 1 because the integer part of an entry plays no role after exponentiation; e.g. $\frac{5}{4} \equiv \frac{1}{4}$) given by:

$$S'A = \log S_8 = \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 & 2 & 1 & 3 \\ 0 & 2 & 3 & 1 & 0 & 2 & 3 & 1 \\ 0 & 0 & 2 & 2 & 1 & 1 & 3 & 3 \\ 0 & 1 & 0 & 1 & 2 & 3 & 2 & 3 \\ 0 & 3 & 0 & 3 & 2 & 1 & 2 & 1 \\ 0 & 2 & 2 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 2 & 3 & 3 & 1 & 1 \end{bmatrix} \tag{17}$$

with the corresponding Hadamard matrix given by

$$S_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mathbf{i} & -\mathbf{i} & 1 & -1 & \mathbf{i} & -\mathbf{i} \\ 1 & -1 & -\mathbf{i} & \mathbf{i} & 1 & -1 & -\mathbf{i} & \mathbf{i} \\ 1 & 1 & -1 & -1 & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} \\ 1 & \mathbf{i} & 1 & \mathbf{i} & -1 & -\mathbf{i} & -1 & -\mathbf{i} \\ 1 & -\mathbf{i} & 1 & -\mathbf{i} & -1 & \mathbf{i} & -1 & \mathbf{i} \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} \end{bmatrix} \tag{18}$$

□

Having described *how to produce* the matrix S_8 the remaining questions are whether S_8 is *new (i.e. not already included in the catalogue* [10]*)*, and whether *any parametric family of complex Hadamard matrices* stems from S_8.

We will first proceed to show that S_8 is *not Dita-type* (nor is it its transpose). This is a delicate matter, as not many criteria are known to decide inequivalence of Hadamard matrices. The Haagerup condition with the invariant set $\Lambda := \{h_{ij}\overline{h}_{kj}h_{kl}\overline{h}_{il}\}$ (see [4] and Lemma 2.5 in [10]) cannot be used here. Also, the elegant characterization of equivalence classes of Kronecker products of Fourier matrices [9] does not apply to S_8. The 'regular' structure of a Dita-type matrix must be exploited in some way. The key observation relies on the following

Definition 3.1. *Let L be an $N \times N$ real matrix. For an index set $I = \{i_1, i_2, \dots, i_n\} \subset \{1, 2, \dots, N\}$ two rows (or columns) $\mathbf{s}$ and $\mathbf{q}$ are called I-equivalent, in notation $\mathbf{s} \sim_I \mathbf{q}$, if the fractional part of the entry-wise differences $s_i - q_i$ are the same for every $i \in I$ (we need to consider fractional parts as the entries of a log-Hadamard matrix are defined only mod 1). Two rows (or columns) $\mathbf{s}$ and $\mathbf{q}$ are called (d)-n-equivalent if there exist n-element disjoint sets of indices $I_1, \dots, I_d$ such that $\mathbf{s} \sim_{I_j} \mathbf{q}$ for all $j = 1, \dots, d$.*

We have the following trivial observation.

Proposition 3.1. *Let L be an $N \times N$ complex Hadamard matrix. Assume that there exist an index set $I = \{i_1, i_2, \dots, i_n\} \subset \{1, 2, \dots, N\}$ and m different rows (resp. columns) $\mathbf{r}_{s_1}, \dots \mathbf{r}_{s_m}$ in the log-Hadamard matrix $\log L$ such that each two of them are I-equivalent. Let M be any complex Hadamard matrix equivalent to L. Then the same property holds for $\log M$, i.e. there exist an index set $J = \{j_1, j_2, \dots, j_n\} \subset \{1, 2, \dots, N\}$ and m different rows (resp. columns) $\mathbf{r}_{k_1}, \dots \mathbf{r}_{k_m}$ such that each two of them are J-equivalent. (Of course, the index sets I and $\{s_1, \dots s_m\}$ might not be the same as J and $\{k_1, \dots k_m\}$.)*

Proof. It follows from the definition of the equivalence of Hadamard matrices that $\log M$ is obtained from $\log L$ by permutation of rows and columns, and addition of constants to rows and columns. It is clear that such operations preserve the existing equivalences between rows and columns (with the index sets being altered according to the permutations used). □

The essence of the proposition is that *"existing equivalences between rows and columns are retained"*. The next main point is that there are many equivalences among the rows of a Dita-type matrix and we will see that such equivalences are not present in $\log S_8$.

