2004

Constructive Lower Bounds on Classical Multicolor Ramsey Numbers

Xiaodong Xu
National University of Defense Technology

Xie Zheng
National University of Defense Technology

Geoffrey Exoo
Indiana State University

Stanislaw Radziszowski
Rochester Institute of Technology

Constructive Lower Bounds on Classical Multicolor Ramsey Numbers

Xu Xiaodong and Xie Zheng
Department of Mathematics and System Science
National University of Defense Technology
Changsha, 410073 P.R. China
xxdmaths@sina.com

Geoffrey Exoo
Department of Mathematics and Computer Science
Indiana State University
Terre Haute, IN 47809
g-exoo@indstate.edu

Stanisław P. Radziszowski
Department of Computer Science
Rochester Institute of Technology
Rochester, NY 14623
spr@cs.rit.edu

Submitted: Apr 5, 2004; Accepted: May 19, 2004; Published: Jun 4, 2004
MR Subject Classification: 05C55

Abstract

This paper studies lower bounds for classical multicolor Ramsey numbers, first by giving a short overview of past results, and then by presenting several general constructions establishing new lower bounds for many diagonal and off-diagonal multicolor Ramsey numbers. In particular, we improve several lower bounds for $R_k(4)$ and $R_k(5)$ for some small k, including $415 \leq R_3(5)$, $634 \leq R_4(4)$, $2721 \leq R_4(5)$, $3416 \leq R_5(4)$ and $26082 \leq R_5(5)$. Most of the new lower bounds are consequences of general constructions.

1 Introduction and Notation

In this paper we study undirected loopless graphs and edge-colorings, where, technically, an edge joining u and v is a set $\{u, v\}$. Often however we will denote the same edge by (u, v), or equivalently by (v, u).

A $(k_1, k_2, \ldots, k_r)$-*coloring*, $r, k_i \geq 1$, is an assignment of one of r colors to each edge in a complete graph, such that it does not contain any monochromatic complete subgraph K_{k_i} in color i, for $1 \leq i \leq r$. Similarly, a $(k_1, k_2, \ldots, k_r; n)$-*coloring* is a $(k_1, \ldots, k_r)$-coloring of the complete graph on n vertices K_n. Let $\mathcal{R}(k_1, \ldots, k_r)$ and $\mathcal{R}(k_1, \ldots, k_r; n)$ denote the set of all $(k_1, \ldots, k_r)$- and $(k_1, \ldots, k_r; n)$-colorings, respectively. The Ramsey number $R(k_1, \ldots, k_r)$ is defined to be the least $n > 0$ such that $\mathcal{R}(k_1, \ldots, k_r; n)$ is empty. In the diagonal case $k_1 = \ldots = k_r = m$ we will use simpler notation $\mathcal{R}_r(m)$ and $\mathcal{R}_r(m; n)$ for sets of colorings, and $R_r(m)$ for the Ramsey numbers.

In the case of 2 colors ($r = 2$) we deal with classical *graph* Ramsey numbers, which have been studied extensively for 50 years. Much less has been done for multicolor numbers ($r \geq 3$). Another area of major interest has been the study of generalized Ramsey colorings, wherein the forbidden monochromatic subgraphs are not restricted to complete graphs. Stanisław Radziszowski maintains a regularly updated survey [22] of the most recent results on the best known bounds on various types of Ramsey numbers.

In Section 2 we given an overview of previous results on bounds for multicolor numbers, focusing mostly on recursive lower bound constructions. Section 3 reviews an old construction described by Giraud in 1968, which produces Schur and cyclic colorings from smaller colorings, and which seems to have been nearly forgotten. In Section 4 we present a sequence of new general constructions, and in Section 5, we describe some K_3-avoiding constructions. Section 6 presents some lower bounds implied by explicit constructions obtained from heuristic computer searches.

Many specific new lower bounds are obtained throughout the paper as corollaries to general constructions, for example $634 \leq R_4(4)$, $2721 \leq R_4(5)$, $15202 \leq R_4(6)$, $62017 \leq R_4(7)$, $3416 \leq R_5(4)$, and $26082 \leq R_5(5)$. All of these bounds improve lower bounds listed in the survey *Small Ramsey Numbers* [22]. All lower bounds discussed in this paper, including several off-diagonal cases, are gathered and indexed in Section 7.

2 Previous Work

In 1955, Greenwood and Gleason [14] proved the general upper bound

$$R(k_1, \ldots, k_r) \leq 2 - r + \sum_{i=1}^{r} R(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r). \tag{1}$$

Inequality (1) is strict if the right hand side is even and at least one of the terms in the summation is even. It is suspected that this upper bound is never tight for $r \geq 3$ and $k_i \geq 3$, except for the case $r = k_1 = k_2 = k_3 = 3$, since $R(2, 3, 3) = R(3, 2, 3) =$

$R(3,3,2) = R(3,3) = 6$ and $R_3(3) = 17$. The latter is the only known nontrivial value of a classical multicolor Ramsey number, determined in the same paper by Greenwood and Gleason [14]. It was later proved by Kalbfleisch and Stanton that there are exactly two nonisomorphic $(3,3,3;16)$-colorings [18]. One of them is a well known coloring with vertices in $GF(2^4)$, where the edge $\{u,v\}$ has color i, $0 \leq i < 3$, if $u - v$ is in the i-th cubic residue class. Interestingly, while the other coloring doesn't have such nice algebraic description, each of the colors in both colorings induces, up to isomorphism, the same graph.

The general lower bound inequality

$$R_r(3) \geq 3R_{r-1}(3) + R_{r-3}(3) - 3, \quad r \geq 4, \tag{2}$$

obtained constructively by Chung [5], implies the best known lower bound for the 4 color case, $51 \leq R_4(3)$. It is known that $R_4(3) \leq 62$ [9], while (1) gives an upper bound of only 66. It seems that any further improvements would require a breakthrough in what we know about upper bounds, since we believe that the true value of $R_4(3)$ is much closer, if not equal to, 51.

Perhaps the only open case of a classical multicolor Ramsey number, for which we can anticipate exact evaluation in the not-too-distance future is $R(3,3,4)$. It is known that this number is equal to either 30 or 31 [17][21]. Note that (1) only gives us $R(3,3,4) \leq 34$. Both above improvements over (1) were obtained with the help of complicated and intensive computations. No other cases of upper bounds which improve on (1) are known.

One of the most successful techniques in deriving lower bounds are constructions based on Schur partitions, and closely related cyclic and linear colorings. We now define these concepts.

A *sum free set* S of integers is a set in which $x, y \in S$ implies that $x + y \notin S$. A Schur partition of the integers from 1 to n, $[1,n]$, is a partition into sum free sets. The Schur number, $s(r)$, is the maximum n for which there exists a Schur partition of $[1,n]$ into r sets. A Schur partition is *symmetric* if x and $n - x$ are always in the same set. Schur partitions give rise to K_3-free colorings of complete graphs as follows. Given a Schur partition, $S_1 \ldots S_r$ of of $[1,n]$, we can construct the coloring of K_{n+1} by associating color i with the set S_i and by identifying the vertices of K_{n+1} with the integers from 0 to n. Given a pair of vertices u and v, determine the set S_j containing $|u-v|$ and color the edge joining u and v with the associated color j. Such a coloring is also called a *linear coloring*. A linear coloring is a *cyclic coloring* if the associated Schur partition is symmetric.

The Schur numbers are known for $1 \leq r \leq 4$, the values being 1, 4, 13, and 44, respectively. For larger values of r, we have only some lower bounds. In 1994, a construction method that produced the best known lower bound $160 \leq s(5)$, yielding immediately $162 \leq R_5(3)$, was described by Exoo [7]. For the record, we have discovered that if, in the resulting $(3,3,3,3,3;161)$-coloring, the colors 2 and 3 are merged, then we obtain a $(3,5,3,3;161)$-coloring, and thus $162 \leq R(3,3,3,5)$. This improves the previous bound of 137. No other combination of colors gives an improvement. In 2000, Fredricksen and Sweet [11] worked with higher parameters establishing lower bounds $538 \leq R_6(3)$ and $1682 \leq R_7(3)$, also by improving on previous Schur partition constructions.

In 1983 Chung and Grinstead [6] proved an inequality equivalent to (3), valid for any fixed $t \geq 1$. They also showed that the limit of $R_r(3)^{1/r}$ exists and is at least 3.16, though it might be infinite. This is in contrast to the famous open problem whether the limit of $R(k,k)^{1/k}$ exists.

$$(2s(t)+1)^{\frac{1}{t}} = c_t < \lim_{r\to\infty} R_r(3)^{\frac{1}{r}}. \tag{3}$$

Note that $c_1 = c_2 = c_3 = 3$ and $c_4 \approx 3.07$. The bound 3.16 for the constant c_5 in [6] follows from $157 \leq s(5)$, which at the time was the best known bound on $s(5)$. A slight improvement

$$3.199 < \lim_{r\to\infty} R_r(3)^{\frac{1}{r}}$$

can be derived by using $536 \leq s(6)$ obtained in [11].

