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CONSTRUCTIVENESS OF HIRONAKA'S RESOLUTION

By Orlando VILLAMAYOR ([1])

Introduction

In [9] Hironaka develops the notion of *local idealistic* presentation for an algebraic scheme X embedded in a regular scheme W. Here we take those results as starting point and we exhibit a *constructive resolution of singularities* (*see* 2.2)

Roughly speaking, an upper semicontinuous function is defined on a fixed Samuel stratum such that

(i) the function determines the center of a permissible transformation $\pi_1: X_1 \to X$.

(ii) for $\pi_1: X_1 \to X$ as before, an upper semicontinuous function can be defined at X_1 [as in (i)] such that either there is an improvement of the Hilbert-Samuel functions at X_1, or there is an improvement on these functions. Repeating (i) and (ii) a finite number of times, say

$$X_r \overset{\pi_r}{\to} X_{r-1} \to \ldots \to X_1 \overset{\pi_1}{\to} X$$

one can force an improvement (at X_r) of the Hilbert-Samuel function.

In section 1 we introduce the notation and some results (without proofs) required for the *construction*. We refer the reader mainly to [9] for more details and proofs. The definition of constructive resolutions and the development of these are given in section 2.

I thank Prof. Jean Giraud for important suggestions on this work.

§ 1. Throughout this article W will denote a regular algebraic scheme admitting a finite cover by affine sets. Each restriction to these being the spectrum of an algebra of finite type over a fixed field k of characteristic zero. And all patching maps being k-algebra maps.

A map $W_1 \to W$ will always mean a morphism of finite type.

([1]) Supported by the Alexander von Humboldt-Stiftung.

We consider pairs of the form (J, b) where b is a positive integer and $J \subset O_W$ is a coherent sheaf of ideals for which $J_x \neq 0$, $\forall x \in W$ (J_x denotes the stalk at x).

Given a valuation ring A and a principal ideal $J \subset A$ let ord (J) denote the value of J with respect to the valuation associated with A.

DEFINITION 1.1. — Assume that (J_1, b_1) and (J_2, b_2) are two pairs as before with the property that for any morphism h: Spec(A) $\rightarrow$ W, where A is a noetherian valuation ring, the following equality holds:

$$\frac{\operatorname{ord}(J_1 A)}{b_1} = \frac{\operatorname{ord}(J_2 A)}{b_2}. \quad (\text{at } \mathbb{Q}).$$

$J_i A$ the ideal induced by J_i via h at A.

This condition defines an equivalence relation among such pairs. We shall say that $(J_1, b_1) \sim (J_2, b_2)$ and the equivalence class of a pair (J, b), say $\mathscr{A} = ((J, b))$ is called an idealistic exponent at W (*see* Def. 3, p. 56 [9]).

Assume that $(J_1, b_1) \sim (J_2, b_2)$ and let π: $W_1 \rightarrow W$ be any morphism of regular schemes, then $(J_1 O_{W_1}, b_1) \sim (J_2 O_{W_2}, b_2)$ So we define for a given idealistic exponent $\mathscr{A} = ((J, b))$ at W, the idealistic exponent $\pi^{-1}(\mathscr{A})$ as:

$$\pi^{-1}(\mathscr{A}) = ((JO_{W_1}, b)).$$

DEFINITION 1.2. — Let (J_1, b_1) and (J_2, b_2) be two equivalent pairs at W corresponding to the idealistic exponent $\mathscr{A}$. If $x \in W$ then

$$c = \frac{\nu_x(J_1)}{b_1} = \frac{\nu_x(J_2)}{b_2},$$

where $\nu_x(J_i)$ denotes the order of the stalk $J_{i,x}$ at the local regular ring $O_{W,x}$. We define *the order of* $\mathscr{A}$ *at* x to be $\nu_x(\mathscr{A}) = c$ and *the order of* $\mathscr{A}$ to be $\operatorname{ord}(\mathscr{A}) = \max_{x \in W} \{\nu_x(\mathscr{A})\}$.

DEFINITION 1.3. — Given a pair (J, b) at W as in Def. 1.1 we define a reduced subscheme:

$$\operatorname{Sing}^b(J) = \{x \in W \mid \nu_x(J) \geqq b\}$$

A transformation π: $W_1 \rightarrow W$ is said to be *permissible for* (J, b) if it is the blowing up with center C, where C is a regular subscheme of W contained in $\operatorname{Sing}^b(J)$.

In this case there is a coherent sheaf of ideals $\bar{J} \subset O_{W_1}$ such that $JO_{W_1} = \bar{J} P^b$ where P denotes the sheaf of ideals $O(-\pi^{-1}(C)) \subset O_{W_1}$.

We define *the transform of* (J, b) *by* π to be the pair $(\bar{J}, b)$ at W_1.

One can check that if $(J_1, b_1) \sim (J_2, b_2)$ at W then:

(i) $\operatorname{Sing}^{b_1}(J_1) = \operatorname{Sing}^{b_2}(J_2)$ and if $(\bar{J}_i, b_i)$ denotes the transform of (J_i, b_i), $i=1, 2$ by a permissible map $\pi: W_1 \to W$, then:

(ii) $(\bar{J}_1, b_1) \sim (\bar{J}_2, b_2)$ at W_1.

So now let (J, b) be a pair at W, $\pi: W_1 \to W$ permissible for (J, b) and $\mathcal{A} = ((J, b))$, then we define the subscheme of *singular points:*

$$\operatorname{Sing}(\mathcal{A}) = \operatorname{Sing}^b(J) \subset W$$

A transformation $\pi: W_1 \to W$ is said to be *permissible for* $\mathcal{A}$ if it is permissible for (J, b) and the *transform of* $\mathcal{A}$ by *the permissible transformation* π to be $\mathcal{A}_1 = ((\bar{J}, b))$ at W_1 where $(\bar{J}, b)$ is the transform of (J, b). Finally a *sequence of permissible transformation of* $\mathcal{A}$ over W is a sequence

$$\begin{array}{ccccc} W = W_0 \overset{\pi_1}{\leftarrow} W_1 & \overset{\pi_2}{\leftarrow} W_2 & \dots & \overset{\pi_r}{\leftarrow} W_r \\ \mathcal{A} = \mathcal{A}_0 \quad \mathcal{A}_1 & \mathcal{A}_2 & & \mathcal{A}_r \end{array}$$

where each π_i is permissible for $\mathcal{A}_{i-1}$ and $\mathcal{A}_i$ is the transform of $\mathcal{A}_{i-1}$.

DEFINITION 1.4. — We define on W_1 for some index set Λ

$$E_\Lambda = \{E_\lambda \mid \lambda \in \Lambda\}$$

each E_λ being a smooth hypersurface of W or the empty set. We also assume that these hypersurfaces have only normal crossings *i. e.* $\bigcup_{\lambda \in \Lambda} E_\lambda (\subset W)$ is a subscheme with only normal crossings.

A monoidal transformation $\pi: W_1 \to W$ is said to be *permissible for* (W, E_Λ), if it is the blowing up at a center C which is regular and has only normal crossings with $\bigcup_{\lambda \in \Lambda} E_\lambda$.

In this case the *transform* of (W, E_Λ) is defined as (W_1, E_{Λ_1}), where $\Lambda_1 = \Lambda \cup \{\beta\}$ and

(i) for each $\lambda \in \Lambda \subset \Lambda_1$, E'_λ is the strict transform of $E_\lambda \subset W$, by this we mean the strict transform of the components of E_λ which are not components of C. $E'_\lambda = \varnothing$ if $E_\lambda = \varnothing$, also if $E_\lambda = C$.

(ii) $E'_\beta = \pi^{-1}(C)$.

It is clear that $\bigcup_{\alpha \in \Lambda_1} E'_\alpha$ consists of hypersurfaces with only normal crossings.

A *permissible tree* is a data of the form:

$$T: \begin{array}{cccc} W = W_0 \overset{\pi_1}{\leftarrow} W_1 \leftarrow & \dots & W_{r-1} \overset{\pi_r}{\leftarrow} W_r \\ E_\Lambda = E_{\Lambda_0} \quad E_{\Lambda_1} & & E_{\Lambda_{r-1}} \quad E_{\Lambda_r} \\ C = C_0 \quad C_1 & & C_{r-1} \end{array}$$

each π_i permissible for $(W_{i-1}, E_{\Lambda_{i-1}})$ and (W_i, E_{Λ_i}) being the corresponding transform.

DEFINITION 1.5. — An isomophism $\Gamma=(\theta, \gamma)$: $(W, E_\Lambda) \to (W', E_{\Lambda'})$ consists of:

(i) A bijection $\gamma: \Lambda \to \Lambda'$.

(ii) An isomorphism θ: $W \to W'$ inducing by restriction an isomorphism

$$\theta:\quad E_\lambda \to E_{\gamma(\lambda)}$$

for seach $\lambda \in \Lambda$.

Remark 1.6. – Given an isomorphism of pairs Γ: $(W, E_\Lambda) \to (W', E_{\Lambda'})$ as before, and a transformation π_1: $W_1 \to W$ permissible for (W, E_Λ) (Def. 1.4) with center C, then $\theta(C) \subset W'$ has only normal crossings with $\bigcup_{\lambda \in \Lambda'} E_\lambda$ and if π'_1 denotes the corresponding transformation then there is a unique isomorphism $\Gamma_1=(\theta_1, \gamma_1)$ of the transforms (W_1, E_{Λ_1}) and (W'_1, E_{Λ_1}) such that the diagram

$$\begin{array}{ccc} W_1 & \xrightarrow{\theta_1} & W'_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi'_1} \\ W & \xrightarrow{\theta} & W' \end{array}$$

is commutative.