By formula (1), the structure of an $N \times N$ Dita-type matrix D (where $N = nk$) implies for the log-Hadamard matrix $\log D$ that there exists a partition of indices to n-element sets $I_1 = \{1, 2, \dots n\}, \dots, I_k = \{(k-1)n + 1, \dots kn\}$ and k-tuples of rows $R_j = \{\mathbf{r}_j, \mathbf{r}_{j+n} \dots \mathbf{r}_{j+(k-1)n}\}$

$(j = 1, \dots n)$ such that any two rows in a fixed k-tuple are equivalent with respect to any of the I_m's, i.e. $\mathbf{r}_{j+(i-1)n} \sim_{I_m} \mathbf{r}_{j+(s-1)n}$ for all $j = 1, \dots n$, and $i, s, m = 1, \dots k$. In other words, in any k-tuple R_j any two rows are (k)-n-equivalent with respect to the I_m's. We will use the terminology *(k)-n-Dita-type* for such matrices D. Naturally, the same property holds for the transposed of a (k)-n-Dita-type matrix, with the role of rows and columns interchanged.

This observation makes it possible to prove the following

Proposition 3.2. *S_8 and its transposed are not Dita-type.*

Proof. The matrix size being 8×8 the only possible values for n are 2 and 4 (with k being 4 and 2, respectively). Therefore we only need to check existing (2)-4-equivalences and (4)-2-equivalences in $\log S_8$ and its transposed.

First, let us assume that $n = 4, k = 2$ and look for (2)-4-equivalences among the rows of $\log S_8$. If S_8 were (2)-4-Dita type, there should be a partition of indices to two 4-element sets I_1, I_2 such that in $\log S_8$ four pairs of rows are equivalent with respect to I_1, I_2. The first row $\mathbf{r}_1$ of $\log S_8$ consists of zeros only, therefore it must be paired with a row containing only two different values. There is only one such row $\mathbf{r}_7$ and then the index sets must correspond to the position of 0's and 2's in $\mathbf{r}_7$, i.e. $I_1 = \{1, 4, 6, 7\}$ and $I_2 = \{2, 3, 5, 8\}$. However, there should exist *three further pairs of rows* which are equivalent with respect to the same set of indices I_1, I_2. It is easy to check that such pairs do not exist (e.g. the second row $\mathbf{r}_2$ is not (2)-4-equivalent with respect to I_1, I_2 to any other row), and hence S_8 cannot be (2)-4-Dita type.

To check the transposed matrix we interchange the role of rows and columns and see that the first column $\mathbf{c}_1$ of $\log S_8$ (all zeros) should be paired with a column containing two values only. But such column does not exist, therefore $\mathbf{c}_1$ is not (2)-4-equivalent to any other column, and hence the transposed of S_8 cannot be (2)-4-Dita type.

Let us turn to the case $n = 2, k = 4$. If S_8 were (4)-2-Dita type, there should be a partition of indices to four 2-element sets I_1, I_2, I_3, I_4 such that in $\log S_8$ two 4-tuples of rows $R_1 = \{\mathbf{r}_{s_1}, \dots, \mathbf{r}_{s_4}\}$ and $R_2 = \{\mathbf{r}_{s_5}, \dots, \mathbf{r}_{s_8}\}$ are equivalent with respect to I_1, I_2, I_3, I_4. Assume, without loss of generality that $1 \in I_1$ (i.e. $I_1 = \{1, m\}$ for some m) and that $\mathbf{r}_{s_1} = \mathbf{r}_1$. Then $\mathbf{r}_{s_2}, \mathbf{r}_{s_3}, \mathbf{r}_{s_4}$ are I_1-equivalent to $\mathbf{r}_1$ which implies that there should be a 4×2 block of 0's in $\log S_8$ corresponding to R_1 and I_1, i.e. $[\log S_8]_{i,j} = 0$ for all $i \in R_1$ and $j \in I_1$. Such block of 0's does not exist, therefore S_8 is not (4)-2-Dita-type.

