

CONSTRUCTIONS OF SMALL SYMPLECTIC 4-MANIFOLDS USING LUTTINGER SURGERY

SCOTT BALDRIDGE & PAUL KIRK

Abstract

Luttinger surgery is used to produce minimal symplectic 4-manifolds with small Euler characteristics. We construct a minimal symplectic 4-manifold which is homeomorphic but not diffeomorphic to $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P}^2$, and which contains a genus two symplectic surface with trivial normal bundle and simply-connected complement. We also construct a minimal symplectic 4-manifold which is homeomorphic but not diffeomorphic to $3\mathbb{C}P^2 \# 5\overline{\mathbb{C}P}^2$, and which contains two disjoint essential Lagrangian tori such that the complement of the union of the tori is simply-connected.

These examples are used to construct minimal symplectic manifolds with Euler characteristic 6 and fundamental group \mathbb{Z} , \mathbb{Z}^3 , or $\mathbb{Z}/p \oplus \mathbb{Z}/q \oplus \mathbb{Z}/r$ for integers p, q, r . Given a group G presented with g generators and r relations, a symplectic 4-manifold with fundamental group G and Euler characteristic $10 + 6(g + r)$ is constructed.

1. Introduction

In this article we construct a number of small (as measured by the Euler characteristic e) simply connected and non-simply connected smooth 4-manifolds which admit symplectic structures. Specifically, we construct examples of:

- A minimal symplectic manifold X homeomorphic but not diffeomorphic to $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P}^2$ containing symplectic genus 2 surface with simply connected complement and trivial normal bundle, and a disjoint nullhomologous Lagrangian torus (Theorem 13).
- A minimal symplectic manifold B homeomorphic but not diffeomorphic to $3\mathbb{C}P^2 \# 5\overline{\mathbb{C}P}^2$ containing a disjoint pair of symplectic tori with simply connected complement and trivial normal bundle (Theorem 18). This provides a smaller substitute for the elliptic

The first author gratefully acknowledges support from the NSF grant DMS-0507857 and NSF Career Grant DMS-0748636. The second author gratefully acknowledges support from the NSF grant DMS-0604310.

Received 03/01/2007.

surface $E(1)$ in many 4-dimensional constructions.

- A minimal symplectic manifold X_1 with fundamental group \mathbb{Z} , Euler characteristic $e(X_1) = 6$, signature $\sigma(X_1) = -2$ containing a symplectic torus T with trivial normal bundle such that the inclusion $X_1 - T \subset X_1$ induces an isomorphism on fundamental groups and so that the inclusion $T \subset X_1$ kills one generator of $\pi_1(T)$ (Theorem 22). This also provides a smaller substitute for $E(1)$ when only one generator is to be killed.

Variations on these constructions quickly provide many more examples of small simply connected minimal symplectic manifolds, including manifolds homeomorphic but not diffeomorphic to $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$, $\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$, $3\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$, $3\mathbb{C}\mathbb{P}^2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$, and $5\mathbb{C}\mathbb{P}^2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$. Constructions of small manifolds can also be found in [2, 3, 6, 19, 27, 29, 32, 33, 34].

The manifolds X, B, X_1 form building blocks which we use to prove a number of results, including the following.

- There exists an infinite family of pairwise non-diffeomorphic smooth simply connected manifolds each homeomorphic to $\mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$.
- If a group G has a presentation with g generators and r relations, then there exists a symplectic 4-manifold M with fundamental group G , $e(M) = 10 + 6(g+r)$ and $\sigma(M) = -2(g+r+1)$ (Theorem 24).
- For any pair of non-negative integers m, n there exists a minimal symplectic manifold which is homeomorphic but not diffeomorphic to $(1 + 2m + 2n)\mathbb{C}\mathbb{P}^2 \# (3 + 6m + 4n)\overline{\mathbb{C}\mathbb{P}^2}$ (Corollary 19).
- For any integers p, q, r , there exists a symplectic manifold $X_{p,q,r}$ with fundamental group $\mathbb{Z}/p \oplus \mathbb{Z}/q \oplus \mathbb{Z}/r$ with $e = 6$ and $\sigma = -2$ (Corollary 30).
- If an abelian group G is generated by n elements with n even, then there exists a symplectic 4-manifold with fundamental group G , $e = \frac{1}{2}n^2 + \frac{19}{2}n + 36$ and $\sigma = -\frac{5}{2}n - 8$. (Theorem 31).
- For any non-negative integer n , there exists a symplectic 4-manifold with fundamental group free of rank n , $e = 10$ and $\sigma = -2$. (Theorem 25).
- For any symplectic manifold M containing a symplectic surface H of genus 1 or 2 with trivial normal bundle so that the homomorphism $\pi_1(H) \rightarrow \pi_1(M)$ induced by inclusion is trivial, there exists infinitely many smooth manifolds M_n with $\pi_1(M_n) = \pi_1(M)$, $e(M_n) = e(M) + 2 + 4 \text{ genus}(H)$, $\sigma(M_n) = \sigma(M) - 2$, and the Seiberg-Witten invariants of M_n are different from those of M_m if $n \neq m$ (Corollary 21).

We refer the reader to the body of the article for more precise statements of these theorems and further results. One particular feature of our constructions is that they contain nullhomologous Lagrangian tori for which the method of [10] allow us to produce families of infinitely many non-diffeomorphic but homeomorphic manifolds.

Our main tools are Luttinger and torus surgery [21, 1, 13], Gompf's symplectic sum construction [15], and, most importantly, the Seifert-Van Kampen theorem, which we use to prove our central result, Theorem 11. This is then combined with Freedman's theorem [14] and fundamental results from Seiberg-Witten theory [36, 37, 39, 23] in the applications.

A problem which motivates our investigations concerns uniqueness of the diffeomorphism type of a symplectic manifold which has the smallest Euler characteristic among symplectic manifolds with a fixed fundamental group. For example, for the trivial group, the "symplectic Poincaré conjecture" (cf. [5]) asks whether a symplectic manifold homeomorphic to $\mathbb{C}P^2$ is diffeomorphic to $\mathbb{C}P^2$. Many constructions on 4-manifolds are simpler to carry out when the Euler characteristic is large, and this has motivated the problem of finding interesting (e.g. exotic) simply connected or non-simply connected 4-manifolds with small Euler characteristic. As one works with smaller manifolds, it becomes difficult to alter the smooth structure without changing the fundamental group or destroying the existence of a symplectic structure.

Another question which motivates these results concerns, for a given group G , the gap between the smallest possible Euler characteristic of smooth 4-manifolds with fundamental group G , and the smallest possible Euler characteristic of symplectic 4-manifolds with fundamental group G , or the smallest possible Euler characteristic of complex surfaces with fundamental group G . For example, one can construct a smooth 4-manifold with fundamental group the finite cyclic group \mathbb{Z}/n and $e = 2$; this is the smallest possible. Corollary 30 establishes the existence of a symplectic 4-manifold with fundamental group \mathbb{Z}/n and $e = 6$, this is smallest currently known although it is possible that a smaller one exists. The smallest known complex surface with finite cyclic fundamental group has $e = 10$.

The paper is organized as follows. In Section 2 we describe Luttinger surgery and calculate the fundamental group of the complement of some tori in the 4-torus. In Section 3 we construct the three main building blocks needed for all subsequent constructions. In Section 4 we prove our main result, Theorem 11, which computes the fundamental group (and all meridians and Lagrangian push offs) of the complement of six Lagrangian tori and a symplectic genus two surface in a certain symplectic manifold Z satisfying $e(Z) = 6, \sigma(Z) = -2$ and $H_1(Z) = \mathbb{Z}^6$. With this result in place we construct the simply connected examples

described above and in Section 5 we construct the non-simply connected examples.

To the extent that the methods of the present article focus on quite involved calculations of fundamental groups, we take great care with our use of the Seifert-Van Kampen theorem, choice of representative loops, and choices of base points. Some of the fundamental group assertions we prove are perhaps not surprising. However, the introduction of unwanted conjugation at any stage can easily lead to a loss of control over fundamental groups, in particular leading to plausible but unverifiable calculations. Given the usefulness of our theorems and that such methods are not so common in 4-dimensional topology, we feel the care we take is justified.

Acknowledgments. The authors thank A. Akhmedov, R. Fintushel, C. Judge, C. Livingston, and J. Yazinsky for helpful discussions, and thank the referees for their careful reading and helpful suggestions.

2. The fundamental group of the complement of some tori in the 4-torus

2.1. Luttinger surgery. Given any Lagrangian torus T in a symplectic 4-manifold M , the Darboux-Weinstein theorem [22] implies that there is a parameterization of a tubular neighborhood of T , $T^2 \times D^2 \cong \text{nb}d(T) \subset M$, such that the image of $T^2 \times \{d\}$ is Lagrangian for all $d \in D^2$. Choosing any point $d \neq 0$ in D^2 determines a push off $F_d : T \rightarrow T^2 \times \{d\} \subset M - T$ called the *Lagrangian push off* or *Lagrangian framing*. Given any embedded curve $\gamma \subset T$, its image $F_d(\gamma)$ is called the *Lagrangian push off* of γ . The smooth isotopy class of the Lagrangian push off $F_d : T \rightarrow M - T$ depends only on the symplectic structure in a neighborhood of T . As is common we will abuse terminology slightly and call the isotopy class of $F_d(\gamma)$ for any $d \neq 0$ the Lagrangian push off of γ . Any curve isotopic to $\{t\} \times \partial D^2 \subset \partial(\text{nb}d(T))$ will be called a *meridian* of T and denoted by μ .

Fix $d \in \partial D$. If x, y are loops in T generating $H_1(T)$, let $m = F_d(x)$ and $\ell = F_d(y)$. Then the triple μ, m, ℓ generate $H_1(\partial(\text{nb}d(T)))$. Since the 3-torus has abelian fundamental group we may choose a base point t on $\partial(\text{nb}d(T))$ and unambiguously refer to $\mu, m, \ell \in \pi_1(\partial(\text{nb}d(T)), t)$.

The push offs and meridians are used to specify coordinates for a p/q torus surgery on T along γ . This is the process of removing a tubular neighborhood of T in M and re-gluing it so that the embedded curve representing $\mu^p F_d(\gamma)^q$ bounds a disk. The diffeomorphism type of the resulting manifold depends only on the isotopy class of the identification $T^2 \times D^2 \rightarrow \text{nb}d(T)$, and not on the particular point d or the specific choice of μ . Its fundamental group is isomorphic to

$$(1) \quad \pi_1(M - T)/N(\mu^p F_d(\gamma)^q)$$

where $N(\mu^p F_d(\gamma)^q)$ denotes the normal subgroup generated by $\mu^p F_d(\gamma)^q$.

When the base point of M is chosen off the boundary of the tubular neighborhood of T , the based loops μ and γ are to be joined to the base point *by the same path* in $M - T$. Then Equation (1) holds with respect to this choice of basing.

Note that if one fixes generating curves x, y on T , then the embedded curve γ can be expressed in $\pi_1(T)$ in the form $\gamma = x^a y^b$ for some relatively prime pair of integers a, b . In that case the fundamental group of the manifold obtained by p/q torus surgery on T along γ is

$$\pi_1(M - T)/N(\mu^p m^{aq} \ell^{bq})$$

where, as above, $m = F_d(x)$ and $\ell = F_d(y)$.

The special case of $p = 1, q = k$ is called $1/k$ *Luttinger surgery on T along the embedded curve $\gamma \subset T$* . This yields a symplectic manifold ([21, 1]). The symplectic form is unchanged away from a neighborhood of T . The fundamental group of the manifold obtained by $1/k$ Luttinger surgery on T along an embedded curve γ is isomorphic to

$$\pi_1(M - T)/N(\mu F_d(\gamma)^k)$$

where $N(\mu F_d(\gamma)^k)$ denotes the normal subgroup generated by $\mu F_d(\gamma)^k$.

It is sometimes convenient to adopt the language of 3-dimensional topology and call the process of gluing $T \times D^2$ to $M - nbd(T)$ a $1/k$ *Luttinger filling*, or, more generally, a p/q *torus filling*.

When $p \neq \pm 1$ there is no reason why the symplectic form should extend over the neighborhood of T , and typically the smooth manifold obtained by p/q surgery admits no symplectic structure when $p \neq \pm 1$.

2.2. The complement of two Lagrangian tori in the product of two punctured tori. Let \hat{H} and \hat{K} denote a pair of 2-tori, endowed with the standard symplectic form. Removing an open disk from \hat{H} and \hat{K} yields punctured tori $H = \hat{H} - D$ and $K = \hat{K} - D$. View $H \times K$ as a codimension 0 symplectic submanifold of $T^4 = \hat{H} \times \hat{K}$ with its standard product symplectic form. The product $H \times K$ should be considered as the complement of a tubular neighborhood of the (singular) union of two symplectic tori $(\hat{H} \times \{u_K\}) \cup (\{u_H\} \times \hat{K}) \subset \hat{H} \times \hat{K}$ (where u_H and u_K denote the centers of the disks removed.)

Choose a pair of curves x, y representing a standard generating set for $\pi_1(H)$ and a pair of curves a, b representing a standard generating set for $\pi_1(K)$. Let X, Y be parallel push offs of x and y in H and let A_1, A_2 be parallel push offs of a in K , as illustrated in the following figure. Let h be the intersection point of x and y and let k be the intersection point of a and b . Give $H \times K$ the base point (h, k) .

We define two disjoint tori T_1, T_2 in $H \times K$ as follows.

$$T_1 = X \times A_1 \text{ and } T_2 = Y \times A_2.$$

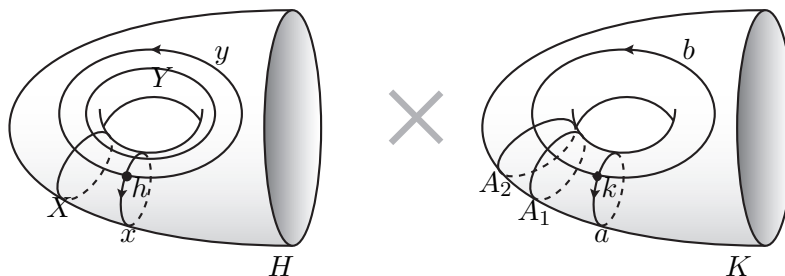


Figure 1. The surface $H \times K$.

Then the tori T_1 and T_2 are Lagrangian and the surfaces $H \times \{p\}$ and $\{q\} \times K$ are symplectic for any $p \in K$ and $q \in H$.

Let A'_i denote a push off of A_i into $K - (A_1 \cup A_2)$, $i = 1, 2$. Then the parallel tori $T'_1 = X \times A'_1$ and $T'_2 = Y \times A'_2$ are Lagrangian, and so the Lagrangian push off of a curve on T_i is its image in $H \times K - (T_1 \cup T_2)$ using this push off $T_i \rightarrow T'_i$. Sometimes it is preferable to use the push offs using the parallel tori $X' \times A_1$ or $Y' \times A_2$ where X' and Y' are parallel copies in H of X and Y . As we explained above, the manifolds resulting from torus and Luttinger surgery are well defined up to diffeomorphism.

The boundary of the tubular neighborhood of T_i is a 3-torus. Therefore $H_1(\partial(\text{nbhd}(T_i))) = \mathbb{Z}^3$, with generating set $\{\mu_i, m_i, \ell_i\}$, where μ_i is the meridian and m_i and ℓ_i the Lagrangian push offs of two generators of $H_1(T_i)$.

We specify notation for certain explicit loops in $H \times K$ based at (h, k) .

- 1) The loop $x \times \{k\} : I \rightarrow H \times \{k\}$ based at (h, k) will be denoted simply by x . This loop misses $T_1 \cup T_2$.
- 2) The loop $y \times \{k\} : I \rightarrow H \times \{k\}$ based at (h, k) will be denoted by y . This loop misses $T_1 \cup T_2$.
- 3) The loop $\{h\} \times a : I \rightarrow \{h\} \times K$ based at (h, k) will be denoted by a . This loop misses $T_1 \cup T_2$.
- 4) The loop $\{h\} \times b : I \rightarrow \{h\} \times K$ based at (h, k) will be denoted by b . This loop misses $T_1 \cup T_2$.

In [6, Section 2] we proved the following theorem.

Theorem 1. *There exist paths in $H \times K - (T_1 \cup T_2)$ from the base point (h, k) to the boundary of the tubular neighborhoods $T_1 \times \partial D^2$ and $T_2 \times \partial D^2$ with the following property.*

Denote by μ_i, m_i, ℓ_i the loops in $H \times K - (T_1 \cup T_2)$ based at (h, k) obtained by following the chosen path to the boundary of the tubular neighborhood of T_i , then following (respectively) the meridian of T_i and the two Lagrangian push offs of the generators on T_i , then returning to the base point along the chosen path.

Then in $\pi_1(H \times K - (T_1 \cup T_2), (h, k))$:

$$\mu_1 = [b^{-1}, y^{-1}], m_1 = x, \ell_1 = a,$$

and

$$\mu_2 = [x^{-1}, b], m_2 = y, \ell_2 = bab^{-1}.$$

where x, y, a, b are the loops described above.

Moreover, $\pi_1(H \times K - (T_1 \cup T_2), (h, k))$ is generated by x, y, a, b and the relations

$$[x, a] = 1, [y, a] = 1, [y, bab^{-1}] = 1$$

as well as

$$[[x, y], b] = 1, [x, [a, b]] = 1, [y, [a, b]] = 1$$

hold in $\pi_1(H \times K - (T_1 \cup T_2), (h, k))$.

q.e.d.