Moreover if T is any permissible tree for (W, E_Λ), then via Γ, T induces a permissible tree over $(W', E_{\Lambda'})$ and the isomorphism Γ can be "lifted" by T.

Remark 1.7. – Let $\mathbb{A}=\mathrm{Spec}(k[X])$ and P_n: $W_n=W\times\mathbb{A}^n \to W$ the natural projection $(n \geqq 0)$. Given a pair (W, E_Λ) as in Def. 1.4 we define on each W_n a set $(E_n)_\Lambda$, which consists for each $\lambda \in \Lambda$ of $(E_n)_\lambda=P_n^{-1}(E_\lambda)$.

An isomorphism $\Gamma=(\theta: \gamma)$: $(W, E_\Lambda) \to (W', E_{\Lambda'})$ (Def. 1.5) induces natural isomorphisms

$$\Gamma_n=(\theta_n: \gamma_n): (W_n, (E_n)_\Lambda) \to (W'_n, (E_n)_{\Lambda'})$$

for all $n \geqq 0$.

DEFINITION 1.8. — Consider now a 3-tuple $(W, \mathscr{A}, E_\Lambda)$ where $\mathscr{A}$ is an idealistic exponent on W and (W, E_Λ) is as in Def. 1.4.

A tree T is said to be *permissible for* $(W, \mathscr{A}, E_\lambda)$ when the two following conditions hold:

(*a*) T is permissible for (W, E_Λ) (Def. 1.4)

(*b*) the induced sequence of transformation

$$W=W_0 \xleftarrow{\pi_1} W_1 \leftarrow \dots \leftarrow W_{r-1} \xleftarrow{\pi_r} W_r$$

is permissible for $(W, \mathscr{A})$ in the sense of Def. 1.3.

If π_1: $W_1 \to W$ is permissible for $(W, \mathscr{A}, E_\Lambda)$, let $\mathscr{A}_1$ denote the transform of $\mathscr{A}$ (Def. 1.3) and (W_1, E_{Λ_1}) the transform of (W, E_Λ) (Def. 1.4), then $(W_1, \mathscr{A}_1, E_{\Lambda_1})$ is called the *transform of* $(W, \mathscr{A}, E_\Lambda)$.

The *grove of* $(W, \mathscr{A}, E_\Lambda)$ consists of all possible permissible trees for $(W, \mathscr{A}, E_\Lambda)$.

Let P_n: $W_n = W \times \mathbb{A}^n \to W$ be as in Remark 1.7 then the *polygrove of* $(W, \mathscr{A}, E_\Lambda)$ consists of the groves of $(W_n, P_n^{-1}(\mathscr{A}), (E_n)_\Lambda)$ for each $n \geqq 0$. $P_n^{-1}(\mathscr{A})$ as in Def. 1.1

An *idealistic situation* is a 3-tuple $(W, \mathscr{A}, E_\Lambda)$ as before, together with its polygrove.

DEFINITION 1.9. — An *isomorphism from the idealistic situation* $(W, \mathscr{A}, E_\Lambda)$ to $(W', \mathscr{A}', E_{\Lambda'})$ consists of an isomorphism

$$\Gamma = (\theta: \gamma): (W, E_\Lambda) \to (W', E_{\Lambda'}) \quad \text{(Def. 1.5)}$$

such that the induced isomorphism

$$\Gamma_n = (\theta_n: \gamma_n): (W_n, (E_n)_\Lambda) \to (W'_n, (E_n)_{\Lambda'}), \qquad n \geqq 0$$

(Remark 1.7) establish a bijection between those trees of the grove of $(W_n, P_n^{-1}(\mathscr{A}), (E_n)_\Lambda)$ and those of the grove of $(W'_n, P_n^{-1}(\mathscr{A}'), (E_n)_{\Lambda'})$ for all $n \geqq 0$. The correspondence of trees via an isomorphism being as in Remark 1.6.

DEFINITION 1.10. — Consider at W an idealistic situation $(W, \mathscr{A}, E_\Lambda)$ and an etale map

$$e: \quad W_1 \to W$$

then the *restriction by e of* $(W, \mathscr{A}, E_\Lambda)$ is the idealistic situation $(W_1, e^{-1}(\mathscr{A}), (E_1)_\Lambda)$ where:

(*a*) for each $\lambda \in \Lambda$, $(E_1)_\lambda = e^{-1}(E_\lambda)$

(*b*) if $\mathscr{A}$ is the class of (J, b), then $e^{-1}(\mathscr{A})$ is the class of (JO_{W_1}, b) (Def. 1.1).

Given a closed point $x \in \operatorname{Sing}(\mathscr{A})$, then an *etale neighbourhood of* $(W, \mathscr{A}, E_\Lambda)$ at x consists of an etale map e: $W_1 \to W$, an idealistic situation $(W_1, e^{-1}(\mathscr{A}), (E_1)_\Lambda)$ as before, and a point $y \in \operatorname{Sing}(e^{-1}(\mathscr{A}))$ such that $e(y) = x$.

Given two idealistic situations $(W_1, \mathscr{A}_1, E_{\Lambda_1})$, $(W_2, \mathscr{A}_2, E_{\Lambda_2})$ and closed points $x_1 \in \operatorname{Sing}(\mathscr{A}_1)$, $x_2 \in \operatorname{Sing}(\mathscr{A}_2)$, then x_1 is said to be *equivalent to* x_2 if there are etale neighbourhoods at x_1 and x_2 which are isomorphic *i. e.* there are etale maps e_i: $W'_i \to W_i$, $i = 1, 2$, restrictions $(W'_i, e_i^{-1}(\mathscr{A}_i), e^{-1}(E)_{\Lambda_i})$, $i = 1, 2$, closed points $y_i \in \operatorname{Sing}(e_i^{-1}(\mathscr{A}_i))$, $i = 1, 2$ and an isomorphism of idealistic situations (Def. 1.9)

$$\Gamma = (\theta, \gamma): (W'_1, e_1^{-1}(\mathscr{A}_1), (e_1^{-1}(E))_{\Lambda_1}) \to (W'_2, e_2^{-1}(\mathscr{A}_2), e_2^{-1}(E)_{\Lambda_2})$$

such that $\theta(y_1) = y_2$.

Remark 1.10.1. – Let the notation and assumptions be as in Def. 1.9.

Let e: $W'_1 \to W'$ be an etale map and

$$\begin{array}{ccc} W_1 & \xrightarrow{\theta_1} & W'_1 \\ {\scriptstyle e_1}\downarrow & & \downarrow{\scriptstyle e} \\ W & \xrightarrow{\theta} & W' \end{array}$$

the commutative diagram arising from the fiber product of $\theta: W \rightarrow W'$ and $e: W'_1 \rightarrow W'$.

Then e_1 is etale and θ_1 induces an isomorphism between the restricted situations (Def. 1.10).

This follows from the definition of excellence.

1.11 . – Let $(Z, \bar{E}_\Lambda)$, (W, E_Λ) be as in Def. 1.4 and $i: Z \rightarrow W$ be an immersion of regular schemes Assume furthermore that the following condition holds:

$$\forall \lambda \in \Lambda: \quad \bar{E}_\lambda = E_\lambda \cap Z. \tag{1.11.1}$$

In this case it is clear that a permissible tree T for $(Z, \bar{E}_\Lambda)$ induces a permissible tree for (W, E_Λ), say $i(T)$. And the final transform of $(Z, \bar{E}_\Lambda)$ and (W, E_Λ) by T and $i(T)$ still satisfy 1.11.1.

Let $\mathbb{A}$ $(= \mathrm{Spec}(k[X])$, $W_n = W \times \mathbb{A}^n$, $Z_n = Z \times \mathbb{A}^n$ and $(E_n)_\Lambda$, $(\bar{E}_n)_\Lambda$ be as in Remark 1.7. If $i: Z \rightarrow W$ is such that condition 1.11.1 is satisfied, then the same will hold for the natural immersions $Z_n \overset{i_n}{\hookrightarrow} W_n$.

Definition 1.11. — Let $(Z, \mathscr{A}, \bar{E}_\Lambda)$, $(W, \mathscr{B}, E_\Lambda)$ be two idealistic situations (Def. 1.8), assume that Z is a subscheme of W, $i: Z \hookrightarrow W$, and that $\bar{E}_\Lambda$ and E_Λ satisfy 1.11.1. Then i is said to be a *strong immersion* if $Z_n \hookrightarrow W_n$ induces a bijection between the grove of $(Z_n, P_n^{-1}(\mathscr{A}), (\bar{E}'_n)_\Lambda)$ and that of $(W_n, P_n^{-1}(\mathscr{B}), (E_n)_\Lambda)$ for all $n \geqq 0$.

Theorem 1.12. — *Let*

$$(Z_1, \mathscr{A}_1, (\bar{E}_1)_\Lambda) \overset{i_1}{\rightarrow} (W, \mathscr{B}, E_\Lambda) \quad \textit{and} \quad (Z_2, \mathscr{A}_2, (\bar{E}_2)_\Lambda) \overset{i_2}{\rightarrow} (W, \mathscr{B}, E_\Lambda)$$

be two strong immersions (Def. 1.11), *and let x_i be a closed point at* $\mathrm{Sing}(\mathscr{A}_i) \subset Z_i$ $(i=1, 2)$ *such that* $i_1(x_1) = i_2(x_2)$.

If $\dim(Z_1)_{x_1} = \dim(Z_2)_{x_2}$ *then x_1 is equivalent to x_2* (Def. 1.10).