In the transposed case there exists such a 2×4 block of zeros, corresponding to the row indices $I_1 = \{1, 7\}$ and column indices $C_1 = \{1, 4, 6, 7\}$. This means that there should be further two-element index sets I_2, I_3, I_4 such that the columns $\{c_1, c_4, c_6, c_7\}$ are equivalent with respect to I_2, I_3, I_4. It is trivial to check that such indices do not exist. This concludes the proof that S_8 and its transposed are not Dita-type. ☐

The significance of this fact is that the *only known 8×8 parametric family of complex Hadamard matrices so far is the one constructed by Dita's method* (see [10]). It is an affine family $F_8^{(5)}(a, b, c, d, e)$ containing 5 free parameters. We have established that this family does not go through S_8, therefore S_8 is indeed *new*. In particular, the matrix S_8 cannot be equivalent to any of the well-known tensor products of Fourier-matrices $F_2 \otimes F_2 \otimes F_2$, $F_4 \otimes F_2$, F_8 which are all contained in the family $F_8^{(5)}(a, b, c, d, e)$.

Now, applying to S_8 the linear variation of phases method of [10] one can hope to obtain new parametric families of complex Hadamard matrices. Indeed, we have been able to obtain[1] the following maximal affine 4-parameter family (the notation is used as in [10], i.e. the symbol $\circ$ denotes the Hadamard product of two matrices $[H_1 \circ H_2]_{i,j} = [H_1]_{i,j} \cdot [H_2]_{i,j}$, and the symbol EXP denotes the entrywise exponential operation $[EXP\ H]_{i,j} = exp([H]_{i,j})$): $S_8^{(4)}(a,b,c,d) = S_8 \circ EXP(\mathbf{i}R_8^{(4)}(a,b,c,d)$, where

$$R_8^{(4)}(a,b,c,d) = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & d & a & a-d & d & \bullet & a-d & a \\ \bullet & d & a & a-d & d & \bullet & a-d & a \\ \bullet & d & d & \bullet & b & b-d & b-d & b \\ \bullet & c & d & c-d & d & c-d & \bullet & c \\ \bullet & c & d & c-d & d & c-d & \bullet & c \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & d & d & \bullet & b & b-d & b-d & b \end{bmatrix} \tag{19}$$

We do not claim that each matrix in $S_8^{(4)}(a,b,c,d)$ is non-Dita-type (in fact, it is not hard to see that the orbit $S_8^{(4)}(a,b,c,d)$ contains the only real 8×8 Hadamard matrix H_8, which *is* Dita-type, so the families $F_8^{(5)}(a,b,c,d,e)$ and $S_8^{(4)}(a,b,c,d)$ *intersect* each other at H_8). However, this is certainly true in a small neighbourhood of S_8 as the set of Dita-matrices is closed.

Example 2. We now turn to $N = 16$ and the group $G_3 = \mathbb{Z}_{2\cdot 2} \times \mathbb{Z}_{4\cdot 2} \times \mathbb{Z}_{2\cdot 4}$ (we leave $N = 12$ last, as the discussion is slightly different there).

The construction described above yields the following matrices:

$$A_{G_3} = \frac{1}{8}\begin{bmatrix} 0&0&0&0&0&0&0&0&2&2&2&2&2&2&2&2 \\ 0&0&1&1&2&2&3&3&0&0&1&1&2&2&3&3 \\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \end{bmatrix}, \tag{20}$$

and we give S'_{G_3} in transposed layout to save space

$$\left(S'_{G_3}\right)^T = \begin{bmatrix} 0&0&0&0&0&0&0&1&2&2&2&2&2&2&2&3 \\ 0&1&3&4&5&6&7&2&0&0&2&4&4&6&6&2 \\ 0&4&4&0&4&0&4&0&1&5&4&0&4&0&4&0 \end{bmatrix}, \tag{21}$$

[1] The authors are grateful to W. Tadej who extended the 3-parameter family $S_8(a,b,c)$ communicated to him.

