J. Math. Kyoto Univ. (JMKYAZ)
23-1 (1983) 1-11

Constructions of eigenfunctions for the Sturm-Liouville operator by comparison method

By

Sadao MIYATAKE

(Received Oct. 2, 1981)

§1. Introduction

This paper is concerned with constructions of eigenfunctions for the Sturm-Liouville operator $L=-\frac{d^2}{dx^2}+q(x)$ in $(-\infty, \infty)$. Here we assume that the real valued function $q(x)$ satisfies the following conditions:

$$\text{(C)}\quad \begin{cases} q(x) \text{ is piecewise continuous and has the minimum value at } x=x_0, \\ m=q(x_0)=\inf_{-\infty<x<\infty} q(x)<M=\varliminf_{x\to\infty} q(x), \quad (M=\infty \text{ is included.}) \end{cases}$$

Especially we consider concrete constructions of eigenfunctions corresponding to eigenvalues in (m, M), relying upon comparison theorems which assure the existence of bounded solutions $u_+(x, \lambda)$ and $u_-(x, \lambda)$ of $\frac{d^2}{dx^2}u=(q(x)-\lambda)u$ in neighbourhoods of $+\infty$ and $-\infty$ respectively. Namely we try to consider the Sturm's method of comparison even in the case of infinite demain $(-\infty, \infty)$. As we see later, this consideration motivates originally comparison theorems of type stated in Section 2, which are generalized in [2] and [3]. Incidentally we show that there exists a continuous monotone increasing function $\Phi(\lambda)$ satisfying $-\pi<\Phi(m)<0$ such that λ is eigenvalue if and only if $\Phi(\lambda)=(n-1)\pi$, $(n=1, 2, 3,\dots)$. In order to see that appearance of eigenvalues more precisely we need some estimates for $\Phi(\lambda)$. For this purpose we write

$$\Phi(\lambda)=\int_{\Omega(\lambda)}(\lambda-q(x))^{1/2}dx+R(\lambda),$$

where $\Omega(\lambda)=\{x; \lambda-q(x)>0\}$ and obtain a suitable estimate for $R(\lambda)$, to show that $R(\lambda)$ is a remainder term as compared with the first term. In many books of physics (for example [1], [5] etc.) we find the following type of formula: $\int_{\Omega(\lambda_n)}(\lambda_n-q(x))^{1/2}dx=\left(n-\frac{1}{2}\right)\pi$, which was explained by the so-called W. K. B. method. As for mathematics Tichmarsh [6] showed that there exists a constant C such that

$|R(\lambda_n)|<C$ for all λ_n if $q(x)$ is convex. And many authors treated the related problems under various assumptions on $q(x)$, $q'(x)$ and $q''(x)$, (for example see [4]). Here we use comparison theorems related with only the value of $q(x)$ and exhibit an estimate for $R(\lambda)$ of type $\underline{E}(\lambda)\leq R(\lambda)\leq \bar{E}(\lambda)$, where $\underline{E}(\lambda)$ and $\bar{E}(\lambda)$ concern the logarithmic order of variations of $q(x)$ in $\Omega(\lambda)$. The results are stated in Section 2 more precisely. In Section 3, to make our argument smooth we verify that all the eigenvalues of L in L^∞ space are real, and then we prove some simple lemmas concerning the comparison theorems which will be used later. Theorems 1 and 2 are proved in Section 4 and 5. In the last section we generalize the estimate for eigenvalues to the case where $\Omega(\lambda)$ is not necessarily one interval, thus clarifying some properties of solutions in the case of tunnel effects.

§ 2. Statements of results

First we mention the definition of eigenvalues and eigenfunctions of $L=-\frac{d^2}{dx^2}+q(x)$ in the space L^∞ of bounded measurable functions defined in $(-\infty, \infty)$. We say λ and $u(x)$ eigenvalue and eigenfunction of L in L^∞ respectively, if there exists $(\lambda, u(x))$ belonging to $C\times L^\infty$ such that $Lu=\lambda u$ in $(-\infty, \infty)$, $u(x)$ not being identically zero. We have

Lemma 1. *Suppose the conditions* (C) *in Introduction. Then all the eigenvalues of L in L^∞ are real and contained in (m, ∞).*

A simple and direct proof of Lemma 1 will be given in next section. In view of Lemma 1 we suppose that the parameter λ is real and larger than m. Especially we restrict λ in (m, M). As we will see later, eigenfunctions in L^∞ and those in L^2 coincide for eigenvalues in (m, M). So we do not mention function spaces hereafter. Now we put for $\lambda\in[m, M)$

$$\begin{aligned} x_+(\lambda) &= \inf\{x_1;\ q(x)-\lambda>0 \quad \text{in} \quad (x_1, \infty)\}, \\ x_-(\lambda) &= \sup\{x_1;\ q(x)-\lambda>0 \quad \text{in} \quad (-\infty, x_1)\}. \end{aligned} \tag{2.1}$$

Then from (2.1), $x_0\leq x_+(\lambda_1)\leq x_+(\lambda_2)$ and $x_-(\lambda_2)\leq x_-(\lambda_1)\leq x_0$ if $m\leq\lambda_1\leq\lambda_2<M$. We have