Various authors used similar techniques even earlier for studying general constructions and lower bound asymptotics of $R_r(k)$ for fixed $k \geq 3$ and $r \to \infty$. Namely, Abbott and Moser [3] in 1966, Giraud [12], [13] in 1968, Abbott and Hanson [2] in 1972, and Fredricksen [10] in 1975. In particular, in [2] it is shown that for each k there exists a positive constant d_k such that

$$R_r(k) \geq d_k(2k-3)^r, \tag{4}$$

which when combined with the proof of another old result by Abbott [1],

$$R_r(pq+1) > (R_r(p+1)-1)(R_r(q+1)-1), \tag{5}$$

leads to bounds similar to (3) for avoiding K_k instead of K_3. Song En Min in [27] obtained (6), which generalizes (5) as follows

$$R(p_1q_1+1,...,p_rq_r+1) > (R(p_1+1,...,p_r+1)-1)(R(q_1+1,...,q_r+1)-1). \tag{6}$$

Since we are not aware of any discussion of inequalities (5) or (6) in many years, we note a special, yet illustrative, case of the Abbott-Song construction for $r = 4$, and $p_i, q_i = 2$ for $i = 1,2,3,4$ (general cases are discussed further in later sections). This case leads to the lower bound $2501 \leq R_4(5)$, which improves the bound of 1833 given by Mathon [20]. Fix any $(3,3,3,3;50)$-coloring C on 50 vertices, for example to one found by Chung [5]. We build a $(5,5,5,5;2500)$-coloring D on 2500 vertices formed by pairs of vertices in C as follows. For vertices x and y let $C(x,y)$ denote the color of the edge $\{x,y\}$ in C. We define $D((p,q),(r,q)) = C(p,r)$, and in the remaining cases for $q \neq s$ let $D((p,q),(r,s)) = C(q,s)$. Note that D is formed by 50 vertex-disjoint Chung's colorings C at the "lower" level, with many of the same on top of them treated as $K_{50,50,...,50}$. In any monochromatic K_5 in D at most 2 vertices can belong to the same lower level block of 50 vertices inducing C, so there must be a triangle at the "higher" level, which is a contradiction. Thus D has no monochromatic K_5. In Section 5 we improve further this bound to $2721 \leq R_4(5)$.

A simpler, but weaker form of (6) can be obtained as follows [26]. Consider $r = s + t$, and let $p_1 = ... = p_s = q_{s+1} = ... = q_r = k - 1$ and $q_1 = ... = q_s = p_{s+1} = ... = p_r = 1$, then (6) becomes (7), and thus (7) is weaker, though more concise.

$$R_{s+t}(k) > (R_s(k) - 1)(R_t(k) - 1) \tag{7}$$

For the sake of completeness, we note that a different approach was used by Mathon, who constructed some colorings based on association schemes [20].

Finally, we include lower bound recurrences (8), (9), (10), and (11) found by Robertson in [23] and [24]. In (11) we require $k_1 < k_2$. Then for $r, k, l, k_i \geq 3$ we have:

$$R(3, k, l) \geq 4R(k, l - 2) - 3, \tag{8}$$

$$R(3, 3, 3, k_1, k_2, \ldots, k_r) \geq 3R(3, 3, k_1, \ldots, k_r) + R(k_1, k_2, \ldots, k_r) - 3, \tag{9}$$

$$R(k_1, k_2, \ldots, k_r) > (k_1 - 1)(R(k_2, k_3, \ldots, k_r) - 1), \tag{10}$$

and

$$R(k_1, k_2, \ldots, k_r) > (k_1 + 1)(R(k_2 - k_1 + 1, k_3, \ldots, k_r) - 1). \tag{11}$$

Constructions (8) and (9) lead to some best known lower bounds for specific parameters (cf. [22]). (10) can be easily obtained from (6). Our Theorem 7 in Section 5 improves (8) significantly.

Some theorems in this paper are based on results described by Xu Xiaodong in the manuscript [28]. The latter contains even a few further sharpenings of lower bounds, but at the price of increasing the complexity of the assumptions, and not necessarily leading to interesting improvements of specific lower bounds.

3 Giraud's Cyclic Construction

A cyclic $(k_1, \ldots, k_r; n)$-coloring of K_n over $\mathcal{Z}_n$ will be represented by a partition $\{C_i\}_{i=1}^{r}$ of $\{1, \ldots, n-1\}$ with the property that $j \in C_i$ implies $n - j \in C_i$, for all $1 \leq i \leq r$ and $1 \leq j < n$. The color of the edge (j_1, j_2), $0 \leq j_1 < j_2 < n$, is equal to i if and only if $j_2 - j_1 \in C_i$. Note that C_i can be thought of as a set of distances in $\mathcal{Z}_n$, where all the edges between vertices with circular distance $d \in C_i$ are assigned color i.

Let $L(k_1, \ldots, k_r)$ denote the maximal order of any cyclic $(k_1, \ldots, k_r)$-coloring. It can be considered as a special case of generalized Schur partitions defining symmetric Schur numbers discussed in previous section. Many lower bounds for specific Ramsey numbers were established by cyclic colorings using the inequality $R(k_1, \ldots, k_r) \geq L(k_1, \ldots, k_r) + 1$. Similarly, the recurrence in Theorem 1 below can be applied to derive lower bounds for multicolor Ramsey numbers. The original 1968 construction for generalized Schur partitions is due to Giraud [12], who later observed [13] (both papers in French) that it also yields a recursive construction for cyclic colorings. Our proof of the same does not explicitly use Schur partitions, and we believe that it is the first one in English.

Theorem 1 [Giraud 1968] *For $k_i \geq 3$, $i = 1, \ldots, r$,*

$$L(k_1, \ldots, k_r, k_{r+1}) \geq (2k_{r+1} - 3)L(k_1, \ldots, k_r) - k_{r+1} + 2.$$

Proof. Consider any cyclic $(k_1, \ldots, k_r; n)$-coloring G over $\mathcal{Z}_n$ given by the partition $\{C_i\}_{i=1}^{r}$ of $\{1, \ldots, n-1\}$. We construct a cyclic $(k_1, \ldots, k_r, k_{r+1}; m)$-coloring H on $m = (2k_{r+1} - 3)n - k_{r+1} + 2$ vertices over $\mathcal{Z}_m$ by defining the corresponding partition $\{D_i\}_{i=1}^{r+1}$ of $\{1, \ldots, m-1\}$ as follows. For $1 \leq i \leq r$ we let

$$D_i = \{j + \lambda(2n-1) \mid (j \in C_i) \wedge (0 \leq \lambda < k_{r+1} - 1)\},$$

and

$$D_{r+1} = \{j + \lambda(2n-1) \mid (n \leq j \leq 2n-1) \wedge (0 \leq \lambda < k_{r+1} - 2)\}.$$

One can easily check that D_i's form a partition of $\{1, \ldots, m-1\}$, and that $j \in D_i$ implies $m - j \in D_i$. We have to show that H does not contain any monochromatic K_{k_i} in color i. Suppose that $S \subset \mathcal{Z}_m$, $|S| = s$, induces all the edges in color i. Without loss of generality we may assume that $0 \in S$, since otherwise we can subtract $\min(S)$ from all elements without changing colors between vertices.

We first consider colors i for $1 \leq i \leq r$. For $x > y$, if $x, y \in S$ then $x - y \in D_i$, and hence $0 \neq (x - y) \pmod{2n-1} \in C_i$. Since $C_i \subset \mathcal{Z}_n$, by taking $y = 0 \in S$ in the latter we obtain s distinct values of $x \in S$ modulo $2n - 1$, furthermore all in $\mathcal{Z}_n$. Consequently, the set $T = \{x \pmod{2n-1} \mid x \in S\} \subset \mathcal{Z}_n$ induces a complete graph K_s of color i in G, and thus $s < k_i$. To complete the proof, consider color $r+1$, and let $S = \{0, x_1, \ldots, x_{s-1}\}$. Observe that $x_t \in D_{r+1}$ for $1 \leq t \leq s-1$, i.e. we can write $x_t = j_t + \lambda_t(2n-1)$, for some $n \leq j_t \leq 2n-1$ and $0 \leq \lambda \leq k_{r+1} - 3$, since $(0, x_t)$ has color $r+1$. For $x_{t_1}, x_{t_2} \in S$ the edge (x_{t_1}, x_{t_2}) cannot have color $r+1$ unless $|x_{t_1} - x_{t_2}| \geq n$, and hence there exists at most one $x_t \in S$ for each fixed λ_t. This implies that $s - 1 \leq k_{r+1} - 2$, and finishes the proof. $\diamondsuit$

Corollary 1 $R_4(4) \geq 634$ *and* $R(3, 6, 6) \geq 303$.