The two important things to note in this theorem are, first, the homotopy class of the loops x, y, a , and b based at (h, k) generate $\pi_1(H \times K - (T_1 \cup T_2), (h, k))$. Second, the explicit expressions for μ_i, m_i, ℓ_i allows us to list relations that hold in the fundamental group of the manifold obtained from torus surgery on the T_i in $H \times K$. For example, the relations

$$[b^{-1}, y^{-1}]x^2a^6 = 1 \text{ and } [x^{-1}, b]ba^{-1}b^{-1} = 1$$

hold in the fundamental group of the manifold obtained from $H \times K$ by performing $1/2$ surgery on T_1 along $m_1\ell_1^3$ and $-1/1$ surgery on T_2 along ℓ_2 .

We will also need the following result.

Lemma 2. Consider the manifold L obtained from $T^4 = \hat{H} \times \hat{K}$ by performing Luttinger surgeries on T_1 along m_1 and T_2 along either m_2 or ℓ_2 . Then L is aspherical.

Proof. Suppose L is obtained from T^4 by performing $1/k_1$ Luttinger surgery on T_1 along m_1 and $1/k_2$ Luttinger surgery on T_2 along γ , where $\gamma = m_2$ or ℓ_2 .

In the case when $\gamma = m_2$, it is straightforward to see that L is diffeomorphic to $U \times S^1$, where U is the 3-manifold that fibers over S^1 with fiber \hat{H} and monodromy $D_Y^{k_2} \circ D_X^{k_1}$, where D_X and D_Y are the positive Dehn twists along X and Y in \hat{H} . This is explained carefully in [1, pg. 189]. Thus the universal cover of L is \mathbb{R}^4 .

In the case when $\gamma = \ell_2$, it is not hard to show (see [4]) that L is diffeomorphic to a non-trivial S^1 bundle over U , where U is the 3-manifold that fibers over S^1 with fiber \hat{H} and monodromy $D_X^{k_1}$, and the first Chern class of the bundle is $k_2 \cdot PD_U([Y])$ where PD_U denotes Poincaré duality in U . Thus again the universal cover of L is \mathbb{R}^4 .

In either case L is aspherical.

q.e.d.

By symmetry, Lemma 2 holds as well if both surgeries are performed along ℓ_i .

3. Three small building blocks

3.1. Our first and simplest building block is the symplectic manifold $W_1 = (T^2 \times S^2) \# 4\overline{\mathbb{C}\mathbb{P}^2}$, containing a symplectic genus 2 surface F_1 with trivial normal bundle.

We construct the surface F_1 by starting with the union of two parallel copies $T^2 \times \{p_1\}$, $T^2 \times \{p_2\}$ of the torus factor and one copy of $\{q\} \times S^2$ in $T^2 \times S^2$. Each of these three surfaces is an embedded symplectic submanifold, and $\{q\} \times S^2$ intersects each of the tori in one point. We symplectically resolve the two double points (cf. [15]), to obtain a symplectic genus 2 surface F_1 of square $(2[T] + [S])^2 = 4$ in T^4 . Recall that, topologically, symplectically resolving corresponds to locally replacing a pair of transversely intersecting discs by an annulus.

Blowing up $T^2 \times S^2$ four times at points which lie on F_1 and taking the proper transform yields the desired $F_1 \subset W_1$. The surface F_1 has trivial normal bundle and W_1 contains an embedded -1 sphere intersecting F_1 transversally in exactly one point.

Let $\phi : F_1 \rightarrow W_1 - \text{nb}(F_1)$ be a push off of F_1 , and choose a base point $w \in \phi(F_1)$. Since W_1 contains a -1 sphere intersecting F_1 transversally, the meridian of F_1 is nullhomotopic in $W_1 - \text{nb}(F_1)$. Hence the inclusion $W_1 - \text{nb}(F_1) \rightarrow W_1$ induces an isomorphism on fundamental groups by transversality and the Seifert-Van Kampen theorem.

The two circle coordinates of T^2 define classes $s, t \in H_1(W_1)$. Given any base point in W_1 , we may unambiguously write $\pi_1(W_1) = \mathbb{Z}s + \mathbb{Z}t$, since $\pi_1(W_1)$ is abelian.

One can choose four loops s_1, t_1, s_2, t_2 on $\phi(F_1)$ based at w which generate $\pi_1(\phi(F_1), w)$ and so that $[s_1, t_1][s_2, t_2] = 1$ in $\pi_1(\phi(F_1), w)$ in such a way that the composite

$$\pi_1(F_1) \xrightarrow{\phi_*} \pi_1(W_1 - \text{nb}(F_1)) \cong \pi_1(W_1) \cong H_1(W_1)$$

takes s_1 to s , s_2 to s^{-1} , t_1 to t , and t_2 to t^{-1} . Thus we adopt the notation:

- 1) The loop $s_1 : I \rightarrow \phi(F_1) \subset \partial(W_1 - \text{nb}(F_1)) \subset W_1 - \text{nb}(F_1)$ based at w is a representative loop for the based homotopy class $s \in \pi_1(W_1, w)$.
- 2) The loop $t_1 : I \rightarrow \phi(F_1) \subset \partial(W_1 - \text{nb}(F_1)) \subset W_1 - \text{nb}(F_1)$ based at w is a representative loop for the based homotopy class $t \in \pi_1(W_1, w)$.

Then the following proposition holds.

Proposition 3. *The symplectic surface F_1 intersects an embedded sphere transversally in one point and the inclusion $W_1 - \text{nb}(F_1) \subset W_1$*

induces an isomorphism on fundamental groups. The inclusion $\phi(F_1) \subset W_1 - nbd(F_1)$ induces a surjection on fundamental groups.

Moreover, the loops s_1, t_1, s_2, t_2 on $\phi(F_1)$ can be chosen so that $\pi_1(W_1 - nbd(F_1), w) = \mathbb{Z}s \oplus \mathbb{Z}t$, where s, t are just the loops s_1, t_1 viewed as loops in $W_1 - F_1$, and so that the inclusion $\phi(F_1) \rightarrow W_1 - nbd(F_1)$ induces the homomorphism $s_1 \mapsto s, t_1 \mapsto t, s_2 \mapsto s^{-1}$, and $t_2 \mapsto t^{-1}$. Every -1 sphere in W_1 intersects F_1 .

Proof. The assertions about the fundamental group are explained above.

The four exceptional spheres all meet F_1 since the blowup was performed on F_1 . Denote by T, S, E_1, E_2, E_3 , and E_4 the five generators of $H_2(W_1)$, where $T = T^2 \times \{p\}$, $S = \{q\} \times S$, and the E_i are the exceptional classes. Thus $F_1 = 2T + S - E_1 - E_2 - E_3 - E_4$. The Hopf exact sequence $\pi_2(W_1) \rightarrow H_2(W_1) \rightarrow H_2(\pi_1(W_1)) \rightarrow 0$ shows that the spherical classes are spanned by S, E_1, E_2, E_3, E_4 (since T maps to the generator of $H_2(\pi_1(W_1))$). Consideration of the intersection form shows that a -1 sphere must have the form $aS \pm E_i$. Then $(aS \pm E_i) \cdot F_1 = 2a \pm 1 \neq 0$. Thus every -1 sphere intersects F_1 . q.e.d.

Suppose that P is any symplectic 4-manifold containing a symplectic surface G of genus 2 with trivial normal bundle. Then the symplectic sum, S , of W_1 and P along F_1 and G (cf. [15]) is a symplectic manifold described topologically as the union of $W_1 - nbd(F_1)$ and $P - nbd(G)$ along their boundary using a fiber preserving diffeomorphism $F_1 \times S^1 \rightarrow G \times S^1$ of the boundary of their tubular neighborhoods. The diffeomorphism type of the manifold S may depend on the choice of such a diffeomorphism, which can be specified up to isotopy by choosing trivializations of the tubular neighborhoods of F_1 and G and a diffeomorphism $\phi : F_1 \rightarrow G$. One then glues $W_1 - nbd(F_1)$ to $P - nbd(G)$ using the gluing diffeomorphism

$$\begin{aligned} \tilde{\phi} : \partial(W_1 - nbd(F_1)) &= F_1 \times S^1 \cong \partial(P - nbd(G)) = G \times S^1, \\ \tilde{\phi}(f, z) &= (\phi(f), z). \end{aligned}$$

Then

$$S = W_1 - nbd(F_1) \cup_{\tilde{\phi}} P - nbd(G).$$

The symplectic sum is defined more generally when G and F have normal bundles with opposite Euler class, i.e. if $[G]^2 = -[F]^2$. For our purposes it will suffice to consider symplectic sums along square zero surfaces. Moreover, the framings we use will either be explicit, or unimportant to the fundamental group calculations.

Assume that the base point p of P lies on G , and that $\phi : F_1 \rightarrow G$ is base point preserving, $\phi(w) = p$. Denote by N the subgroup of $\pi_1(P, p)$ normally generated by $\phi(s_1s_2), \phi(t_1t_2)$, and $\phi([s_1, t_1])$. Then

Proposition 3 and the Seifert-Van Kampen theorem imply that $\pi_1(S, w)$ is a quotient of $\pi_1(P, p)/N$.

More generally, one can replace W_1 and F_1 by any appropriate pair W, F . We state this formally:

Lemma 4. *Suppose the 4-manifold W contains a genus 2 surface F with trivialized normal bundle, and the 4-manifold P contains a genus 2 surface G with trivialized normal bundle. Let $\phi : F \rightarrow G$ be a diffeomorphism, and let $\tilde{\phi} = \phi \times Id : F \times S^1 \rightarrow G \times S^1$.*

Suppose that

- 1) F meets a sphere in W transversally in one point,
- 2) The inclusion $F \rightarrow W$ induces a surjection on fundamental groups.

Let

$$S = (W - \text{nb}(F)) \cup_{\tilde{\phi}} (P - \text{nb}(G)).$$

Then there is a surjection

$$\pi_1(P) \rightarrow \pi_1(S)$$

whose kernel contains $\phi(r)$, for every loop r in $\ker \pi_1(F) \rightarrow \pi_1(W)$.

q.e.d.

In our applications of this lemma we will typically use it to show S is simply connected, or use it when P is simply connected. In either of these cases base point issues will not matter. Notice also that choice of trivializations of the normal bundle do not affect the conclusion.

We state a similar but easier fact whose proof can be safely left to the reader.

Lemma 5. *Suppose the 4-manifold W contains a genus 2 surface F with trivialized normal bundle, with $W - F$ simply connected. Let P be a 4-manifold containing a genus 2 surface G with trivialized normal bundle. Let $\phi : F \rightarrow G$ be a diffeomorphism, and let $\tilde{\phi} = \phi \times Id : F \times S^1 \rightarrow G \times S^1$. Let*

$$S = (W - \text{nb}(F)) \cup_{\tilde{\phi}} (P - \text{nb}(G))$$

Then there is a surjection

$$\pi_1(P) \rightarrow \pi_1(S)$$

whose kernel contains the image of $\pi_1(F) \rightarrow \pi_1(P)$.

q.e.d.

3.2. Our second building block W_2 is similar to W_1 but starts with T^4 instead of $T^2 \times S^2$:

$$W_2 = T^2 \times T^2 \# 2\overline{\mathbb{C}\mathbb{P}^2}.$$

We use the calculations of Section 2 to identify two Lagrangian tori T_1 and T_2 in W_2 and calculate the fundamental group of $W_2 - (T_1 \cup T_2)$, as well as their meridians and Lagrangian push offs.

Recall from Section 2 that \hat{H}, \hat{K} are 2-tori, H is the complement of a small disk in \hat{H} , and K is the complement of a small disk in $\hat{K} = T^2$.

Denote $T^4 \# 2\overline{\mathbb{C}\mathbb{P}^2} = (\hat{H} \times \hat{K}) \# 2\overline{\mathbb{C}\mathbb{P}^2}$ by W_2 . Then W_2 contains a symplectic surface F_2 of genus 2 with trivial normal bundle. The construction is similar to that of $F_1 \subset W_1$. Start with the symplectic surface $\hat{H} \times \{k\} \cup \{h\} \times \hat{K} \subset T^4$. Symplectically resolve the double point to obtain a symplectic surface $F_2 \subset T^4$ of square $([H] + [K])^2 = 2$. Blow up at two points on F_2 to obtain W_2 and denote again by $F_2 \subset W_2$ the proper transform.

Notice that $W_2 - \text{nb}d(F_2)$ contains the two Lagrangian tori $T_1 = X \times A_1$ and $T_2 = Y \times A_2$ from Theorem 1. These Lagrangian tori miss the two exceptional spheres, since $\hat{H} \times \hat{K}$ is blown up at points on F_2 , which misses T_1 and T_2 .

Recall that $\pi_1(H \times K - (T_1 \cup T_2), (h, k))$ is generated by four loops, denoted by x, y, a, b in Section 2. The loops x, y lie on $H \times \{k\}$ and form a basis of $\pi_1(\hat{H})$ and the loops a, b lie on $\{h\} \times K$ and form a basis of $\pi_1(\hat{K})$. Choose a small 4-ball neighborhood $B_{(h,k)}$ of (h, k) . Since F_2 is constructed by desingularizing $\hat{H} \times \{k\} \cup \{h\} \times \hat{K}$, we may assume that F_2 coincides with $\hat{H} \times \{k\} \cup \{h\} \times \hat{K}$ outside $B_{(h,k)}$. One can choose loops s_1, t_1, s_2, t_2 on F_2 based at point w in $F_2 \cap B_{(h,k)}$ which form the standard generators of $\pi_1(F_2, w)$ (in particular the relation $[s_1, t_1][s_2, t_2] = 1$ holds) and which coincide with the loops x, y, a, b outside a small ball neighborhood of (h, k) .

Proposition 6. *The symplectic surface $F_2 \subset W_2$ intersects an embedded sphere transversally in one point. This sphere is disjoint from $T_1 \cup T_2$, and hence the inclusion $W_2 - \text{nb}d(F_2 \cup T_1 \cup T_2) \subset W_2 - \text{nb}d(T_1 \cup T_2)$ induces an isomorphism on fundamental groups, as does $Q - \text{nb}d(F) \subset Q$ for any manifold Q obtained by any torus surgeries on T_1 and T_2 in W_2 .*

The fundamental group $\pi_1(W_2 - \text{nb}d(T_1 \cup T_2), w)$ is generated by the loops s_1, t_1, s_2, t_2 , which lie on F_2 . The relations

$$[s_1, s_2] = 1, [t_1, s_2] = 1$$

as well as

$$[s_1, t_1] = 1, [s_2, t_2] = 1$$

hold in $\pi_1(W_2 - \text{nb}d(T_1 \cup T_2), w)$.

Moreover, one can choose paths in $W_2 - \text{nb}d(T_1 \cup T_2)$ from w to the boundary of the tubular neighborhoods of T_1 and T_2 so that the meridian and the two Lagrangian push offs of T_1 are represented in the fundamental group $\pi_1(W_2 - \text{nb}d(T_1 \cup T_2), w)$ as

$$\mu_1 = [t_2^{-1}, t_1^{-1}], m_1 = s_1, \ell_1 = s_2$$

and of T_2 are

$$\mu_2 = [s_1^{-1}, t_2], m_2 = t_1, \ell_2 = t_2 s_2 t_2^{-1} = s_2.$$

Proof. The assertions all follow from the construction and Theorem 1, except the relations $[s_1, t_1] = 1$ and $[s_2, t_2] = 1$. The relation $[s_1, t_1] = 1$ holds in $W_2 - (T_1 \cup T_2)$ because the loops s_1 and t_1 agree with the two generators x, y of the fundamental group of the torus $\hat{H} \times \{k\} \subset W_2 - (T_1 \cup T_2)$ outside a small 4-ball neighborhood of the base point. The relation $[s_2, t_2] = 1$ follows from the surface relation $[s_1, t_1][s_2, t_2]$, since F_2 lies in $W_2 - (T_1 \cup T_2)$. q.e.d.

Since F_2 meets a sphere in $W_2 - nbd(T_1 \cup T_2)$ transversally in one point, and $\pi_1(F_2) \rightarrow \pi_1(W_2 - nbd(T_1 \cup T_2))$ is surjective, Lemma 4 applies to the pair $(W_2 - nbd(T_1 \cup T_2), F_2)$. Thus if P is any manifold containing a genus 2 surface G with trivialized normal bundle, and $\phi : F_2 \rightarrow G$ is a diffeomorphism, then the sum

$$S = (W_2 - nbd(F_2)) \cup_{\tilde{\phi}} (P - nbd(G))$$

has fundamental group a quotient of $\pi_1(P, \phi(w))$, as does

$$S - nbd(T_1 \cup T_2) = (W_2 - nbd(F_2 \cup T_1 \cup T_2)) \cup_{\tilde{\phi}} (P - nbd(G)).$$

Applying Proposition 6 we conclude that

- 1) The kernel of the surjection $\pi_1(P, \phi(w)) \rightarrow \pi_1(S - nbd(T_1 \cup T_2), w)$ contains the classes

$$\phi([s_1, s_2]), \phi([t_1, s_2]), \phi([t_1, t_2 s_2 t_2^{-1}]), \phi([s_1, t_1]), \phi([s_2, t_2]).$$

- 2) The meridians and Lagrangian push offs of T_1 and T_2 in S with respect to appropriate paths to the boundary of their tubular neighborhood are given by the images of

$$\mu_1 = \phi([t_2^{-1}, t_1^{-1}]), m_1 = \phi(s_1), \ell_1 = \phi(s_2)$$

and

$$\mu_2 = \phi([s_1^{-1}, t_2]), m_2 = \phi(t_1), \ell_2 = \phi(s_2)$$

under the surjection

$$\pi_1(P, \phi(w)) \rightarrow \pi_1(S - nbd(T_1 \cup T_2), w)$$

and hence if S' is obtained from $1/k_i$ Luttinger surgery on T_i along $\gamma_i = m_i^{p_i} \ell_i^{q_i}$ for $i = 1, 2$, then the kernel of the corresponding surjection

$$\pi_1(P, \phi(w)) \rightarrow \pi_1(S')$$

contains the classes $\phi([t_2^{-1}, t_1^{-1}](s_1^{p_1} s_2^{q_1})^{k_1})$ and $\phi([s_1^{-1}, t_2](t_1^{p_2} s_2^{q_2})^{k_2})$.