Proposition 1. *Assume* (C). *Then there exists a continuous function $\theta_+(x_1, \lambda)$, (resp. $\theta_-(x_1, \lambda)$) defined on $(-\infty, \infty)\times[m, M)$ which has the following properties: (I) The solution of $u''=(q(x)-\lambda)u$ satisfying $u(x_1)=u_0$ and $u'(x_1)=u_1$ is bounded in (x_1, ∞), (resp. $(-\infty, x_1)$) if and only if $\frac{u_1}{u_0}=\tan\theta_+(x_1, \lambda)$, (resp. $\frac{u_1}{u_0}=\tan\cdot\theta_-(x_1, \lambda)$), (II) $\theta_\pm(x, \lambda)$ satisfies*

$$\begin{cases} -\frac{\pi}{2}<\theta_+(x, \lambda)<0 & \text{for} \quad x\in(x_+(\lambda), \infty), \\ 0<\theta_-(x, \lambda)<\frac{\pi}{2} & \text{for} \quad x\in(-\infty, x_-(\lambda)), \end{cases} \tag{2.2}$$

and the following differential equation respectively

(2.3) $$\frac{d\theta}{dx}=\frac{(q(x)-\lambda)-\tan^2\theta}{1+\tan^2\theta}, \qquad \text{for all} \quad \lambda\in[m, M),$$

(*III*) $\theta_+(x, \lambda)$, (*resp.* $\theta_-(x, \lambda)$) *is monotone increasing*, (*resp. decreasing*) *in* λ *at every point* $x\in(-\infty, \infty)$.

Remark 2.1. From the above property (II) and the uniqueness of the solution of (2.3), $\theta_+(x, \lambda)=\theta_-(x, \lambda)+k\pi$ in $(-\infty, \infty)$, (k, integer), if $\theta_+(x_1, \lambda)=\theta_-(x_1, \lambda)+k\pi$ at a point $x_1\in(-\infty, \infty)$.

Now we put

(2.4) $$\Phi(x, \lambda)=\theta_+(x, \lambda)-\theta_-(x, \lambda)$$

Then we have the following theorem.

Theorem 1. *Suppose* (C). *Then there exists a continuous function* $\Phi(x, \lambda)$ *defined on* $(-\infty, \infty)\times[m, M)$, *which satisfies* (1) $\Phi(x, \lambda)$ *is monotone increasing in* λ *at every* $x\in(-\infty, \infty)$, (2) $-\pi<\Phi(x, m)<0$, *and* $\Phi(x, \lambda_n)=(n-1)\pi$ *in* $(-\infty, \infty)$ *if* $\Phi(x_1, \lambda_n)=(n-1)\pi$ *at a point* $x_1\in(-\infty, \infty)$. *At that time* λ_n, $(n=1, 2,\dots,)$ *are eigenvalues if and only if* $\Phi(x, \lambda_n)=(n-1)\pi$. *Corresponding eigenfunctions* $u_n(x)$ *are* equal to *the solutions of* $u''=(q(x)-\lambda_n)u$ *with* $u(x_+(\lambda_n))=1$ *and* $u'(x_+(\lambda_n))=\tan\theta_+(x_+(\lambda_n), \lambda_n)$. *Incidentally* u_n *has* $(n-1)$ *roots. Moreover* $u'_n(x)$ *has just* n *zeros if* $q(x)$ *is assumed to be monotone in* $(-\infty, x_0)$ *and* (x_0, ∞), *where* x_0 *is a point satisfying* $q(x_0)=\min_{-\infty<x<\infty} q(x)$.

Now we put

(2.5) $$R(x, \lambda)=\Phi(x, \lambda)-\int_{-\infty}^{\infty}Q(s, \lambda)ds, \text{ where}$$

$$Q(x, \lambda)=\begin{cases}(\lambda-q(x))^{1/2}, & \text{if} \quad 0\le\lambda-q(x),\\ 0, & \text{if} \quad \lambda-q(x)<0.\end{cases}$$

Remark that $R(x, \lambda_n)$ is constant: $(n-1)\pi-\int_{-\infty}^{\infty}Q(s, \lambda_n)ds$. Put

(2.6) $$R(\lambda)=R(x_0, \lambda).$$

We show the estimates for $R(x, \lambda)$ and $R(\lambda_n)$, in order to assure the actual appearance of λ_n in each given case.

Theorem 2. *Suppose* (C) *and that* $q(x)$ *is monotone in* (x_0, ∞) *and* $(-\infty, x_0)$. *Then we have the following estimates for* $\lambda\in[m, M)$:

$$\underline{E}(\lambda)-2\pi<R(x, \lambda)<\bar{E}(\lambda)+\pi,$$

$$\underline{E}(\lambda_n)-\pi<R(\lambda_n)<\bar{E}(\lambda_n), \quad (\lambda_n, \textit{eigenvalues of } L)$$

where $\underline{E}(\lambda)=\int_{-\infty}^{\infty}\underline{E}(s,\lambda)ds$, $\bar{E}(\lambda)=\int_{-\infty}^{\infty}\bar{E}(s,\lambda)ds$ *and*

$$\underline{E}(x,\lambda)=\begin{cases}-\frac{1}{2}|(\log Q(x,\lambda))'| & \text{if } 1\leq\lambda-q(x),\\ -Q(x,\lambda)(1-Q(x,\lambda)) & \text{if } 0\leq\lambda-q(x)<1,\\ 0 & \text{if } \lambda-q(x)<0,\end{cases}$$

$$\bar{E}(x,\lambda)=\begin{cases}\frac{1}{2}|(\log Q(x,\lambda))'| & \text{if } 1\leq\lambda-q(x),\\ 1-Q(x,\lambda) & \text{if } 0\leq\lambda-q(x)<1,\\ 0 & \text{if } \lambda-q(x)<0.\end{cases}$$

Remark. Theorem 2 will be generalized in Section 6 to the case where $q(x)$ is not necessarily monotone in $(-\infty, x_0)$ and (x_0, ∞).