and the arising log-Hadamard matrix (containing 8th roots of unity):

$$S'_{G_3}A_{G_3} = \log S_{16} = \frac{1}{8}\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&4&1&5&2&6&3&7&0&4&1&5&2&6&3&7\\ 0&4&3&7&6&2&1&5&0&4&3&7&6&2&1&5\\ 0&0&4&4&0&0&4&4&0&0&4&4&0&0&4&4\\ 0&4&5&1&2&6&7&3&0&4&5&1&2&6&7&3\\ 0&0&6&6&4&4&2&2&0&0&6&6&4&4&2&2\\ 0&4&7&3&6&2&5&1&0&4&7&3&6&2&5&1\\ 0&0&2&2&4&4&6&6&2&2&4&4&6&6&0&0\\ 0&1&0&1&0&1&0&1&4&5&4&5&4&5&4&5\\ 0&5&0&5&0&5&0&5&4&1&4&1&4&1&4&1\\ 0&4&2&6&4&0&6&2&4&0&6&2&0&4&2&6\\ 0&0&4&4&0&0&4&4&4&4&0&0&4&4&0&0\\ 0&4&4&0&0&4&4&0&4&0&0&4&4&0&0&4\\ 0&0&6&6&4&4&2&2&4&4&2&2&0&0&6&6\\ 0&4&6&2&4&0&2&6&4&0&2&6&0&4&6&2\\ 0&0&2&2&4&4&6&6&6&6&0&0&2&2&4&4 \end{bmatrix} \tag{22}$$

$\square$

Proposition 3.3. *S_{16} and its transposed are not Dita-type.*

Proof. By checking existing I-equivalences between rows (and columns) it is elementary (but tedious) to show that S_{16} (and its transposed) is *not Dita-type*. To find possible index sets I and I-equivalences between rows (resp. columns, in the transposed case) it is perhaps most convenient to note the position of 0's in $\log S_{16}$ and look for 2×8, 4×4 and 8×2 blocks of 0's as in the last part of the proof concerning S_8. Then each of these I-patterns can be excluded by looking at further rows (resp. columns). $\square$

The significance of this fact, once again, is that *the only known* 16×16 *parametric family so far is the one constructed with Dita's method* (see [10]). It is an affine family $F_{16}^{(17)}(a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,r)$ containing 17 free parameters. We have established that this family does not go through S_{16}. In particular, the matrix S_{16} cannot be equivalent to any of the well-known tensor products of Fourier-matrices $F_2\otimes F_2\otimes F_2\otimes F_2$, $F_4\otimes F_2\otimes F_2$, $F_4\otimes F_4$, $F_8\otimes F_2$, F_{16} which are all contained in the family $F_{16}^{(17)}$.

By applying the linear variation of phases method of [10] we have been able to find the following 11-parameter affine family stemming from S_{16}. Again, we can claim that the members of this family are not Dita-type in a neighbourhood of S_{16}. However, in this case we do not know whether this affine family is maximal or further parameters can be introduced.

$$S_{16}^{(11)}(a,b,c,d,e,f,g,h,i,j,k) = S_{16}\circ EXP(\mathbf{i}R_{16}^{(11)}(a,b,c,d,e,f,g,h,i,j,k)), \quad \text{where} \tag{23}$$

$$R_{16}^{(11)}(a,b,c,d,e,f,g,h,i,j,k) = \tag{24}$$

$$\begin{bmatrix}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & b & b & d & d & b+j & b+j & \bullet & \bullet & b & b & d & d & b+j & b+j \\
\bullet & \bullet & c & c & d & d & c+j & c+j & \bullet & \bullet & c & c & d & d & c+j & c+j \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & b & b & d & d & b+j & b+j & \bullet & \bullet & b & b & d & d & b+j & b+j \\
\bullet & h & h+i & i & \bullet & h & h+i & i & h & \bullet & i & h+i & h & \bullet & i & h+i \\
\bullet & \bullet & c & c & d & d & c+j & c+j & \bullet & \bullet & c & c & d & d & c+j & c+j \\
\bullet & \bullet & i & i & \bullet & \bullet & i & i & g & g & g+i & g+i & g & g & g+i & g+i \\
\bullet & a & k & a & \bullet & a & k & a & \bullet & a & k & a & \bullet & a & k & a \\
\bullet & a & k & a & \bullet & a & k & a & \bullet & a & k & a & \bullet & a & k & a \\
\bullet & h & h+i & i & \bullet & h & h+i & i & h & \bullet & i & h+i & h & \bullet & i & h+i \\
\bullet & f & k & f & \bullet & f & k & f & \bullet & f & k & f & \bullet & f & k & f \\
\bullet & f & k & f & \bullet & f & k & f & \bullet & f & k & f & \bullet & f & k & f \\
\bullet & e & i & e+i & \bullet & e & i & e+i & \bullet & e & i & e+i & \bullet & e & i & e+i \\
\bullet & e & i & e+i & \bullet & e & i & e+i & \bullet & e & i & e+i & \bullet & e & i & e+i \\
\bullet & \bullet & i & i & \bullet & \bullet & i & i & g & g & g+i & g+i & g & g & g+i & g+i
\end{bmatrix}$$