3.3. The final and most complicated building block M is a product $\hat{H} \times \Sigma$ of a torus \hat{H} with a genus 2 surface Σ . Give M the product symplectic form. We will identify four Lagrangian tori T_1, T_2, T_3 , and T_4 and a genus two symplectic surface F in M which are pairwise disjoint and compute the fundamental group of $M - \text{nb}d(F \cup_{i=1}^4 T_i)$ and all meridians and Lagrangian push offs.

In contrast to W_1 and W_2 , M contains no exceptional spheres since $\pi_2(M) = 0$. In particular the inclusion

$$M - \text{nb}d(F \cup_{i=1}^4 T_i) \subset M - \text{nb}d(\cup_{i=1}^4 T_i)$$

does not induce an isomorphism on fundamental groups. Thus we will have to be extremely careful when choosing generating loops and computing the fundamental groups of symplectic sums with M .

Our approach is to view M as the union of two copies of $\hat{H} \times K$ from Section 2. The main technical difficulty which arises is that of identifying the generators of the fundamental group of the boundary of a tubular neighborhood of F to the generators constructed from Theorem 1. This is critical in order to properly set up the use of the Seifert-Van Kampen theorem.

Let D be a disk with center u in \hat{H} and identify the complement of D with the surface H of Section 2. Thus we have curves x, y, X, Y and the point h in \hat{H} for Figure 1. To each point $q \in \hat{H}$, write

$$\Sigma_q = \{q\} \times \Sigma.$$

The surface Σ_u corresponding to the center u of the disk D will play a special role in the following, so that we denote it by F :

$$F = \{u\} \times \Sigma.$$

The surfaces Σ_q are symplectic for all q . Moreover, if q misses $X \cup Y$ then Σ_q misses all the T_i . Fix h' in the boundary of the disk D and choose an arc α in \hat{H} joining h' to h , as in Figure 2.

Next view the genus 2 surface Σ as the union of two copies of K along their boundary, $\Sigma = K_1 \cup_{\partial K_1 = \partial K_2} K_2$. Thus we have curves a_1, b_1 on K_1 and a_2, b_2 on K_2 . Choose arcs β_1 (resp. β_2) from a point k' on the circle separating K_1 and K_2 in Σ to the intersection point k_1 of a_1 and b_1 (resp. k_2 of a_2 and b_2). Use the β_i to define the corresponding based homotopy classes which satisfy $[a_1, b_1][a_2, b_2] = 1$ in $\pi_1(K, k')$. Choose two loops A_1, A_2 parallel to a_1 in K_1 and A_3, A_4 parallel to a_2 in K_2 .

The notation is illustrated in Figure 2.

The product $M = \hat{H} \times \Sigma$ contains four disjoint Lagrangian tori $T_1 = X \times A_1, T_2 = Y \times A_2, T_3 = X \times A_3$ and $T_4 = Y \times A_4$ and the symplectic surface $F = \Sigma_u$. These five surfaces are pairwise disjoint.

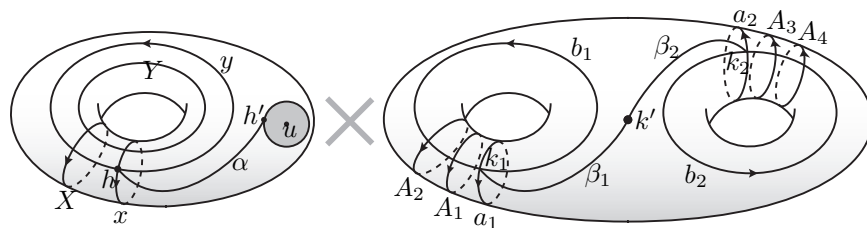


Figure 2. The surface $\hat{H} \times \Sigma$.

The boundary $\partial D \times \Sigma$ of the tubular neighborhood of F in M contains the push off $\Sigma_{h'}$ of F , as well as a meridian $\mu_F = \partial D \times \{k'\}$. We think of $\partial(\text{nb}d(F))$ as $\mu_F \times \Sigma_{h'}$, with base point (h', k') .

The work we do in the rest of this subsection amounts to finding loops on $\Sigma_{h'}$ and paths between the different base points to allow us to understand the homomorphism

$$\pi_1(\partial(\text{nb}d(F))) \rightarrow \pi_1(M - \text{nb}d(F \cup_{i=1}^4 T_i))$$

explicitly.

For convenience denote by N the open tubular neighborhood in M of the union of F and the Lagrangian tori:

$$N = (D \times \Sigma) \cup \text{nb}d(T_1 \cup T_2 \cup T_3 \cup T_4).$$

Give M the base point $p = (h', k')$ on the boundary of the tubular neighborhood of F . We define six loops in $M - N$ based at p .

- 1) The loop $x \times \{k'\}$ lies on $H \times \{k'\} \subset M - N$ and is based at (h, k') . We conjugate this by the path $\alpha \times \{k'\}$ to define a loop \tilde{x} based at $p = (h', k')$:

$$\tilde{x} = (\alpha * x * \alpha^{-1}) \times \{k'\} : I \rightarrow H \times \{k'\} \subset M - N.$$

- 2) The loop $y \times \{k'\}$ lies on $H \times \{k'\} \subset M - N$ and is based at (h, k') . We conjugate this by the path $\alpha \times \{k'\}$ to define a loop \tilde{y} based at p :

$$\tilde{y} = (\alpha * y * \alpha^{-1}) \times \{k'\} : I \rightarrow H \times \{k'\} \subset M - N.$$

- 3) The loops a_1, b_1, a_2, b_2 on Σ defined above determine loops on $\Sigma_{h'} \subset \partial(D \times \Sigma)$ based at p :

$$\tilde{a}_1 = \{h'\} \times a_1$$

$$\tilde{b}_1 = \{h'\} \times b_1$$

$$\tilde{a}_2 = \{h'\} \times a_2$$

$$\tilde{b}_2 = \{h'\} \times b_2.$$

Thus the loops $\tilde{a}_1, \tilde{b}_1, \tilde{a}_2$, and \tilde{b}_2 lie on the push off $\Sigma_{h'}$ of F in the boundary of the tubular neighborhood of F . Together with the loop $\mu_F = \partial D \times \{k'\}$, they generate the fundamental group of $\partial(\text{nb}d(F)) = \partial D \times \Sigma_{h'}$ based at p .

By contrast, away from the base point, the loops \tilde{x} and \tilde{y} lie in the interior of $M - N$. However, their commutator $[\tilde{x}, \tilde{y}]$ equals μ_F in the group $\pi_1(M - N, p)$, since the punctured torus $H \times \{k'\} \subset M - N$ has boundary μ_F .

At first glance, the following proposition may appear to be a direct application of the Seifert-Van Kampen applied to two copies of the manifold of Theorem 1. However, the base point in Theorem 1 does not lie on the boundary. Thus we must change base point *and* homotope appropriate loops into the boundary of $M - N$, being careful not to homotope the loops through N in the process.

Proposition 7. *The fundamental group $\pi_1(M - \text{nb}d(F \cup_{i=1}^4 T_i), p)$ is generated by $\tilde{x}, \tilde{y}, \tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2$ and the relations*

$$1 = [\tilde{x}, \tilde{a}_1] = [\tilde{y}, \tilde{a}_1] = [\tilde{y}, \tilde{b}_1 \tilde{a}_1 \tilde{b}_1^{-1}] = [\tilde{x}, \tilde{a}_2] = [\tilde{y}, \tilde{a}_2] = [\tilde{y}, \tilde{b}_2 \tilde{a}_2 \tilde{b}_2^{-1}]$$

hold in this group. With respect to certain paths to the boundary of the tubular neighborhoods of the T_i , the meridian and two Lagrangian push offs are given by

- 1) $T_1 : \mu_1 = [\tilde{b}_1^{-1}, \tilde{y}^{-1}], m_1 = \tilde{x}, \ell_1 = \tilde{a}_1,$
- 2) $T_2 : \mu_2 = [\tilde{x}^{-1}, \tilde{b}_1], m_2 = \tilde{y}, \ell_2 = \tilde{b}_1 \tilde{a}_1 \tilde{b}_1^{-1},$
- 3) $T_3 : \mu_3 = [\tilde{b}_2^{-1}, \tilde{y}^{-1}], m_3 = \tilde{x}, \ell_3 = \tilde{a}_2,$
- 4) $T_4 : \mu_4 = [\tilde{x}^{-1}, \tilde{b}_2], m_4 = \tilde{y}, \ell_4 = \tilde{b}_2 \tilde{a}_2 \tilde{b}_2^{-1}.$

The loops $\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2$ lie on the genus 2 surface $\Sigma_{h'}$ and form a standard set of generators (so $[\tilde{a}_1, \tilde{b}_1][\tilde{a}_2, \tilde{b}_2] = 1$). These four loops and a meridian μ_F generate the fundamental group of the boundary of the tubular neighborhood of F , and μ_F is homotopic to $[\tilde{x}, \tilde{y}]$ in the group $\pi_1(M - \text{nb}d(F \cup_{i=1}^4 T_i), p)$.

(Please see the remark which follows the proof.)

Proof. First notice that the punctured torus $H \times \{k'\}$ misses the tubular neighborhood N . Since the path $\alpha \times \{k'\}$ lies in $H \times \{k'\}$, the boundary of this punctured torus represents the same based homotopy class as $[\tilde{x}, \tilde{y}]$ in $\pi_1(M - N, p)$. This represents the meridian μ_F .

The boundary of the tubular neighborhood of F is trivialized by the push off $\Sigma_{h'}$. The curves $\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2$ lie on this push off and so these four loops and μ_F generate the fundamental group of the boundary of the tubular neighborhood of F , based at p .

Let $S \subset \Sigma$ denote the circle separating Σ into the two punctured tori K_1 and K_2 . Cutting $M - \text{nb}d(F)$ along $H \times S$ exhibits $M - \text{nb}d(F)$ as the union of two copies of $H \times K$, where H and K are punctured tori.

The first copy $H_1 \times K_1$ contains the two Lagrangian tori T_1 and T_2 and the other contains the tori T_3 and T_4 .

After cutting $M - nbd(F)$, the surface $H \times \{k'\}$ appears as the codimension 1 submanifold $H_1 \times \{k'_1\}$ of $\partial(H_1 \times K_1)$ and as the submanifold $H_2 \times \{k'_2\}$ of $\partial(H_2 \times K_2)$. Call the copies of \tilde{x} and \tilde{y} that appear in $H_1 \times \{k'_1\}$ \tilde{x}_1 and \tilde{y}_1 , and in the other component \tilde{x}_2 and \tilde{y}_2 . The copy of $\{h'\} \times S$ (oriented and based) in $H_1 \times K_1$ represents $[\tilde{a}_1, \tilde{b}_1]$ and in $H_2 \times K_2$ represents $[\tilde{a}_2, \tilde{b}_2]^{-1}$.

The Seifert-Van Kampen theorem shows that the fundamental group $\pi_1(M - nbd(F \cup_{i=1}^4 T_i), p)$ is the quotient of the free product

$$\pi_1(H_1 \times K_1 - (T_1 \cup T_2), (h'_1, k'_1)) * \pi_1(H_2 \times K_2 - (T_3 \cup T_4), (h'_2, k'_2))$$

by the normal subgroup generated by $\tilde{x}_1 \tilde{x}_2^{-1}$, $\tilde{y}_1 \tilde{y}_2^{-1}$ and $[\tilde{a}_1, \tilde{b}_1][\tilde{a}_2, \tilde{b}_2]$. In particular, the loops $\tilde{x}, \tilde{y}, \tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2$ generate the fundamental group $\pi_1(M - nbd(F \cup_{i=1}^4 T_i), p)$.

We reduce the proof to Theorem 1 by working one side at a time, and so, to ease eye strain, we drop the subscripts 1, 2. Here is what is to be shown: We have loops $\tilde{x}, \tilde{y}, \tilde{a}, \tilde{b}$ in $H \times K - (T_1 \cup T_2)$ based at (h', k') defined earlier in this section, and loops x, y, a, b based at (h, k) defined in the paragraph preceding the statement of Theorem 1. The loops x, y, a, b satisfy the conclusions which we will show the $\tilde{x}, \tilde{y}, \tilde{a}, \tilde{b}$ satisfy.

We first move from $p = (h', k')$ to (h, k') . Recall we have the path α from h' to h in H . We let $\tilde{\alpha}$ denote the path $\alpha \times \{k'\}$. Then conjugation by the path $\tilde{\alpha}^{-1}$ defines an isomorphism

$$\begin{aligned} \Psi_1 : \pi_1(H \times K - (T_1 \cup T_2), p) &\rightarrow \pi_1(H \times K - (T_1 \cup T_2), (h, k')), \\ \Psi_1(\gamma) &= \tilde{\alpha}^{-1} * \gamma * \tilde{\alpha}. \end{aligned}$$

From the definition preceding the statement of Proposition 7 we see that $\Psi_1(\tilde{x})$ and $\Psi_1(\tilde{y})$ are homotopic rel (h, k') to the loops $x \times \{k'\}$ and $y \times \{k'\}$, since e.g.

$$\Psi_1(\tilde{x}) = \tilde{\alpha}^{-1} * \tilde{x} * \tilde{\alpha} = \tilde{\alpha}^{-1} \tilde{\alpha} * (x \times \{k'\}) * \tilde{\alpha}^{-1} \tilde{\alpha} \sim x \times \{k'\}.$$

Recall that \tilde{a} takes the form $\{h'\} \times (\beta * a * \beta^{-1})$, where β is the given path in K from k' to k , and similarly for \tilde{b} .

The free homotopy $t \mapsto \{\alpha(t)\} \times (\beta * a * \beta^{-1})$ from \tilde{a} to $\{h\} \times (\beta * a * \beta^{-1})$ misses $T_1 \cup T_2$ and drags the base point along $\tilde{\alpha}$. Hence $\Psi_1(\tilde{a})$ is represented by the loop $\{h\} \times (\beta * a * \beta^{-1})$ which lies on $\{h\} \times K$. Similarly $\Psi_1(\tilde{b})$ is represented by the loop $\{h\} \times (\beta * b * \beta^{-1})$.

Now we use conjugation by the path $\tilde{\beta} = \{h\} \times \beta$ to define an isomorphism

$$\Psi_2 : \pi_1(H \times K - (T_1 \cup T_2), (h, k')) \rightarrow \pi_1(H \times K - (T_1 \cup T_2), (h, k))$$

$$\Psi_2(\gamma) = \tilde{\beta}^{-1} * \gamma * \tilde{\beta}.$$

This takes the loop $\Psi_1(\tilde{a}) = \{h\} \times (\beta * a * \beta^{-1})$ to $\{h\} \times a$:

$$\Psi_2(\Psi_1(\tilde{a})) = \tilde{\beta}^{-1} * (\{h\} \times (\beta * a * \beta^{-1})) * \tilde{\beta} \sim \{h\} \times a.$$

Similarly $\Psi_2(\Psi_1(\tilde{b})) = \{h\} \times b$. These are the loops simply denoted by a and b in Theorem 1.

The free homotopy $t \mapsto x \times \{\beta(t)\}$ starts at $x \times \{k'\} = \Psi_1(\tilde{x})$ and ends at $x \times \{k\}$, which is the loop labeled by x in Theorem 1. Moreover, the loop $x \times \{\beta(t)\}$ misses $T_1 \cup T_2$, since β avoids A_1 and A_2 . Since this free homotopy drags the base point along $\{h\} \times \beta = \tilde{\beta}$, it shows that $\Psi_2(\Psi_1(\tilde{x})) \sim x \times \{k\}$. Similarly $\Psi_2(\Psi_1(\tilde{y})) \sim y \times \{k\}$.

Thus we have found a path $\tau = \tilde{\beta} * \tilde{\alpha}$ in $H \times K - (T_1 \cup T_2)$ from (h, k) to (h', k') and proven that the isomorphism

$$\pi_1(H \times K - (T_1 \cup T_2), (h', k')) \rightarrow \pi_1(H \times K - (T_1 \cup T_2), (h, k))$$

given by conjugating by τ^{-1} takes (the based homotopy classes of) $\tilde{x}, \tilde{y}, \tilde{a}, \tilde{b}$ to (the based homotopy classes of) x, y, a, b . Hence any relation satisfied by x, y, a, b in $\pi_1(H \times K - (T_1 \cup T_2), (h, k))$ is also satisfied by $\tilde{x}, \tilde{y}, \tilde{a}, \tilde{b}$ in $\pi_1(H \times K - (T_1 \cup T_2), (h', k'))$.

Moreover, if one takes the paths from (h', k') to the boundary of the tubular neighborhood of T_i to be the composite of τ and the path given in Theorem 1, then e.g. the meridian of T_1 with respect to this path is

$$\tau * \mu_1 * \tau^{-1} = (\Psi_2 \circ \Psi_1)^{-1}(\mu_1) = (\Psi_2 \circ \Psi_1)^{-1}([b^{-1}, y^{-1}]) = [\tilde{b}^{-1}, \tilde{y}^{-1}].$$

A similar argument establishes the calculations for the other meridian and the Lagrangian push offs.

Applying the argument on each half $H_i \times K_i$ $i = 1, 2$ and using the Seifert-Van Kampen theorem finishes the proof. q.e.d.

Remark. To simplify notation, for the rest of this paper we drop the decorations, and so we will denote \tilde{x} simply by x and similarly for the others. Thus the explicit loops in $M - N$ based at $p = (h', k')$ defined prior to Proposition 7 will be denoted by x, y, a_1, b_1, a_2, b_2 . a_1, b_1, a_2, b_2 are loops that lie on $\Sigma_{h'}$ and together with μ_F generate the fundamental group of the boundary of the tubular neighborhood of F .