The proofs of Proposition 1 and Theorems 1 and 2 rely upon the following comparison theorems which we state as lemmas.

Lemma 2. *Let* $q(x)$ *and* $\tilde{q}(x)$ *be piecewise continuous functions such that* $q(x)>\tilde{q}(x)$ *on* $[0, \infty)$. *Suppose that there exists a solution* $v(x)$ *of* $v''=\tilde{q}(x)v$ *in* $(0, \infty)$ *satisfying* $v(0)=1$ *and* $0<v(x)$ *in* $(0, \infty)$. *Then* $u''=q(x)u$ *has a solution* $u(x)$ *satisfying* $0<u(x)<v(x)$ *in* $(0, \infty)$ *and* $u(0)=1$. *It holds* $u'(0)<v'(0)$.

Lemma 3. *Assume the same conditions as in Lemma 2. Moreover suppose that* $v(x)$ *is a unique bounded solution in* $(0, \infty)$ *of* $v''=\tilde{q}(x)v$ *satisfying* $v(0)=1$. *Then the bounded solution in* $(0, \infty)$ *of* $u''=q(x)u$ *with* $u(0)=1$ *is unique.*

Now we introduce a notation.

Notation By $t(q)$ we denote $u'(0)$, when $u''=q(x)u$ has a unique positive bounded solution $u(x)$ in $(0, \infty)$ satisfying $u(0)=1$.

Lemma 4. *Let* $\{q_\mu(x)\}_{0<\mu<1}$ *be a family of functions satisfying the condition* (C). *Assume that* q_μ *converges to* q_{μ_0} *uniformly in each compact set of* $[0, \infty)$ *as* μ *tends to* $\mu_0\in(0, 1)$. *Suppose that conditions in Lemmas 2 and 3 are fulfilled replacing* q *by* q_μ *for all* $\mu\in(0, 1)$. *Then* $t(q_\mu)$ *converges to* $t(q_{\mu_0})$ *as* μ *tends to* μ_0.

Finally we point out some examples.

Example 1. If $q(x)>0$ in $(0, \infty)$, it follows that $\tilde{q}(x)\equiv 0$ satisfies all the conditions in Lemmas 2 and 3. Then applying Lemmas 2 and 3 in (x_1, ∞) for all $x_1\in[0, \infty)$, we see that $u''=q(x)u$ has a bounded solution $u(x)$ satisfying $\frac{u'}{u}<0$ in $(0, \infty)$. Similarly $q(x)>\tilde{q}(x)>0$ in $(0, \infty)$ implies $\frac{u'}{u}<\frac{v'}{v}$ in $(0, \infty)$, where u and v are bounded solutions $u''=q(x)u$ and $v''=\tilde{q}(x)v$ respectively.

Remark. Let us confine ourselves to simple cases: $q>\tilde{q}$ or $q<\tilde{q}$, where

$q>\tilde{q}$ means $q(x)>\tilde{q}(x)$ on $[0, \infty]$. Then in Example 1, $0<\tilde{q}<q$ equals to $\frac{u'}{u}<\frac{v'}{v}<0$. $\frac{u'}{u}$ converges to $\frac{v'}{v}$ if and only if q converges to $\tilde{q}$ in the sense of Lemma 4.

Example 2. As an application of the above Remark, we consider the harmonic oscillator: $q(x)=x^2$. Put $v=\exp(-ax^k)$, then $\tilde{q}(x)=\frac{v''}{v}=(ak)^2x^{2k-2}-ak(k-1)x^{k-2}$. Taking $k=2$ and $a=\frac{1}{2}$ we see $\lambda_1=1$ and $u_1=\exp\left(-\frac{1}{2}x^2\right)$. Next put $v=x^k\exp\left(-\frac{1}{2}x^2\right)$, then similarly $\lambda_2=3$ and $v_2=x\exp\left(-\frac{1}{2}x^2\right)$ follow. Step by step we have $\lambda_n=2n-1$ and $u_n=\varphi_n(x)\exp\left(-\frac{1}{2}x^2\right)$ where $\varphi_n(x)$ is a polynominal of degree $n-1$ with $n-1$ real roots. Thus we approach to a motivation of Hermite polynomials.

Finally we comment on Lemmas stated above.

Remark. In Lemmas 2, 3, 4 and Example 1, we have the same results even if we replace $q(x)>\tilde{q}(x)$ by $q(x)\geq\tilde{q}(x)$, where $q(x)$ is not identically equal to $\tilde{q}(x)$. The proof is accomplished in the same way as in next section.