Proof. – Argue as in Theorem 11.1 [8] and construct a retraction from W to Z, locally at some etale neighbourhood of $i_1(x_1) = i_2(x_2)$ which induces an isomorphism of the restricted idealistic situations (Def. 1.10).

Theorem 1.13.1. — *Let x_i be a closed singular point of an idealistic situation* $(Z_i, \mathscr{A}_i, E_{\Lambda_i})$ $i=1, 2$ (Def. 1.8). *If x_1 and x_2 are equivalent* (Def. 1.10) *then*

$$\nu_{x_1}(\mathscr{A}_1) = \nu_{x_2}(\mathscr{A}_2) \quad \text{(Def. 1.2)}$$

Proof. – (*see* Prop. 8, p. 68 [9].

1.13.2. – We now refer to Definition 1.9, p. 59 [9] for the notion of *tangent vector space* of an idealistic exponent $\mathscr{A}$ at a closed point $x \in \mathrm{Sing}(\mathscr{A}) \subset W$ (say $T_{\mathscr{A}, x}$). This is a subspace of $T_{W, x}$ (the tangent-space of W at x) and we shall denote its codimension by $\tau(\mathscr{A}, x)$.

THEOREM 1.13.2. — *Let* $(Z_i, \mathscr{A}_i, E_{\Lambda_i})$ $i=1, 2$ *and* x_i $i=1, 2$ *be as in the last theorem. Then*

$$\tau(\mathscr{A}_1, x_1)=\tau(\mathscr{A}_2, x_2)$$

and $\tau(\mathscr{A}_1, x_1)\geqq 0$ *iff* $\nu_{x_1}(\mathscr{A}_1)=1$ (Def. 1.2).

Proof. — The proof of this fact is similar to that of Theorem 1.13.1.

1.14. Let $Z \overset{i}{\hookrightarrow} W$ be as before a closed immersion of regular schemes and $Z_n=Z\times \mathbb{A}^n \hookrightarrow W_n=W\times \mathbb{A}^n$ the induced immersions.

Let $(W, \mathscr{A}, E_\Lambda)$ be an idealistic situation and

$$\begin{array}{cccc} W\times \mathbb{A}^n=(W_n)_0 & \overset{\pi_1}{\leftarrow} (W_n)_1 \ldots & \leftarrow (W_n)_r \\ (E_n)_\Lambda=(E_n)_{\Lambda_0} & (E_n)_{\Lambda_1} & (E_n)_{\Lambda_r} \\ C_0 & C_1 & \end{array}$$

a tree over W_n, permissible for $(W_n, P_n^{-1}(\mathscr{A}), (E_n)_\Lambda)$ (*see* Def. 1.8). For any such tree let $(Z_n)_i\subset(W_n)_i$ denote the strict transform of $Z_n\,(\subset W_n=(W_n)_0)$.

DEFINITION 1.14. — With the notation as before, a regular subscheme $Z\subset W$ is said to have *maximal contact* with the idealistic situation $(W, \mathscr{A}, E_\Lambda)$ if, for any fix $n\geqq 0$ and any tree T of the grove of $(W_n, P_n^{-1}(\mathscr{A}), (E_n)_\Lambda)$ one has that $C_i\subset(Z_n)_i$ $0\leqq i<r$, or equivalently if $\mathscr{A}_i$ denotes the transform at $(W_n)_i$ of $\mathscr{A}_0=P_n^{-1}(\mathscr{A})$, then $\operatorname{Sing}(\mathscr{A}_i)\subset(Z_n)_i$, $\forall n\geqq 0$.

THEOREM 1.15. — Let $(W, \mathscr{A}, E_\Lambda)$ be an idealistic situation (Def. 1.8), $Z \overset{i}{\hookrightarrow} W$ a regular subscheme having maximal contact with $\mathscr{A}$, and $(Z, \bar{E}_\Lambda)$ as in Def. 1.4. If the condition 1.11.1 holds for $(Z, \bar{E}_\Lambda)$ and (W, E_Λ) then, locally at any closed point $x\in\operatorname{Sing}(\mathscr{A})$, either

(*a*) $\operatorname{Sing}\mathscr{A}=Z$ or

(*b*) for a convenient restriction of $(Z, \bar{E}_\Lambda)$ at a Zariski neighbourhood of x (as in Def. 1.10), say $(Z, \bar{E}_\Lambda)$, there is an idealistic situation $(Z, \mathscr{B}, \bar{E}_\Lambda)$ such that $i: Z\hookrightarrow W$ is a strong immersion (Def. 1.11).

Proof. — *See* theorem 5, p. 111 [9].

DEFINITION 1.15. — If (*a*) ever holds at x, we shall say that x is a *regular point* of $\operatorname{Sing}(\mathscr{A})$.

THEOREM 1.16.1. — *Let* $(W, \mathscr{A}, E_\Lambda)$ *be an idealistic situation and assume that* $\operatorname{ord}(\mathscr{A})=1$ (Def. 1.2). *Then, locally at any closed point* $x\in\operatorname{Sing}(\mathscr{A})$, *there is a regular hypersurface* H *having maximal contact with the restricted idealistic situation* (Def. 1.10 *and* Def. 1.14).

COROLLARY 1.16.1. — Assume that $x\in W$ is not a point at which (locally) $\operatorname{Sing}(\mathscr{A})$ is regular of codimension one (Def. 1.15). And assume also that H is a hypersurface of maximal contact, $(H, \bar{E}_\Lambda)$ is as in Def. 1.4 and that $(H, \bar{E}_\Lambda)$ and (W, E_Λ) satisfy the condition 1.11.1. Then, after restricting to a convenient Zariski neighbourhood of x, there is an idealistic situation $(H, \mathscr{B}, \bar{E}_\Lambda)$ such that $i: H \hookrightarrow W$ is a strong immersion (Def. 1.11).

THEOREM 1.16.2. — *Let* $\pi: W_1 \to W$ *be permissible for an idealistic situation* $(W, \mathscr{A}, E_\Lambda)$ (Def. 1.8), *assume that* $\operatorname{ord}(\mathscr{A})=1$ *and let* $(W_1, \mathscr{A}_1, E_{\Lambda_1})$ *be the transform. Then either* $\operatorname{Sing}(\mathscr{A}_1)=\varnothing$ *or* $\operatorname{ord}(\mathscr{A}_1)=1$. *If* x *is any closed point of* $\operatorname{Sing}(\mathscr{A}_1)$:

$$\tau(\mathscr{A}, \pi(x)) \leqq \tau(\mathscr{A}, x)$$

DEFINITION 1.16.3. — Let $(W, \mathscr{A}, E_\Lambda)$ be an idealistic situation, we define

$$\tau(\mathscr{A}) = \inf_{x \in \operatorname{Sing}(\mathscr{A})} \{\tau(\mathscr{A}, x)\}$$

and

$$R(\tau)(\mathscr{A}) = \{x \in \operatorname{Sing}(\mathscr{A}) \mid \tau(\mathscr{A}, x) = \tau(\mathscr{A}) \text{ and } x \text{ is a regular point of } \operatorname{Sing}(\mathscr{A}) \text{ (Def. 1.15)}\}.$$

PROPOSITION 1.16.4 (with the same notation as before). — (*a*) *The set* $R(\tau)(\mathscr{A})$ *is a regular subscheme of* W, *of codimension* $\tau(\mathscr{A})$ *at any point, and every irreducible component of* $R(\tau)(\mathscr{A})$ *is a connected component of* $\operatorname{Sing}(\mathscr{A})$.

(*b*) *Let* $\pi: W_1 \to W$ *be permissible for* $(W, \mathscr{A}, E_\Lambda)$ (Def. 1.8) *and let* $(W_1, \mathscr{A}_1, E_{\Lambda_1})$ *be its transform, then at a closed point* $x \in \operatorname{Sing}(\mathscr{A}_1)$ *both conditions:*

(i) x *is regular at* $\operatorname{Sing}(\mathscr{A}_1)$ (*in the sense of* Def. 1.15).

(ii) $\tau(\mathscr{A}_1, x) = \tau(\mathscr{A})$

will hold if and only if $\pi(x) \in R(\tau(\mathscr{A}))$.

Theorem 1.16.1, 1.16.2 and Prop. 1.16.4 follow from Theorem 1 p. 104 [9].

1.17. WEIGHTED IDEALISTIC SITUATIONS. — Let (W, E_Λ) be as in Def. 1.4 and P_λ the sheaf of ideals $(\subset O_W)$ defining E_λ (*i.e.* $P_\lambda = O(-E_\lambda)$) for each $\lambda \in \Lambda$.

DEFINITION 1.17.1. — A *weighted idealistic situation* is an idealistic situation $(W, \mathscr{A}, E_\Lambda)$ (Def. 1.8) together with:

(i) a set A_Λ consisting for each $\lambda \in \Lambda$, of a locally constant function

$\alpha(\lambda): E_\lambda \to (Q \geqq 0)$ (non negative rational numbers) such that if $\mathscr{A} = ((J, b)$ and $x \in \operatorname{Sing}^b(J)$, then at $O_{W, x}$:

$$J_x = \prod_{\{\lambda \mid x \in E_\lambda\}} P_{\lambda, x}^{\beta(\lambda)(x)} . \bar{J}_x, \quad \bar{J}_x \not\subset P_{\lambda, x}, \qquad \forall \lambda / x \in E_\lambda$$

and $\beta(\lambda)(x) = b.(\alpha(\lambda)(x)) \in (\mathbb{Z} \geqq 0)$, for some coherent sheaf of ideals $\bar{J} (\subset O_W)$.