Example 3. Finally, we turn to the case $N = 12$ and the group $G_2 = \mathbb{Z}_{2\cdot 2} \times \mathbb{Z}_{2\cdot 2} \times \mathbb{Z}_{3\cdot 3}$. Here our construction yields the following matrices:

$$A_{G_2} = \frac{1}{36}\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 9 & 9 & 9 & 9 & 9 & 9 \\
0 & 0 & 0 & 9 & 9 & 9 & 0 & 0 & 0 & 9 & 9 & 9 \\
0 & 4 & 8 & 0 & 4 & 8 & 0 & 4 & 8 & 0 & 4 & 8
\end{bmatrix} \tag{25}$$

and S'_{G_2} in transposed layout

$$\left(S'_{G_2}\right)^T = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 3 \\
0 & 0 & 1 & 2 & 3 & 2 & 0 & 0 & 0 & 2 & 2 & 2 \\
0 & 6 & 3 & 6 & 3 & 0 & 1 & 4 & 7 & 3 & 6 & 0
\end{bmatrix} \tag{26}$$

and the arising log-Hadamard matrix (containing 36th roots of unity):

$$\log S_{12} = \frac{1}{36}\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 24 & 12 & 0 & 24 & 12 & 0 & 24 & 12 & 0 & 24 & 12 \\
0 & 12 & 24 & 9 & 21 & 33 & 0 & 12 & 24 & 9 & 21 & 33 \\
0 & 24 & 12 & 18 & 6 & 30 & 0 & 24 & 12 & 18 & 6 & 30 \\
0 & 12 & 24 & 27 & 3 & 15 & 0 & 12 & 24 & 27 & 3 & 15 \\
0 & 0 & 0 & 18 & 18 & 18 & 9 & 9 & 9 & 27 & 27 & 27 \\
0 & 4 & 8 & 0 & 4 & 8 & 18 & 22 & 26 & 18 & 22 & 26 \\
0 & 16 & 32 & 0 & 16 & 32 & 18 & 34 & 14 & 18 & 34 & 14 \\
0 & 28 & 20 & 0 & 28 & 20 & 18 & 10 & 2 & 18 & 10 & 2 \\
0 & 12 & 24 & 18 & 30 & 6 & 18 & 30 & 6 & 0 & 12 & 24 \\
0 & 24 & 12 & 18 & 6 & 30 & 18 & 6 & 30 & 0 & 24 & 12 \\
0 & 0 & 0 & 18 & 18 & 18 & 27 & 27 & 27 & 9 & 9 & 9
\end{bmatrix} \tag{27}$$

□

The difference in the discussion of this case lies in the fact that there are several parametric families known already for $N = 12$. The catalogue [10] lists seven 9-parameter families stemming from F_{12}, and only one of them is certain to be constructed with Dita's method.

(We remark that possible permutational equivalences between these families are still unclear.) Also, there are other 12×12 families listed in [10], all of which are constructed with Dita's method. We will now prove the following

Proposition 3.4. *The matrix S_{12} is not included (even up to equivalence) in any of the known 12×12 families listed in* [10].

Proof. By checking existing I-equivalences between rows (and columns) it is elementary to show that S_{12} (and its transposed) is *not Dita-type*. To find possible index sets I and I-equivalences between rows (resp. columns, in the transposed case) it is perhaps most convenient to note the position of 0's in $\log S_{16}$ and look for 2×6, 3×4, 4×3 and 6×2 blocks of 0's as in the last part of the proof concerning S_8. In this case such blocks do not exist at all which immediately implies that S_{12} is not Dita-type. Therefore S_{12} is not contained in any of the Dita-type families in [10].