§ 3. Proofs of Lemmas

Proof of Lemma 1. Let $\chi(x)$ be a real valued C^1 function satisfying $\chi(x)>0$ in $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\chi(x)=0$ in $\left(-\infty, -\frac{1}{2}\right)\cup\left(\frac{1}{2}, \infty\right)$, $\chi(x)=\chi(-x)$, $\chi(x_1)\geq(x_2)$ for $0\leq x_1\leq x_2$ and

$$\chi'(x)^2\leq C\chi(x) \quad \text{in} \quad (-\infty, \infty), \quad \text{where} \quad C>0. \tag{3.1}$$

For example define $\chi(x)$ by $\left(x+\frac{1}{2}\right)^2 H\left(x+\frac{1}{2}\right)$ in $\left(-\infty, -\frac{1}{3}\right)$, $\left(-x+\frac{1}{2}\right)^2\cdot H\left(-x+\frac{1}{2}\right)$ in $\left(\frac{1}{3}, \infty\right)$ and a suitable positive function in $\left(\frac{1}{3}, \frac{1}{3}\right)$, where $H(x)=1$ for $x>0$ and $H(x)=0$ for $x\leq 0$. Denote $(u, v)=\int_{-\infty}^{\infty} u(x)\overline{v(x)}dx$ and

$$(u, v)_\varepsilon=(\chi(\varepsilon x)u, v) \quad \text{and} \quad L_\varepsilon=\chi(\varepsilon x)L. \tag{3.2}$$

Let u be eigenfunction in L^∞ corresponding to eigenvalue λ:

$$\begin{cases} (L_\varepsilon u, u)=\lambda(u, u)_\varepsilon \\ \sup|u|\leq C_1. \end{cases} \tag{3.3}$$

On the other hand the integration by parts yields

$$(L_\varepsilon u, u)=(u', u')_\varepsilon+(qu, u)_\varepsilon+\varepsilon\int_{-\infty}^{\infty}\chi'(\varepsilon x)u'(x)\overline{u(x)}dx. \tag{3.4}$$

From Schwartz inequality and (3.1) it follows

$$(3.5)\qquad \left|\int \chi'(\varepsilon x)u'(x)\overline{u(x)}dx\right| \leq C_1\varepsilon^{-1/2}\left(\int\left|\chi'(\varepsilon x)u'(x)\right|^2 dx\right)^{1/2}$$

$$\leq CC_1\varepsilon^{-1/2}(u', u')_\varepsilon^{1/2}$$

From (3.3) and (3.4) we have

$$(3.6)\qquad |\mathrm{Im}\,\lambda| \leq CC_1\varepsilon^{1/2}(u', u')_\varepsilon^{1/2}/(u, u)_\varepsilon \equiv J_\varepsilon(u),$$

$$(3.7)\qquad \mathrm{Re}\,\lambda - \{(u', u')_\varepsilon + (qu, u)_\varepsilon\}/(u, u)_\varepsilon \geq -J_\varepsilon(u).$$

Since u' is not identically zero it follows

$$(3.8)\qquad \lim_{\varepsilon\to 0} \varepsilon^{1/2}(u', u')_\varepsilon^{-1/2} = 0.$$

Therefore (3.7) makes

$$(3.9)\qquad \mathrm{Re}\,\lambda - m \geq \{(1 - CC_1\varepsilon^{1/2}(u', u')_\varepsilon^{-1/2})(u', u')_\varepsilon + ((q-m)u, u)_\varepsilon\}/(u, u)_\varepsilon.$$

Now making ε sufficiently small we have $\mathrm{Re}\,\lambda - m > 0$. Then (3.6) and (3.9) give

$$(3.10)\qquad \frac{\mathrm{Im}\,\lambda}{\mathrm{Re}\,\lambda - m} \leq \frac{CC_1\varepsilon^{1/2}(u', u')_\varepsilon^{-1/2}}{1 - CC_1\varepsilon^{1/2}(u', u')_\varepsilon^{-1/2}}.$$

Tending ε to zero we have $\mathrm{Im}\,\lambda = 0$.

Remark. The results of Lemma 1 and the outline of the above proof are also valid even if we replace $-\frac{d^2}{dx^2} + q(x)$ in R by $-\sum \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + q(x)$ in R^n. We will discuss it elsewhere.

Proof of Lemma 2. Let $u(x; t)$ be solution of $u'' = q(x)u$, $u(0) = 1$ and $u'(0) = t$. Denote

$$U(q) = \{t \in (-\infty, v'(0)),\ ^\exists x_1 > 0,\ 0 < u(x; t) < v(x) \quad \text{in} \quad (0, x_1), \quad u(x_1; t) = v(x_1)\}$$

$$D(q) = \{t \in (-\infty, v'(0)),\ ^\exists x_1 > 0,\ 0 < u(x; t) < v(x) \quad \text{in} \quad (0, x_1), \quad u(x_1; t) = 0\}$$

Then $U(q)$ and $D(q)$ are open as follows. If $t \in U(q)$, then $u'(x_1; t) \geq v'(x_1)$ follows. Now suppose $u'(x_1; t) = v'(x_1)$, then $u''(x_1; t) = q(x_1)u(x_1; t) > \tilde{q}(x_1) \cdot v(x_1) = v''(x_1)$ contradicts $0 < u(x; t) < v(x)$ in $(0, x_1)$. Hence $u'(x_1; t) > v'(x_1)$. Therefore $U(q)$ is open. The similar argument is valid for $D(q)$. Now let us show that $U(q)$ contains an open interval $(v'(0) - \delta, v'(0))$. We have $u(x; v'(0)) > v(x)$ holds in a neighbourhood of $x = 0$, because of $u''(0; v'(0)) = q(0) > \tilde{q}(0)$. Therefore the above assertion follows from the continuity of $u(x; t)$ in t. On the other hand an interval $(-\infty, -t_0 + \varepsilon)$ belongs to $D(q)$ for sufficient large t_0. Hence the connected interval $(-t_0, v'(0))$ involves at least a point α such that $\alpha \notin U(q) \cup D(q)$. Namely we have $0 < u(x; \alpha) < v(x)$ in $(0, \infty)$.