(ii) at each closed point $x \in \operatorname{Sing}^b(J)$ define $\Lambda_x = \{\lambda \in \Lambda \mid x \in E_\lambda\}$. Since these hypersurfaces have only normal crossings at W it follows that $\# \Lambda_x \leqq \dim W$. We assume the existence of a total order at any such Λ_x, say $\underset{x}{<}$, subject to the following conditions:

(1.17.1.1) Given two closed points $\{x_1, x_2\} \subset E_{\alpha_1} \cap E_{\alpha_2}$ then $\alpha_1 \underset{x_1}{\lesssim} \alpha_2$ if and only if $\alpha_1 \underset{x_2}{\lesssim} \alpha_2$. We denote this weighted idealistic situation by $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$.

We also define the *weighted order of* $\mathscr{A}$ *at* x

$$w-\nu_x(\mathscr{A}) = \frac{\nu_x(J)}{b} \quad \text{(check consistency).}$$

The *weighted order of* $\mathscr{A}$:

$$w\text{-ord}(\mathscr{A}) = \max_{x \in \operatorname{Sing} \mathscr{A}} \{ w-\nu_x(\mathscr{A}) \}.$$

And the *weighted singularities of* $\mathscr{A}$:

$$w\text{-Sing}(\mathscr{A}) = \{ x \in \operatorname{Sing}(\mathscr{A}) \mid w-\nu_x(\mathscr{A}) = w\text{-ord}(\mathscr{A}) \}$$

which is a closed subset of $\operatorname{Sing}(\mathscr{A})$.

DEFINITION 1.17.2 (notation as in Definition 1.9). — Two weighted idealistic situations $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$ and $(W', \mathscr{A}', E_{\Lambda'} A_{\Lambda'})$ are said to be *isomorphic* if there is an isomorphism of the underlying idealistic situation $(W, \mathscr{A}, E_\Lambda)$ and $(W', \mathscr{A}', E_{\Lambda'})$, induced by an isomorphism

$$\Gamma : (\theta, \gamma) : (W, E_\Lambda) \to (W', E_{\Lambda'}) \quad \text{(Def. 1.9)}$$

such that:

(i) for each $\lambda \in \Lambda$ let $\alpha(\lambda) \in A_\Lambda$ and $\alpha'(\gamma(\lambda)) \in A_{\Lambda'}$ be the corresponding functions, then

$$\alpha(\lambda) = \alpha'(\gamma(\lambda)) \circ (\theta|_{E_\lambda}) : E_\lambda \to (Q \geqq 0)$$

(ii) at any closed point $x \in \operatorname{Sing}(\mathscr{A})$, $\lambda_1 \underset{x}{<} \lambda_2$ (at Λ_x) if and only if $\gamma(\lambda_1) \underset{\theta(x)}{<} \gamma(x_2)$ (at $\Lambda'_{\theta(x)}$).

(From Theorem 1.13.1 it follows that only (ii) must be checked)

DEFINITION 1.17.3 (notation as in Def. 1.10). — Consider a weighted idealistic situation $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$ and an etale map $e : W_1 \to W$ then the *restriction* by e consists of:

(i) the restriction of the idealistic situation

$$(W_1, e^{-1}(\mathscr{A}), (E_1)_\Lambda) \quad \text{(Def. 1.10)}$$

(ii) $(e^{-1}(A))_\Lambda = \{ \alpha'(\lambda) \mid \lambda \in \Lambda \}$ where

$$\alpha'(\lambda) = \alpha(\lambda) \circ e|_{e^{-1}(E_\lambda)}, \qquad \forall \lambda \in \Lambda$$

(iii) At a closed point $x \in \operatorname{Sing}(e^{-1}(\mathscr{A}))$, given $\lambda_1, \lambda_2 \in \Lambda_x$, define $\lambda_1 \underset{x}{<} \lambda_2$ if and only if $\lambda_1 \underset{e(x)}{<} \lambda_2$. The restriction by e of $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$ is again a weighted idealistic situation.

Given two weighted idealistic situations $(W_i, \mathscr{A}_i, E_{\Lambda_i}, A_{\Lambda_i})$ $i=1,2$ and closed points $x_i \in \operatorname{Sing}(\mathscr{A}_i)$, then x_1 and x_2 are said to be *equivalent* (as singular points of *weighted* idealistic situations) if there are restrictions at etale neighbourhoods of x_i $(i=1,2)$ and an isomorphism as in Def. 1.10 which is also isomorphism of weighted idealistic situations (Def. 1.17.2).

Remark. — So far we have not defined a notion of transform of weighted idealistic situations, at least not as *weighted* idealistic situations.

DEFINITION 1.17.4. — Let $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$ be as before. A transformation $\pi: W_1 \to W$ is said to be *w*-permissible if:

(i) π is permissible for the idealistic situation $(W, \mathscr{A}, E_\Lambda)$ (Def. 1.8).

(ii) In the case that $w\text{-ord}(\mathscr{A})>0$ (Def. 1.17.1), and if π is the blowing up at center $C \subset W$ then $C \subset w\text{-Sing}(\mathscr{A})$.

If $\pi: W_1 \to W$ is a *w*-permissible transformation as before and (W_1, E_{Λ_1}) is the transform of (W, E_Λ) (*see* Def. 1.4), then $\Lambda_1 = \Lambda \cup \{\beta\}$ and we define now A_{Λ_1} as follows:

(i) for each $\lambda \in \Lambda \subset \Lambda_1$, let $\alpha'(\lambda) = \alpha(\lambda) \circ \pi|_{E'_\lambda}$ where E'_λ is the strict transform of E_λ (Def. 1.4).

(ii) $\alpha'(\beta)|_{\pi^{-1}(c_i)} = \sum_{\{\lambda \mid c_i \subset E_\lambda\}} \alpha'(\lambda) \circ \pi + w\text{-ord}(\mathscr{A})$

where the c_i are the connected components of C, so $\alpha'(\beta): \pi^{-1}(C) \to Q$ is a locally constant function. Now we define at each closed point $x \in \operatorname{Sing}(\mathscr{A}_1)$ [$\mathscr{A}_1$ the transform of $\mathscr{A}$ (Def. 1.3)] a total order at $(\Lambda_1)_x$:

(i) If $\beta \in (\Lambda_1)_x$ [*i. e.* if $x \in \pi^{-1}(C)$] and $\beta \neq \alpha \in (\Lambda_1)_x$ then $\beta \underset{x}{<} \alpha$.

(ii) Given $\alpha_1 \neq \beta \neq \alpha_2$, then $\alpha_1 \underset{x}{<} \alpha_2$ if and only if $\alpha_1 \underset{\pi(x)}{<} \alpha_2$.

$(W, \mathscr{A}_1, E_{\Lambda_1}, A_{\Lambda_1})$ is now a weighted idealistic situation called the *transform* of $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$ *by* π, which we also denoted by $(W_1, \mathscr{A}_1, E_{\Lambda_1}, A_{\Lambda_1}) \xrightarrow{\pi} (W, \mathscr{A}, E_\Lambda, A_\Lambda)$.

Remark 1.17.5. — Let $\Gamma: (\theta, \gamma): (W, \Lambda) \to (W', \Lambda')$ define an isomorphism of the weighted idealistic situations $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$ and $(W', \mathscr{A}', E_{\Lambda'}, A_{\Lambda'})$ (Def. 1.17.2). Let $\pi: W_1 \to W$ be a *w*-permissible transformation for $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$ (Def. 1.17.4). Then there exists a unique isomorphism of weighted idealistic situations Γ' such that the diagram

$$\begin{array}{ccc} (W, \mathscr{A}_1, E_{\Lambda_1}, A_{\Lambda_1}) & \overset{\Gamma'}{\underset{\sim}{\to}} & (W'_1, \mathscr{A}'_1, E'_{\Lambda_1}, A'_{\Lambda_1}) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ (W, \mathscr{A}, E_\Lambda, A_\Lambda) & \overset{\Gamma}{\underset{\sim}{\to}} & (W', \mathscr{A}', E_{\Lambda'}, A_{\Lambda'}) \end{array}$$

commuts, where π' corresponds to π via Γ and $(W'_1, \mathscr{A}'_1, E_{\Lambda'_1}, A_{\Lambda'_1})$ is the transform of $(W', \mathscr{A}', E_{\Lambda'}, A_{\Lambda'})$.

Remark 1.17.6. — With the notion as in Def. 1.17.1.

Let $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$ be a weighted idealistic situation and $t = w\text{-ord}(\mathscr{A})$. If $\mathscr{A} = ((J, b))$ then:

(*a*) $t_1 = b.t = \max_{x \in W} \{ v_x(\bar{J}) \}$ and

(*b*) $w\text{-Sing}(\mathscr{A}) = \{ x \in \text{Sing}(\mathscr{A}) \mid v_x(\bar{J}) = t_1 \}$.

When $t > 0$ we attach to (J, b) a new idealistic pair $w(J, b)$ as follows:

If $t \geqq 1$, then: $w(J, b) = (\bar{J}, t_1)$.

If $0 < t < 1$, then: $w(J, b) = (\langle (\Pi P_\lambda^{\beta(\lambda)})^{t_1}, \bar{J}^{b - t_1} \rangle, t_1(b - t_1))$ where $t_1 = tb$, and $\bar{J}$ and $P_\lambda^{\beta(\lambda)}$ are as in Def. 1.17.1. Now we can check:

(i) If $(J, b) \sim (J', b') \Rightarrow w(J, b) \sim w(J', b')$ (check first that $(\bar{J}, b) \sim (\bar{J}', b')$, notation as before).