The loops x, y lie on the surface $H \times \{k'\}$ (and in particular in the interior of $M - N$ away from p). The meridian μ_F equals $[x, y]$ in $\pi_1(M - N, p)$, and the loops x, y, a_1, b_1, a_2, b_2 generate $\pi_1(M - N, p)$, with relations, meridians, and Lagrangian push offs as given in Proposition 7.

4. Constructions of small simply symplectic manifolds

4.1. We start, as a warm up, with a construction of a minimal symplectic manifold homeomorphic but not diffeomorphic to $\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$. Such examples are known [27, 25]; we include it because our construction illustrates the kind of fundamental group calculations we will do below in a simple case.

Theorem 8. *One can perform two Luttinger surgeries on the symplectic sum of W_1 and W_2 along F_1 and F_2 to produce a minimal symplectic manifold U homeomorphic but not diffeomorphic to $\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$.*

Proof. Form the symplectic sum $S = W_1 - \text{nb}d(F_1) \cup_{\tilde{\phi}} W_2 - \text{nb}d(F_2)$ using the gluing diffeomorphism $\phi : F_1 \rightarrow F_2$ which take the loops denoted by s_1, t_1, s_2, t_2 on F_1 to their namesakes on F_2 . The Lagrangian tori T_1, T_2 in W_2 remain Lagrangian in S ([16, Theorem 10.2.1]).

Lemma 4 shows that $\pi_1(S - (T_1 \cup T_2))$ is a quotient of the group $\pi_1(W_2 - (T_1 \cup T_2))$ and the kernel of the surjection contains the classes $s_1 s_2, t_1 t_2$, and $[s_1, t_1]$. Applying Proposition 6 we see that the group $\pi_1(S - (T_1 \cup T_2))$ is a quotient of the group generated by s_1, t_1 and the relation $[s_1, t_1] = 1$ holds, i.e. $\pi_1(S - (T_1 \cup T_2))$ is a quotient of $\mathbb{Z}s_1 \oplus \mathbb{Z}t_1$. Moreover, the meridians and Lagrangian push offs of the tori T_i are given by

$$\mu_1 = 1, m_1 = s_1, \ell_1 = s_1^{-1}$$

and

$$\mu_2 = 1, m_2 = t_1, \ell_2 = s_1^{-1}.$$

We perform Luttinger surgeries on T_1 and T_2 in S or, equivalently, Luttinger fillings on $S - \text{nb}d(T_1 \cup T_2)$. Then $-1/1$ Luttinger surgery on T_1 along m_1 kills s_1 and $-1/1$ Luttinger surgery on T_2 along m_2 then kills t_1 , yielding a simply connected symplectic manifold U . Notice that Luttinger surgery does not change the Euler characteristic nor the signature of a 4-manifold. Thus we have

$$e(U) = e(W_1) + e(W_2) + 4 = 10$$

and, using Novikov additivity,

$$\sigma(U) = \sigma(W_1 - \text{nb}d(F_1)) + \sigma(W_2 - \text{nb}d(F_2)) = \sigma(W_1) + \sigma(W_2) = -6.$$

Freedman's theorem [14] then implies that the 4-manifold U is homeomorphic to $\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$.

We showed that every -1 sphere in W_1 meets F_1 in Proposition 3. Let W'_2 be the manifold obtained from W_2 by performing the Luttinger surgeries as described, so that U is the symplectic sum of W_1 and W'_2 . To see that every -1 sphere in W'_2 intersects F_2 takes a bit more work. Notice that W'_2 is obtained by performing the two Luttinger surgeries on T_1 and T_2 in $T^4 = \hat{H} \times \hat{K}$ and then blowing up twice along F_2 .

These are Luttinger surgeries along m_1 and m_2 , hence by Lemma 2, W'_2 is obtained by blowing up an aspherical manifold L twice along points on F_2 .

Since L is aspherical and $\pi_1(W'_2) \rightarrow \pi_1(L)$ is an isomorphism, the Hopf sequence (see [9, pg. 265])

$$\pi_2(W'_2) \rightarrow H_2(W'_2) \rightarrow H_2(L) \rightarrow 0$$

is exact, where $W'_2 \rightarrow L$ is the map that collapses the two exceptional spheres (i.e. the blow-down map). The kernel of $H_2(W'_2) \rightarrow H_2(L)$ is clearly generated by the two exceptional spheres E_1 and E_2 , and therefore every spherical class in $H_2(W'_2)$ has the form $aE_1 + bE_2$. In particular, the only -1 spheres are $\pm E_1$ and $\pm E_2$, and both of these intersect F .

If W_1 were an S^2 bundle with section F_1 , then $\pi_1(W_1)$ could not be \mathbb{Z}^2 . If W'_2 were an S^2 bundle with section F_2 , then the exact sequence in homotopy groups of a fibration would show that $\pi_2(W'_2)$ equals $\pi_2(S^2) = \mathbb{Z}$. But we showed in the previous paragraph that the rank of the image of the Hurewicz map equals 2. (One can also compute directly that $H_1(W'_2)$ is generated by s_2 and t_2 , so that $\pi_1(W'_2) \not\cong \pi_1(F_2)$.) Thus W'_2 cannot be an S^2 bundle over F_2 . Applying Usher’s theorem [39], we conclude that U is a minimal symplectic 4-manifold.

By results of Taubes, [38, Theorem 0.2], a minimal symplectic 4-manifold cannot contain a smoothly embedded -1 sphere, but the manifold $\mathbb{C}P^2 \# 7\overline{\mathbb{C}P}^2$ contains smoothly embedded -1 spheres, namely, the exceptional spheres. Hence U cannot be diffeomorphic to $\mathbb{C}P^2 \# 7\overline{\mathbb{C}P}^2$, since this would contradict the minimality of U . q.e.d.

4.2. Our next example is more involved. We take a symplectic sum of W_1 and M and perform four Luttinger surgeries to produce a minimal symplectic manifold homeomorphic but not diffeomorphic to $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P}^2$ (cf. [29, 2]).

Consider the surface H and its curves x, y, X, Y and base point h in Figure 1. Let D_X and D_Y denote the Dehn twists along X and Y . We leave the proof of the following simple lemma to the reader.

Lemma 9. *The composite of Dehn twists along X and Y given by $D_X D_Y D_X$ takes x to y^{-1} and y to xyx^{-1} in $\pi_1(H, h)$. q.e.d.*

Note Lemma 9 holds in $\pi_1(H, h')$ if we replace x, y by $\alpha * x * \alpha^{-1}$, $\alpha * y * \alpha^{-1}$ where α is the path in H from the boundary point h' to h which misses X and Y . Moreover, $D_X D_Y D_X$ is the identity on the boundary, so that Lemma 9 extends to higher genus surfaces.

Consider the surface $F_1 \subset W_1$ with base point w and $\Sigma_{h'} \subset M$ with base point $p = (h', k')$. Denote by $\phi : F_1 \rightarrow \Sigma_{h'}$ the base point preserving diffeomorphism obtained by composing the diffeomorphism

of Lemma 9 (applied to $H = F_1$) with a diffeomorphism $F_1 \rightarrow \Sigma_{h'}$ taking s_i to a_i and t_i to b_i . This induces the isomorphism on fundamental groups

$$\phi : \pi_1(F_1, w) \rightarrow \pi_1(\Sigma_{h'}, p), (s_1, t_1, s_2, t_2) \mapsto (b_1^{-1}, b_1 a_1 b_1^{-1}, a_2, b_2)$$

and extends to $\tilde{\phi} : F_1 \times S^1 \rightarrow \Sigma_{h'} \times S^1$. The symplectic sum

$$V' = (W_1 - nbd(F_1)) \cup (M - nbd(\Sigma_{h'}))$$

contains the four Lagrangian tori T_1, T_2, T_3, T_4 and a symplectic surface F so that these five surfaces are pairwise disjoint.

Using Lemma 4 and the facts that $s_1 = s_2^{-1}$, $t_1 = t_2^{-1}$, and $[s_1, t_1]$ in $\pi_1(W_1)$, we see that $b_1 = a_2$ in $\pi_1(V' - nbd(F \cup_{i=1}^4 T_i), w)$. The relation $[s_1, t_1] = 1$ implies that $1 = [b_1^{-1}, b_1 a_1 b_1^{-1}] = [a_1, b_1^{-1}]$ and hence $[a_1, a_2] = [a_1, b_1] = 1$ and $b_2 = a_1^{-1}$ hold in $\pi_1(V' - nbd(F \cup_{i=1}^4 T_i), w)$.

Therefore, $\pi_1(V' - nbd(F \cup_{i=1}^4 T_i), w)$ is generated by x, y, a_1, a_2 and the relation

$$[a_1, a_2] = 1$$

holds. The relations

$$[x, a_1] = 1, [y, a_1] = 1, [x, a_2] = 1, [y, a_2] = 1$$

coming from $M - nbd(F \cup_{i=1}^4 T_i)$ were established in Proposition 7, and by the Seifert-Van Kampen theorem also hold in $\pi_1(V' - nbd(F \cup_{i=1}^4 T_i))$.

Furthermore, it follows from Proposition 7 that the meridian of F is

$$\mu_F = [x, y].$$

and the meridians and Lagrangian push offs of the tori T_i are given by

- $T_1 : \mu_1 = [a_2^{-1}, y^{-1}] = 1, m_1 = x, \ell_1 = a_1.$
- $T_2 : \mu_2 = [x^{-1}, a_2] = 1, m_2 = y, \ell_2 = a_2 a_1 a_2^{-1} = a_1.$
- $T_3 : \mu_3 = [a_1, y^{-1}] = 1, m_3 = x, \ell_3 = a_2.$
- $T_4 : \mu_4 = [x^{-1}, a_1^{-1}] = 1, m_4 = y, \ell_4 = a_1^{-1} a_2 a_1 = a_2.$

Theorem 10. *There exists a minimal symplectic manifold V homeomorphic but not diffeomorphic to $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$ containing a symplectic genus 2 surface F with simply connected complement and trivial normal bundle.*

Proof. We do four Luttinger surgeries on $T_1, T_2, T_3,$ and T_4 in $V' - F$. First, $-1/1$ Luttinger surgery on T_1 along m_1 kills x . Then $-1/1$ surgery on T_2 along ℓ_2 kills a_1 . Next, $-1/1$ surgery on T_3 along ℓ_3 kills a_2 . Finally, $-1/1$ surgery on T_4 along m_4 kills y .

Thus we have produced a symplectic 4-manifold V such that $V - F$ is simply connected. The Seifert-Van Kampen theorem shows that V is simply connected. The Euler characteristic and signature of V are the same as V' since they are unchanged by Luttinger surgery. Hence

$$e(V) = e(W_1) + e(M) + 4 = 4 + 0 + 4 = 8$$

and

$$\sigma(V) = \sigma(W_1) + \sigma(M) = -4 + 0 = -4.$$

Since V has an indefinite, odd intersection form (its signature is not a multiple of 8), Freedman’s theorem [14] then implies that V is homeomorphic to $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$.

Let M' be the manifold obtained from $M = \hat{H} \times \Sigma$ by doing the four Luttinger surgeries described above. Minimality of V follows from Usher’s theorem [39] just as in the proof of Theorem 8 once we show that M' is minimal, and not an S^2 bundle with section $\Sigma_{h'}$. Note that M' is the symplectic sum of two manifolds M'_1 and M'_2 along the torus \hat{H} .

Indeed, view $M = \hat{H} \times \Sigma$ as the union of $\hat{H}_1 \times K_1$ and $\hat{H}_2 \times K_2$ along $\hat{H} \times S$ as in the proof of Proposition 7. This exhibits M as the symplectic sum of $\hat{H}_1 \times \hat{K}_1$ and $\hat{H}_2 \times \hat{K}_2$ along the symplectic surfaces $\hat{H}_1 \times \{v_1\}$ and $\hat{H}_2 \times \{v_2\}$, where $v \in \hat{K}$ is the center of the disk whose complement is K . So let M'_1 denote the manifold obtained by Luttinger surgery on T_1 and T_2 in the 4-torus $\hat{H}_1 \times \hat{K}_1$ and M'_2 denote the manifold obtained by Luttinger surgery on T_3 and T_4 in the 4-torus $\hat{H}_2 \times \hat{K}_2$.

Lemma 2 implies that M'_1 and M'_2 are aspherical, hence minimal. In $H_1(M'_1)$, the generators x and a_1 are zero, so that $H_1(M'_1)$ is generated by y and b_1 . In particular, $\hat{H}_1 \times \{v_1\} \subset M'_1$ cannot be a section of an S^2 fiber bundle structure on M'_1 since $\pi_1(\hat{H}_1 \times \{v_1\})$ is generated by x and y . Thus Usher’s theorem implies that M' is minimal. Finally, M' is not an S^2 bundle with section $\Sigma_{h'}$ for similar reasons: the loops a_1, b_1, a_2, b_2 are generators of $\pi_1(\Sigma_{h'})$ but a_1 and a_2 are trivial in $H_1(M')$, as one can readily check. As explained above, it follows that V is minimal.

Using the same argument as in the proof of Theorem 8 one concludes from Taubes’s results that V is not diffeomorphic to $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$. q.e.d.

Since V contains an appropriate Lagrangian torus, the argument of Corollary 14 below applies to the manifold V as well to produce infinitely many smooth (but not symplectic) pairwise non-diffeomorphic manifolds homeomorphic to V .

4.3. We next put together the manifolds W_2 and M . The construction is based on the example of a minimal symplectic manifold homeomorphic to $\mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$ which we constructed in [6]. However, the extra information obtained by keeping track of the surface F will allow us to produce many more examples which we will use in constructing small non-simply connected symplectic 4-manifolds in the next section.

Recall that the manifold W_2 contains a genus 2 square zero symplectic surface F_1 and two Lagrangian tori T_1 and T_2 .

The manifold $M = \hat{H} \times \Sigma$ of Proposition 7 contains a symplectic surface F with trivial normal bundle and four Lagrangian tori T_1, T_2, T_3, T_4 ,

with the fundamental group of $M - nbd(F \cup T_1 \cup T_2 \cup T_3 \cup T_4)$ generated by the loops x, y, a_1, b_1, a_2, b_2 based at p satisfying all the conclusions of Proposition 7.

To avoid notational confusion, we denote the two Lagrangian tori in W_2 by T'_1 and T'_2 . We preserve the notation T_1, T_2, T_3, T_4 for the four Lagrangian tori in M .

The parallel (and symplectic) push off $\Sigma_{h'}$ of F lies in the boundary of the tubular neighborhood $D \times \Sigma$ of F and carries the base point and the loops a_1, b_1, a_2, b_2 . We have a framing of $\Sigma_{h'} \subset M$ defined by taking a nearby push off Σ_z for a point z near h' . Choose some identification of the tubular neighborhood of $F_2 \subset W_2$ with $F_2 \times D^2$.

We form the symplectic sum Z of W_2 and M along F_2 and $\Sigma_{h'}$:

$$(2) \quad Z = W_2 - nbd(F_2) \cup_{\tilde{\phi}} M - nbd(\Sigma_{h'}).$$

Observe that we have taken the symplectic sum along $\Sigma_{h'}$, not F . Thus F survives as a symplectic genus 2 surface in Z .

We choose the base point preserving diffeomorphism $\phi : F_2 \rightarrow \Sigma_{h'}$ to form this sum as follows. Using Lemma 9 in each half of the decomposition of Σ into two punctured tori, we conclude that there is a base point preserving diffeomorphism (a composition of six Dehn twists) $\phi_1 : \Sigma_{h'} \rightarrow \Sigma_{h'}$ that induces the isomorphism a_1, b_1, a_2, b_2 to $b_1^{-1}, b_1 a_1 b_1^{-1}, b_2^{-1}, b_2 a_2 b_2^{-1}$ on $\pi_1(\Sigma_{h'}, h')$. Composing this with the diffeomorphism $\phi_2 : F_2 \rightarrow \Sigma_{h'}$ that takes s_1, t_1, s_2, t_2 to a_1, b_1, a_2, b_2 yields the desired diffeomorphism $\phi : F_2 \rightarrow \Sigma_{h'}$. Hence ϕ induces the isomorphism

$$(3) \quad \begin{aligned} \phi : \pi_1(F_2, w) &\rightarrow \pi_1(\Sigma_{h'}, h'), \\ (s_1, t_1, s_2, t_2) &\mapsto (b_1^{-1}, b_1 a_1 b_1^{-1}, b_2^{-1}, b_2 a_2 b_2^{-1}). \end{aligned}$$

The symplectic manifold Z contains the surface F and six Lagrangian tori $T'_1, T'_2, T_1, T_2, T_3, T_4$. These seven surfaces are pairwise disjoint. For convenience denote the union of these seven surfaces by $R \subset Z$. Lemma 4 and the discussion following Proposition 6 shows that $\pi_1(Z - R)$ is a quotient of $\pi_1(M - (F \cup_{i=1}^4 T_i))$. Thus $\pi_1(Z - R)$ is generated by x, y, a_1, b_1, a_2, b_2 .