Proof of Lemma 3. It sufficies to prove that the solution $u_1(x)$ of $u_1'' = q(x)u_1$, $u_1(0) = 0$ and $u_1'(0) = 1$ is not bounded in $(0, \infty)$. First we show that $v_1(x)$ satisfying $v_1'' = \tilde{q}(x)v_1$, $v_1(0) = 0$ and $v_1'(0) = 1$ is positive and unbounded. In fact, if $v_1(a) = 0$ and

$0<v_1(x)$ in $(0, a)$ for some $a>0$, then we can find positive numbers $b \in (0, a)$ and c such that $0<v_1(x)<cv(x)$ in $(0, b)$, $v_1(b)=cv(b)$ and $v_1'(b)=cv'(b)$. However this contradicts the uniqueness theorem on the solution of $v''=\tilde{q}(x)v$. Since the bounded solution $v(x)$ satisfying $v''=\tilde{q}(x)v$ in $(0, \infty)$ and $v(0)=1$ is unique, $v_1(x)$ is not bounded in $(0, \infty)$. Let us prove $u_1(x)\geq v_1(x)$ in $(0, \infty)$, which equals to $u_1(x)> sv_1(x)$ in $(0, \infty)$ for any $s \in (0, 1)$. Suppose $u_1(a)=sv_1(a)$ for some $a>0$, and $u_1(x)> sv_1(x)$ in $(0, a)$. Then there exist $b \in (0, a)$ and $c>0$ such that $u_1(x)<sv_1(x)+cv(x)$ in $(0, b)$, $u_1(b)=sv_1(b)+cv(b)$ and $u_1'(b)=sv_1'(b)+cv'(b)$. However this is a contradiction since it holds $u_1''(b)=q(b)u_1(b)>\tilde{q}(b)(sv_1(b)+cv(b))=sv_1''(b)+cv_1''(b)$ from $q(x)>\tilde{q}(x)$. Hence we have $u_1(x)>sv_1(x)$ in $(0, \infty)$. Therefore $u_1(x)$ is positive and unbounded in $(0, \infty)$. This means the uniqueness of the bounded solution of $u''=q(x)u$ satisfying $u(0)=1$.

Proof of Lemma 4. From Lemma 2 it holds $t(q_\mu)<t(\tilde{q})$ for all $\mu \in (0, 1)$. Let $\varepsilon_0=t(\tilde{q})-t(q_{\mu_0})$. For given positive number ε less than ε_0, from Lemma 3 we have $t(q_{\mu_0})+\varepsilon \in U(q_{\mu_0})$ and $t(q_{\mu 0})-\varepsilon \in D(q_{\mu_0})$. Let us recall the property of $U(q_{\mu_0})$: if $t \in U(q_{\mu_0})$, then there exists a positive number a such that the solution $u(x)$ of $u''= q_{\mu_0}(x)u$ with $u(0)=1$ and $u'(0)=t$ satisfies $0<u(x)<v(x)$ in $(0, a)$, $u(a)=v(a)$ and $u'(a) >v'(a)$. Then from the continuity of solutions of $u''=q(x)u$ for $q(x)$, there exists a positive number δ which is smaller than $\min\{\mu_0, 1-\mu_0\}$, such that $t(q_{\mu_0})+\varepsilon \in U(q_\mu)$ and $t(q_{\mu_0})-\varepsilon \in D(q_\mu)$ for all μ belonging to $(\mu_0-\delta, \mu_0+\delta)$. Therefore from Lemma 3 we have $t(q_\mu)\in(t(q_{\mu_0})-\varepsilon, t(q_{\mu_0})+\varepsilon)$ for all $\mu \in (\mu_0-\delta, \mu_0+\delta)$. This completes the proof of Lemma 4.

§4. Proof of Theorem 1

First we state a preliminary lemma for the proof of Proposition 1.

Lemma 4.1. *Let $f(x, y)$ and $g(x, y)$ be continuous in R^2 and satisfy the Lipschitz condition in y. Assume $f(x, y)\geq g(x, y)$ in R^2. Then $u(a)>v(a)$, (reps. $u(a)<v(a)$) implies $u(x)>v(x)$ in (a, ∞), (resp. $u(x)<v(x)$ in $(-\infty, a)$).*

Proof. In the case of $f(x, y)>g(x, y)$ in R^2, the result is easily verified by method of contradiction. We prove the general case in the following way. Suppose $u(x_1)\leq v(x_1)$ at $x=x_1 \in (a, \infty)$. Let $u_\varepsilon(x)$ be solution of $u_\varepsilon''=f(x, u_\varepsilon)+\varepsilon$ with $u_\varepsilon(x_1)=u(x_1)-\varepsilon$. Then form the result in the above case we have $u_\varepsilon(x)< v(x)$ in $(-\infty, x_1)$. Therefore $u(a)=\lim_{\varepsilon\to 0} u_\varepsilon(a)\leq v(a)$ contradicts the assumption.