(ii) If $w(\mathscr{A})$ denotes $(w(J, b))$, then $\text{Sing}(w(\mathscr{A})) = w\text{-Sing}(\mathscr{A})$. So $\pi : W_1 \to W$ is w-permissible for $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$ if and only if it is permissible for $(W, w(\mathscr{A}), E_\Lambda)$ (Def. 1.17.4 and Def. 1.8).

(iii) Let $\pi : W_1 \to W$ be as in (ii) and let $(W_1, \mathscr{A}_1, E_{\Lambda_1}, A_{\Lambda_1})$ be the transform of $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$(Def. 1.17.4). Then:

$$w\text{-ord}(\mathscr{A}_1) \leqq w\text{-ord}(\mathscr{A})$$

and if the equality holds, then $w(\mathscr{A}_1)$ is the transform (simply as idealistic situation) of $w(\mathscr{A})$ (Def. 1.8).

Remark 1.17.7. — Given a weighted idealistic situation $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$, assume $w\text{-ord}(\mathscr{A}) > 0$, and let $w(\mathscr{A})$ be as before, then: $\text{ord}(w(\mathscr{A})) = 1$.

Remark 1.17.8. — If $(W, \mathscr{A}, E_\Lambda)$ is an idealistic situation (Def. 1.8) and $\text{ord}(\mathscr{A}) = 1$ (Def. 1.2) then it can be given a structure of weighted idealistic situation, taking A_Λ to consists of the functions $\alpha(\lambda)$ which are constantly equal to zero along E_λ for each $\lambda \in \Lambda$ (Def. 1.17.2).

Note also that in this case $w\text{-Sing}(\mathscr{A}) = \text{Sing}(\mathscr{A})$. So the notions of w-permissibility and of permissibility coincide (Def. 1.17.4 and Def. 1.8).

If $\pi : W_1 \to W$ is permissible for $(W, \mathscr{A}, E_\Lambda)$ [w-permissible for $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$] and $(W, \mathscr{A}_1, E_{\Lambda_1})$ $((W_1, \mathscr{A}_1, E_{\Lambda_1}, A_{\Lambda_1}))$ denotes the transform. Then again A_{Λ_1} consists of functions $\alpha(\lambda) : E_\lambda \to Q$ such that $\alpha(\lambda)(x) = 0 \ \forall x \in E_\lambda, \ \forall \lambda \in \Lambda_1$.

1.18. Idealistic spaces

Definition 1.18.1. — By $(C(m), \Lambda)$ we denote a category, where the objects are those weighted idealistic situations $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$ where $\dim W = m$ (Def. 1.17.1) and a map $(W_1, \mathscr{A}_1, E_{\Lambda_1}, A_{\Lambda_1}) \to (W, \mathscr{A}, E_\Lambda, A_\Lambda)$ is an etale map $e : W_1 \to W$ such that id_{W_1} induces an isomorphism (Def. 1.17.2) between $(W_1, \mathscr{A}_1, E_{\Lambda_1}, A_{\Lambda_1})$ and the restriction of $(W, \mathscr{A}, E_\Lambda, A_\Lambda)$ by e (Def. 1.17.3).

To simplify the notation, given an object $\alpha \in C(m, \Lambda)$ we denote

$$\alpha = (W(\alpha), \mathscr{A}(\alpha), E_{\Lambda_\alpha}, A_{\Lambda_\alpha}).$$

A *subset* C *of* $C(m, \Lambda)$ consists, for each $\alpha \in C(m, \Lambda)$ of a locally closed subset $C(\alpha) \subset \operatorname{Sing}(Q(\alpha)) \subset W(\alpha)$ subject to the following conditions:

1. Given $\alpha \xrightarrow{j} \beta$ in $C(m, \Lambda)$, then $e(j)^{-1}(C(\beta)) = C(\alpha)$ where $e(j): W(\alpha) \to W(\beta)$ is the associated etale map.

2. Given $\alpha_1, \alpha_2 \in C(m, \Lambda)$ and closed points $x_i \in W(\alpha_i)$, if x_1 and x_2 are equivalent (Def. 1.17.3), then $x_1 \in C(\alpha_1) \Leftrightarrow x_2 \in C(\alpha_2)$.

DEFINITION 1.18.2. — An *idealistic space of dimension m* is a map χ from a set I to $C(m, \Lambda)$ ($\dim \chi = m$).

A *closed subset* C of χ consists of a subset C of $C(m, \Lambda)$ such that for each $\alpha \in I$ $C(\chi(\alpha)) (\subset W(\chi(\alpha)))$ is a closed subset. A closed subset C of χ is said to be *permissible* for χ if $C(\chi(\alpha))$ is *w*-permissible for $\chi(\alpha)$ in the sense of Def. 1.17.4. In such case the *transform of* χ *by* C is defined by $\chi': I \to C(m, \Lambda)$ where $\chi'(\alpha)$ is the transform of $\chi(\alpha)$ by $C(\alpha)$ (Def. 1.17.4). This we denote by $\chi' \to \chi$ and π is said to be a *permissible transformation* with center C.

A point $x \in \chi$ consists of a closed point $x_\alpha \in \operatorname{Sing}(\mathscr{A}(\chi(\alpha)) \subset W(\chi(\alpha))$ (for some $\alpha \in I$) together with all those $x_\beta \in \operatorname{Sing}(\mathscr{A}(\chi(\beta)) \subset W(\chi(\beta)) (\beta \in I)$ such that x_α and x_β are equivalent (Def. 1.17.3).

DEFINITION 1.18.3. — A *m*-dimensional idealistic space $\chi: I \to C(m, \Lambda)$ is said to be *restrictive to an n-dimensional idealistic space* if $n \leq m$ and there are idealistic spaces $\chi_m: \bar{I} \to C(m, \Lambda)$ and $\chi_n: \bar{I} \to C(n, \Lambda)$ such that:

1. Points of χ are locally equivalent to points of χ_m and the converse also holds (local equivalence always as in Def. 1.17.3).

2. For each $\alpha \in \bar{I}$ there is a strong immersion (Def. 1.11), disregarding the weighted structure, induced by $W(\chi_n(\alpha)) \overset{i(\alpha)}{\subsetneq} W(\chi_m(\alpha))$ such that two points

$$x_i \in \operatorname{Sing}(\mathscr{A}(\chi_n(\alpha_i)) \subset W(\chi_n(\alpha_i)),$$

$i = 1,2$ are equivalent points at $C(n, \Lambda)$ (Def. 1.18.2) if and only if $i(\alpha_i)(x_i)$ are equivalent as points of χ_m [at $C(m, \Lambda)$].

Remark 1.18.4. – Given χ_n and χ_m as before, permissible center for χ_n and χ_m coincide (via *i*) and if $\chi'_m \to \chi_m$ and $\chi'_n \to \chi_n$ are the permissible transforms at an identified center, then (1) and (2) hold for χ'_n and χ'_m.

Remark 1.18.5. – Suppose that for each $\alpha \in I$,

$$\chi_m(\alpha) = (W(\chi_m(\alpha)), \mathscr{A}(\chi_m(\alpha)), E_{\Lambda\,\alpha)}, A_{\Lambda\,\alpha)})$$

is such that all functions $\alpha(\lambda)$ (Def. 1.17.1) [for all $\lambda \in \Lambda(\alpha)$] are constant functions equal to zero *i. e.*

$\alpha(\lambda): E_\lambda \to \mathbb{Q}$ is such that $\alpha(\lambda)(x)=0$, $\forall x \in E_\lambda$, $\forall \lambda \in \Lambda(\alpha)$. Assume that this also holds for any $\alpha \in \bar{I}$ at $\chi_n(\alpha)$, then (2) of Def. 1.18.3 can be replaced by:

(2′) For each $\alpha \in \bar{I}$ there is a strong immersion, disregarding the weighted structure, induce by:

$$W(\chi_n(\alpha)) \underset{i(\alpha)}{\subsetneq} W(\chi_m(\alpha))$$

1.19. When we consider a fixed idealistic space $\chi: I \to C(m, \Lambda)$, and $\alpha \in I$ we denote $\chi(\alpha)=(W(\chi(\alpha)), \mathscr{A}(\chi(\alpha)), E_{\Lambda_{\chi(\alpha)}}, A_{\Lambda_{\chi(\alpha)}})$ by $(W(\alpha), \mathscr{A}(\alpha), E_{\Lambda\alpha}, A_{\Lambda\alpha})$.

Definition 1.19.1. — An idealistic space $\chi: I \to C(m, \Lambda)$ is said to be *quasi-compact* if there is a finite subset $\{\alpha_1, \ldots, \alpha_n\} \subset I$ such that for any $\alpha \in I$ and any closed point $x \in \mathrm{Sing}(\mathscr{A}(\alpha)) \subset W(\alpha)$ there is an index i, $1 \leqq i \leqq n$ and a point $y \in \mathrm{Sing}(\mathscr{A}(\alpha_i))$ such x and y are locally equivalent (Def. 1.17.3).

If x is a point of χ (Def. 1.18.2), say that $x_1 \in W(\alpha_1)$ belongs to the class of x, then we define *the order of* χ *at* x

$$\mathrm{ord}_x(\chi) = \nu_{x_1}(\mathscr{A}(\alpha_1)) \quad \text{(Def. 1.2)}$$

and

$$\tau(\chi, x) = \tau(\mathscr{A}(\alpha_1), x_1), \quad \text{(Def. 1.13.2)}$$

the consistency of these definitions are given by Theorems 1.13.1 and 1.13.2.