Moreover, the relations of Proposition 6 and Lemma 4 imply that in $\pi_1(Z - R)$,

$$[\phi(s_1), \phi(s_2)] = [\phi(t_1), \phi(s_2)] = [\phi(s_1), \phi(t_1)] = [\phi(s_2), \phi(t_2)] = 1.$$

Rewriting this in terms of the a_i, b_i using Equation (3) one obtains

$$1 = [b_1^{-1}, b_2^{-1}], 1 = [b_1 a_1 b_1^{-1}, b_2^{-1}], 1 = [b_1^{-1}, b_1 a_1 b_1^{-1}], 1 = [b_2^{-1}, b_2 a_2 b_2^{-1}].$$

Notice that $[b_1^{-1}, b_1 a_1 b_1^{-1}] = [a_1, b_1]$ and $[b_2^{-1}, b_2 a_2 b_2^{-1}] = [a_2, b_2]$. Moreover, $[r, s] = 1$ implies $[r^{-1}, s^{-1}] = 1$ and $[r, s^{-1}] = 1$. Hence this set of

relations simplifies to

$$(4) \quad 1 = [b_1, b_2], 1 = [a_1, b_2], 1 = [b_1, a_1], 1 = [b_2, a_2].$$

Since $\pi_1(Z - R)$ is a quotient of $\pi_1(M - nbd(F \cup_{i=1}^4 T_i))$, the relations coming from $M - nbd(F \cup_{i=1}^4 T_i)$ also hold in $\pi_1(Z - R)$, hence

$$[x, a_1] = [y, a_1] = [y, b_1 a_1 b_1^{-1}] = [x, a_2] = [y, a_2] = [y, b_2 a_2 b_2^{-1}] = 1.$$

These simplify, using Equation (4), to

$$(5) \quad 1 = [x, a_1], 1 = [y, a_1], 1 = [x, a_2], 1 = [y, a_2].$$

The following is the result we have been aiming towards. Its usefulness will be illustrated in most of the subsequent constructions in this article.

Theorem 11. *Let Z denote the symplectic sum of W_2 and M along the surfaces F_2 and $\Sigma_{h'}$ using the diffeomorphism $\phi : F_2 \rightarrow \Sigma_{h'}$ inducing the isomorphism of Equation (3) on fundamental groups. Let R denote the union the seven surfaces in Z : the symplectic surface F and the six Lagrangian tori $T'_1, T'_2, T_1, T_2, T_3, T_4$.*

Then the fundamental group of $Z - R$ is generated by the six loops x, y, a_1, b_1, a_2, b_2 . The relations

$$1 = [b_1, b_2], 1 = [a_1, b_2], 1 = [b_1, a_1], 1 = [b_2, a_2]$$

and

$$1 = [x, a_1], 1 = [y, a_1], 1 = [x, a_2], 1 = [y, a_2]$$

hold in $\pi_1(Z - R)$.

The meridian of the surface F is given by $\mu_F = [x, y]$. For the six Lagrangian tori, the meridians and Lagrangian push offs (with appropriate paths to the base point) in $\pi_1(Z - R)$ are given by:

- $T'_1 : \mu'_1 = [a_2^{-1}, a_1^{-1}], m'_1 = b_1^{-1}, \ell'_1 = b_2^{-1}.$
- $T'_2 : \mu'_2 = [b_1, a_2], m'_2 = b_1 a_1 b_1^{-1} = a_1, \ell'_2 = b_2^{-1}.$
- $T_1 : \mu_1 = [b_1^{-1}, y^{-1}], m_1 = x, \ell_1 = a_1.$
- $T_2 : \mu_2 = [x^{-1}, b_1], m_2 = y, \ell_2 = a_1.$
- $T_3 : \mu_3 = [b_2^{-1}, y^{-1}], m_3 = x, \ell_3 = a_2.$
- $T_4 : \mu_4 = [x^{-1}, b_2], m_4 = y, \ell_4 = a_2.$

Proof. Proposition 7 implies that the meridian of F in $\pi_1(Z - R)$ is given by

$$\mu_F = [x, y].$$

Propositions 6 and 7 give the meridians and Lagrangian push offs of the six tori in terms of these generators.

Proposition 6 shows that $\mu'_1 = [b_2 a_2^{-1} b_2^{-1}, b_1 a_1^{-1} b_1^{-1}]$. Equation (4) can be used to simplify this: $[b_2 a_2^{-1} b_2^{-1}, b_1 a_1^{-1} b_1^{-1}] = [a_2^{-1}, a_1^{-1}]$, and hence μ'_1 is equal to $[a_2^{-1}, a_1^{-1}]$. Similarly $[b_1, b_2 a_2 b_2^{-1}] = [b_1, a_2]$, showing that $\mu'_2 = [b_1, a_2]$. Next, $(b_2 a_2 b_2^{-1}) b_2^{-1} (b_2 a_2 b_2^{-1})^{-1} = b_2^{-1}$, showing that

$\ell'_2 = b_2^{-1}$. Continuing, $b_1 a_1 b_1^{-1} = a_1$ and so $\ell_2 = a_1$ and $b_2 a_2 b_2^{-1} = a_2$, so that $\ell_4 = a_2$.

The rest of the argument was described before the statement of the Theorem. q.e.d.

We determine the basic homological properties of Z .

Proposition 12. *The first homology $H_1(Z) \cong \mathbb{Z}^6$, generated by the loops x, y, a_1, b_1, a_2, b_2 .*

The second homology $H_2(Z) \cong \mathbb{Z}^{16}$. There exist six disjoint tori $R'_1, R'_2, R_1, R_2, R_3$, and R_4 , and a genus 2 surface H_3 with trivial normal bundles in Z which are geometrically dual to the six Lagrangian tori $T'_1, T'_2, T_1, T_2, T_3$, and T_4 and the surface F , in the sense that T'_i intersects R'_i transversally in one point and similarly for T_i, R_i and F, H_3 , and all other intersections are pairwise empty.

There are two disjoint tori H_1, H_2 with square -1 which intersect F transversally once and are pairwise disjoint from the T'_i, R'_i, T_i, R_i , and H_3 . The sixteen surfaces $T'_i, R'_i, T_i, R_i, F, H_1, H_2$ and H_3 generate $H_2(Z)$. Hence the intersection form of Z is an orthogonal sum of six hyperbolic summands generated by the pairs T_i, R_i (and T'_i, R'_i) and one 4-dimensional summand spanned by H_1, H_2, H_3 and F with matrix

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

The signature $\sigma(Z)$ equals -2 . The Euler characteristic $e(Z)$ equals 6.

Moreover, the symplectic manifold Q obtained by performing any six Luttinger surgeries on the T'_i and T_i satisfies $e(Q) = 6$ and $\sigma(Q) = -2$, and contains a symplectic genus 2 surface F with trivial normal bundle and six Lagrangian tori.

Proof. Since Z is the symplectic sum of $W_2 = T^4 \# 2\overline{\mathbb{C}\mathbb{P}^2}$ and $M = T^2 \times \Sigma$ along genus 2 surfaces,

$$e(Z) = e(T^4 \# 2\overline{\mathbb{C}\mathbb{P}^2}) + e(T^2 \times \Sigma) + 4 = 6$$

and Novikov additivity implies that

$$\sigma(Z) = \sigma(T^4 \# 2\overline{\mathbb{C}\mathbb{P}^2}) + \sigma(T^2 \times \Sigma) = -2.$$

Luttinger surgery does not affect e and σ , and so $e(Q) = 6$ and $\sigma(Q) = -2$. The core torus $T \times \{0\} \subset T^2 \times D^2$ glued in to form a Luttinger surgery is Lagrangian [21]. Since the T'_i and T_i miss F , the assertions about Q are verified.

The homomorphism $H_1(F_2) \rightarrow H_1(W_2 - F_2) = H_1(W_2) \cong \mathbb{Z}^4$ induced by any push off is an isomorphism. Also $M - nbd(\Sigma_{h'}) = H \times \Sigma$, where

H is a punctured torus. Thus from the Mayer-Vietoris sequence for the decomposition

$$Z = W_2 - nbd(F_2) \cup_{\tilde{\phi}} M - nbd(\Sigma_{h'})$$

one easily sees that $H_1(M - nbd(\Sigma_{h'})) \rightarrow H_1(Z)$ is an isomorphism. Since $H_1(M - nbd(\Sigma_{h'})) = H_1(H \times \Sigma) \cong \mathbb{Z}^6$, and $\pi_1(H \times \Sigma)$ is generated by the six loops x, y, a_1, b_1, a_2, b_2 , $H_1(Z) \cong \mathbb{Z}^6$ generated by these loops. Since $e(Z) = 6$, it follows that $H_2(Z) \cong \mathbb{Z}^{16}$.

We find 16 embedded surfaces that generate $H_2(Z)$ and calculate the intersection form.

The torus T_1 has a geometrically dual torus $R_1 = y \times b_1$ in $M = \hat{H} \times \Sigma$ which misses T_2, T_3, T_4 and also F and its parallel $\Sigma_{h'}$ (since the points u and h' are different from h). Similarly T_2 has a dual torus $R_2 = x \times b_1$, T_3 has a dual torus $R_3 = y \times b_2$, and T_4 has a dual torus $R_4 = x \times b_2$. These all miss F and $\Sigma_{h'}$ and so survive in Z . It is also easy to push the R_i off each other. Notice also that the T_i and R_i miss $\hat{H} \times \{q\}$ for most points q .

Similarly one can find disjoint dual tori R'_1 and R'_2 to the classes T'_1 and T'_2 in $W_2 - F_2$. The tori T'_i and R'_i may be assumed to miss the 2 exceptional curves E_1 and E_2 , and also miss one of the vertical tori $\{r\} \times T^2$.

Let $H_1 \subset Z$ be the torus formed by joining up the exceptional curve E_1 to $H \times \{q\}$ for the appropriate q in the symplectic sum. Similarly define H_2 using the other exceptional curve, and let H_3 be a genus 2 surface formed by joining $H \times \{r\}$ to one of the vertical tori.

The 12 tori T_i, R_i, T'_i, R'_i can be isotoped by a small isotopy in Z so each meets its dual transversally once and all other intersections are empty. Moreover, they each miss F, H_1, H_2 , and H_3 . Notice that each H_i intersects F once (geometrically and algebraically), that $F^2 = 0$, $H_1^2 = -1$, $H_2^2 = -1$, and $H_3^2 = 0$. Finally H_i is disjoint from H_j when $i \neq j$. One can check that the classes $F = H_0, H_1, H_2, H_3$ span a primitive subspace of $H_2(Z)$ by calculating that the determinant of $H_i \cdot H_j$ is equal to 1.

Thus the intersection form of Z is the orthogonal sum of 6 hyperbolic planes spanned by each T_i, R_i pair, and a 4 dimensional space spanned by F, H_1, H_2, H_3 with matrix as asserted in the statement.

Notice moreover that the subspace spanned by F and the T_i is a 7 dimensional isotropic subspace, and that H_2 and H_3 span a subspace with intersection form $2(-1)$. q.e.d.

We can now do Luttinger surgery on the six tori to obtain interesting symplectic 4-manifolds containing a symplectic genus 2 surface F . The effect of Luttinger surgery on the homology is easy to understand. Let T denote one of the T_i or T'_i , and μ, m, ℓ its meridian and two Lagrangian

push offs. Theorem 11 shows that m is sent to one of the generators x, y, a_1, b_1, a_2, b_2 in $H_1(Z)$, and similarly for ℓ . The meridian μ is trivial in $H_1(Z)$. Therefore,

- 1) 1/0 surgery on T along any curve γ does not change anything, one removes a neighborhood of T and re-glues it the same way.
- 2) For $k \neq 0$, $1/k$ surgery on T along the curve $\gamma = m^p \ell^q$ kills $k(p[m]+q[\ell])$ in $H_1(Z)$. The surgery decreases the rank of $H_1(Z; \mathbb{Q})$ by one. Since the Euler characteristic is unchanged, the rank of $H_2(Z; \mathbb{Q})$ decreases by two. If $k = \pm 1$, then the rank of $H_2(Z; \mathbb{Z})$ decreases by two.

Notice that $\pm 1/1$ Luttinger surgery on T only changes the manifold near T . Letting \tilde{T} denote the core $T \times \{0\}$ of $T \times D^2$ in the *surgered* manifold, we see that \tilde{T} is nullhomologous, since the other surfaces T_i, R_i, H_i and F are disjoint from T and its dual and hence are not affected. In other words, the effect on second homology of ± 1 Luttinger surgery on T_i or T'_i is to kill the hyperbolic summand spanned by T_i and its dual.

Computing fundamental groups is harder, but the following refinement of the main result of [6] indicates why we chose the gluing map ϕ as we did.

Theorem 13. *There exists a minimal symplectic 4-manifold X homeomorphic but not diffeomorphic to $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ containing a symplectic surface F of genus 2 with simply connected complement and trivial normal bundle.*

Moreover, X contains a nullhomologous Lagrangian torus T disjoint from F , and a curve λ on the boundary of the tubular neighborhood of T such that $\lambda = 0$ in $H_1(X - T)$ and λ is isotopic in $\text{nb}d(T)$ to an embedded essential curve on T .

Proof. We perform six Luttinger surgeries on the six Lagrangian tori in $Z - F$, referring to Theorem 11. First $1/1$ surgery on T'_1 along m'_1 imposes the relation $b_1 = [a_2^{-1}, a_1^{-1}]$. Then $-1/1$ surgery on T_1 along m_1 imposes the relation $x = [b_1^{-1}, y^{-1}] = [[a_1^{-1}, a_2^{-1}], y^{-1}] = 1$. The last equality comes from Equation (5).

Next perform $-1/1$ Luttinger surgery on T_2 along ℓ_2 . This imposes $a_1 = [x^{-1}, b_1] = 1$. Returning to our first surgery we see that $b_1 = [a_2^{-1}, a_1^{-1}] = 1$. Continuing, $1/1$ surgery on T'_2 along ℓ'_2 sets $b_2 = [b_1, a_2] = 1$.

Next perform $-1/1$ surgery on T_3 along ℓ_3 to set $a_2 = 1$. We finish with $-1/1$ surgery on T_4 along m_4 which sets $y = 1$.

This kills all the generators, so the resulting symplectic 4-manifold X satisfies $\pi_1(X - F) = 1$.

It was explained in the paragraphs preceding Theorem 13 that the six Luttinger fillings kill the first homology, and $H_2(X) = \mathbb{Z}^4$, and that any of the six core tori in X are nullhomologous. Hence we let T denote the core of the neighborhood glued in the last Luttinger surgery.

Since $T \times D^2$ is glued along $T \times \partial D^2$ so that ∂D^2 is identified with the curve $\mu_4 m_4^{-1}$, we can choose an essential embedded curve λ on T so that $\lambda \times \{s\}$ is sent to $\mu_4 = [x^{-1}, b_2] = 1$ in $\pi_1(X - T)$, and hence $\mu_4 = 0$ in $H_1(X - T)$. (Note that since each Luttinger surgery kills one generator of first homology, $\pi_1(X - T) = H_1(X - T) = \mathbb{Z}y$.)

The minimality of X follows just as in the proofs of Theorems 8 and 10, using Usher’s theorem. Note that $e(X) = e(Z) = 6$ and $\sigma(X) = \sigma(Z) = -2$, so that by Freedman’s theorem [14] X is homeomorphic to $\mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$. As in the proof of Theorem 8, Taubes’s results imply that V is not diffeomorphic to $\mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$. This completes the proof. q.e.d.

Referring to Proposition 12, we see that the four classes generating $H_2(X)$ and diagonalizing the intersection form are $F + H_1, H_1, H_2 - H_3, H_3 - H_1 - F$. These are represented by smoothly embedded surfaces of genus 3, 1, 3, and 5 respectively. Notice that H_1 has minimal genus, since X cannot contain a smoothly embedded -1 sphere. Similarly H_2 has minimal genus.

Remark. The last Luttinger surgery in the proof of Theorem 13, $-1/1$ surgery on T_4 , kills the last loop $m_4 = y$. But one can leave the setting of symplectic manifolds, and define a family of smooth manifolds as follows. Denote by X_0 the symplectic manifold constructed in the proof of Theorem 13 in the penultimate step. Thus X_0 is a symplectic manifold with $\pi_1(X_0) = \mathbb{Z}$, generated by y , and $b^+(X_0) = 2, b^-(X_0) = 4$. The torus T_4 in X_0 is Lagrangian. The boundary of the tubular neighborhood of T_4 is a 3-torus whose fundamental group is generated by the loops μ_4, m_4 and ℓ_4 . In $\pi_1(X_0 - T_4)$, $\mu_4 = 1, m_4 = y$, and $\ell_4 = 1$.

Let Y denote the manifold obtained from $X_0 - nbd(T_4)$ by gluing $T^2 \times D^2$ in such a way that $\alpha = S^1 \times \{1\} \times \{1\}$ is sent to ℓ_4 , $\beta = \{1\} \times S^1 \times \{1\}$ is sent to μ_4 , and $\mu_Y = \{(1, 1)\} \times \partial D^2$ is sent to m_4^{-1} . Then Y is simply connected, since $m_4 = y$ is killed, but Y need not be symplectic. Let $\tilde{T} \subset Y$ denote the resulting core torus.

With these coordinates α, β, μ_Y , one sees that

- 1) X_0 is obtained by $0/1$ surgery on \tilde{T} along β .
- 2) β is nullhomologous (indeed nullhomotopic) in $Y - \tilde{T} = X_0 - T_4$, since $\beta = \mu_4$.
- 3) The manifold Y_n obtained from $1/n$ surgery on \tilde{T} along β is simply connected, and hence homeomorphic to $\mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$. (This manifold can be described as $-n/1$ surgery on T_4 along m_4 in X_0 , and so $\mu_4^n m_4^{-1} = y^{-1}$ is killed. Thus $X = Y_1$.)

- 4) The torus \tilde{T} is nullhomologous in Y . The reason is the same as in the construction of X : the last torus surgery kills one generator of H_1 , hence two generators of H_2 , the class of T_1 and its dual R_1 .
- 5) X_0 is symplectic, minimal, and has $b^+ = 2$, hence has non-trivial Seiberg-Witten invariant by [37].

The Morgan-Mrowka-Szabo formula [23] (as formulated in [35, Theorem 3.4]) can be used to prove that the family $\{Y_n\}$ contains infinitely many diffeomorphism types, detected by Seiberg-Witten invariants (see [10, Theorem 5.3]). This proves the following.