Collary of Lemma 4.1. *In the statements of Lemma* 4.1 *we can replace $u(a)>v(a)$, $u(x)>v(x)$ etc. by $u(a)\geq v(a)$, $u(x)\geq v(x)$ etc. respectively, by virtue of the continuity of solutions.*

Proof of Proposition 1. From the definition of $x_+(\lambda)$, it holds

(4.1) $$q(x)-\lambda>0 \text{ in } (x_1, \infty), \text{ where } (x_1, \lambda)\in(x_+(\lambda), \infty)\times[m, M).$$

Take $\tilde{q}(x)\equiv 0$ and apply Lemma 2 replaced $(0, \infty)$ and $q(x)$ by (x_1, ∞) and $q(x)-\lambda$

respectively. Then we know that for each $\lambda \in [m, M)$, there exists a negative valued function $I_+(x_1, \lambda)$ defined in $(x_+(\lambda), \infty)$, such that the non-zero solution $u(x)$ of $u'' = (q(x)-\lambda)u$ is bounded in (x_1, ∞) if and only if $u'(x_1)/u(x_1) = I_+(x_1, \lambda)$. From Lemma 3, $I_+(x, \lambda)$ is continuous in $(x_+(\lambda), \infty)$ for each $\lambda \in [m, M)$. Put

$$\theta_+(x, \lambda) = \mathrm{Tan}^{-1} I_+(x, \lambda), \tag{4.2}$$

then $\theta_+(x, \lambda)$ satisfies $-\frac{\pi}{2} < \theta_+(x, \lambda) < 0$ in $(x_+(\lambda), \infty)$ and

$$\theta' = \frac{(q(x)-\lambda) - \tan^2\theta}{1+\tan^2\theta} \tag{4.3}$$

in $(x_+(\lambda), \infty)$. The equation (4.3) and

$$u'' = (q(x)-\lambda)u \tag{4.4}$$

are translated each other by putting $u(x) = u(x_0)\exp\int_{x_0}^{x} \tan\theta(s, \lambda)ds$ or $\theta = \tan^{-1}\frac{u'}{u}$, so long as $u \neq 0$ or $\theta \neq \left(n+\frac{1}{2}\right)\pi$, ($n$: integers). Since the right hand side of (4.3) is bounded in any compact set in $R^2 \ni (x, \theta)$ and that of (4.4) is linear, definition domains of solutions θ and u can be extended to $(-\infty, \infty)$ so that the relation $\theta = \tan^{-1}\frac{u'}{u}$ holds even if u takes zero. In this way we define $\theta_+(x, \lambda)$ on $(-\infty, \infty) \times [m, M)$. Then $\theta_+(x, \lambda)$ is continuous in $\lambda \in [m, M)$ at every fixed $x \in (-\infty, \infty)$. In fact from Lemma 4 $\theta_+(x, \lambda)$ is continuous in λ at every x in $(x_+(M-\delta), \infty)$ if λ is restricted in $[m, M-\delta)$, where δ is an arbitrary small constant. Hence we have the desired result from the continuity of solutions of (4.3) for initial data. Moreover from Lemma 2 and Lemma 4.1 $\theta_+(x, \lambda)$ is a monotone increasing function in $\lambda \in [m, M)$ at every $x \in (-\infty, \infty)$. From this monotony follows the continuity of $\theta_+(x, \lambda)$in (x, λ). Similarly we can define $\theta_-(x, \lambda)$ and the corresponding properties.

Proof of Theorem 1. From (2.2) it follows $0 < \theta_-(x_1, m) < \frac{\pi}{2}$ for $x_1 \in (-\infty, x_-(m))$. Now we compare $\theta_-(x, m)$ with $\tilde{\theta} \equiv 0$ which is a solution of $\tilde{\theta}' = \frac{-\tan^2\tilde{\theta}}{1+\tan^2\tilde{\theta}}$, applying Lemma 4.1 with $u = \theta_+(x, m)$, $v \equiv 0$, $f(x, y) = \frac{(q(x)-m)-\tan^2 y}{1+\tan^2 y}$ and $g(x, y) = \frac{-\tan^2 y}{1+\tan^2 y}$. Then $0 < \theta_-(x_1, m)$ implies $0 < \theta_-(x, m)$ in (x_1, ∞). Thus we have $0 < \theta_-(x, m)$ in $(-\infty, \infty)$. Now we remark that $\theta_-(x, \lambda)$ is decreasing when $\theta_-(x, \lambda) = \frac{\pi}{2} (\mathrm{mod}\ \pi)$, because $\theta_-' = -1$ when $\tan^2\theta_- = \infty$ in (2.3). Therefore we have $0 < \theta_-(x, m) < \frac{\pi}{2}$ in $(-\infty, \infty)$. Similarly follows $-\frac{\pi}{2} < \theta_+(x, \lambda) < 0$ in $(-\infty, \infty)$. Hence by definition $-\pi < \Phi(x_0, m) < 0$. And in the same time $-\pi < \Phi(x, m) < 0$ in $(-\infty, \infty)$. For simplicity we consider $\Phi(x, \lambda)$ at $x = x_0$. From Proposition 1, $\Phi(x_0, \lambda)$ is increasing continuous function. If $\lim_{\lambda \to M} \Phi(x_0, \lambda) > (n-1)\pi$ we define λ_n by $\Phi(x_0, \lambda_n) = (n-1)\pi$, $(n = 1, 2, \ldots)$. We have