The *order of* χ is:

$$\mathrm{ord}\,\chi = \max_{\alpha \in I} \{\mathrm{ord}\,\mathscr{A}(\alpha)\} \quad \text{(Def. 1.2)}$$

The *weighted order of* χ is:

$$w\text{-ord}(\chi) = \max_{\alpha \in I} \{w\text{-ord}(\mathscr{A}(\alpha))\} \quad \text{(Def. 1.17.1)}$$

and

$$\tau(\chi) = \inf_{\alpha \in I} \{\tau(\mathscr{A}(\alpha), x) \mid x \in \mathrm{Sing}(\mathscr{A}(\alpha))\}.$$

1.19.2. One can check that the following are closed subsets of χ in the sense of Definition 1.18.2.

1. Sing χ: $(\mathrm{Sing}\,\chi)(\alpha) = \mathrm{Sing}(\chi(\alpha)) = \mathrm{Sing}(\mathscr{A}(\alpha)) \subset W(\alpha)$, $\forall \alpha \in I$.
2. w-Sing χ: $(w\text{-Sing}\,\chi)(\alpha) = w\text{-Sing}(\mathscr{A}(\alpha)) \subset W(\alpha)$, $\forall \alpha \in I$.
3. If $\tau = \tau(\chi)$ then $F(\tau)(\chi)$:

$$F(\tau)(\chi)(\alpha) = \{x \in \mathrm{Sing}\,\mathscr{A}(\alpha) \mid \tau(\mathscr{A}(\alpha), x) = \tau\}$$

4. If $\tau=\tau(\chi)$ then $R(\tau)(\chi)$:

$$R(\tau)(\chi)(\alpha)=\{x\in \operatorname{Sing}\mathscr{A}(\alpha) \mid \tau(\mathscr{A}(\alpha), x)=\tau$$

and

$$x \text{ is regular at } \operatorname{Sing}\mathscr{A}(\alpha) \text{ (Def. 1.15)}\}.$$

Remark 1.19.2. — $R(\tau)(\chi)$ is a component of Sing χ in the sense that $\forall\alpha\in I$, $R(\tau)(\chi)(\alpha)$ is a union of connected components of $(\operatorname{Sing}\chi)(\alpha)=\operatorname{Sing}(\mathscr{A}(\alpha))$ (*see* Proposition 1.16.4).

DEFINITION 1.19.3. — Given $\chi: I\to C(m, \Lambda)$ such that w-ord$(\chi)>0$ (Def. 1.19.1), define $w(\chi): I\to C(m, \Lambda)$ by:

$$w(\chi)(\alpha)=(W(\alpha), w(\mathscr{A}(\alpha)), E_{\Lambda\alpha}, A'_{\Lambda\alpha})$$

$w(\mathscr{A}(\alpha))$ as in 1.17.6 and all functions of $A'_{\Lambda\alpha}$ being constantly equal to zero (*see* Remark 1.17.8).

Now one can check that $w(\chi)$ is an idealistic space for which:

(i) $\operatorname{ord}(w(\chi))=1$ (Def. 1.19.1).

(ii) $\operatorname{Sing}(w(\chi))=w$-$\operatorname{Sing}(\chi)$.

(iii) If $\pi:\chi_1\to\chi$ is a permissible transformation (Def. 1.18.2) then w-ord $\chi_1\leqq w$-ord χ.

(iv) If the equality holds at (iii) then naturally $\pi: w(\chi_1)\to w(\chi)$ is a permissible transformation.

THEOREM 1.20. — *Let* $\chi: I\to C(m, \Lambda)$ *be a quasi-compact m-dimensional idealistic space of order* 1 (Def. 1.19.1). *If* $E_{\Lambda\alpha}=\varnothing\,\forall\alpha\in I$, *then* $\tau(\chi)>1$, *and* χ *is restrictive to a quasi-compact idealistic space of dimension* $m-1$ (Def. 1.18.3).

Proof. — Follows from theorems 1.16.1 and 1.12.

§ 2. Constructive Resolutions

2.1. Recall from 1.19.3 that if $\pi:\chi_1\to\chi$ is a permissible transformation of idealistic spaces, then

$$w\text{-ord}(\chi_1)\leqq w\text{-ord}(\chi).$$

DEFINITION 2.1. — Fix a sequence of idealistic spaces and permissible transformations (1.18.2):

$$(2.1.1)\qquad \chi_0 \overset{\pi_1}{\leftarrow} \chi_1 \overset{\pi_2}{\leftarrow} \chi_2 \leftarrow \dots \overset{\pi_r}{\leftarrow} \chi_r$$

and assume that w-ord$(\chi_0)=w$-ord$(\chi_r)>0$, we shall say that χ_0 is a *new* space and χ_0 is the *birth* of χ_r.

In this case [(2.1.1) being fixed], we define $\tau(w\chi_r))$ to be $\tau(w(\chi_0))$ [$\tau(\chi_0)$ as in Def. 1.19.1 and $w(\chi_i)$ as in 1.19.3].

Let $\chi_0: \mathrm{I} \to \mathrm{C}(m, \Lambda)$, then (2.1.1) induces for each $\alpha \in \mathrm{I}$ a sequence of w-permissible transformations of weighted idealistic situations

$$(\mathrm{W}^{(0)}(\alpha), \mathscr{A}^{(0)}(\alpha), \mathrm{E}^{(0)}_{\Lambda(\alpha)}, \mathrm{A}^{(0)}_{\Lambda(\alpha)}) \overset{\pi_1}{\leftarrow} (\mathrm{W}^{(1)}(\alpha), \mathscr{A}^{(1)}(\alpha), \mathrm{E}^{(1)}_{\Lambda(\alpha)}, \mathrm{A}^{(1)}_{\Lambda(\alpha)}) \dots$$
$$\overset{\pi_r}{\leftarrow} (\mathrm{W}^{(r)}(\alpha), \mathscr{A}^{(r)}(\alpha), E^{(r)}_{\Lambda(\alpha)}, \mathrm{A}^{(r)}_{\Lambda(\alpha)})$$

For each $\alpha \in \mathrm{I}$ we define $(\mathrm{E}^{(r)}_{\Lambda(\alpha)})^+$, $(\mathrm{E}^{(r)}_{\Lambda(\alpha)})^-$ such that

$$\mathrm{E}^{(r)}_{\Lambda(\alpha)} = (\mathrm{E}^{(r)}_{\Lambda(\alpha)})^+ \cup (\mathrm{E}^{(r)}_{\Lambda(\alpha)})^-.$$

(i) $(\mathrm{E}^{(r)}_{\Lambda(\alpha)})^-$ consists of the strict transform at $\mathrm{W}^{(r)}(\alpha)$ of elements of $\mathrm{E}^{(0)}_{\Lambda(\alpha)}$ [as in (i) of Def. 1.4].

(ii) $(\mathrm{E}^{(r)}_{\Lambda(\alpha)})^+$ consists of the strict transforms at $\mathrm{W}^{(r)}(\alpha)$ of the exceptional locus of π_j, $j = 1, 2, \dots, r$ [as in (ii) Def. 1.4].

A *partial resolution* of χ consists of a sequence of permissible transformations

$$\chi = \chi_0 \overset{\pi_1}{\leftarrow} \chi \overset{\pi_2}{\leftarrow} \chi_2 \dots \overset{\pi_r}{\leftarrow} \chi_r \overset{\pi_{r+1}}{\leftarrow} \chi_{r+1}$$

such that $w\text{-ord}(\chi) = w\text{-ord}(\chi_r) > w\text{-ord}(\chi_{r+1})$. And a *resolution* is a sequence

$$\chi_0 \leftarrow \dots \leftarrow \chi_s$$

of permissible transformations, and $\mathrm{Sing}\,\chi_s = \varnothing$.

2.2. At this point we want to establish the meaning of a *constructive resolution of quasi compact idealistic spaces of dimension m*.

On any partially ordered set (D, <) consider the discrete topology, then a constructive resolution of χ consists of:

(i) An upper semicontinuous function $\varphi: \mathrm{Sing}\,\chi \to \mathrm{D}$ such that

$$\underline{\mathrm{Max}}\,\varphi = \{ x \in \mathrm{Sing}\,\chi \mid \varphi(x) \text{ is maximum} \}$$

is the center of a permissible transformation

$$\pi_1: \quad \chi_1 \to \chi.$$

(ii) If $\pi_1: \chi_1 \to \chi$ [as in (i)] is not a resolution of χ (Def. 2.1), then there is an upper semicontinuous function $\varphi_1: \mathrm{Sing}\,\chi_1 \to \mathrm{D}$, such that:

(*a*) $\varphi(\pi_1(x)) \geqq \varphi_1(x)$, $\forall x \in \mathrm{Sing}\,\chi_1$

(*b*) If $\pi(x) \notin \underline{\mathrm{Max}}\,\varphi$ then $\varphi_1(x) = \varphi(\pi(x))$

(*c*) $\underline{\mathrm{Max}}\,\varphi_1$ is permissible at χ_1

(iii) Assume that a sequence

$$\chi=\chi_0 \leftarrow \chi_1 \leftarrow \chi_2 \ldots \leftarrow \chi_r$$

has been defined, that Sing $\chi_r \neq \varnothing$, and also that the functions $\varphi_i : \chi_i \rightarrow D$ are given $i=0, \ldots, r$. Then $\underline{\mathrm{Max}}(\varphi_r)$ is the center of a permissible transformation say π_{r+1}:

$$\chi_r \overset{\pi_{r+1}}{\leftarrow} \chi_{r+1}$$

such that either χ_{r+1} is a resolution of χ_r or there is an upper semicontinuous function $\varphi_{r+1} : \chi_{r+1} \rightarrow D$ and conditions (*a*), (*b*) and (*c*) of (ii) (with the obvious adjustement of subindices) hold.