Corollary 14. *Among the manifolds $Y_n, n \in \mathbb{Z}$ there exists an infinite family of smooth manifolds homeomorphic to $\mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$ which are pairwise non-diffeomorphic (detected by Seiberg-Witten invariants). Each Y_n contains a smoothly embedded genus 2 surface F with trivial normal bundle and simply connected complement. q.e.d.*

As remarked in [33], Seiberg-Witten invariants cannot be used to distinguish more than two minimal symplectic manifolds homeomorphic to $\mathbb{C}\mathbb{P}^2 \# n\overline{\mathbb{C}\mathbb{P}^2}$ for $n < 9$. Thus at most two of the Y_n are minimal symplectic manifolds.

4.4. The manifold X is very useful. We illustrate this with a number of constructions next. A quick corollary is the following (cf. [2]).

Corollary 15. *The symplectic sum of X with W_2 along F in X and F_2 in W_2 is a minimal symplectic manifold homeomorphic but not diffeomorphic to $3\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$.*

Proof. The map $\pi_1(F_2) \rightarrow \pi_1(W_2)$ induced by inclusion is onto, and hence the symplectic sum S of W_2 and X is simply connected by Lemma 5. Since $e(S) = e(X) + e(W_2) + 4 = 12$ and $\sigma(S) = \sigma(X) + \sigma(W_2) = -4$, Freedman's theorem [14] implies that S is homeomorphic to $3\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$.

The Seiberg-Witten invariants of a connected sum of manifolds with $b^+ > 1$ vanishes ([31]). On the other hand the Seiberg-Witten invariants for a symplectic manifold with $b^+ > 1$ are non-trivial ([37]). Hence S cannot be diffeomorphic to $3\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$. q.e.d.

Similarly one obtains the following (cf. [34]).

Corollary 16. *The fiber sum of X with W_1 along F and F_1 is a minimal symplectic manifold homeomorphic but not diffeomorphic to $3\mathbb{C}\mathbb{P}^2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$. q.e.d.*

And also

Corollary 17. *The fiber sum of two copies of X along F is a minimal symplectic manifold homeomorphic but not diffeomorphic to $5\mathbb{C}\mathbb{P}^2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$. q.e.d.*

In Corollaries 15, 16, and 17 the manifolds constructed contain null-homologous Lagrangian tori with an appropriate curve to surger, and so the argument given above for X applies to find infinitely many smooth homeomorphic but not diffeomorphic (and not necessarily symplectic) examples in each case.

4.5. Any simply connected symplectic 4-manifold (M, ω) containing a symplectic torus of square zero either has $e \geq 12$ or $b^+ \geq 3$. Indeed, suppose that $b^+(M) = 1$ and $e < 12$. Let $t \in H^2(M; \mathbb{R})$ denote the Poincaré dual to T and k the canonical class. Since T is symplectic, $\omega \cdot t > 0$ and $k \cdot t = 2g(T) - 2 + t \cdot t = 0$. Since $k^2 = 2e + 3\sigma$, it follows from $e < 12$ that $k^2 > 0$. Clearly $\omega^2 > 0$. These facts easily imply that the intersection form has a 2-dimensional positive definite subspace, contradicting $b^+(M) = 1$.

Thus the small simply connected symplectic manifold X with $b^+ = 1$ constructed above cannot contain a square zero torus with simply connected complement.

The elliptic surface $E(1)$, which has $e = 12$ and $\sigma = -8$, does contain a symplectic square zero torus with simply connected complement, namely the fiber in any elliptic fibration with cusp fibers. The following theorem establishes the existence of a fake $3\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$ which contains a square zero symplectic torus with simply connected complement. In the statement X denotes the manifold of Theorem 13 and M the manifold of Proposition 7. Recall that X contains a square zero genus 2 surface F with simply connected complement. Also, M contains a square zero genus two surface, also denoted F , and four homologically essential Lagrangian tori.

Theorem 18. *One can do Luttinger surgeries on the symplectic sum of X with M to obtain a minimal symplectic manifold B homeomorphic but not diffeomorphic to $3\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$. This manifold contains a pair of tori with trivial normal bundles T_3, T_4 , with $B - (T_3 \cup T_4)$ simply connected. The tori T_3 and T_4 can be assumed to be either Lagrangian or symplectic.*

Proof. Call the four Lagrangian tori in the sum

$$A = X - \text{nb}d(F) \cup M - \text{nb}d(F)$$

T_1, T_2, T_3, T_4 (these lie in the M side). Since $X - \text{nb}d(F)$ is simply connected, and the boundary of $M - \text{nb}d(F)$ carries the loops a_1, b_1, a_2, b_2 , $\pi_1(A)$ is generated by the loops x and y in M . Proposition 7 shows that the meridians of T_1 and T_2 are trivial. Then $-1/1$ Luttinger surgery along m_1 in T_1 and m_2 in T_2 (and no surgery, i.e. $1/0$, surgery on T_3 and T_4) yields a simply connected, symplectic manifold B with $e = 10$ and $\sigma = -2$.

Notice that $\pi_1(B - (T_3 \cup T_4)) = 1$. The Lagrangian torus T_3 is homologically essential in B , since it intersects the $y \times b_2$ torus transversally once. Thus one can perturb the symplectic structure on B so that T_3 becomes symplectic. Similarly for T_4 .

The manifold X is minimal by Theorem 13. Since it is simply connected, it is not an S^2 bundle over F . The symplectic manifold M' obtained from the Luttinger surgeries on T_1 and T_2 in M is minimal by the same argument as in the proof of Theorem 10. Since B is the fiber sum of X and M' , Usher's theorem implies that B is minimal. q.e.d.

The nullhomologous tori in B coming from T_1 and T_2 can be used to produce infinitely many non-diffeomorphic smooth manifolds homeomorphic to $3\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$, using the argument of Corollary 14. Moreover, the tori T_3 and T_4 with simply connected complement can be used to perform knot surgery in the sense of [11].

More simply connected examples can be obtained by taking the symplectic sums with V or B . For example, the symplectic sum of B with W_1 along a parallel copy of F and F_1 is simply connected by Lemma 4. It is minimal, symplectic, and homeomorphic but not diffeomorphic to $5\mathbb{C}\mathbb{P}^2 \# 11\overline{\mathbb{C}\mathbb{P}^2}$. Similarly the symplectic sum of B with W_2 gives a minimal symplectic exotic $5\mathbb{C}\mathbb{P}^2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$, the symplectic sum of V with W_1 gives a minimal symplectic exotic $3\mathbb{C}\mathbb{P}^2 \# 11\overline{\mathbb{C}\mathbb{P}^2}$, and the symplectic sum of V with W_2 gives a minimal symplectic exotic $3\mathbb{C}\mathbb{P}^2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$ (cf. [26]).

This process can be iterated. The following corollary gives one such example.

Corollary 19. *Given non-negative integers m and n , let $C_{m,n}$ denote the symplectic sum of X with m copies of W_1 and n copies of W_2 along parallel copies of F . Then $C_{m,n}$ is a simply connected minimal symplectic manifold homeomorphic but not diffeomorphic to the manifold $(1 + 2m + 2n)\mathbb{C}\mathbb{P}^2 \# (3 + 6m + 4n)\overline{\mathbb{C}\mathbb{P}^2}$.*

Proof. Lemma 4 and induction proves that $C_{m,n}$ is simply connected. We have

$$e(C_{m,n}) = e(X) + me(W_1) + ne(W_2) + 4(m+n) = 6 + 8m + 6n$$

and

$$\sigma(C_{m,n}) = \sigma(X) + m\sigma(W_1) + n\sigma(W_2) = -2(1 + 2m + n).$$

The intersection form of $C_{m,n}$ is odd for any n . To see this, consider the class represented by $\hat{H} \times \{s\}$ in $M = \hat{H} \times \Sigma$. We form $C_{m,n}$ by taking the symplectic sum of M with m copies of W_1 and $n+1$ copies of W_2 along $m+n+1$ parallel copies of F , say $\{p_i\} \times \Sigma$, $i = 1, \dots, m+n+1$. We can arrange that the first symplectic sum (of M with W_2) is a

relative symplectic sum ([15]) which lines up the punctured \hat{H} with one of the exceptional spheres in W_2 . This produces a torus with square -1 in X . We can then take relative symplectic sums so that this torus lines up with vertical (square zero) tori in $W_1 = T^2 \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$ and $W_2 = T^4 \# 2\overline{\mathbb{C}\mathbb{P}^2}$ (rather than the exceptional curves), so that $C_{m,n}$ contains an embedded genus $m + n + 1$ surface of square -1 . Hence the intersection form of $C_{m,n}$ is odd.

Using Freedman’s theorem we see that $C_{m,n}$ is homeomorphic to the manifold $(1 + 2m + 2n)\mathbb{C}\mathbb{P}^2 \# (3 + 6m + 4n)\overline{\mathbb{C}\mathbb{P}^2}$. Minimality is proved as before using Usher’s theorem. Taubes’s results then imply that $C_{m,n}$ is not diffeomorphic to $(1 + 2m + 2n)\mathbb{C}\mathbb{P}^2 \# (3 + 6m + 4n)\overline{\mathbb{C}\mathbb{P}^2}$. q.e.d.

The following theorem is useful to construct manifolds with different smooth structures.

Theorem 20. *There exists a minimal symplectic manifold B_1 containing a pair of Lagrangian tori T_3 and T_4 and a square zero symplectic genus 2 surface F so that T_3, T_4 and F are pairwise disjoint and*

- 1) $\pi_1(B_1 - (F \cup T_3 \cup T_4)) = \mathbb{Z}^2$, generated by t_1 and t_2 .
- 2) The inclusion $B_1 - (F \cup T_3 \cup T_4) \subset B_1$ induces an isomorphism on fundamental groups. In particular the meridians μ_F, μ_3, μ_4 all vanish in $\pi_1(B_1 - (F \cup T_3 \cup T_4))$.
- 3) The Lagrangian push offs m_3, ℓ_3 of $\pi_1(T_3)$ are sent to 1 and t_2 respectively in the fundamental group of $B_1 - (F \cup T_3 \cup T_4)$.
- 4) The Lagrangian push offs m_4, ℓ_4 of $\pi_1(T_4)$ are sent to t_1 and t_2 respectively in the fundamental group of $B_1 - (F \cup T_3 \cup T_4)$.
- 5) The push off $F \subset B_1 - (F \cup T_3 \cup T_4)$ induces a map on fundamental groups with image the subgroup generated by t_2 .

The symplectic structure may be perturbed so that one or both of the tori T_3, T_4 are symplectic.

Proof. Construct B_1 by starting with the manifold Z of Theorem 11, and doing Luttinger surgeries along four of the Lagrangian tori, just as was done in the proof of Theorem 13. Explicitly, we do $1/1$ Luttinger surgery on T'_1 along m'_1 , $-1/1$ surgery on T_1 along m_1 , then $-1/1$ surgery on T_2 along ℓ_2 , and finally $1/1$ surgery on T'_2 along ℓ'_2 .

The resulting manifold B_1 has two remaining Lagrangian tori, T_3 and T_4 , and the same calculation as in the proof of Theorem 13 shows that $\pi_1(B_1 - (F \cup T_3 \cup T_4))$ is generated by a_2 and y and $[a_2, y] = 1$. Moreover, the meridian $\mu_F = 1$ and also the other two meridians $\mu_3 = \mu_4 = 1$. The Lagrangian push offs are given by

$$m_3 = 1, \ell_3 = a_2, m_4 = y, \ell_4 = a_2$$

using Theorem 11.

Thus $\pi_1(B_1 - (F \cup T_3 \cup T_4))$ is a quotient of $\mathbb{Z}a_2 \oplus \mathbb{Z}y$. But Proposition 12 shows that $H_1(B_1 - (F \cup T_3 \cup T_4)) = \mathbb{Z} \oplus \mathbb{Z}$, and so the group $\pi_1(B_1 - (F \cup T_3 \cup T_4)) = \mathbb{Z}a_2 \oplus \mathbb{Z}y$.

The tori T_3 and T_4 have dual tori R_3 and R_4 by Proposition 12, and so they are homologically essential and linearly independent in second homology. Hence the symplectic form can be perturbed slightly so that one or both of T_3 and T_4 are symplectic ([15]).

Setting $t_1 = y$ and $t_2 = a_2$ completes the proof. q.e.d.

The following corollary shows that at the expense of a small stabilization, any symplectic manifold containing a square zero symplectic genus 1 or 2 surface can be used to produce infinite families of manifolds with different Seiberg-Witten invariants. Note that if the surface G has genus 1, then a symplectic sum with $E(1)$ and the knot surgery procedure of Fintushel-Stern [11] gives a similar result, at the cost of 12 to e and -8 to σ . Thus this result can be viewed as an extension (it includes genus 2) and an improvement (the cost to e is 6 or 10). (See also [12].)

Corollary 21. *Suppose M is any symplectic manifold which contains a square zero genus 1 or 2 symplectic surface G such that the inclusion $G \subset M$ induces the trivial map on fundamental groups. Then there exists an infinite family of smooth manifolds M_n so that $\pi_1(M_n) = \pi_1(M)$, $e(M_n) = e(M) + 2 + 4 \text{ genus}(G)$, $\sigma(M_n) = \sigma(M) - 2$, and the Seiberg-Witten invariants of M_n are different from the Seiberg-Witten invariants of M_m if $m \neq n$. Moreover M_1 is symplectic.*

Proof. This is an application of the gluing formula for Seiberg-Witten invariants of Mrowka-Morgan-Szabo [23], as explained in [35, Theorem 3.4] (see also [10]) and is similar to the argument given in Corollary 14 above.

Starting with M , take the symplectic sum of M with the manifold B_1 of Theorem 20 along F if G has genus 2, and along T_3 (after making T_3 symplectic by perturbing the symplectic structure) if G has genus 1. Call the result S . Then S is a symplectic manifold satisfying $e(S) = e(M) + 6 + 4(\text{genus}(G) - 1)$ and $\sigma(M_n) = \sigma(M) - 2$.

The Seifert-Van Kampen theorem and Theorem 20 implies that the group $\pi_1(S)$ is isomorphic to $\pi_1(M) * \mathbb{Z}t_1$. The inclusion $S - T_4 \subset S$ induces an isomorphism on fundamental groups. The Lagrangian push offs of T_4 are $m_4 = t_1$ and $\ell_4 = 1$, and the meridian is $\mu_4 = 1$.

We follow the argument of Corollary 14. Let M_0 denote the manifold obtained from $S - nbd(T_4)$ by torus filling by $T^2 \times D^2$ in such a way that $\alpha = S^1 \times \{(1, 1)\}$ is sent to ℓ_4 , $\beta = \{1\} \times S^1 \times \{1\}$ is sent to μ_4 , and $\mu = \{(1, 1)\} \times \partial D^2$ is sent to m_4^{-1} . Then $\pi_1(M_0) = \pi_1(M)$ since t_1 is killed. This is not a Luttinger surgery, so M_0 need not be symplectic.

Let $\tilde{T} \subset M_0$ denote the resulting core torus. Then $M_0 - nbd(\tilde{T}) = S - nbd(T_4)$. With the coordinates α, β, μ , S is obtained from M_0 by $0/1$ surgery on \tilde{T} along $\beta = \mu_4$. Moreover, β is nullhomologous in $M_0 - \tilde{T}$ since $\mu_4 = 0$ in $H_1(B_1 - (F \cup T_3 \cup T_4))$. Note that S is a symplectic manifold. Moreover, using the surfaces described in Proposition 12, since T_3 has a dual torus R_3 and either T_4 also has a dual torus R_4 (in case G has genus 2) or F has a dual genus 2 surface H_3 (when G has genus 1), the intersection form of S contains at least 2 hyperbolic pairs, and thus $b^+(S) > 1$. Hence the Seiberg-Witten invariants of S are non-zero by Taubes's theorem [37].

Since the surgery on T_4 producing M_0 from S kills the generator t_i of H_1 , it also kills the hyperbolic pair containing T_4 , i.e. $\tilde{T} = 0$ in $H_2(M_0)$.

Let M_n denote the manifold obtained from $1/n$ surgery on \tilde{T} in M_0 along β . Note that M_1 can also be viewed as $-1/1$ Luttinger surgery on T_4 in S along m_4 . Thus M_1 is symplectic.

The formula [35, Theorem 3.4] can be used as in [10] to prove that the family M_n obtained from $1/n$ surgery on \tilde{T} along β contains infinitely many diffeomorphism types. q.e.d.

5. Non-trivial fundamental group

5.1. Fundamental group \mathbb{Z} . We turn now to a useful example of a symplectic 4-manifold X_1 with fundamental group \mathbb{Z} .

Theorem 22. *There exists a minimal symplectic 4-manifold X_1 with $\pi_1(X_1) = \mathbb{Z}$, $e(X_1) = 6$ and $\sigma(X_1) = -2$. Moreover, X_1 contains a symplectic surface F of genus 2 and square zero and symplectic torus T of square zero disjoint from F so that $\pi_1(X_1 - (T \cup F)) \cong \mathbb{Z}$. The homomorphisms induced by inclusion, $\pi_1(X_1 - (T \cup F)) \rightarrow \pi_1(X_1 - F)$ and $\pi_1(X_1 - T) \rightarrow \pi_1(X_1)$, are isomorphisms.*

Furthermore, the homomorphism $\pi_1(T) \rightarrow \pi_1(X_1 - F)$ induced by inclusion takes one generator $t_1 \in \pi_1 T$ to the generator of $\pi_1(X_1 - F)$ and the other t_2 to the trivial element 1. The homomorphism induced by inclusion $\pi_1(F) \rightarrow \pi_1(X_1)$ is trivial.

Proof. We follow the beginning of the proof of Theorem 13, performing Luttinger surgeries on the Lagrangian tori in $Z - F$, where Z is the manifold of Equation (2). We refer to the calculations of Theorem 11.