Proof. Using Theorem 1, a cyclic $(4, 4, 4; 127)$-coloring described by Hill and Irving [15] implies the existence of a cyclic $(4, 4, 4, 4; m)$-coloring for $m = 5 \cdot 127 - 2 = 633$, which in turn gives the lower bound $R_4(4) \geq 634$. Similarly, using a cyclic $(6, 6; 101)$-coloring found by Kalbfleisch [16] Theorem 1 implies the second bound. $\diamondsuit$

4 Building-Up Colorings

We start this section with a theorem which actually is a special case of the result by Song [27] with a suitable choice of p_i's and q_i's equal to 1 in (6). This special case is interesting in itself, and furthermore enhancements of this construction will appear later in the paper.

Theorem 2 *If $k_j \geq 2$ for $1 \leq j \leq r$, then for all $i = 1, \ldots, r-1$*

$$R(k_1, \ldots, k_r) > (R(k_1, \ldots, k_i) - 1)(R(k_{i+1}, \ldots, k_r) - 1). \qquad (12)$$

Proof. Let $s = R(k_1, \ldots, k_i) - 1$ and $t = R(k_{i+1}, \ldots, k_r) - 1$. Consider any i-coloring $C_1 \in \mathcal{R}(k_1, \ldots, k_i; s)$ of K_s with the vertex set $U = \{u_1, \ldots, u_s\}$, and any $(r-i)$-coloring $C_2 \in \mathcal{R}(k_{i+1}, \ldots, k_r; t)$ of K_t with the vertex set $V = \{v_1, \ldots, v_t\}$. Let $C_1(u_p, u_q) \in \{1, ..., i\}$ and $C_2(v_p, v_q) \in \{i+1, ..., r\}$ be the colors of the corresponding edges in C_1 and C_2, respectively. We define an r-coloring $F \in \mathcal{R}(k_1, \ldots, k_r; st)$ of K_{st} on the vertex set $U \times V$ as follows:

$$F((u_{p_1}, v_{q_1}), (u_{p_2}, v_{q_2})) = \begin{cases} C_2(v_{q_1}, v_{q_2}) & \text{if } p_1 = p_2, \\ C_1(u_{p_1}, u_{p_2}) & \text{otherwise.} \end{cases}$$

Observe that in F the edges which receive one of the first i colors induce an s-partite $K_{t,\ldots,t}$, colored accordingly to C_1, and the edges of colors from the set $\{i+1, ..., r\}$ induce s disjoint copies of K_t, all of them colored as in C_2. Consequently, no forbidden monochromatic K_{k_j} in color j is created for any j, and thus $F \in \mathcal{R}(k_1, \ldots, k_r; st)$. The theorem follows. $\diamondsuit$

There is a similarity between the recurrences (6) and (12), and between their constructive proofs as well. They differ in that (6) keeps the number of colors fixed when increasing orders of forbidden cliques, while (12) increases the number of colors but preserves clique orders.

Using the value $R_2(4) = 18$ [14] and lower bounds $128 \leq R_3(4)$ [15], $102 \leq R_2(6)$ and $205 \leq R_2(7)$ (cf. [20], [22], [25]), Theorem 2 implies $2160 \leq R_5(4)$, $10202 \leq R_4(6)$ and $41617 \leq R_4(7)$. Of course for such diagonal cases, inequality (7) would have been sufficient. The last two bounds were also known to Richard Beekman [4] in 2000. By using $51 \leq R_4(3)$ and $162 \leq R_5(3)$ in (5) with $p = q = 2$ for $r = 4$ and $r = 5$, we obtain $2501 \leq R_4(5)$ and $25922 \leq R_5(5)$, respectively, which are better than the bounds which could be obtained by using Theorem 2. Still better bounds for $R_4(5)$ and $R_5(5)$ will be derived later.

The following construction for 2-color Ramsey numbers was presented in [29]. Given a $(k, p; n_1)$-graph G and a $(k, q; n_2)$-graph H, such that G and H both contain an induced subgraph isomorphic to some K_{k-1}-free graph M on m vertices, the authors construct a $(k, p+q-1; n_1+n_2+m)$-graph. For $k \geq 3$ and $p, q \geq 2$, this implies $R(k, p+q-1) \geq$

$R(k,p)+R(k,q)+m-1$. The next theorem extends this idea to multiple colors and employs product graphs, thereby improving Abbott's inequality (5).

Theorem 3 *If $p,q,r \geq 2$ and $p \geq q$, then*

$$R_r(pq+1) \geq R_r(p+1)(R_r(q+1)-1).$$

Proof. Consider any colorings $G \in \mathcal{R}_r(p+1;s)$ with $VG=\{u_1,...,u_s\}$, $s=R_r(p+1)-1$, and $H \in \mathcal{R}_r(q+1;t)$ with $VH=\{v_1,...,v_t\}$, $t=R_r(q+1)-1$. In order to prove the theorem, we will construct an r-coloring $F \in \mathcal{R}_r(pq+1;st+t-1)$ with the vertex set $VF=(VG \times VH) \cup (VH \setminus \{v_1\})$. Note that F has the right number of vertices since $(st+t-1)+1=R_r(p+1)(R_r(q+1)-1)$. The structure of F induced on $VG \times VH$ is similar to that in the proof of Theorem 2. In addition, F contains a recolored copy of H with one vertex deleted, and the connecting edges.

More formally, the coloring of the edges of F is constructed as follows. We begin by letting $F((u_{i_1},v_{j_1}),(u_{i_2},v_{j_2}))$ to be the same as $G(u_{i_1},u_{i_2})$ if $j_1=j_2$, and $H(v_{j_1},v_{j_2})$ otherwise. Observe that at this stage of the definition $F[VG \times VH] \in \mathcal{R}_r(pq+1;st)$ is as in the Abbott's construction (5). Let $U_i=\{(u_i,v_j) \mid 1 \leq j \leq t\}$ and $V_j=\{(u_i,v_j) \mid 1 \leq i \leq s\}$. Note that $H_i=F[U_i]$ is isomorphic to H for each $1 \leq i \leq s$, and $G_j=F[V_j]$ is isomorphic to G for each $1 \leq j \leq t$. Actually, F contains at least s^t subcolorings isomorphic to H, namely those induced by any t-set containing exactly one element in each V_j.

We recolor all edges induced in U_1 by applying any permutation π without fixed points to colors $\{1,\ldots,r\}$. After this recoloring it still holds that $H_1 \in \mathcal{R}_r(q+1;t)$, but now every edge in H_1 has different color than the corresponding edge in H. No edges in all other H_i's and G_j's were recolored. Next, we color the edges of $F[VH \setminus \{v_1\}]$ with the same colors as the corresponding edges in H_1 (after recoloring), namely $F(v_j,v_m)=\pi(H(v_j,v_m))=H_1(v_j,v_m)$ for $2 \leq j,m \leq t$. We complete the coloring of F by defining $F((u_i,v_j),v_m)=G(u_1,u_i)$ for all $2 \leq i \leq s$, and $F((u_1,v_j),v_m)=H_1(v_1,v_m)$, for all $1 \leq j \leq t$ and $2 \leq m \leq t$.

We will prove that F does not contain any monochromatic K_{pq+1}. Suppose that $D \subset VF$, $|D|=d$, induces all edges in the same color c, for some $1 \leq c \leq r$. Partition D into $D_1 \cup D_2 \cup D_3$ by defining

$$\begin{aligned}
D_1 &= D \cap (\{u_1\} \times VH), d_1=|D_1|,\\
D_2 &= D \cap (VH \setminus \{v_1\}), d_2=|D_2|,\\
D_3 &= D \cap ((VG \setminus \{u_1\}) \times VH), d_3=|D_3|.
\end{aligned}$$

Since D_i's form a partition of D we have $d=d_1+d_2+d_3$. Our goal is to show $d \leq pq$. Observe that $F[D_1]$ and $F[D_2]$ are subcolorings of $H_1 \in \mathcal{R}_r(q+1;t)$, which implies $d_1,d_2 \leq q$. Hence we can further suppose that $d_3 \geq 1$, since otherwise $d \leq 2q \leq pq$. We consider four cases.

Case 1: $d_1 + d_2 \le 1$**.** In this case $d_1 = 0$ or $d_2 = 0$, so D induces in F a subcoloring on $d \le d_3 + 1$ vertices similar to the construction of Theorem 2, since all the edges in K_d on D have colors as before recoloring H_1. Hence $d \le pq$.

Case 2: $d_1 \ge 2, d_2 = 0$**.** For each $1 \le i \le t$, let us denote $n_i = |D \cap V_i|$, $m_i = |D_1 \cap V_i|$, and $k_i = |D_3 \cap V_i|$, and introduce the corresponding sets of indices for nonempty intersections $I_1 = \{i \mid m_i > 0\}$ and $I_3 = \{i \mid k_i > 0\}$. Clearly, we have $m_i + k_i = n_i$, $m_i \in \{0, 1\}$, and $k_i \le p$. Note further that $d_1 \ge 2$ and recolored H_1 enforce $k_i = 0$ whenever $m_i = 1$, so $I_1 \cap I_3 = \emptyset$. This means that D is partitioned into $|I_1| = d_1$ singletons in U_1 and $|I_3|$ sets of at most p elements each. Any set formed by a singleton in $D_1 \cap V_i$ and $|I_3|$ representatives one from each nonempty $D_3 \cap V_i$ induces a subcoloring in an isomorph of H, hence $|I_3| + 1 \le q$. Putting it together, and using $p \ge q$, we have

$$d = \sum_{i=1}^{t} n_i = \sum_{i \in I_1} m_i + \sum_{i \in I_3} k_i \le |I_1| + |I_3| p \le q + (q-1)p \le pq.$$

Case 3: $d_1 = 0, d_2 \ge 2$**.** The reasoning is the same as in the Case 2, if the roles of D_1 and D_2 are interchanged. The bound on d would still hold even if the vertex v_1 were included in VF.