(iv) For some r, Sing $\chi_r = \varnothing$ *i.e.*

$$\chi=\chi_0 \overset{\pi_1}{\leftarrow} \chi_1 \overset{\pi_2}{\leftarrow} \ldots \overset{\pi_r}{\leftarrow} \chi_r$$

is a resolution (Def. 2.1).

(v) Suppose that $\mathrm{ord}(\chi)=1$, that $\mathrm{Sing}(\chi)=R(\tau)(\chi)$ (1.19.2) and $\chi \overset{\pi_1}{\leftarrow} \chi_1 \leftarrow \ldots \overset{\pi_r}{\leftarrow} \chi_r$ have been constructed, and assume that only hypersurfaces arising as exceptional locus from this sequence of permissible transformations intersect $\mathrm{Sing}(\chi_r)$ [which is also regular (Prop. 1.16.4)], then

$$\underline{\mathrm{Max}}\, \varphi_r = \mathrm{Sing}\, \chi_r$$

i. e. φ_r is constant at Sing χ_r.

Remark 2.2.1. — Let χ_r be as in (v) then φ_r is constantly equal to some $c \in D$. If

$$\chi_r \overset{\pi_r}{\leftarrow} \chi_{r+1}$$

is any permissible transformation and Sing $\chi_{r+1} \neq \varnothing$ *then all conditions on* χ_r hold also on χ_{r+1}, and if we define $\varphi_{r+1} : \mathrm{Sing}\, \chi_{r+1} \rightarrow D$ by $\varphi_{r+1}=c$ (the constant function), then condition (iii) still holds.

Remark 2.2.2. — On a ordered set (D, $\leq$) we may assume the existence of an element $\infty_D \in D$ such that $\lambda < \infty_D$, $\forall \lambda \in D$, $\lambda \neq \infty_D$. If not we can "add" such an element to D.

Given D_1 and D_2 as before we consider on $D_1 \times D_2$ the lexicographic order, then $\infty_{D_1 \times D_2} = (\infty_{D_1}, \infty_{D_2})$.

$\mathbb{Z}$ (or $\mathbb{Z} \cup \{\infty\}$) will be considered with the usual order.

2.3. We begin by constructing an upper semicontinuous function T from which φ will derive.

First we consider the case of an idealistic space of dimension m, say $\chi : I \rightarrow C(m, \Lambda)$ and weighted order zero (Def. 1.19.1).

2.3.1. Case $\dim \chi = m$ and $w-\mathrm{ord}\, \chi = 0$.

At each closed point $x \in \operatorname{Sing}\chi$ define $\Lambda_x = \{\alpha \in \Lambda \mid x \in E_\alpha\}$ [*see* Def. 1.17.1 (ii)] and recall that $\# \Lambda_x \leqq m$.

Let now $T: \operatorname{Sing}\chi \to \mathbb{Z}^3 \times \Lambda^m$ be defined as follows

$$T(1)(x) = 0$$

$$T(2)(x) = -\mathscr{B}(x) \quad \text{where} \quad \mathscr{B}(x) = \min\{k \mid \exists i_1 < i_2 < \ldots < i_k \mid$$
$$i_j \in \Lambda_x j = 1, 2, \ldots, k \quad \text{and} \quad \alpha(i_1)(x) + \ldots + \alpha(i_k)(x) \geqq 1\}.$$

If $\mathscr{B} = \mathscr{B}(x)$ then

$$T(3)(x) = \max\{\alpha(i_1)(x) + \ldots + \alpha(i_{\mathscr{B}})(x) \mid i_1 < \ldots < i_\beta$$

and

$$E_{i_j} \in \Lambda_x, i = 1, 2, \ldots, \mathscr{B}\}.$$

Now consider $\Lambda_x^{\mathscr{B}} = \Lambda_x \times \ldots \times \Lambda_x$ ($\mathscr{B}$-times) with the lexicographic ordering, and:

$$\beta = (\bar{\beta}_1, \ldots, \bar{\beta}_{\mathscr{B}}) = \max\{(\beta_1 \ldots \beta_{\mathscr{B}}) \mid \beta_1 \underset{x}{>} \beta_2 \ldots \underset{x}{>} \beta_{\mathscr{B}}, \beta_i \in \Lambda_x$$
$$i = 1, 2, \ldots, \mathscr{B} \quad \text{and} \quad \alpha(\beta_1)(x) + \ldots + \alpha(\beta_{\mathscr{B}})(x) = T(3)(x)\}.$$

Define:

$$T(4)(x) = (\beta, \infty) \in \Lambda^m (\beta \in \Lambda_x^{\mathscr{B}} \subset \Lambda^{\mathscr{B}} \text{ and } \infty = \infty_{\Lambda^{m-\mathscr{B}}} \in \Lambda^{m-\mathscr{B}})$$

We shall now define at $\operatorname{Img} T \subset \mathbb{Z}_3 \times \Lambda^m$ a partial order, without a notion of order at Λ, but extending the lexicographic order at $\mathbb{Z}^3$.

It suffices to defines a notion of $T(x) < T(y)$ at closed points $x, y \in \operatorname{Sing}\chi$ for which $T(j)(x) = T(j)(y) = a_j$, $j = 1$, 2 and 3 ($a_1 = 0$ by assumption).

Let $J = \{x \in \operatorname{Sing}\chi \mid T(j)(x) = a_j, j = 1, 2, 3\}$. One can check (at each $\alpha \in I$) that the irreducible components of J are open subset of irreducible components of $\operatorname{Sing}\chi$ of dimension $m + a_2$ [at $W(\alpha)$]. Now we say that $T(4)(x) < T(4)(y)$ if there are closed points $\{x_0 = x_1, \ldots, x_2 = y\} \subset J$ such that:

(*a*) $T(4)(x_i) \in \Lambda_{x_{i+1}}^{-a_2}$, $i = 0, \ldots, l-1$

(*b*) for some i as before $T(4)(x_i) < T(4)(x_{i+1})$ at $\Lambda_{x_{i+1}}^{-a_2}$.

The consistency of this definition follows from (1.17.1.1) and Def. 1.17.2 (ii).

This order is not a total order at Img T, and the existence of maximal elements follows from the hypothesis of quasi-compactness on χ.

The maximal elements might not be unique as shown in the following examples:

Examples. – Consider at $W = \operatorname{Spec}(C[x, y, z])$ hypersurfaces

$$E_1 = \{x = 0\}, \quad E_2 = \{x = 1\}, \quad E_3 = \{y = 0\}, \quad E_4 = \{z = 0\},$$

and given $\{i, j\} \in \Lambda_x$, let $i \underset{x}{<} j$ iff $i < j$ (at $\mathbb{Z}$).

Define also $T_{ij}=E_i\cap E_j$.

Example 1. — Let (J, b) be defined at W by $J=\langle x(x-1)z\rangle$ and $b=2$. Then $\mathrm{Sing}^{(b)}(J)=T_{14}\cup T_{24}$, T is maximal along $\mathrm{Sing}^b(J)$ and

$$\max T=\{(0,\ -2, 1, (1, 4, \infty));\ (0,\ -2, 1, (2, 4, \infty))\}.$$

Example 2:

$$J=\langle x(x-1).y.z\rangle,\qquad b=2.$$

$$\mathrm{Sing}^b J=T_{14}\cup T_{24}\cup T_{34}\cup T_{13}\cup T_{23}$$

in this case $\max T=\{(0,\ -2,\ 1,\ (3,\ 4,\ \infty))\}$ is reached exactly along T_{34}.

Remark 2.3.1. — One can check that T is upper semicontinuous, moreover for a fixed $d\in\mathbb{Z}^3\times\Lambda^m$ the condition $T>d$ is closed at $\mathrm{Sing}\,\chi$.

Recall now from Def. 1.17.4 the notion of total order at Λ_x after a permissible transformation and check that $T=\varphi$ satisfies all conditions of 2.2.

2.3.2. Case of $\dim\chi=m$ and $w-\mathrm{ord}(\chi)>0$. Consider χ together with a fixed sequence

$$\chi^{(-r)}\overset{\pi_{-r}}{\leftarrow}\chi^{(-r+1)}\leftarrow\ldots\ \chi^{(-1)}\overset{\pi_{-1}}{\leftarrow}\chi^{(0)}=\chi$$

in the conditions of the sequence (2.1.1) of Def. 2.1, so that $\chi^{(-r)}$ is the birth of χ and $E_\Lambda=E_\Lambda^+\cup E_\Lambda^-$ ($E_\Lambda(\alpha)=E_\Lambda^+(\alpha)+E_\Lambda^-(\alpha)$, $\forall\alpha\in I$) are defined.

Now let $T: w\text{-Sing}\,\chi\to\mathbb{Z}^3\times\Lambda^m$ be defined for each $x\in w\text{-Sing}\,\chi$ by:

$$T(1)(x)=w-\mathrm{ord}(\chi)\quad(\text{Def. }1.9.1)$$

$$T(2)(x)=\begin{cases}0 & \text{if}\quad x\in R(\tau)(w(\chi))\quad(1.19.2\text{ and }1.19.4)\\ 1 & \text{if}\quad x\notin R(\tau)(w(\chi))\end{cases}.$$

OBSERVATION 2.3.2. — $R(\tau)(w\chi)$ is a "component" of $w-\mathrm{sing}\,\chi$ (Remark 1.19.2), this fact can be checked at any $\mathrm{Sing}(w(\mathscr{A}\alpha)))\subset W(\alpha)(\alpha\in I)$). Moreover the definitions of $\tau(\chi)$ (Def. 2.1) together with Proposition 1.16.4 and 1.19.3 assert that a point $x\in R(\tau)(w(\chi))$ if and only if the final imagen of such point at $\chi^{(-r)}$ is a point of $R(\tau)(w(\chi^{(-r)}))$.