$$m < \lambda_1 < \lambda_2, \ldots, (< M).$$

Let us put $u_n^+ = \exp\int_{x_+(\lambda_n)}^{x} \theta_+(s, \lambda_n)ds$ in $(x_+(\lambda_n), \infty)$ and $u_n^- = \exp\int_{x_-(\lambda_n)}^{x} \theta_-(s, \lambda_n)ds$

in $(-\infty, x_-(\lambda_n))$. Then u_n^+ satisfies $(u_n^+)''=(q(x)-\lambda_n)u_n^+$. We extend u_n^+ as solutions of this equation. Then $\tan\theta_\pm(x, \lambda_n)=\dfrac{u_n^{\pm\prime}(x, \lambda_n)}{u_n^\pm(x, \lambda_n)}$. $\Phi(x_0, \lambda_n)=(n-1)\pi$ equals $\theta_+(x, \lambda_n)=\theta_-(x, \lambda_n)+(n-1)\pi$, i.e. $(\log u_n^+)'=(\log u_n^-)'$. Thus we have $u_n^+(x)=Cu_n^-(x)$. Put $u_n(x)=u_n^-(x)$. Then $u_n(x)$ is bounded in $(-\infty, \infty)$ and satisfies $Lu_n=\lambda_n u_n$. From (2.2) and $\Phi(x, \lambda_n)=(n-1)\pi$ in $(-\infty, \infty)$ we see that $u_n(x)$ has $(n-1)$ zero in $(-\infty, \infty)$ since $\theta'_\pm=-1$ if $\tan^2\theta_\pm=\infty$. Moreover, if we assume that $q(x)$ is monotone in $(-\infty, x_0)$ and in (x_0, ∞), then u'_n has n zero in $(-\infty, \infty)$, because $\theta'_-(x, \lambda)$ is decreasing in $(x_-(\lambda), x_+(\lambda))$.

§5. Proof of Theorem 2

From the definition of $R(x, \lambda)$ in (2.5) it follows

$$(5.1)\quad R(x, \lambda)=-\int_{x_-(\lambda)}^{x}(\theta'_-(s, \lambda)+Q(s, \lambda))ds-\int_{x}^{x_+(\lambda)}(\theta'_+(s, \lambda)+Q(s, \lambda))ds$$
$$+\{\theta_+(x_+(\lambda), \lambda)-\theta_-(x_-(\lambda), \lambda)\}.$$

Remark that the third term is estimated from (2.2), (2.3) and Lemma 4.1 as follows.

$$(5.2)\qquad -\pi<\{\theta_+(x_+(\lambda), \lambda)-\theta_-(x_-(\lambda), \lambda)\}<0.$$

However the integrations of $|\theta'_\pm+Q|$ are not small in general. Therefore we define a modification of $\theta_\pm$; $\tilde\theta_\pm$ in the following way such that the integrations of $|\tilde\theta'_\pm+Q|$ become smaller:

$$(5.3)\qquad \tan\tilde\theta_\pm(x, \lambda)=\tan(\theta_\pm(x, \lambda)/\hat{Q}(x, \lambda)),$$
$$\sup|\tilde\theta_\pm(x, \lambda)-\theta_\pm(x, \lambda)|<\pi/2 \quad \text{for}\quad (x, \lambda)\in(-\infty, \infty)\times[m, M),$$

where
$$\tilde{Q}(x, \lambda)=\begin{cases} Q(x, \lambda) & \text{if}\quad q(x)-\lambda\le -1,\\ 1 & \text{if}\quad -1<q(x)-\lambda.\end{cases}$$

First we notice that $\tilde\theta_+(x, \lambda)$ satisfies

$$(5.4)\quad \begin{aligned} &\text{(i)}\quad |\tilde\theta_\pm(x, \lambda)-\theta_\pm(x, \lambda)|<\frac{\pi}{2},\\ &\text{(ii)}\quad \tilde\theta_\pm(x, \lambda)=\frac{k\pi}{2} \quad \text{if}\quad \theta_\pm(x, \lambda)=\frac{k\pi}{2},\quad (k\text{: integers}),\\ &\text{(iii)}\quad \tilde\theta_\pm(x, \lambda)=\theta_\pm(x, \lambda) \quad \text{if}\quad -1\le q(x)-\lambda,\\ &\text{(iv)}\quad \tilde\theta_+(x, \lambda_n)-\tilde\theta_-(x, \lambda_n)=\theta_+(x, \lambda_n)-\theta(x, \lambda_n)=(n-1)\pi.\end{aligned}$$

Let us put

$$(5.5)\qquad \tilde\Phi(x, \lambda)=\tilde\theta_+(x, \lambda)-\tilde\theta_-(x, \lambda).$$