Case 4: $d_1, d_2, d_3 \ge 1$**.** Consider vertices $x = (u_1, v_{j_1}) \in D_1$, $y = v_{j_2} \in D_2$ and $z = (u_i, v_{j_3}) \in D_3$. By the assumption we know that $F(x, y) = F(x, z) = F(y, z) = c$. From the construction we see that $H_1(v_1, y) = c$, and thus all the vertices in $D_2 \cup \{v_1\}$ span a monochromatic clique in H_1. This implies that $d_2 \le q - 1$, and so $d_1 + d_2 \le 2q - 1$. Note that $(u_1, v_{j_3}) \notin D$. Next observe that $F((u_1, v_{j_3}), (u_i, v_{j_3})) = c$, consequently for $j = j_3$, and similarly for all j, $(D_3 \cap V_j) \cup \{(u_1, v_j)\}$ induces a monochromatic subcoloring in G_j, and therefore we have $|D_3 \cap V_j| + 1 \le p$.

Define $I_2 = \{j \mid v_j \in D_2\}$, and let I_1 and I_3 be as in the Case 2. Similarly as before, we have a partition of D into $d_1 + d_2 \le 2q - 1$ singletons and at most $q - 1$ blocks $D_3 \cap V_j$. Now, however, each of the latter blocks can have at most $p - 1$ elements. Hence, using $p \ge q$, we obtain

$$d = |I_1| + |I_2| + |I_3|(p-1) \le q + (q-1) + (q-1)(p-1) \le pq.$$

This completes the proof. $\diamondsuit$

This theorem improves Abbott's construction (5) by the term $t - 1$. In particular, using $p = q = 2$ and the lower bounds $R_4(3) \ge 51$ and $R_5(3) \ge 162$ (cf. [22]), we obtain new lower bounds $R_4(5) \ge 2550$ (which will be improved again by Corollary 5 in Section 5) and $R_5(5) \ge 26082$, respectively.

We cannot always improve over Song's generalization (6) of (5), because of the way the recoloring of H was used in the proof. We can however do so in the following restricted case.

Theorem 4 *If $p_i, q, r \geq 2$, $p_i \geq q_i$ and $q_i \in \{1, q\}$ for $1 \leq i \leq r$, then*

$$R(p_1q_1 + 1, \ldots, p_rq_r + 1) \geq R(p_1 + 1, \ldots, p_r + 1)(R(q_1 + 1, \ldots, q_r + 1) - 1).$$

Proof. This is a simple generalization of the proof of previous theorem. $\diamondsuit$

Theorem 5 *For $k, l \geq 3$, let G be a $(k, l; 2n)$-graph, and suppose that for some partition $VG = V_1 \cup V_2$ the induced subgraphs $G[V_1]$ and $G[V_2]$ are isomorphic. Then, given any $(s, 3; m)$-graph, we have*

$$R(s, k, l) \geq mn + 1.$$

Proof. Consider graph H with $VH = V_1 = \{v_1, \ldots, v_n\}$ isomorphic to $G[V_1]$ and $G[V_2]$, and any $(s, 3; m)$-graph P with the vertices $U = \{u_1, \ldots, u_m\}$. We build from H, P and an isomorphism $\phi : G[V_1] \rightarrow G[V_2]$, $\phi(v) = v'$, a 3-coloring $F \in \mathcal{R}(s, k, l; mn)$ with the vertex set $U \times VH$ by defining the colors of edges as follows. For each fixed i, $1 \leq i \leq m$, the edge $((u_i, v_{j_1}), (u_i, v_{j_2}))$ has color 2 if $(v_{j_1}, v_{j_2}) \in EH$, otherwise it has color 3. Next, if $(u_{i_1}, u_{i_2}) \in EP$ then we set $F((u_{i_1}, v_{j_1}), (u_{i_2}, v_{j_2})) = 1$. Finally, for $i_1 < i_2$ and $(u_{i_1}, u_{i_2}) \notin EP$ we use colors 2 or 3 depending on the adjacency in G, namely $F((u_{i_1}, v_{j_1}), (u_{i_2}, v_{j_2}))$ has color 2 if $(v_{j_1}, v'_{j_2}) \in G$, otherwise it has color 3.

One can easily see, as in the previous proofs, that F has no K_s in color 1. Since P has no $\overline{K}_3$, any K_p in color 2 or K_q in color 3, by the construction, may involve vertices with at most two distinct coordinates u_i. However, in this case such a monochromatic clique is induced in an isomorph of G, hence $p < k$ and $q < l$. This completes the proof. $\diamondsuit$

We won't give any new bounds that follow immediately from Theorem 5, but we will have two strong new bounds from a generalization, which follows.

Theorem 6 *Let $G \in \mathcal{R}(k_1, \ldots, k_r; 2n + 1)$, for $k_i \geq 3$, $1 \leq i \leq r$, and suppose that for some partition $VG = V_1 \cup V_2 \cup \{w\}$ the induced subcolorings $G[V_1]$ and $G[V_2]$ are isomorphic. Then, given any $(3, s_1, \ldots, s_t; m)$-coloring with a vertex of degree d in color 1, we have constructively*

$$R(s_1, \ldots, s_t, k_1, \ldots, k_r) \geq mn + d + 1.$$

Proof. We only outline the proof, which is a generalization of Theorem 5 to more colors, with an observation permitting the use of $n + 1$ points of G, instead of n, d times. The color avoiding K_s is now split into t colors with clique bounds $s_1, \ldots, s_t$, and instead of two colors avoiding K_k and K_l we now have r colors with bounds $k_1, \ldots, k_r$. Observe further that we can add extra d points to the construction by augmenting d copies of $H \cong G[V_1]$, corresponding to d neighbors of a vertex x in P in color 1, by a copy of vertex w. Since P has no triangles in color 1, we will be adding edges between different copies of H with only at most one of them being augmented by w, so we can still follow the original structure of G as in the proof of Theorem 5. $\diamondsuit$

Corollary 2 *$R_5(4) \geq 3416$ and $R_4(6) \geq 15202$*

Proof. In both cases we will use Theorem 6 applied to a cyclic coloring G on $2n+1$ vertices, which can easily be split into parts as required. Consider the $(4,4,4;127)$-coloring from [15], and thus setting $n = 63$, and the $(3,4,4;54)$-coloring found in [19], which has vertices of degree 13 in color 1. Theorem 6 implies $R_5(4) \geq 54 \cdot 63 + 13 + 1 = 3416$. For $R_4(6)$, consider two colorings used in the proof of Corollary 1 in Section 3; a $(6,6;101)$-coloring setting $n = 50$ and a $(3,6,6;302)$-coloring built there. The latter has vertices of degree 101 in color 1. Applying Theorem 6 gives $R_4(6) \geq 302 \cdot 50 + 101 + 1 = 15202$ $\diamondsuit$

5 Avoiding Triangles and Merging Colors

Theorem 7 *If $k \geq 3$ and $l \geq 5$, then $R(3,k,l) \geq 4R(k,l-1) - 3$.*

Proof. For suitable k and l, let $m = R(k,l-1)-1$, and consider any $(k,l-1;m)$-graph G with $VG = \{v_1, v_2, ..., v_m\}$. We will prove the theorem by establishing a $(3,k,l)$-coloring on $4R(k,l-1)-4 = 4m$ vertices. Using colors from the set $\{1,2,3\}$, we construct a 3-coloring $F \in \mathcal{R}(3,k,l;4m)$ on the vertex set $VF = \bigcup_{i=1}^{4} U_i$, where $U_i = \{(u_i, v_j) \mid 1 \leq j \leq m\}$. For each i, $1 \leq i \leq 4$, we color the edges induced by U_i with colors 2 and 3 according to G, namely, $F((u_i,v_{j_1}),(u_i,v_{j_2})) = 2$ if $(v_{j_1},v_{j_2}) \in EG$, otherwise $F((u_i,v_{j_1}),(u_i,v_{j_2})) = 3$. Next, we color the edges between U_{i_1} and U_{i_2}, for $i_1 \in \{1,2\}$ and $i_2 \in \{3,4\}$ by letting

$$F((u_{i_1},v_{j_1}),(u_{i_2},v_{j_2})) = \begin{cases} 1, & \text{if } j_1 = j_2; \\ 2, & \text{if } (v_{j_1},v_{j_2}) \in EG; \\ 3, & \text{if } (v_{j_1},v_{j_2}) \notin EG, \text{ and } j_1 \neq j_2, \end{cases}$$

and for $(i_1,i_2) \in \{(1,2),(3,4)\}$ by

$$F((u_{i_1},v_{j_1}),(u_{i_2},v_{j_2})) = \begin{cases} 3, & \text{if } j_1 = j_2; \\ 1, & \text{if } j_1 \neq j_2. \end{cases}$$

Clearly, F has the right number of vertices. From the construction, it is straightforward to observe that F does not contain any triangles in color 1, nor K_k in color 2. We need to prove that the coloring F does not contain any K_l in color 3.