We must also show that it does not belong to the families stemming from F_{12}, as listed in [10]: $F_{12A}^{(9)}$, $F_{12B}^{(9)}$, $F_{12C}^{(9)}$, $F_{12D}^{(9)}$, $(F_{12B}^{(9)})^T$, $(F_{12C}^{(9)})^T$, $(F_{12D}^{(9)})^T$. The key observation is that in each of these families some rows (and columns) are left without parameters. In particular, in each of the above families either the 1st and 7th or the 1st, 5th and 9th rows remain unchanged. Therefore, in any matrix contained in these families there are either two rows which are (2)-6-equivalent, or three rows which are pairwise (3)-4-equivalent. It is easy to check (by a short computer program, rather than by hand) that there are no such rows in S_{12}. This means that S_{12} is indeed not contained in any of the known 12×12 orbits. □

By applying the linear variation of phases method of [10] we have been able to find the following 5-parameter affine family stemming from S_{12}. (Again, we can claim that the members of this family are not Dita-type in a neighbourhood of S_{12}. We do not know whether this affine family is maximal or further parameters can be introduced).

$$S_{12}^{(5)}(a,b,c,d,e) = S_{12} \circ EXP(\mathbf{i}R_{12}^{(5)}(a,b,c,d,e)), \quad \text{where} \tag{28}$$

$$R_{12}^{(5)}(a,b,c,d,e) = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & e & e & e & \bullet & \bullet & \bullet & e & e & e \\ \bullet & \bullet & \bullet & d & d & d & \bullet & \bullet & \bullet & d & d & d \\ \bullet & \bullet & \bullet & e & e & e & \bullet & \bullet & \bullet & e & e & e \\ \bullet & \bullet & \bullet & d & d & d & \bullet & \bullet & \bullet & d & d & d \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & c & c & c & c & c & c \\ \bullet & a & b & \bullet & a & b & \bullet & a & b & \bullet & a & b \\ \bullet & a & b & \bullet & a & b & \bullet & a & b & \bullet & a & b \\ \bullet & a & b & \bullet & a & b & \bullet & a & b & \bullet & a & b \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & c & c & c & c & c & c \end{bmatrix} \tag{29}$$

4. Conclusion

In this paper we have used the connection between tiling of Abelian groups and complex Hadamard matrices to *recover the general construction of Dita* [2], and also to *obtain new parametric families of order 8, 12 and 16* which complement the recent catalogue [10]. The construction of the new families was based on a spectral-set analogue of a tiling method of Szabó [8]. In principle, the method of [8] works in any finite Abelian group $G = \mathbb{Z}_{p_1q_1} \times \mathbb{Z}_{p_2q_2} \times \mathbb{Z}_{p_3q_3}$ and the corresponding spectral sets yield complex Hadamard matrices of size $p_1p_2p_3$ for any $p_1, p_2, p_3 \geq 2$. It is not clear whether different choices of q_1, q_2, q_3 lead to non-equivalent matrices. In this paper we have only included the cases where $p_1p_2p_3 \leq 16$, and for which we could prove that the arising matrices are new and thus complement the catalogue [10]. The next smallest dimension in which the method works is $p_1p_2p_3 = 2 \cdot 3 \cdot 3 = 18$. Also, it would be interesting to see a *conceptual* proof that the Hadamard matrices constructed with this method are *never Dita-type* (for the matrices S_8, S_{12}, S_{16} above we have proved this by a case-by-case analysis of the rows and columns).

The *correspondence between tiling and complex Hadamard matrices* is interesting in its own right and may well lead to new families of Hadamard matrices in the future. To achieve this, one would need any new tiling construction (different from that of [6] and [8] which have been used in this paper), and use the spectral set analogue of the construction to produce new Hadamard matrices.

Finally, let us emphasize that our results may find direct application in various problems of quantum information theory, since previously unknown complex Hadamard matrices allow to construct new teleportation and dense coding schemes and to find previously unknown bases of maximally entangled states.

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