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CONSTRUCTIVE GENERALIZED QUANTIFIERS

The syntactic categorization of ordinary quantifiers in contemporary (model-theoretic) theories of quantification, namely that of expressions which combine with predicates to yield complete sentences, is an inheritance from Frege, whereas the standard notation

(1) $Qx\varphi$

is borrowed from Peano via Russell (in preference to Frege's two-dimensional symbolism). These Founding Fathers of modern logic operated with interpreted formal languages where each formula had its fixed meaning. Thus Fregean quantifiers were intended to range over the universe of 'all objects' and hence, since all quantifications concern the same domain, there was no practical or theoretical need for Frege to include explicit information concerning the domain of quantification in the quantifier-notation.

Frege's notion of a universe of all objects is highly problematic, though. The ontological category of objects (*Gegenstände*) is obtained via the semantical category of proper names (*Eigennamen*). An object is what can serve as the *Bedeutung* of a proper name. Among the elements of this all-inclusive universe are certain *abstract* objects, the proper names of which are obtained through the use of such second-level operators as "the class of x such that $\varphi(x)$" and the cardinality-operator "the number of x such that $\varphi(x)$". The predicate, however, is allowed to take on also the form (1) and this means that the totality of objects has not been sharply delineated: objects are the referents of proper names and some of the latter are formed in such a way as to involve a universal quantification over the universe of objects in an essential way. The viciousness of this circle of impredicativity gets acute in, for example, the paradoxes of Russell and Cantor, respectively.[1]

Owing to these difficulties, one insists that the quantifiers are to range over a totality whose existence and extension are secured prior to, and independently of, its service as a domain of quantification. Thus one might prima facie think that post-Fregean theories of

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quantification would use a notation that explicitly indicates the domain A of quantification such as

(2) $(Qx\colon A)\varphi$

or, in another notation

(3) $Q(A, (x)\varphi)$ (Here $(x)\varphi$ indicates that the variable x is bound in the predicate φ.)

The syntactic category of such a quantifier is that of an expression which combines with a saturated expression indicating a set and a predicate to yield a complete sentence. This we find in Russell who worked with an interpreted formal language and who chose to use *types*, that is, predicational ranges of significance, as quantificational domains (even though 'typical ambiguity' made the use of (1) possible in certain circumstances). The reversal back to the Fregean syntax (1) comes about through the abandonment of a central Fregean feature, namely that the formalism should have one fixed interpretation. After the separation between syntax and semantics in the 1930s, when logicians came to adopt a metamathematical attitude in the study of *un*interpreted formal systems, model theoretic semantics reigned supreme. Then the domain of quantification has no syntactic role, but belongs to semantics solely. This preference may further have been enforced by the central role that *pure* predicate calculus came to occupy within the metamathematical studies; here there is no 'intended interpretation' and all domains of quantification stand on an equal footing. In view of the impredicative obstacles built into Frege's universe it is worthwhile to remark here that even though the syntax is the Fregean (1), in any particular interpretation the effect of the demand that the domain of quantification by specifiable independently of its use as such a domain is, on the level of semantics, to transform the form (1) into (2).

In the model-theoretic tradition the syntax of the generalized quantifiers is Fregean. Examples are

(4) Most $x\varphi$ – most things are φ

and its binary version

(5) Most $x(\varphi(x), \psi(x))$ – most φ are ψ,

where in both cases an underlying domain is presupposed in the semantics. Thus, considered semantically, the real forms of (4) and (5) are (even model theoretically)

(6) (Most x: A)$\varphi(x)$ – most A are φ, or, along the lines of (3), Most(A, $(x)\varphi$)

and

(7) (Most x: A)($\varphi(x)$; $\psi(x)$) – most φ in A are ψ, or Most (A, $(x)\varphi$, $(x)\psi$).

From a constructive point of view, on the other hand, the use of an uninterpreted formalism is unnatural and (6) and (7) are also syntactically the proper versions. One can, however, consider quantifiers that combine with more sets than one, such as

(8) More (A, B) – There are more A than B

and

(9) More (A, $(x)\varphi$; B, $(x)\psi$) – There are more φ in A than there are ψ in B.

The possible forms are:

(i) combine a domain with one or more domains;
(ii) combine a domain with one or more one- or many-place predicates;
(iii) combine two or more domains with one or more one- or many-place predicates.

Since classical model theoretic semantics has to be ruled out from a constructive point of view, the meaning of a constructive generalized quantifier has to be given via either a real or a nominal definition. The former alternative consists in taking the generalized quantifiers as new primitives and providing them with direct meaning explanations along the lines of Heyting's familiar pattern. The second alternative consists in giving explicit definitions of the quantifiers using known constructive abstractions.