Suppose otherwise, and let $S \subset VF$ be the set of l vertices inducing a monochromatic K_l in color 3. Let $S_i = S \cap U_i$, and denote $s_i = |S_i|$, for $i = 1,2,3,4$. We have $|S| = \sum_{i=1}^{4} s_i = l$, and observe that $l \geq 5$ implies $s_i > 1$ for some i. By symmetry, without loss of generality, we may assume that $s_1 \geq 2$, which in turn by construction immediately implies $s_2 = 0$. We next claim that $s_1 + s_3 < l - 1$. Write $S_1 = \{(u_1,v_{p_1}),(u_1,v_{p_2}),...,(u_1,v_{p_{s_1}})\}$ and $S_3 = \{(u_3,v_{q_1}),(u_3,v_{q_2}),...,(u_3,v_{q_{s_3}})\}$, and denote the corresponding sets of indices by $P = \{p_1,...,p_{s_1}\}$, $Q = \{q_1,...,q_{s_3}\}$. Now, $F((u_1,v),(u_3,v)) = 1$ implies that $|P \cap Q| = \emptyset$, and we see that $\{v_j \in VG \mid j \in P \cup Q\}$ is an independent set in the graph G. Consequently, $s_1 + s_3 = |P \cup Q| < l-1$. In the same way one can argue that $s_1 + s_4 < l-1$. Recall that $s_2 = 0$. Hence, if one of s_3, s_4 is equal to zero, then $|S| < l-1$. Finally, the colors of the edges between U_3 and U_4 imply that if both s_3 and s_4 are nonzero, then $s_3 = s_4 = 1$, and thus $|S| = s_1 + s_3 + s_4 \leq l-1$, a contradiction. $\diamondsuit$

We note that Theorem 7 is a significant improvement over (8). Using lower bounds on $R(3,k)$ from [22], we obtain new bounds $R(3,3,10) \geq 141$, $R(3,3,11) \geq 157$, $R(3,3,12) \geq 181$, and $R(3,3,13) \geq 205$. Using a different reasoning, one can actually prove that Theorem 7 holds also for $l = 4$ and $l = 3$. This extension, however, leads only to rather weak specific bounds.

One can generalize Theorem 7 by allowing more colors in place of color 2 on the left hand side $R(3,k,l)$.

Corollary 3 *For $k_1 \geq 5$ we have*

$$R(3, k_1, k_2, ..., k_r) \geq 4R(k_1 - 1, k_2, ..., k_r) - 3. \tag{13}$$

Proof. Consider k_1 playing the role of l in Theorem 7, and $k_2, ..., k_r$ being constraints on cliques in new colors instead of single k. Then, under a suitable permutation of colors, the statement (13) follows as a straightforward generalization of the proof of Theorem 7. $\diamondsuit$

For example, using $R(3,3,10) \geq 141$ in (13) with $k_1 = 11$ and $k_2 = k_3 = 3$, gives an improved bound $R(3,3,3,11) \geq 561$.

Another lower bound for the numbers of the form $R(3,k,k)$ is given in the next theorem. It is based on the well known *Paley graphs* Q_p defined for primes p of the form $p = 4t+1$. Let $QR(p)$ ($\overline{QR}(p)$) denote the set of quadratic residues (nonresidues) modulo p. In Q_p the vertex set is equal to $\mathcal{Z}_p$, and the vertices x and y are joined by an edge if and only if $x - y \in QR(p)$. The condition $p \equiv 1 \pmod 4$ implies that -1 is a quadratic residue, and thus Q_p is a well defined cyclic graph. We further note that $QR(p)$ and $\overline{QR}(p)$ each have $(p-1)/2$ elements, they partition $\mathcal{Z}_p \setminus \{0\}$, and both are closed under multiplication by any element in $QR(p)$, in particular under $f(x) = (-1)x \equiv p - x \pmod p$. On the other hand multiplication by any nonresidue in $\overline{QR}(p)$ swaps elements between $QR(p)$ and $\overline{QR}(p)$. Using elementary number theory one can also easily prove that Q_p is edge-transitive and self-complementary.

If α_p denotes the order of the largest clique in Q_p, then we clearly have $R(\alpha_p + 1, \alpha_p + 1) > p$. Shearer [25], and later but independently Mathon [20], described a construction "doubling" Q_p, which yields a graph H_p on $2p+2$ vertices in $\mathcal{R}(\alpha_p + 2, \alpha_p + 2)$. This construction gives the best known lower bounds for several diagonal Ramsey numbers, in particular $R(7,7) \geq 205$ based on the Paley graph Q_{101}. The Shearer-Mathon construction cannot be iterated, since the graph H_p has no longer structure of a Paley graph, in particular it doesn't have to be cyclic or self-complementary. Paley graphs yield also good lower bounds when used as a starting point for the Giraud construction of Theorem 1 discussed in Section 3. Giraud construction requires the starting graph to be cyclic, and thus it cannot be used after Shearer-Mathon doubling. The other order of extensions, as is, is not feasible either. However, Theorem 8 below shows that a special way of merging the Giraud and Shearer-Mathon constructions works. We first formulate and prove it only for avoiding triangles in the new color, which apparently is the case producing some of the strongest known lower bounds.

Theorem 8 *For a prime p of the form $4t+1$, let α_p be the order of the largest clique in the Paley graph Q_p. Then*

$$R(3, \alpha_p + 2, \alpha_p + 2) \geq 6p + 3.$$

Proof. Let $k = \alpha_p + 1$, and denote $Qr = QR(p)$, $Q\bar{r} = \overline{QR}(p)$. Consider Q_p to be a cyclic $(k, k; p)$-coloring used as a starting coloring in the construction of the proof of Theorem 1 with $r = 2$ and $k_3 = 3$. Let H be the resulting $(k, k, 3; 3p-1)$-coloring with the vertex set $\mathcal{Z}_{3p-1}$. H is cyclic and the distance sets of the three colors are:

$$\begin{aligned} D_1 &= \{j, j+2p-1 \mid j \in Qr\}, \\ D_2 &= \{j, j+2p-1 \mid j \in Q\bar{r}\}, \\ D_3 &= \{j \mid p \leq j < 2p\}. \end{aligned}$$

Let H' be a vertex disjoint isomorphic copy of H, so that the vertex x' of H' corresponds to the vertex x of H. In order to prove the theorem, we construct a 3-coloring $F \in \mathcal{R}(k+1, k+1, 3; 6p+2)$ with the vertex set $VF = VH \cup VH' \cup W$, where $W = \{w_1, w_2, w_3, w_4\}$. The connections between H, H' and W are defined similarly as in the Shearer-Mathon construction, but here, having one more color, we manage to use 4 external vertices of W instead of 2 in the original method.

The edges of $F[VH]$ and $F[VH']$ are colored the same as the corresponding edges of H and H', and of $F[W]$ by setting $F(w_1, w_2) = F(w_3, w_4) = 3$ and $F(w_1, w_3) = F(w_1, w_4) = F(w_2, w_3) = F(w_2, w_4) = 2$. For $x, y \in VH$, and thus for corresponding x', y' in VH', we define

$$F(x, y') = F(x', y) = \begin{cases} 1 & \text{if } H(x, y) = 2, \\ 2 & \text{if } H(x, y) = 1 \text{ or } x = y, \\ 3 & \text{if } H(x, y) = 3. \end{cases}$$

The special $3p-1$ matching edges of color 2 for $x = y$ could be defined alternatively in color 1, or in any mixture of colors 1 and 2. Finally we complete the definition of the coloring F. For the edges between W and $VH \cup VH'$ assign $F(w_i, x) = 2$, $F(w_i, x') = 1$ for $i \in \{1, 2\}$, and $F(w_i, x) = 1$, $F(w_i, x') = 2$ for $i \in \{3, 4\}$.

We have to show that F is a $(k+1, k+1, 3)$-coloring.

By Theorem 1 we know that H and H' are $(k,k,3;3p-1)$-colorings. Considering all possible triples of vertices, one can easily see from the construction that F does not contain any triangle in color 3. Suppose $S \subset VF$, $s = |S|$, induces all edges in color c, where c is color 1 or color 2. We will prove that $s < k+1$.