Then from (5.4) we have

$$(5.6)\qquad -\pi<\Phi(x, \lambda)-\tilde\Phi(x, \lambda)<\pi,$$

$$\Phi(x, \lambda_n) = \tilde{\Phi}(x, \lambda_n) = (n-1)\pi. \tag{5.7}$$

We rewrite (5.1) as follows

$$(5.1)' \quad R(x, \lambda) = -\int_{x_-(\lambda)}^{x} (\tilde{\theta}'_-(s, \lambda) + Q(s, \lambda))ds - \int_{x}^{x_+(\lambda)} (\tilde{\theta}'_+(s, \lambda) + Q(s, \lambda))ds$$
$$+ \{\theta_+(x_+(\lambda), \lambda) - \theta_-(x_-(\lambda), \lambda)\} + \{\Phi(x, \lambda) - \tilde{\Phi}(x, \lambda)\}.$$

From (5.3) we have

$$\tilde{\theta}'_\pm(x, \lambda) = \begin{cases} -Q(x, \lambda) - (Q'/Q)(QI_\pm/(Q^2 + I_\pm^2)), & \text{if } q(x) - \lambda \le -1, \\ (q(x) - \lambda - I_\pm^2)/(1 + I_\pm^2), & \text{if } -1 < q(x) - \lambda, \end{cases} \tag{5.8}$$

where $Q = Q(x, \lambda) = (\lambda - q(x))^{1/2}$ and $I_\pm = \tan\theta_\pm$.
Then the integrands in (5.1)′ are estimated as follows.

$$|\tilde{\theta}'_\pm + Q| \le \frac{1}{2}|(\log Q)'|, \qquad \text{if } q(x) - \lambda \le -1, \tag{5.9}$$

$$Q - 1 \le \theta'_\pm + Q \le Q(1 - Q), \qquad \text{if } -1 < q(x) - \lambda. \tag{5.10}$$

From (5.1)′, (5.2), (5.6), (5.9) and (5.10) we have Theorem 2.

§6. Generalization of Theorem 2

Here we consider Theorem 2 without the monotony of $q(x)$ in $(-\infty, x_0)$ and in (x_0, ∞). First we describe

Lemma 6.1. *Let $q(x)$ be continuous and positive in (x_1, x_2). Suppose that the solution $\theta(x)$ of $\theta' = \frac{q(x) - \tan^2}{1 + \tan^2}$ satisfies $n\pi \le \theta(x_3) \le \left(n + \frac{1}{2}\right)\pi$ at $x_3 \in (x_1, x_2)$. Then it follows*

$$n\pi < \theta(x) < \left(n + \frac{1}{2}\right)\pi \qquad \text{in} \quad (x_3, x_2).$$

Proof. $F(x, \theta) = \frac{q(x) - \tan^2\theta}{1 + \tan^2\theta}$ is negative if $\theta = \left(n + \frac{1}{2}\right)\pi$ and $F(x, \theta)$ is positive if $\theta = n\pi$ and $q(x) > 0$. Therefore we have

$$n\pi < \theta(x) < \left(n + \frac{1}{2}\right)\pi \qquad \text{in} \quad (x_3, x_2).$$

Corollary. *Suppose $q(x) > 0$ in (x_1, x_2). Then the variation of $\theta(x)$ in (x_1, x_2) is less than π.*

Now we decompose $\Omega_+(\lambda) = \{x;\, q(x) - \lambda > 0\}$ as follows

$$\Omega_+(\lambda) = (-\infty, x_-(\lambda)) \cup ((x_+(\lambda), \infty) \bigcup_{i=1}^{k} \omega_i(\lambda), \tag{6.1}$$

where $\omega_i(\lambda)$ are connected open finite intervals. From (5.8) we have

(6.2) $$-1 \leq \theta'_{\pm}(x, \lambda) \leq q(x)-\lambda \quad \text{if} \quad 0<q(x)-\lambda.$$

From the above Corollary and (6.2) we have

Theorem 2′. *Assume* (C). *Then we have the estimates*:

$$\underline{E}(\lambda)-\underline{F}(\lambda)-2\pi<R(x, \lambda)<\bar{E}(\lambda)+\bar{F}(\lambda)+\pi,$$

$$\underline{E}(\lambda_n)-F(\lambda_n)-\pi<R(\lambda_n)<\bar{E}(\lambda_n)+F(\lambda_n),$$

where $\underline{F}(\lambda)$ *and* $\bar{F}(\lambda)$ *are given below and other notations are the same ones as in the precedent sections. Incidentally* $u'_n(x)$ *has at most* $n+2k$ *zeros.*

$$\underline{F}(\lambda)=\sum_{i=1}^{k} \min\left\{\int_{\omega_i(\lambda)} (q(x)-\lambda)dx, \pi\right\}$$

$$\bar{F}(\lambda)=\sum_{i=1}^{k} \min\left\{\int_{\omega_i(\lambda)} 1\, dx, \pi\right\}.$$

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] L. D. Landau-E. M. Lifshitz, Quantum mechanics, Pergamon Press, 1967.
[2] S. Miyatake, Théorèmes de comparaison pour les équations defférentielles du second ordre, C. R. Acad. Sci., (5) **289** (1979), 345–347.
[3] S. Miyakate, Comparison theorem for differential equations of second order and its application to the Thomas — Fermi problem, to appear.
[4] F. W. J. Olver, Asymptotics and special functions, Academic Press, New York, 1973.
[5] W. Pauli, Pauli's lecture on physics, Quantum mechanics, Kodansha, Tokyo, 1976.
[6] E. C. Titchmarsh, Eigenfunction expansions associated with second order differential equations, Clarendon Press, Oxford, 1946.