In this note I shall adopt the latter perspective, using the type theory of Martin-Löf (1975, 1984) as my constructivist framework. Following Martin-Löf's current practice the (1984) I-rules are to be reformulated along the lines of (1975) and I shall not make any difference between judgmental identity $=$ and definitional equality $\equiv$. Furthermore, I shall use a colon in place of Martin-Löf's epsilon. In (1986) I have given an informal presentation of the type theory of Martin-Löf (whose initial section in (1975) also gives sufficient background). Aczel (1980) is another accessible presentation.

Of crucial importance in my treatment is that $\Sigma(A, (x)\varphi)$ does triple duty in the type theory: it serves as generalized union, existential quantifier, and set abstraction. An element of the set of elements in A that have the property φ is from a constructivist perspective nothing but a pair (a, b), where $a: A$ and b is a proof-object for $\varphi(a)$, i.e., a proof that a has the property φ.

In the type theory, canonical R-element sets M_r, $r = 0, 1, 2, \ldots$, are present. Here I shall need these sets presented as a *family* $M(r)$ over the natural numbers N. Martin-Löf has given an elegant treatment which unfortunately remains unpublished and here I shall adapt the M_r-sequence offered by Aczel (1980). First put

$$f(a) = R(a, I(N, 0, 0), (x, y)(y + I(N, 0, 0))) : U$$

for all $a : N$.

Here U is the universe of small types and + is disjoint sum (rather than addition on the natural numbers). Then

$$f(0) = I(N, 0, 0) : U$$

and

$$f(s(a)) = f(a) + I(N, 0, 0) : U \quad (a : N).$$

Finally put

$$M(a) = R(a, \bot, (x, y)f(x)) : U \quad (a : N);$$

then

$$\begin{aligned} &M(0) = \bot : U \\ &M(1) = I(N, 0, 0) : U \\ &M(s(s(a))) = M(s(a)) + I(N, 0, 0) : U \quad (a : N), \end{aligned}$$

as required by Aczel.

Furthermore, I shall use various sorts of mappings between sets. Consider first the case of *injections*.

(10) $\dfrac{f : A \to B}{f \text{ is an injection} : \text{Prop}}$

(11) $\dfrac{f : A \to B}{f \text{ is an injection} = (\forall x : A)(\forall y : A)(I(B, ap(f, x), ap(f, y)) \supset I(A, x, y)) : \text{PROP}}$

The first of these two rules says that f ***is an injection*** is a proposition, given that f is a function from A to B and the second says which proposition it is, given the same presupposition. This is the form that nominal definitions will take in this note. In the sequel I shall compress the two rules into one. Surjections are dealt with similarly.

(12) $\dfrac{f : A \to B}{f \text{ is a surjection} : \text{Prop}}$
$f \text{ is a surjection} = (\forall y : B)(\exists x : A)I(B, ap(f, x), y).$

Finally, *bijections* are defined

(13) $\dfrac{f : A \to B}{f \text{ is a bijection} : \text{Prop}}$
$f \text{ is a bijection} = f \text{ is an injection} \;\&\; f \text{ is a surjection} : \text{Prop}.$[2]

The last of our preliminaries deals with finiteness.

(14) $\dfrac{A : \text{set}}{\text{Finite } (A) : \text{Prop}}$
$\text{Finite } (A) = (\exists k : N)(\exists f : M(k) \to A)\ (f \text{ is a bijection}) :$ Prop.

Here we see the use of the *family* $M(k)$; a set is finite if it is a bijective image of a k-element set.

(15) $\dfrac{A : \text{set}}{\text{Infinite } (A) : \text{Prop}}$
$\text{Infinite } (A) = (\exists f : N \to A)(f \text{ is an injection}) : \text{Prop}.$

We are now ready to treat of a number of generalized quantifiers.

(i) *Finately many A are* φ.

(16) $\dfrac{A : \text{set} \quad \varphi(x) : \text{Prop}(x : A)}{\text{FIN}(A, (x)\varphi) : \text{Prop}}$
$\text{FIN}(A, (x)\varphi) = (\exists k : N)(\exists f : M(k) \to A)(f \text{ is an injection} \;\&\; (\forall x : A)(\varphi(x) \leftrightarrow (\exists y : M(k))I(A, x, ap(f, y))))) : \text{Prop}.$

This definition says that the elements of A that have the property are an injective image of a k-element set for some natural number k.

(II) *Infinitely many A are* φ.