Observe that each w in W, for fixed color 1 or 2, has edges only to one of VH or VH'. Note further that $F[W]$ has no edges in color 1, and for each edge such that $F(w_i, w_j) = 2$ the neighborhoods of w_i and w_j in color 2 are disjoint. Hence there is at most one vertex w in $S \cap W$, or otherwise $S \subset W$, in which case $s \leq 2$ and we are done. If there is such w, then S is disjoint from VH or from VH'. Since no cliques in color 1 or 2 in H and H' have k vertices, then $w \in S$ implies $s < k+1$, Thus in the sequel we will assume that $S \cap W = \emptyset$.

If for some $x \in VH$, $\{x, x'\} \subset S$ (this must be the case of $c = 2$), then we easily see that $S = \{x, x'\}$. So we assume that no matching edge $\{x, x'\}$ is in S, which makes further reasoning identical for $c = 2$ and $c = 1$. If S intersects only one of VH or VH', then clearly $s < k$. Since H is cyclic we can assume that $0 \in S \cap VH$, and we can write $S = \{0, x_1, \ldots, x_m, y'_1, \ldots, y'_n\}$, where $X = \{x_1, \ldots, x_m\} \subset VH$ and $Y' = \{y'_1, \ldots, y'_n\} \subset VH'$. Consider the set $Y = \{y_1, \ldots, y_n\} \subset VH$. Previous comments about $\{x, x'\}$ imply that $X \cap Y = \emptyset$. From the construction of F we see that $X \subset D_1$ and $Y \subset D_2$, where D_i's are distance sets of color i in H. Similarly as in the proof of Theorem 1 we can argue that no two elements of X or Y can be the same modulo $2p - 1 \in D_3$. Consequently, the sets $X(p)$ and $Y(p)$ of values of X and Y, respectively, reduced modulo p, have the same cardinalities as the original sets.

Hence we have $X(p) \subset Qr$, $|X(p)| = m$ and $Y(p) \subset Q\bar{r}$, $|Y(p)| = n$, $X(p) \cap Y(p) = \emptyset$, and furthermore all differences between two elements of $X(p)$ or two elements of $Y(p)$ must be in Qr, while all differences between elements from $X(p)$ and $Y(p)$ must be in $Q\bar{r}$. The final argument is the same as in the Paley doubling construction by Shearer [25]. Consider the set

$$T = \{x^{-1} \pmod{p} \mid x \in X(p) \cup Y(p)\}$$

of $m+n$ elements in $\mathcal{Z}_p^*$. We can show that T forms a clique in the Paley graph Q_p. For any distinct $a^{-1}, b^{-1} \in T$, consider quadratic character of the factors in the representation $a^{-1} - b^{-1} = (b-a)(ab)^{-1}$, all arithmetic performed modulo p. In all cases we can easily see that $a^{-1} - b^{-1} \in QR$. For example, if $a, b \in Y(p)$ then $ab \in Qr$ by the basic property of nonresidues, and $b - a \in Qr$ because of the structure of F as argued above. Therefore

$$s - 1 = m + n = |T| < \alpha_p + 1 = k,$$

which completes the proof. $\diamondsuit$

We illustrate Theorem 8 on the smallest case for which it produces a new bound, namely for $p = 101$. The Paley graph Q_{101} gives first the bound $R(6,6) \geq 102$, after applying Giraud extension we have $R(3,6,6) \geq 303$, and finally enhanced Shearer-Mathon "doubling" gives $R(3,7,7) \geq 609$. Using the latter and a weaker version of Theorem 6 where G has only $2n = 204$ vertices and the construction has no d extra points, we can easily conclude that $R_4(7) \geq 608 \cdot 102 + 1 = 62017$. The bound $R(3,9,9) \geq 1689 = 6 \cdot 281 + 3$ can be obtained similarly by Theorem 8 from the Paley graph $Q_{281} \in \mathcal{R}(8,8;281)$.

Corollary 4 *For a prime p of the form $4t+1$, let α_p be the order of the largest clique in the Paley graph Q_p. Then for $s \geq 3$*

$$R(s, \alpha_p + 2, \alpha_p + 2) \geq 4ps - 6p + 3.$$

Sketch of the proof. We will use the notation of the proof of Theorem 8. The same method works not only for triangles but for all K_s, $s \geq 3$. We first build $H \in \mathcal{R}(k,k,s;(2s-3)p - s + 2)$ using Theorem 1. The coloring F is constructed similarly as before on two copies of H, but now with additional $2(s-1)$ vertices in W, totaling $2((2s-3)p - s + 2) + 2(s-1) = 4ps - 6p + 2$ vertices as required. The edges of a $K_{s-1,s-1}$ in $F[W]$ are in color 2, and the remaining edges of two copies of K_{s-1} are assigned color 3. It is straightforward to prove that F has no K_s in color 3. The sets X and Y are defined similarly as in Theorem 8. Following the proof of Theorem 1 we can show that no two elements of X or Y can be the same modulo $2p-1$. The final steps of reasoning are the same as in the proof of Theorem 8. $\diamondsuit$

While the main focus of this paper is on multicolor Ramsey numbers, we digress to present a 2-color theorem which follows naturally from Theorem 7.

Theorem 9 *For $l \geq 5, k \geq 2$, $R(2k-1, l) \geq 4R(k, l-1) - 3$.*

Proof. We will use exactly the same graph G and 3-coloring F as in the proof of Theorem 7. Consider a 2-coloring (graph) H obtained from F by merging colors 1 and 2, i.e. the edges of H are those colored 1 or 2 in F. Clearly, by Theorem 7, H has no independent sets of order l. We have to prove that H contains no K_{2k-1}.

Suppose that $D \subset VF = VH$, of order $d = |D|$, is a set of vertices inducing K_d in H. We partition D into six sets, and associate with them sets of indices, which are subsets of $\{1, ..., m\}$. First, define $D_1 = D \cap U_1$, $d_1 = |D_1|$, $D_2 = D \cap U_2$, $d_2 = |D_2|$, and write $D_1 = \{(u_1, v_{p_1}), (u_1, v_{p_2}), ..., (u_1, v_{p_{d_1}})\}$, $D_2 = \{(u_2, v_{q_1}), (u_2, v_{q_2}), ..., (u_2, v_{q_{d_2}})\}$. We denote the corresponding sets of indices by $I_1 = \{p_1, ..., p_{d_1}\}$ and $I_2 = \{q_1, ..., q_{d_2}\}$. Define further the remaining sets of the partition by

$$\begin{aligned}
D_3 &= \{(u_3, v_i) \in D \mid i \in I_1\} \cup \{(u_4, v_i) \in D \mid i \in I_1\}, \\
D_4 &= \{(u_3, v_i) \in D \mid i \in I_2\} \cup \{(u_4, v_i) \in D \mid i \in I_2\}, \\
D_5 &= (D \cap U_3) \setminus (D_3 \cup D_4), \\
D_6 &= (D \cap U_4) \setminus (D_3 \cup D_4),
\end{aligned}$$

with the corresponding sets of indices

$$
\begin{aligned}
I_3 &= \{i \in I_1 \mid (u_3, v_i) \in D \vee (u_4, v_i) \in D\}, \\
I_4 &= \{i \in I_2 \mid (u_3, v_i) \in D \vee (u_4, v_i) \in D\}, \\
I_5 &= \{i \in \{1, ..., m\} \setminus (I_1 \cup I_2) \mid (u_3, v_i) \in D\}, \\
I_6 &= \{i \in \{1, ..., m\} \setminus (I_1 \cup I_2) \mid (u_4, v_i) \in D\},
\end{aligned}
$$

and their cardinalities $d_j = |I_j| = |D_j|$. We first claim that $D = \bigcup_{i=1}^{6} D_i$ is really a partition. $F((u_1, v), (u_2, v)) = 3$ implies that $I_1 \cap I_2 = \emptyset$, from which we can conclude that all D_j's are mutually disjoint and cover D. Next, observe that I_1, I_4 and I_5 are mutually disjoint, and thus $\{(u_1, v_j) \mid j \in I_1 \cup I_4 \cup I_5\}$ induces in $H[U_1]$ a complete subgraph of order $t = d_1 + d_4 + d_5$. Since $H[U_1]$ is isomorphic to G, we know that $t \leq k-1$. In the same way we argue that $d_2 + d_3 + d_6 \leq k-1$, and hence $d = \sum_{i=1}^{6} d_i \leq 2(k-1)$. This shows that H doesn't contain K_{2k-1}, and thus it completes the proof of the theorem. $\diamondsuit$

Normally, one would not expect to obtain interesting 2-color Ramsey constructions by merging colors in multicolorings. The method of the proof of Theorem 9 is an exception, since it surprisingly produces some new lower bounds improving on those listed in the 2002 revision of [22]. For $k = 3$ we obtain a general inequality

$$R(5, l) \geq 4R(3, l-1) - 3, \tag{14}$$

which when applied for small l to bounds on $R(3, l-1)$ from [22] gives the following new lower bounds: $R(5, 11) \geq 157$, $R(5, 13) \geq 205$, $R(5, 14) \geq 233$ and $R(5, 15) \geq 261$. For higher values of k, using Theorem 9 we obtain further new bounds, such as $R(6, 9) \geq 169$, $R(6, 13) \geq 317$, $R(7, 11) \geq 405$, $R(8, 9) \geq 317$ and $R(8, 13) \geq 817$ (the entries we list improve over lower bounds previously recorded in the survey [22]).