First $1/1$ surgery on T'_1 along m'_1 imposes the relation $b_1 = [a_2^{-1}, a_1^{-1}]$. Then $-1/1$ surgery on T_1 along m_1 imposes the relation $x = [b_1^{-1}, y^{-1}] = [[a_1^{-1}, a_2^{-1}], y^{-1}] = 1$. The last equality comes from Equation (5).

Next perform $-1/1$ Luttinger surgery on T_2 along ℓ_2 . This imposes $a_1 = [x^{-1}, b_1] = 1$ and hence also $b_1 = 1$. Then $1/1$ surgery on T'_2 along ℓ'_2 sets $b_2 = [b_1, a_2] = 1$. A $-1/1$ surgery on T_3 along ℓ_3 sets $a_2 = [b_2^{-1}, y^{-1}] = 1$.

This leaves T_4 untouched. Call the resulting manifold X_1 . Thus the classes x, a_1, b_1, a_2 , and b_2 equal 1 in $\pi_1(X_1 - (T_4 \cup F))$. Since $H_1(Z) = \mathbb{Z}^6$ and we performed five Luttinger surgeries, $\pi_1(X_1 - (T_4 \cup F))$ is generated by y and $H_1(X_1) = \mathbb{Z}$, so that $\pi_1(X_1 - (T_4 \cup F)) = \pi_1(X_1 - F) = \pi_1(X_1) = \mathbb{Z}y$.

Then the torus T_4 carries the classes y and a_2 , and the surface F is generated by the classes a_1, b_1, a_2, b_2 , and so the assertions about fundamental groups follow. Note that $e(X_1) = e(Z) = 6$ and $\sigma(X_1) = \sigma(Z) = -2$.

Lastly, the torus T_4 meets the torus $Q = x \times b_2$ in M transversally in one point. Since Q misses all other tori in the construction and F , it survives to provide T_4 with a dual class in $H_2(X_1)$. In particular, $[T_4] \neq 0$ in $H_2(X_1)$. Hence the symplectic form on X_1 can be perturbed slightly so that T_4 (which we rename T) becomes symplectic ([15]). We relabel its generators $t_1 = y$ and $t_2 = a_2$.

Minimality follows just as in the proof of Theorem 10. q.e.d.

The intersection form of X_1 is equivalent to $2(1) \oplus 4(-1)$. But X_1 is not diffeomorphic to $(S^1 \times S^3) \# 2\mathbb{C}\mathbb{P}^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$, since X_1 is a minimal symplectic manifold with $b^+ = 2 > 1$, and hence has non-vanishing Seiberg-Witten invariants [37]. On the other hand the Seiberg-Witten invariants of the connected sum $(S^1 \times S^3) \# 2\mathbb{C}\mathbb{P}^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$ must vanish. It is an interesting question whether the manifold X_1 is homeomorphic to $(S^1 \times S^3) \# 2\mathbb{C}\mathbb{P}^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$.

Our interest in the manifold X_1 is two fold. First, it is the smallest known (to us) symplectic 4-manifold with fundamental group \mathbb{Z} , where we measure the size using the Euler characteristic (or equivalently the second Betti number). (Constructions of symplectic manifolds with fundamental group \mathbb{Z} can be found in the literature, e.g. [24, 15, 32].) The other reason is that it can be used as a smaller replacement for the elliptic surface $E(1)$ typically used to control fundamental groups of symplectic 4-manifolds. We will illustrate this in the following theorem, which also refers to the manifold B constructed in Theorem 18.

Theorem 23. *Let L be a symplectic 4-manifold containing a symplectic torus T' with trivial normal bundle such that $\alpha, \beta \in \pi_1(L)$ represent the two generators of $\pi_1(T')$. Then*

- 1) *The symplectic sum of X_1 and L along T and T' , $X_1 \#_T L$, using the gluing map $\phi : T \rightarrow T'$ inducing*

$$t_1 \mapsto \alpha, t_2 \mapsto \beta$$

admits a symplectic structure which agrees with that of X_1 and L away from T, T' and satisfies

$$e(X_1 \#_T L) = e(L) + 6, \quad \sigma(X_1 \#_T L) = \sigma(L) - 2,$$

and

$$\pi_1(X_1 \#_T L) = \pi_1(L)/N(\beta)$$

where $N(\beta)$ denotes the normal subgroup of $\pi_1(L)$ generated by β .

- 2) The symplectic sum of B and L along T_3 and T' , $B \#_T L$ admits a symplectic structure which agrees with that of B and L away from T_3, T' and satisfies

$$e(B \#_T L) = e(L) + 10, \quad \sigma(B \#_T L) = \sigma(L) - 2,$$

and

$$\pi_1(B \#_T L) = \pi_1(L)/N(\alpha, \beta).$$

- 3) Suppose L is a symplectic 4-manifold and $\tilde{F} \subset L$ is a genus 2 symplectic surface with trivial normal bundle. Then the symplectic sum $L \#_F X$ of L and X along their genus 2 surfaces is a symplectic manifold satisfying

$$e(L \#_F X) = e(L) + 10, \quad \sigma(L \#_F X) = \sigma(L) - 2$$

and

$$\pi_1(L \#_F X) = \pi_1(L)/N$$

where N denotes the normal subgroup of $\pi_1(L)$ generated by the image of $\pi_1(\tilde{F}) \rightarrow \pi_1(L)$.

Proof. The assertions about e and σ are straightforward. The fundamental group assertion is proved using the Seifert-Van Kampen theorem. q.e.d.

Theorem 23 can be restated informally by saying that at a cost of 6 to the Euler characteristic one can symplectically kill one class in the fundamental group of a symplectic manifold, provided that class is carried by a symplectic torus. Similarly at a cost of 10 to e one can kill two classes carried on a symplectic torus. Lastly at a cost of 10 to e one can kill four classes carried by a symplectic genus 2 surface.

5.2. Arbitrary fundamental group. The fundamental group G of a closed, orientable 4-manifold M determines all its Betti numbers except $b_2(M)$. Moreover, by the Hopf exact sequence, $b_2(M) \geq b_2(G)$, and hence

$$(6) \quad e(M) = 2 - 2b_1(M) + b_2(M) = 2 - 2b_1(G) + b_2(M) \geq 2 - 2b_1(G) + b_2(G).$$

In particular, thinking of $b_2(M)$ as a measure of the size of a 4-manifold M among manifolds with a given fundamental group G , one sees that minimizing the Euler characteristic is the same as minimizing this size. One can get more subtle lower bounds on $e(M)$ than (6) by studying the algebraic topology (e.g. the ring structure) of $K(G, 1)$; see [17, 18].

As explained in Section 4 of [5], the existence of the symplectic manifold X_1 and its symplectic torus T of Theorem 23 allows us to improve (by 50%) the main result of [5] to the following theorem.

Theorem 24. *Let G be a finitely presented group that has a presentation with g generators and r relations. Then there exists a symplectic 4-manifold M with $\pi_1 M \cong G$, Euler characteristic $e(M) = 10 + 6(g+r)$, and signature $\sigma(M) = -2 - 2(g+r)$.*

Proof. The proof relies on our construction in [5, Theorem 6] of a symplectic 4-manifold N whose fundamental group contains classes $s, t, \gamma_1, \dots, \gamma_{r+g}$ so that

$$G \cong \pi_1(N)/N(s, t, \gamma_1, \dots, \gamma_{r+g})$$

where $N(s, t, \gamma_1, \dots, \gamma_{r+g})$ denotes the normal subgroup generated by the classes $s, t, \gamma_1, \dots, \gamma_{r+g}$.

Moreover, N contains symplectic tori T_0, T_1, \dots, T_{g+r} so that the two generators of $\pi_1(T_0)$ represent s and t , and for $i \geq 1$ the two generators of $\pi_1(T_i)$ represent s and γ_i . The manifold N satisfies $e(N) = 0$ and $\sigma(N) = 0$; in fact N is a product $Y \times S^1$ where Y is a 3-manifold that fibers over S^1 .

Let B denote the manifold of Theorem 18. Take the fiber sum of N with B along T_0 , and $g+r$ copies of the manifold X_1 of Theorem 22 along the tori T_i , $i \geq 1$ using an appropriate gluing map as in Theorem 23, so that s, t and the γ_i are killed. Then a repeated application of Theorem 23 computes

$$\pi_1(M) = \pi_1(N)/N(s, t, \gamma_i) = G, \quad e(M) = 10 + 6(g+r),$$

and

$$\sigma(M) = -2 - 2(g+r).$$

q.e.d.

An examination of the proof of Theorem 6 of [5] shows that Theorem 24 can be improved for certain presentations, namely, if G is the quotient of a surface group, $G = \langle x_i, y_i \mid \prod [x_i, y_i], w_1, \dots, w_r \rangle$, one can find M so that $e(M) = 10 + 6(2g' + r)$ and $\sigma(M) = -2 - 2(2g' + r)$, where g' is the number of generators which appear in some relation w_i with negative exponent.

Indeed, the first step of the construction of [5, Theorem 6] replaces a presentation of G by a new presentation in which every relation is expressed as a word in the generators with only positive exponents. One way to do this is by adding a pair of generators \tilde{a}, \tilde{b} and two relations $a\tilde{a} = 1$ and $\tilde{b} = 1$ for each generator a which appears in the first presentation with negative exponent in some relation. In the new presentation, any occurrence of a with a negative exponent in a relation is replaced by \tilde{a} . This process produces a presentation of G as a quotient of a surface

group with $r + 2g'$ relations in which every generator appears with only positive powers in each relation. The approach used in Theorem 24 is to add a generator \tilde{a} and a relation $a\tilde{a} = 1$ for every generator a of the first presentation.

In particular, if G is a quotient of a surface group with r additional relations in which every generator appears only with positive exponent in each additional relation, then there exists a symplectic 4-manifold M with fundamental group $\pi_1(M) = G$, and $e(M) = 10 + 6r$, $\sigma(M) = -2 - 2r$. Moreover, using Usher's theorem [39] one sees that the manifolds constructed are minimal.

Variations on this idea show that if G is the free product of n finite cyclic groups, then G has a presentation

$$G = \langle x_1, \dots, x_n \mid x_1^{a_1}, \dots, x_n^{a_n} \rangle$$

with all the $a_i > 0$, and there exists a symplectic 4-manifold with fundamental group G and $e = 10 + 6n$.

If one uses $E(1)$ instead of B in the proof of Theorem 24, the resulting manifold has $e = 12 + 6(g + r)$ and $\sigma = -8 - 2(g + r)$, and contains a symplectic torus which lies in a cusp neighborhood. Thus the geography results of J. Park [28] can be improved to find a larger region of the (c_1^2, χ_h) plane for which to each pair of integers in that region one can find infinitely many non-diffeomorphic, homeomorphic minimal symplectic manifolds with fundamental group G .

Another (decidedly minor) improvement concerns groups of the form $G \times \mathbb{Z}$: for a presentation of G as above there exists a symplectic manifold M with $\pi_1(M) = G \times \mathbb{Z}$, $e(M) = 6(g + r + 1)$, and $\sigma(M) = -2(g + r + 1)$. The reason is that one step in the proof of [5, Theorem 6] consists of taking a symplectic sum with $E(1)$ to kill two generators (t and s along T_0 in the notation of the proof of Theorem 24). But to get $G \times \mathbb{Z}$ it suffices to kill t , for which the manifold X_1 can be used instead of B , using Theorem 23. This only adds 6 to e .

Notice that if G is the fundamental group of a 3-manifold Y that fibers over S^1 , then $Y \times S^1$ is a symplectic 4-manifold with fundamental group $G \times \mathbb{Z}$ and Euler characteristic zero.

Suppose M is a symplectic 4-manifold containing a symplectic torus T with trivial normal bundle so that $\pi_1(M - T) = 1$ or \mathbb{Z} and so that pushing T into $M - T$ induces a surjection $\pi_1(T) \rightarrow \pi_1(M - T)$. One can prove (using the argument we gave at the beginning of Section 4.5) that if M is simply connected then $e(M) \geq 6$, and if $6 \leq e(M) < 12$ then $b^+(M) > 1$. The manifold B of Theorem 18 is such a simply connected manifold and has $e(B) = 10$ and $b^+(B) = 3$. A similar argument shows that if $\pi_1(M) \cong \mathbb{Z}$ then $e(M) \geq 3$, and if $3 \leq e(M) < 6$ then $b^+(M) > 1$. The manifold X_1 has $e(X_1) = 6$ and $b^+(X_1) = 2$.

Further improvements in the geography problem for symplectic manifolds will be obtained if such an M is found with $6 \leq e(M) < 10$ in the simply connected case and $3 \leq e(M) < 6$ in the \mathbb{Z} case. The search for such a manifold is a promising direction for future study.

5.3. Free groups.

Theorem 25. *For any n there exists a symplectic 4-manifold D_n with fundamental group free of rank n , $e(D_n) = 10$, and $\sigma(D_n) = -2$.*

Proof. This is explained in [5, Theorems 8 and 11], but here, instead of taking a symplectic sum with $E(1)$, one uses the smaller manifold B of Theorem 18. Specifically, let E be a closed surface of genus n , with standard generators $x_1, y_1, \dots, x_n, y_n$ of $\pi_1(E)$. Let $D : E \rightarrow E$ be the diffeomorphism given by composite of the n Dehn twists along the x_i and let Y be the corresponding 3-manifold which fibers over S^1 . Then $Y \times S^1$ is a symplectic 4-manifold containing a symplectic torus $T' = S \times S^1$ where S is a section of $Y \rightarrow S^1$.

The standard calculation of $\pi_1(Y)$ as an HNN extension shows

$$\pi_1(Y \times S^1) = \langle x_i, y_i, t \mid tx_it^{-1} = x_i, ty_it^{-1} = y_ix_i \rangle \oplus \mathbb{Z}s.$$

By Theorem 23, taking the symplectic sum of $Y \times S^1$ with B along T' and T has fundamental group obtained by killing t and s in $\pi_1(Y \times S^1)$, which also kills the x_i (since the relation $y_i = y_ix_i$ implies that $x_i = 1$), leaving the free group generated by the y_i . Theorem 23 together with $e(Y \times S^1) = 0$ and $\sigma(Y \times S^1) = 0$ gives the result. q.e.d.

Notice that the smooth 4-manifold obtained by taking connected sums of n copies of $S^1 \times S^3$ has fundamental group free of rank n and $e = 2(1-n)$. Kotschick has shown ([20]) that any symplectic 4-manifold with fundamental group free of rank n has $e \geq \frac{6}{5}(1-n)$. Thus the gap in size (measured say by the rank of the second homology) between the smallest smooth and symplectic manifolds must grow linearly with the rank. The manifold of Theorem 25, with fundamental group free of rank n and $e = 10$, is the smallest known symplectic 4-manifold with free fundamental group. With the exception of rank 1 (for which the manifold X_1 has $e = 6$) we do not know of any symplectic 4-manifold M with fundamental group free of rank n which satisfies

$$\frac{6}{5}(1-n) \leq e(M) < 10.$$

5.4. Fundamental groups associated to surface bundles over the circle. By exactly the same proof as Theorem 25 one establishes (see [5, Theorem 8]).

Theorem 26. *If $H : F \rightarrow F$ be an orientation preserving diffeomorphism of a closed orientable surface, fixing a point z , and G is the quotient of $\pi_1(F, z)$ by the normal subgroup generated by the words*

$x^{-1}H_*(x)$, then there exists a symplectic 4-manifold with fundamental group G , Euler characteristic $e = 10$, and signature $\sigma = -2$.

Moreover, there exists a symplectic 4-manifold with fundamental group $G \times \mathbb{Z}$, Euler characteristic $e = 6$, and signature $\sigma = -2$. q.e.d.

We can also produce small symplectic 4-manifolds with the same fundamental group as a fibered 3-manifold.

Theorem 27. *If $H : F \rightarrow F$ be an orientation preserving diffeomorphism of a closed orientable surface and $Y \rightarrow S^1$ the F -bundle over S^1 with monodromy H , then there exists a symplectic 4-manifold D with $\pi_1(D) = \pi_1(Y)$, $e(D) = 6$, and $\sigma(D) = -2$.*

Proof. The manifold $Y \times S^1$ admits a symplectic structure so that the torus $S \times S^1$ is symplectic, where $S \subset Y$ is a section. Theorem 23 shows that taking a symplectic sum of $Y \times S^1$ with X_1 along an appropriate diffeomorphism of tori yields a symplectic manifold D in which the homotopy class of the S^1 factor is killed, and so $\pi_1(D) = \pi_1(Y)$. This manifold has $e = 6$ and $\sigma = -2$. q.e.d.

In [18] it is established that for *any* closed 3-manifold group G and any number σ , there exists a smooth 4-manifold with fundamental group G and $e = 2 + |\sigma|$, and that this is the smallest possible Euler characteristic among all 4-manifolds with fundamental group G and signature σ . In particular, there is a smooth 4-manifold with fundamental group G , $\sigma = -2$, and $e = 4$. We have found a symplectic 4-manifold whose Euler characteristic is within 2 of the smooth minimum for those G which are the fundamental group of a fibered 3-manifold.

5.5. Free abelian groups. The manifold X_1 constructed in Theorem 22 is currently the smallest known symplectic 4-manifold with fundamental group infinite cyclic, with $e = 6$. Producing small smooth 4-manifolds with free abelian fundamental groups poses an interesting challenge [17, 18]. The number of required relations in a presentation grows quadratically in the number of relations, and so one expects many 2-handles in a handlebody presentation.

Finding symplectic examples is harder. For free abelian groups of even rank a nearly complete answer was found in the collection of symplectic (in fact Kähler) manifolds $Sym^2(F_n)$ (F_n a surface of genus n). The manifold $Sym^2(F_n)$ has fundamental group \mathbb{Z}^{2n} and minimizes the Euler characteristic among symplectic manifolds with fundamental group \mathbb{Z}^{2n} except possibly when $n \equiv 2 \pmod{4}$ ([5]).