(17) $$\frac{A:\text{set} \quad \varphi(x):\text{Prop}(x:A)}{\text{INF}(A,(x)\varphi):\text{Prop}}$$

$\text{INF}(A,(x)\varphi) = (\exists f:N\to A)(f \text{ is an injection } \& (\forall k:N)\varphi(ap(f,k))):\text{Prop}.$

The injective image of N lies within φ; thus infinitely many A are φ.

(III) *Countably many A are.*

(18) $$\frac{A:\text{set} \quad \varphi(x):\text{Prop}(x:A)}{\text{COUNT}(A,(x)\varphi):\text{Prop}}$$

$\text{COUNT}(A,(x)\varphi) = (\exists f:N\to A)(f \text{ is an injection } \& (\forall x:A)(\varphi(x) \leftrightarrow (\exists k:N)I(A,x,ap(f,k))))):\text{Prop}.$

(IV) *Uncountably many A are.*

(19) $$\frac{A:\text{set} \quad \varphi(x):\text{Prop}(x:A)}{\text{Uc}(A,(x)\varphi):\text{Prop}}$$

$\text{Uc}(A,(x)\varphi) = (\exists f:\sigma\to A)(f \text{ is an injection } \& (\forall \alpha:\sigma)\varphi(ap(f,\alpha))):\text{Prop}.$

Here σ is the constructive second number class, cf. Martin-Löf (1984, p. 84).

(V) *There are more A than B.*

(20) $$\frac{A:\text{set} \quad B:\text{set}}{\text{More }(A,B):\text{Prop}}$$

$\text{More }(A,B) = (\forall g:B\to A)(\exists x:A)(\forall y:B)\neg I(A,x,ap(g,y)).$[3]

No mapping from B to A is onto; hence there are more A than B.

(VI) *There are at least as many* φ *in A as* ψ *in B.*

$$\frac{A:\text{set} \quad \varphi(x):\text{Prop}(x:A) \quad B:\text{set} \quad \psi(x):\text{Prop}(x:B)}{\text{At least }(A,(x)\varphi;B,(x)\psi):\text{Prop},}$$

$\text{At least }(A,(x)\varphi;B,(x)\psi)$
$= (\exists f:(\Sigma x:B)\psi(x)\to(\Sigma x:A)\varphi(x))$
$((\forall z:\Sigma(B,(x)\varphi))(\forall w:\Sigma(B,(x)\psi))$
$(I(A,p(ap(f,z)),p(ap(f,w)))\supset I(B,p(z),p(w)))):\text{Prop}.$

One might here prefer another definition using a function $f:B\to A$ which is one-one on such Bs as are ψ and which takes ψs into φs. Def.

(21) seems weaker in that it is not clear how to define constructively a function in $B \to A$ with the desired properties given a function in $\Sigma(B, (x)\psi) \to \Sigma(A, (x)\varphi)$ when the ψ and φ are undecidable.

(VII) *Almost all A are* φ.

For finite A it is not obvious that a uniform definition exists. (Compare the cases where A has 11, 98 and 2048 members!). Thus I prefer to work under the presupposition that A is infinite in which case "Almost all A are φ" means the same as "All but finitely many A are φ". This then provides the key to the uniform definition.

(22)
$$\frac{A:\text{Set} \quad \varphi(x):\text{Prop}(x:A) \quad a:\text{Infinite}(A)}{\text{Almost all } (A,(x)\varphi):\text{Prop}}$$
$$\text{Almost all } (A,(x)\varphi) = (\exists k:N)(\exists f:M(k)\to A)(f \text{ is an injection } \& \ (\forall x:A)(\varphi(x) \leftrightarrow (\forall y:M(k)) \neg I(A, y, ap(f, x)))).$$

(What is defined is really "all but finitely many a are φ" and this quantifier is uniformly meaningful *and true* for finite A, whence no use is made of the proof-object a.)

Finally I wish to treat of the most interesting case, namely

(VIII) *Most A are* φ.

which quantifier will be taken in its mathematical sense of "More than half of the A are φ". (Its natural language meaning seems more 'statistical' in nature.) Thus in order to make sense of this I need to impose Finite (A) as a presupposition.

(23)
$$\frac{A:\text{set} \quad \varphi(x):\text{Prop}(x:\text{set}) \quad a:\text{Finite}(A)}{\text{Most } (A,(x)\varphi):\text{Prop.}}$$

The second definitional clause takes some preparation. First we define some primitive recursive functions (whose definitions can all be given in the type theory using the recursor R), namely the signum and the remainder, and integral part, upon division by 2.

$$sg(0) = 1 \qquad \begin{cases} \text{rem } (0/2) = 0 \\ \text{rem } (S(a)/2) = sg(rem(a/2)) \end{cases}$$
$$sg(s(a)) = 0$$

$$[0/2] = 0$$
$$[S(a)/2] = [a/2] + rem(a/2)$$

Now we give the outstanding clauses still under the same presuppositions as before.