Corollary 5 *For $k_1 \geq 5$ and $k_i \geq 2$ we have*

$$R(k_1, 2k_2 - 1, k_3, ..., k_r) \geq 4R(k_1 - 1, k_2, ..., k_r) - 3. \tag{15}$$

Proof. Similarly as (13) was obtained from Theorem 7 by considering more colors, we can think of (15) being a generalization derived from Theorem 9. Or, equivalently, in the construction of coloring corresponding to the left hand side of (13) in Corollary 3 merge colors 1 and 3, with forbidden cliques of orders 3 and k_2, respectively. $\diamondsuit$

Recall that permuting arguments of Ramsey numbers does not change their values. Consequently, we obtain a new lower bound on $R_4(5)$ by applying (15) twice and then using $R(3, 3, 4, 4) \geq 171$ ([8], see also Section 6), as follows:

$$R_4(5) \geq 4R(3, 4, 5, 5) - 3 \geq 4(4R(3, 3, 4, 4) - 3) - 3 = 2721.$$

The previously best known lower bound for $R_4(5)$ cited in [22] was 2501. It could be derived by a method described by Abbott [1] or as discussed in Section 2 and Section 4. Finally, we mention yet another new bound $R_3(9) \geq 13761$, which can be obtained by applying (15) three times and using $R(7,8) \geq 216$ [29].

6 Computer Searches for Colorings

In 1994, a construction method that produced the best known lower bound for the 5th Schur number and for the 5-color Ramsey number of K_3 was described by Exoo [7]. With the availability of faster computers, it has become feasible to apply the method to bigger problems, particularly for K_4 and K_5.

The colorings described here are all linear colorings, i.e., the vertices are numbered from 0 to $n-1$ and the color of the edge joining vertices i and j depends only on the difference $|i-j|$. This class of coloring includes cyclic colorings. Linear colorings possess a useful hereditary property that cyclic colorings do not: given a linear coloring on $n+1$ vertices, we can find a linear subcoloring on n vertices.

The growth method begins with a coloring of a small graph, one far smaller than the one that we ultimately aim to construct. At each stage of the algorithm, we have a target number of vertices on which we are trying to complete a good coloring. When we succeed, we increment the target. The hereditary property gives us a chance to succeed at the larger number of vertices without making an excessive number of changes in our coloring.

Theorem 10 *$R(5,5,5) \geq 415$.*

Proof. The coloring that proves the theorem is given below. In this coloring we identify the vertices with the positive integers from 1 to 414. The color of the edge joining a pair of vertices is determined by their difference. So, for example, edges are given color 1 if the absolute value of their difference is any of the values listed in the first set of integers below.

```
Color 1:
       14,  22,  25,  30,  33,  35,  41,  43,  59,  67,
       75,  81,  89,  90,  98, 102, 110, 114, 116, 117,
      122, 124, 130, 132, 135, 136, 137, 138, 143, 144,
      146, 154, 157, 159, 165, 167, 170, 171, 173, 178,
      179, 181, 185, 186, 187, 189, 190, 192, 193, 194,
      198, 200, 201, 205, 208, 209, 212, 213, 214, 216,
      222, 225, 227, 228, 233, 235, 236, 243, 244, 247,
      249, 255, 257, 260, 265, 268, 270, 271, 277, 278,
```

279, 282, 284, 288, 290, 292, 298, 300, 304, 312,
315, 316, 317, 323, 324, 325, 333, 334, 339, 347,
355, 371, 372, 373, 379, 381, 384, 388, 389, 391,
392, 400, 410.

Color 2:

2, 3, 9, 10, 11, 13, 16, 19, 21, 24,
32, 36, 37, 44, 46, 48, 52, 54, 56, 57,
65, 68, 70, 71, 76, 78, 79, 80, 86, 87,
92, 100, 101, 103, 105, 107, 109, 111, 113, 120,
121, 125, 127, 133, 141, 147, 148, 155, 156, 160,
162, 166, 168, 176, 182, 197, 203, 211, 217, 223,
224, 231, 232, 238, 246, 252, 254, 258, 259, 266,
273, 281, 287, 289, 293, 299, 301, 303, 305, 309,
311, 314, 322, 327, 328, 335, 336, 338, 343, 344,
346, 348, 349, 357, 358, 360, 362, 363, 366, 368,
370, 377, 382, 390, 393, 395, 398, 401, 403, 404,
405, 411, 412.

Color 3:

1, 4, 5, 6, 7, 8, 12, 15, 17, 18,
20, 23, 26, 27, 28, 29, 31, 34, 38, 39,
40, 42, 45, 47, 49, 50, 51, 53, 55, 58,
60, 61, 62, 63, 64, 66, 69, 72, 73, 74,
77, 82, 83, 84, 85, 88, 91, 93, 94, 95,
96, 97, 99, 104, 106, 108, 112, 115, 118, 119,
123, 126, 128, 129, 131, 134, 139, 140, 142, 145,
149, 150, 151, 152, 153, 158, 161, 163, 164, 169,
172, 174, 175, 177, 180, 183, 184, 188, 191, 195,
196, 199, 202, 204, 206, 207, 210, 215, 218, 219,
220, 221, 226, 229, 230, 234, 237, 239, 240, 241,
242, 245, 248, 250, 251, 253, 256, 261, 262, 263,
264, 267, 269, 272, 274, 275, 276, 280, 283, 285,
286, 291, 294, 295, 296, 297, 302, 306, 307, 308,
310, 313, 318, 319, 320, 321, 326, 329, 330, 331,
332, 337, 340, 341, 342, 345, 350, 351, 352, 353,
354, 356, 359, 361, 364, 365, 367, 369, 374, 375,
376, 378, 380, 383, 385, 386, 387, 394, 396, 397,
399, 402, 406, 407, 408, 409, 413.

$\diamondsuit$

Other new lower bounds for off-diagonal multicolor numbers involving triangles were obtained with the help of heuristic algorithms by Geoff Exoo, and most of them are presented at his website [8]. In the following we present previously unpublished constructions establishing the bounds $79 \leq R(3,3,7)$, $93 \leq R(3,3,3,4)$ and $171 \leq R(3,3,4,4)$.

The coloring which gives $79 \leq R(3,3,7)$ follows.

```
Color 1:
        4,   5,   6,  13,  15,  22,  29,  31,  38,  40,
       40,  47,  49,  63,  65,  72,  73,  74,  75.

Color 2:
        1,   3,   9,  14,  16,  24,  35,  37,  39,  41,
       43,  54,  56,  62,  64,  69,  77.

Color 3:
        2,   7,   8,  10,  11,  12,  17,  18,  19,  20,
       21,  23,  25,  26,  27,  28,  30,  32,  33,  34,
       36,  42,  44,  45,  46,  48,  50,  51,  52,  53,
       55,  57,  58,  59,  60,  61,  66,  67,  68,  70,
       71,  76.
```

Next is the coloring which shows $93 \leq R(3,3,3,4)$.

```
Color 1:
        2,   3,   8,   9,  14,  15,  19,  25,  31,  35,
       41,  57,  61,  67,  73,  77,  78,  83,  84,  89,
       90.

Color 2:
        1,   7,  10,  12,  16,  18,  43,  45,  47,  49,
       51,  74,  76,  80,  82,  85,  91.

Color 3:
        4,   5,   6,  13,  20,  21,  22,  29,  37,  38,
       39,  46,  53,  54,  55,  63,  70,  71,  72,  79,
       86,  87,  88.

Color 4:
       11,  17,  23,  24,  26,  27,  28,  30,  32,  33,
       34,  36,  40,  42,  44,  48,  50,  52,  56,  58,
       59,  60,  62,  64,  65,  66,  68,  69,  75,  81.
```

And finally the coloring for $171 \leq R(3,3,4,4)$. Note that in this coloring there are more edges in color 1, a K_3 avoiding color, than in color 3, a K_4 avoiding color. The explanation for this may be that the color 1 chords are larger numbers. We have seen this phenomenon in other colorings.

```
Color 1:
       50,  56,  59,  62,  65,  67,  71,  75,  76, 77,
       79,  80,  81,  82,  83,  84,  85,  86,  87, 88,
       90,  91,  92,  93,  95,  96,  97,  98,  99, 101,
      102, 103, 104, 105, 108, 110, 111, 114, 116, 120,
      122, 128.

Color 2:
        1,   4,   7,  10,  13,  16,  22,  25,  28,  31,
       34,  37,  40,  43,  46,  49,  51,  54,  57,  60,
       63,  69, 107, 113, 119, 121, 124, 127, 130, 133,
      136, 142, 145, 148, 154, 157, 163, 166, 169.

Color 3:
        8,   9,  14,  15,  17,  18,  19,  20,  21,  29,
       30,  32,  33,  35,  36,  41,  42,  58,  64,  66,
       70,  72,  78, 100, 106, 112, 129, 134, 135, 137,
      138, 140, 141, 149, 150, 152, 153, 155, 156, 161,
      162.

Color 4:
        2,   3,   5,   6,  11,  12,  23,  24,  26,  27,
       38,  39,  44,  45,  47,  48,  52,  53,  55,  61,
       68,  73,  74,  89,  94, 109, 115, 117, 118, 123,
      125, 126, 131, 132, 139, 143, 144, 146, 147, 151,
      158, 159, 160, 164, 165, 167, 168.
```

7 Summary of Bounds

The summary of all lower bounds used, derived, or otherwise mentioned in the paper (not necessarily new) is presented in the Table I, together with pointers to references and relevant places in this paper. The bounds which are new and the best for given parameters are marked with a '*' in the column "best new". For example, the lower bound of 162 in the case number 3 is not new and it was established in [7], while a new lower bound of 634 for the case number 7 is obtained using Theorem 2 in Section 3, and it is listed in the abstract.