For odd rank free abelian fundamental group the situation is less clear. We do not know if there exists a symplectic manifold M with fundamental group \mathbb{Z} and $2 \leq e(M) < 6$, nor any reason why such a manifold cannot exist. For rank 3, we have the following result.

Corollary 28. *There exists a symplectic 4-manifold X_3 with fundamental group \mathbb{Z}^3 , $e = 6$ and $\sigma = -2$.*

Proof. Start with the manifold Z of Theorem 11. We do a sequence of Luttinger surgeries, as in the proof of Theorems 13 and 22. We first do $1/1$ surgery on T'_1 along m'_1 to impose the relation $b_1 = [a_2^{-1}, a_1^{-1}]$. Then $-1/1$ surgery on T_1 along ℓ_1 yields $a_1 = [b_1^{-1}, y^{-1}] = [[a_2^{-1}, a_1^{-1}], y^{-1}] = 1$ using Equation (5). Thus also $b_1 = 1$. Next, $1/1$ surgery on T'_2 along ℓ'_2 gives $b_2 = [b_1, a_2] = 1$. The remaining three generators x, y, a_2 commute using Equation (5) and the fact that $[x, y] = 1$ (we did not remove F). Call the result X_3 .

Since each surgery decreases the first Betti number by one, and $H_1(Z) = \mathbb{Z}^6$ it follows that $\pi_1(X_3) = H_1(X_3) = \mathbb{Z}^3$. q.e.d.

The smallest previously known example of a symplectic 4-manifold with fundamental group \mathbb{Z}^3 has $e = 12$. Any such symplectic manifold must have $e \geq 3$ ([5]). We know no reason why an example satisfying $3 \leq e \leq 5$ cannot exist.

More generally, the technique of [5, Theorem 20] allows us to improve the construction of symplectic 4-manifolds with odd rank free abelian groups by taking the fiber sum of $Sym^2(F_n)$ with the manifold X_1 along an appropriate torus, rather than the larger manifold K of [5, Lemma 18]. We refer the interested reader to [5] for details of the proof of the following corollary, which follows by replacing every occurrence of the symbol K by X_1 in the proof of [5, Lemma 18].

Corollary 29. *There exists a symplectic 4-manifold M with $\pi_1(M) = \mathbb{Z}^{2n-1}$ such that $e(M) = 9 - 5n + 2n^2$ and $\sigma(M) = -1 - n$. q.e.d.*

This is an improvement over the best previously known example, ($e(M) = 15 - 5n + 2n^2$ of [5, Theorem 20]), but still far from the lower bound $6 - 7n + 2n^2 \leq \min_{\pi_1(M)=\mathbb{Z}^{2n-1}} e(M)$.

5.6. Abelian groups. Modifying the argument of Corollary 28 gives the following.

Corollary 30. *Given any $p, q, r \in \mathbb{Z}$ there exists a symplectic 4-manifold $X_{p,q,r}$ with fundamental group $\mathbb{Z}/p \oplus \mathbb{Z}/q \oplus \mathbb{Z}/r$, $e = 6$ and $\sigma = -2$.*

Proof. Start with the manifold X_3 of Corollary 28. Thus $\pi_1(X_3) = \mathbb{Z}^3$, generated by x, y , and a_2 , and $b_1 = a_1 = b_2 = 1$. The Lagrangian tori T_2, T_3 and T_4 were not used to construct X_3 . Then $1/p$ Luttinger surgery on T_2 along $m_2 = y$ sets $y^p = 1$. Similarly $1/q$ surgery on T_3 along $m_3 = x$ and $1/r$ surgery on T_4 along $\ell_4 = a_2$ sets $x^q = 1$ and $a_2^r = 1$. q.e.d.

Note that the smallest previously known symplectic 4-manifolds with finite cyclic abelian group are certain complex algebraic surfaces of general type with $e = 10$ [8]. For sums of two or three abelian group the smallest previously known examples had $e = 12$ [15, 5].

The finitely generated abelian group with n generators

$$G = \mathbb{Z}^{n-k} \oplus \mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \oplus \cdots \oplus \mathbb{Z}/d_k$$

(with $d_i > 1$) has a presentation with n generators, and $\binom{n}{2} + k$ relations. Thus Theorem 24 implies that there exists a symplectic manifold N with fundamental group G satisfying

$$e(N) = 10 + 6(n + \binom{n}{2} + k), \sigma(N) = -2(n + \binom{n}{2} + k + 1).$$

The leading term of this expression for e as a function of n is $3n^2$. In other words, if we let

$$p(n) = \min\{e(N) \mid \pi_1(N) \text{ is abelian and is generated by } n \text{ elements}\}$$

one can say that

$$\lim_{n \rightarrow \infty} \frac{p(n)}{n^2} \leq 3.$$

Theorem 2 of [17] implies that

$$\frac{1}{2} \leq \lim_{n \rightarrow \infty} \frac{p(n)}{n^2}.$$

The manifolds $Sym^2(F_n)$ and those constructed in Corollary 29 show that for free abelian groups with n generators, the leading term for e is $\frac{n^2}{2}$. In fact, the following theorem shows this to be true for all finitely generated abelian groups.

Theorem 31. *Assume n is even by setting $d_k = 1$ if necessary. There exists a symplectic manifold N_G with fundamental group*

$$G = \mathbb{Z}^{n-k} \oplus \mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \oplus \cdots \oplus \mathbb{Z}/d_k$$

satisfying

$$e(N_G) = \frac{1}{2}n^2 + \frac{19}{2}n + 36$$

and

$$\sigma(N_G) = -\frac{5}{2}n - 8.$$

Proof. Write $G = \mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \oplus \cdots \oplus \mathbb{Z}/d_n$, where we allow d_n to be any integer. By allowing $d_n = 1$ if necessary, we may assume that n is even, say $n = 2g - 6$ for some $g \geq 3$.

Let F_g denote an compact Riemann surface of genus g . The Kähler surface (hence symplectic 4-manifold)

$$S_g = Sym^2(F_g) = F_g \times F_g / \sim, (x, y) \sim (y, x)$$

has fundamental group isomorphic to \mathbb{Z}^{2g} .

A pair x, y of embedded curves in F_g determine a torus $\tilde{T}(x, y) = x \times y$ in $F_g \times F_g$. This torus descends to an embedded torus $T(x, y) \subset S_g$ when $x \cap y = \phi$. Moreover, in this case $\tilde{T}(x, y)$ is Lagrangian in $F_g \times F_g$ and its image $T(x, y) \subset S_g$ is Lagrangian ([5, Proposition 21], [30]).

Choose embedded curves $a_1, b_1, a_2, \dots, a_g, b_g \subset F_g$ which represent a standard symplectic basis for $H_1(F_g)$. The composite

$$F_g \cong F_g \times \{p\} \rightarrow F_g \times F_g \rightarrow S_g$$

takes $\{a_i, b_i\}$ to a basis for $H_1(S_g)$. Let $x_4, y_4, x_5, y_5, \dots, x_g, y_g$ denote $2g - 6$ parallel copies of the curve a_1 on F_g . Consider the Lagrangian tori

$$(7) \quad T(a_1, b_2), T(b_1, b_3), T(a_2, a_3)$$

and also

$$(8) \quad T(x_4, a_4), T(y_4, b_4), T(x_5, a_5), T(y_5, b_5), \dots, T(x_g, a_g), T(y_g, b_g).$$

These $2g - 3$ tori are pairwise disjointly embedded in S_g . They are each homologically essential since each one intersects a dual torus of the same form (e.g. $T(x_i, a_i)$ intersects $T(b_1, b_i)$ transversally once). Moreover, they are linearly independent since one can check that together with their dual tori they span a hyperbolic subspace of $H_2(S_g)$. Thus [15, Lemma 1.6] implies that the symplectic form on S_g can be perturbed slightly so that these $2g - 3$ tori are symplectic.

Since $\pi_1(S_g)$ is abelian, the Hurewicz map to $H_1(S_g)$ is an isomorphism, and taking symplectic sums with X_1 has the same effect on the fundamental group as on the first homology. In particular, we need not worry about base points: killing a class in $H_1(S_g)$ also kills the corresponding class in $\pi_1(S_g)$.

Now take the symplectic sum of S_g with 3 copies of the manifold B and $2g - 6$ copies of the manifold X_1 as follows. The three copies of B are summed along the three tori of Equation (7). This kills a_1, b_2, b_1, b_3, a_2 and a_3 .

The $2g - 6$ copies of X_1 are summed along the other tori using appropriate diffeomorphisms. To start, take the sum with X_1 along $T \subset X_1$ the diffeomorphism $T \rightarrow T(x_4, a_4)$ which takes the generators t_1, t_2 of $\pi_1(T)$ to $a_4^{-1}, x_4 a_4^{d_1}$. Theorem 23 implies that this kills $x_4 a_4^{d_1}$. Recall that x_4 is a parallel copy of a_1 . Since a_1 was killed before, this sets $a_4^{d_1} = 1$. Continue by setting $b_4^{d_2} = 1$, etc. Note that d_i can be any integer, including 0 or 1.

This produces a symplectic manifold N_G with fundamental group G . Theorem 23 shows that

$$e(N_G) = 30 + 6(2g - 6) + e(S_g) = 12g - 6 + (2g^2 - 5g + 3) = \frac{1}{2}n^2 + \frac{19}{2}n + 36$$

and

$$\sigma(N_G) = -2(2g - 3) + \sigma(S_g) = -4g + 6 + (1 - g) = -\frac{5}{2}n - 8.$$

q.e.d.

References

- [1] D. Auroux, S. K. Donaldson & L. Katzarkov, *Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves*, Math. Ann. **326** (2003), no. 1, 185–203, MR 1981618, Zbl 1026.57020.
- [2] A. Akhmedov, *Exotic smooth structures on $3\mathbb{C}\mathbb{P}^2 \# 7\overline{\mathbb{C}\mathbb{P}^2}$* , preprint (2006) math.GT/0612130.
- [3] A. Akhmedov & B. Doug Park, *Exotic Smooth Structures on Small 4-Manifolds*, Invent. Math. **173** (2008), no. 1, 209–223, MR 2403396, Zbl 1144.57026.
- [4] S. Baldridge, *New symplectic 4-manifolds with $b_+ = 1$* , Math. Ann. **333** (2005), 633–643, MR 2198801, Zbl 1082.53079.
- [5] S. Baldridge & P. Kirk, *On symplectic 4-manifolds with prescribed fundamental group*, Comment. Math. Helv. **82** (2007), no. 4, 845–875, MR 2341842, Zbl pre05229484.
- [6] S. Baldridge & P. Kirk, *A symplectic manifold homomorphic but not diffeomorphic to $\mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$* , Geometry & Topology **12** (2008), 919–940, MR 2403801, Zbl pre05275135.
- [7] S. Baldridge & P. Kirk, *Symplectic 4-manifolds with arbitrary fundamental group near the Bogomolov-Miyaoka-Yau line*, J. Symplectic Geom. **4** (2006), no. 1, 63–70, MR 2240212, Zbl 1105.53067.
- [8] W. Barth, C. Peters & A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 4. Springer-Verlag, Berlin, 1984. x+304 pp., MR 0749574, Zbl 0718.14023.
- [9] J. F. Davis & P. Kirk, *Lecture notes in algebraic topology*, Graduate Studies in Mathematics, 35, American Mathematical Society, Providence, RI, 2001. xvi+367 pp., MR 1841974, Zbl 1018.55001.
- [10] R. Fintushel & R. Stern, *Surgery on nullhomologous tori and simply connected 4-manifolds with $b^+ = 1$* , J. Topol. **1** (2008), no. 1, 1–15, MR 2365649, Zbl pre05254778.
- [11] R. Fintushel & R. Stern, *Knots, links, and 4-manifolds*, Invent. Math. **134** (1998), no. 2, 363–400, MR 1650308, Zbl 0914.57015.
- [12] R. Fintushel & R. Stern, *Families of simply connected 4-manifolds with the same Seiberg-Witten invariants*, Topology **43** (2004), no. 6, 1449–1467, MR 2081432, Zbl 1064.57036.
- [13] R. Fintushel & R. Stern, *Will we ever classify simply-connected smooth 4-manifolds?*, Clay Mathematics Proceedings **5** (2006), Floer Homology, Gauge Theory and Low Dimensional Topology, CMI/AMS Book Series, 225–240, MR 2233609, Zbl 1097.57001.
- [14] M. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), no. 3, 357–453, MR 0679066, Zbl 0528.57011.
- [15] R. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. (2) **142** (1995), no. 3, 527–595, MR 1356781, Zbl 0849.53027.

- [16] R. Gompf & A. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20, American Mathematical Society, Providence, RI, 1999, xvi+558 pp., MR 1707327, Zbl 0933.57020.
- [17] P. Kirk & C. Livingston, *The Hausmann-Weinberger 4-manifold invariant of abelian groups*, Proc. Amer. Math. Soc. **133** (2005), no. 5, 1537–1546, MR 2111955, Zbl 1061.57001.
- [18] P. Kirk & C. Livingston, *The geography problem for 4-manifolds with specified fundamental group*, Trans. Amer. Math. Soc. **361** (2009), 4091–4124.
- [19] D. Kotschick, *On manifolds homeomorphic to $\mathbb{C}P^2 \# 8\overline{\mathbb{C}P}^2$* , Invent. Math. **95** (1989), no. 3, 591–600, MR 0979367, Zbl 0691.57008.
- [20] D. Kotschick, *Minimizing Euler characteristics of symplectic four-manifolds*, Proc. Amer. Math. Soc. **134** (2006), no. 10, 3081–3083, MR 2231635, Zbl 1094.57004.
- [21] K. M. Luttinger, *Lagrangian Tori in \mathbb{R}^4* , J. Differential Geom. **42** (1995), no. 2, 220–228, MR 1366546, Zbl 0861.53029.
- [22] D. McDuff & D. Salamon, *Introduction to symplectic topology*, Second edition. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998. x+486 pp., MR 1698616, Zbl 0844.58029.
- [23] J. Morgan, T. Mrowka & Z. Szabo, *Product formulas along T^3 for Seiberg-Witten invariants*, Math. Res. Lett. **4** (1997), no. 6, 915–929, MR 1492130, Zbl 0892.57021.
- [24] B. Ozbagci & A. Stipsicz, *Noncomplex smooth 4-manifolds with genus 2 Lefschetz fibrations*, Proc. Amer. Math. Soc. **128** (2000), no. 10, 3125–3128, MR 1670411, Zbl 0951.57015.
- [25] P. Ozsváth & Z. Szabó, *On Park’s exotic smooth four-manifolds*, ‘Geometry and topology of manifolds’, 253–260, Fields Inst. Commun., 47, Amer. Math. Soc., Providence, RI, 2005, MR 2189937, Zbl 1095.57027.
- [26] B. Doug Park, *Exotic smooth structures on $3\mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2$* , Proc. Amer. Math. Soc. **128** (2000), no. 10, 3057–3065, MR 1664426, Zbl 0957.5702.
- [27] Jongil Park, *Simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $c_1^2 = 2$* , Invent. Math. **159** (2005), no. 3, 657–667, MR 2125736, Zbl 1080.57030.
- [28] Jongil Park, *The geography of symplectic 4-manifolds with arbitrary fundamental group*, Proc. Amer. Math. Soc. **135** (2007), no. 7, 2301–2307, MR 2299508, Zbl 1116.57023.
- [29] Jongil Park, A. Stipsicz & Z. Szabó, *Exotic smooth structures on $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P}^2$* , Math. Res. Lett. **12** (2005), no. 5-6, 701–712, MR 2189231, Zbl 1081.57025.
- [30] T. Perutz, *A remark on Kähler forms on symmetric products of Riemann surfaces*, Preprint (2005) math.SG/0501547.
- [31] D. Salamon, *Removable singularities and a vanishing theorem for Seiberg-Witten invariants*, Turkish J. Math **20** (1996), no. 1, 61–73, MR 1392663, Zbl 0865.57019.
- [32] I. Smith, *Symplectic geometry of Lefschetz fibrations*, Dissertation, Oxford 1998.
- [33] A. Stipsicz & Z. Szabó, *An exotic smooth structure on $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P}^2$* , Geom. Topol. **9** (2005), 813–832, MR 2140993.
- [34] A. Stipsicz & Z. Szabó, *Small Exotic 4-manifolds with $b_2^+ = 3$* , Bull. London Math. Soc. **38** (2006), 501–506, MR 2239045, Zbl 1094.57027.

- [35] Z. Szabó, *Simply-connected irreducible 4-manifolds with no symplectic structures*, *Invent. Math.* **132** (1998), no. 3, 457–466, MR 1625732, Zbl 0906.57014.
- [36] C. Taubes, *Seiberg-Witten and Gromov invariants*, *Geometry and physics (Aarhus, 1995)*, 591–601, *Lecture Notes in Pure and Appl. Math.*, 184, Dekker, New York, 1997, MR 1423194, Zbl 0873.57017.
- [37] C. Taubes, *The Seiberg-Witten invariants and symplectic forms*, *Math. Res. Lett.* **1** (1994), no. 6, 809–822, MR 1306023, Zbl 0853.57019.
- [38] C. Taubes, *SW \Rightarrow Gr: from the Seiberg-Witten equations to pseudo-holomorphic curves*, *J. Amer. Math. Soc.* **9** (1996), no. 3, 845–918, MR 1362874, Zbl 0867.53025.
- [39] M. Usher, *Minimality and symplectic sums*, *Int. Math. Res. Not.* 2006, Art. ID 49857, 17 pp., MR 2250015, Zbl 1110.57017.

DEPARTMENT OF MATHEMATICS
LOUISIANA STATE UNIVERSITY
BATON ROUGE, LA 70817
E-mail address: sbaldrid@math.lsu.edu

MATHEMATICS DEPARTMENT
INDIANA UNIVERSITY
BLOOMINGTON, IN 47405
E-mail address: pkirk@indiana.edu