(23) $\text{Most}\ (A, (x)\varphi) = (\exists k : N)$
$(k \geqslant [p(a)/2] + 1\ \&\ (\exists f : M(k) \rightarrow A)$
$(f \text{ is an injection } \&\ (\forall y : M(k)\varphi(ap(f, y))))) : \text{Prop}.$

In this definition a is the proof-object that Finite (A). Thus $p(a) : N$ is the cardinal of A and $[p(a)/2] + 1$ is the size of the least possible majority in A.[4]

It remains to discuss some interesting points concerning these definitions and (VIII) in particular. In this context it is worthwhile to recall the treatment offered in my (1986, Section 7) of another quantificational worry, namely the so-called donkey-sentences studied by Geach.

Thus, within the type-theory, a typical donkey-sentence, such as

(*) All men who own a donkey beat it,

is analysed as

(24) $\forall z : (\Sigma x : \text{Man})(\exists y : \text{Donkey})\text{Own}(x, y)\text{Beat}(p(z), p(q(z))).$

Here one makes use of a Σ-representation of the set of men who own a donkey. Martin-Löf independently suggested the more straightforward

(25) $(\forall x : \text{Man})((\Pi z : (\exists y : \text{Donkey})\text{Own}(x, y))\text{Beat}(x, p(z)))$

Here the hanging "Πz" expresses the implication in

(**) Every man is such that *if* he owns a donkey, *then* he beats it.

(*) and (**) are provably equivalent in the type theory (cf. Martin-Löf (1984, p. 48) but they are not synonymous, that is, they are not equal as propositions. Consideration of "Most" gives a ground on which to base a decision as to which treatment is preferable. The donkey-sentence "most men who own a donkey beat it" has to be analysed as "most donkey-owners beat their donkeys" rather than as "most men are such that if they own a donkey they beat it" since the latter can be true even though *no* man who owns a donkey beats it. Consider a case with four donkey-owners only, all kind and caring. Then, for any man m (except the four kind ones), it is false that he owns a donkey and vacuously true that if m owns a donkey, then he beats it. So in this case most men are such that if they own a donkey, then they beat it. Hence

(26) $(\text{Most } z : (\Sigma x : \text{Man})(\exists y : \text{Donkey})$
$\text{Own}(x, y)) \text{ Beat}(p(z), p(q(z))$

is the right analysis. The situation corresponds with the well-known non-reducibility of the binary most to the unary most in the model-theoretic case.

From the above discussion we see that the use of the $\Sigma(A, B)$-representation of the set of As such that B is needed to take care of the anaphora in the donkey-sentences. The use of the subset-type $\{x : A \mid P(x)\text{true}\}$ from Nordström et al. (1986), where the proof-objects are thrown away, is not suitable here; the proof-objects are needed in that we explicitly have to point to the donkey that constitutes (part of) the proof-object that a given donkey-owner owns a donkey.

All of the above nominal definitions are *uniform in the quantificational domain A*. Thus in the case of the donkey-sentences we quantify over a set of the form $\Sigma(A, B)$ and we use a presupposition of the form a: Finite $(\Sigma(A, B))$. What gets counted by the bijection in the definition of Finite $(\Sigma(A, B))$ is not elements of A (that happen to satisfy further conditions), but ordered pairs of a fairly complex structure. A donkey-owner is thus an ordered pair (a, b), where a is a man and b is a proof-object that he owns a donkey, i.e., b is an ordered pair (c, d) where c is a donkey and d is a proof-object that a owns c. Hence, there are two ways in which the cardinal of $(\Sigma x : \text{Man})(\exists y : \text{Donkey})\text{Own}(x, y)$ can differ from that of the subset type

$$\{x : \text{Man} \mid (\exists y : \text{Donkey})\text{Own}(x, y) \text{ true}\}.$$

First, there might be many proof-objects d that a owns c. Here it seems reasonable that there is only one canonical proof-object in analogy with recursive realizability. Secondly, a man who owns two donkeys is counted twice in the $\Sigma(A, B)$-version, but not in the case of $\{x : A \mid B(x)\text{true}\}$. This, perhaps is not always natural, but it is not preposterous. Think, e.g., of car-owners in fiscal contexts. Fiscally, a man who possesses two cars is counted as two car-owners.

The alternative in this last case is to use what one may call

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