For a complete listing of all known related bounds see the dynamic survey paper [22].

case no.	Ramsey number	lower bound	best new	section, reference	(t)heorem/ (c)orollary, etc.
1.	$R_3(3)$	17		2, [14]	
2.	$R_4(3)$	51		2, [5]	
3.	$R_5(3)$	162		2, [7]	
4.	$R_6(3)$	538		2, [11]	
5.	$R_7(3)$	1682		2, [11]	
6.	$R_3(4)$	128		3, [15]	
7.	$R_4(4)$	634	*	3	c1, abstract
8.	$R_5(4)$	2160		4	t2, (7)
9.	$R_5(4)$	3416	*	4	c2, abstract
10.	$R_3(5)$	415	*	6	t10, abstract
11.	$R_4(5)$	1833		2, [20]	after (6)
12.	$R_4(5)$	2501		2	after (6)
13.	$R_4(5)$	2550		4	t3
14.	$R_4(5)$	2721	*	5	c5, (15), abstract
15.	$R_5(5)$	25922		4	(5), after t2
16.	$R_5(5)$	26082	*	4	t3, abstract
17.	$R_4(6)$	10202		4, [4]	t2, (7)
18.	$R_4(6)$	15202	*	4	c2
19.	$R_4(7)$	41617		4, [4]	t2, (7)
20.	$R_4(7)$	62017	*	5	t6, t8
21.	$R_3(9)$	13761	*	5	c5++

Table I. Summary of lower bounds.

case no.	Ramsey number	lower bound	best new	section, reference	(t)heorem/ (c)orollary, etc.
22.	$R(5,11)$	157	*	5	(14)
23.	$R(5,13)$	205	*	5	(14)
24.	$R(5,14)$	233	*	5	(14)
25.	$R(5,15)$	261	*	5	(14)
26.	$R(6,9)$	169	*	5	t9
27.	$R(6,13)$	317	*	5	t9
28.	$R(7,11)$	405	*	5	t9
29.	$R(8,9)$	317	*	5	t9
30.	$R(8,13)$	817	*	5	t9
31.	$R(3,3,4)$	30		2, [17]	after (2)
32.	$R(3,3,7)$	79	*	6	after t10
33.	$R(3,3,10)$	141	*	5	t7
34.	$R(3,3,11)$	157	*	5	t7
35.	$R(3,3,12)$	181	*	5	t7
36.	$R(3,3,13)$	205	*	5	t7
37.	$R(3,4,4)$	54		4, [19]	in proof of c2
38.	$R(3,6,6)$	303	*	3	c1
39.	$R(3,7,7)$	609	*	5	t8
40.	$R(3,9,9)$	1689	*	5	t8
41.	$R(3,3,3,4)$	93	*	6	after t10
42.	$R(3,3,3,5)$	162	*	2	before (3)
43.	$R(3,3,3,11)$	561	*	5	c3
44.	$R(3,3,4,4)$	171	*	6	after t10
45.	$R(3,4,5,5)$	681	*	5	used after c5

Table I (continued). Summary of lower bounds.

References

[1] H. L. Abbott, *Ph. D. thesis*, University of Alberta, Edmonton, 1965.

[2] H. L. Abbott and D. Hanson, A Problem of Schur and Its Generalizations, *Acta Arithmetica*, **20** (1972) 175-187.

[3] H. L. Abbott and L. Moser, Sum-Free Sets of Integers, *Acta Arithmetica*, **11** (1966) 393-396.

[4] R. Beekman, *personal communication* (2000).

[5] F. R. K. Chung, On the Ramsey Numbers $N(3,3,...,3;2)$, *Discrete Mathematics*, **5** (1973) 317-321.

[6] F. R. K. Chung and C. M. Grinstead, A Survey of Bounds for Classical Ramsey Numbers, *Journal of Graph Theory*, **7** (1983) 25-37.

[7] G. Exoo, A Lower Bound for Schur Numbers and Multicolor Ramsey Numbers of K_3, *Electronic Journal of Combinatorics*, #R8, **1** (1994), 3 pages, `http://www.combinatorics.org`.

[8] G. Exoo, Some New Ramsey Colorings (2003), constructions available at `http://isu.indstate.edu/ge/RAMSEY`.

[9] S. Fettes, R. L. Kramer and S. P. Radziszowski, An Upper Bound of 62 on the Classical Ramsey Number $R(3,3,3,3)$, *to appear* in *Ars Combinatoria.*

[10] H. Fredricksen, Five Sum-Free Sets, *Congressus Numerantium*, **14** (1975) 309-314.

[11] H. Fredricksen and M. M. Sweet, Symmetric Sum-Free Partitions and Lower Bounds for Schur Numbers, *Electronic Journal of Combinatorics*, #R32, **7** (2000), 9 pages, `http://www.combinatorics.org`.

[12] G. Giraud, Une généralisation des nombres et de l'inégalité de Schur, *C.R. Acad. Sc. Paris*, Série A, **266** (1968) 437-440.

[13] G. Giraud, Minoration de certains nombres de Ramsey binaires par les nombres de Schur généralisés, *C.R. Acad. Sc. Paris*, Série A, **266** (1968) 481-483.

[14] R. E. Greenwood and A. M. Gleason, Combinatorial Relations and Chromatic Graphs, *Canadian Journal of Mathematics*, **7** (1955) 1-7.

[15] R. Hill and R. W. Irving, On Group Partitions Associated with Lower Bounds for Symmetric Ramsey Numbers, *European Journal of Combinatorics*, **3** (1982) 35-50.

[16] J. G. Kalbfleisch, Construction of Special Edge-Chromatic Graphs, *Canadian Mathematical Bulletin*, **8** (1965) 575-584.

[17] J. G. Kalbfleisch, Chromatic Graphs and Ramsey's Theorem, *Ph.D. thesis*, University of Waterloo, January 1966.

[18] J. G. Kalbfleisch and R. G. Stanton, On the Maximal Triangle-Free Edge-Chromatic Graphs in Three Colors, *Journal of Combinatorial Theory*, **5** (1968) 9-20.

[19] D. L. Kreher, Li Wei and S. P. Radziszowski, Lower Bounds for Multi-Colored Ramsey Numbers From Group Orbits, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **4** (1988) 87-95.

[20] R. Mathon, Lower Bounds for Ramsey Numbers and Association Schemes, *Journal of Combinatorial Theory*, Series B, **42** (1987) 122-127.

[21] K. Piwakowski and S. P. Radziszowski, Towards the Exact Value of the Ramsey Number $R(3,3,4)$, *Congressus Numerantium*, **148** (2001) 161-167.

[22] S. P. Radziszowski, Small Ramsey numbers, *Electronic Journal of Combinatorics*, Dynamic Survey 1, revision #9, July 2002, `http://www.combinatorics.org`.

[23] A. Robertson, New Lower Bounds for Some Multicolored Ramsey Numbers, *Electronic Journal of Combinatorics*, #R12, **6** (1999), 6 pages, `http://www.combinatorics.org`.

[24] A. Robertson, New Lower Bounds Formulas for Multicolored Ramsey Numbers, *Electronic Journal of Combinatorics*, #R13, **9** (2002), 6 pages, `http://www.combinatorics.org`.

[25] J. B. Shearer, Lower Bounds for Small Diagonal Ramsey Numbers, *Journal of Combinatorial Theory*, Series A, **42** (1986) 302–304.

[26] Song En Min, New Lower Bound Formulas for the Ramsey Numbers $N(k,k,...,k;2)$ (in Chinese), *Mathematica Applicata*, **6** (1993) suppl., 113-116.

[27] Song En Min, An Investigation of Properties of Ramsey Numbers (in Chinese), *Mathematica Applicata*, **7** (1994) 216-221.

[28] Xu Xiaodong, Classical Ramsey Theory and Its Application (in Chinese), *manuscript* (2003).

[29] Xu Xiaodong, Xie Zheng and S. P. Radziszowski, A Constructive Approach for the Lower Bounds on the Ramsey Numbers $R(s,t)$, *to appear* in the *Journal of Graph Theory*.