Now define:

$$n(x)=\#\{\alpha\in\Lambda_x\,|\,E_\alpha\in E_\Lambda^-\}$$

$$m(x)=\#\{\alpha\in\Lambda_x\,|\,E_\alpha\in E_\Lambda^-\text{ and }w\text{-Sing}(\chi)\notin E_\alpha\text{ locally at }x\}$$

and finally

$$T(3)(x)=\begin{cases}n(x) & \text{if}\quad x\notin R(\tau)\\ m(x) & \text{if}\quad x\in R(\tau)\end{cases}$$

And $T(4)(x)=\infty\in\Lambda^m$.

The function T_1 takes values at $\mathbb{Q}$, but since we assume that χ is quasi-compact there is $n \in \mathbb{Z}$ such that $\operatorname{Img} T_1 \subset 1/n\,\mathbb{Z} \subset \mathbb{Q}$, and $1/n\,\mathbb{Z} \simeq \mathbb{Z}$ as ordered sets.

Remark 2.3.2. — The fact T is well defined follows from the notion of equivalence of points at weighted idealistic situations (Def. 1.17.3) and Theorems 1.13.1 and 1.13.2.

OBSERVATION 2.3.3. — If $\dim \chi = m = 1$ (Def. 1.18.2) then $T = \varphi$ satisfies all conditions of 2.2.

Remark 2.3.4. — If $w-\operatorname{ord} \chi > 0$ then T reaches a *unique* maximal value along w-Sing(χ). And for a fixed element $d \in \mathbb{Z} \times \Lambda^m$ both $\{x \in w\text{-Sing}\,\chi \mid T(X) \geqq d\}$ and $\{x \in w\text{-Sing}\,\chi \mid T(X) > d\}$ are closed subsets (Def. 1.18.2) included in w-Sing χ. In fact the values of T are taken in the totally ordered discrete subset $\mathbb{Z}^3 \times \infty\,(\subset \mathbb{Z}^3 \times \Lambda^m)$.

DEFINITION 2.4. — *A preparation procedure* of an idealistic space χ of weighted order bigger than zero, constists of a sequence of permissible transformation

$$\chi \overset{\pi_1}{\leftarrow} \chi_1 \ldots \leftarrow \chi_s \overset{\pi_{S+1}}{\leftarrow} \chi_{s+1}$$

such that $w-\operatorname{ord}\,\chi = w-\operatorname{ord}\,\chi_s$ and either $w-\operatorname{ord}\,\chi_{s+1} < w-\operatorname{ord}\,\chi_s$ or, if $w-\operatorname{ord}$ $\chi_{s+1} = w-\operatorname{ord}\,\chi_s$ then $T(3)(x) = 0$, $\forall x \in w-\operatorname{Sing}(\chi_{s+1})$.

DEFINITION 2.5. — Let

$$\beta: \chi^{(-r)} \overset{\pi_{-r}}{\leftarrow} \chi^{(-r+1)} \leftarrow \ldots \leftarrow \chi^{(0)} = \chi$$

be as in 2.3.2, *i. e.* $\chi^{(-r)}$ is the birth of χ (Def. 2.1), and let $\pi: \chi \to \chi^{(-r)}$ denote the composition of the intermediate transformation. Then given $x \in w$-Sing(χ) we define the *birth of* x to be the point $\pi(x) \in w$-Sing$(\chi^{(-r)})$.

2.6. Here we define a notion of an *inductive procedure*. Let the assumptions and notation be as in Def. 2.5. Assume also that $T(3)(x) = 0$, $\forall x \in w$-Sing(χ), and that this condition does not hold at $\chi^{(-1)}$.

Now fix $x \in w$-Sing(χ) and let $y \in w$-Sing$(\chi^{(-r)})$ denote the birth of $x . \chi^{(-r)}: I \to C(m, \Lambda)$. Choose $\alpha \in I$ such that

$$y \in w\text{-Sing}\,(\mathscr{A}^{(-r)}(\alpha)) \subset W^{(-r)}(\alpha).$$

Now w-Sing$(\mathscr{A}^{(-r)}(\alpha)) = \operatorname{Sing}(w(\mathscr{A}^{(-r)}(\alpha))$ (Remark 1.17.6), and $\operatorname{ord}(w(\mathscr{A}^{(-r)}(\alpha))) = 1$ (Remark 1.17.7).

So Theorem 1.16.1 asserts that there is a regular hypersurface H, such that $y \in H \subset W^{(-r)}(\alpha)$, having maximal contact with $W(\mathscr{A}^{(-r)}(\alpha))$ locally at y.

After a convenient restriction assume that H has maximal contact with $W(\mathscr{A}^{(-r)}(\alpha))$.

The sequence of permissible transformations $\beta : \chi^{(-r)} \leftarrow \ldots \leftarrow \chi^{(0)}$ gives rise to:

(1) a sequence of w-permissible transformations over

$(W^{(-r)}(\alpha), \mathscr{A}^{(-r)}(\alpha), E_\Lambda(-r), A_\Lambda(-r))$ (Def. 1.17.4):

$$(W^{(-r)}(\alpha), \mathscr{A}^{(-r)}(\alpha), E_{\Lambda^{(-r)}(\alpha)}, A_{\Lambda^{(-r)}(\alpha)}) \leftarrow \ldots$$
$$\leftarrow (W^{(0)}(\alpha), \mathscr{A}^{(0)}(\alpha), E_{\Lambda(\alpha)}(0), A_{\Lambda(\alpha)}(0)).$$

(2) a sequence of permissible transformations over

$$(W^{(-r)}(\alpha), w(\mathscr{A}^{(-r)}(\alpha)), E_{\Lambda(-r)}) \quad \text{(Def. 1.8).}$$

Since $\operatorname{ord}(w(\mathscr{A}^{(-r)}(\alpha)))=1$ (Remark 1.17.7), it can be interpreted as a sequence of w-permissible transformations (*see* Remark 1.17.8).

$$(W^{(-r)}(\alpha), w(\mathscr{A}^{(-r)}(\alpha)), E_{\Lambda^{(-r)}(\alpha)}, \bar{A}_{\Lambda^{(-r)}(\alpha)}) \leftarrow \ldots$$
$$\leftarrow (W^{(0)}(\alpha), w(\mathscr{A}^{(0)}(\alpha)), E_{\Lambda^{(0)}(\alpha)}, \bar{A}_{\Lambda^{(0)}(\alpha)}).$$

Let H_1 denote the final strict transform of $H(\subset W^{(-r)}(\alpha))$ at $W^{(0)}(\alpha)$, and let $E_{\Lambda^{(0)}(\alpha)} = E^+_{\Lambda^{(0)}(\alpha)} \cup E^-_{\Lambda^{(0)}(\alpha)}$ be as in 2.3.2.

Now we consider two cases

2.6 (*a*) Case $T(2)(y)=1$. In this case, $y \notin R(\tau(w(\mathscr{A}^{(-r)})))$. Since $R(\tau(w(\mathscr{A}^{(-r)})))$ is a connected component of $w\text{-Sing}(\mathscr{A}^{-r}) = \operatorname{Sing}(w(\mathscr{A}^{-r}))$ (Proposition 1.16.4), we may assume after shrinking that $R(\tau(w(\mathscr{A}^{(-r)})) = \varnothing$ (at $W^{(-r)}(\alpha)$).

Now one can check at $W^{(0)}(\alpha)$ that $\bar{E}_\lambda = E_\lambda \cap H_1$ is empty or a smooth hypersurface for $E_\lambda \in E^+_{\Lambda^0(\alpha)}$, and $\bar{E}_\lambda = \varnothing$ if $E_\lambda \in E^-_{\Lambda^0(\alpha)}$ [at least locally at $w\text{-Sing}(\chi)$].

Let $\bar{E}_\Lambda = \{\bar{E}_\lambda \mid \lambda \in \Lambda\}$, then the inclusion $H \subsetneq W^{(0)}(\alpha)$ and $(H_1, \bar{E}_\Lambda)$, $(W^{(0)}(\alpha), E_\Lambda)$ satisfy the condition 1.11.1.

On the other hand H_1 has maximal contact with $w(\mathscr{A}^{(0)}(\alpha))$ at $W^{(0)}(\alpha)$. One can check that the conditions are given for Theorem 1.15, (*b*) to hold, so that there is an idealistic situation (Def. 1.8) $(H_1, \mathscr{B}, \bar{E}_\Lambda)$ such that $i : H_1 \subsetneq W^{(0)}(\alpha)$ is a strong immersion from $(H_1, \mathscr{B}, \bar{E}_\Lambda)$ to $(W^{(0)}(\alpha), w(\mathscr{A}^{(0)}(\alpha)), E_\Lambda)$ (Def. 1.11).

$\mathscr{B}$ might have order bigger than $1 = \operatorname{ord}(w(\mathscr{A}^0(\alpha)))$ (Remark 1.17.7). We define the weighted idealistic situation $(H_1, \mathscr{B}, \bar{E}_\Lambda, \bar{A}_\Lambda)$ where $\bar{A}_\Lambda = \{\alpha(\lambda) \mid \lambda \in \Lambda\}$ such that $\alpha(\lambda)(x)=0$, $\forall x \in \bar{E}_\lambda (\forall \bar{E}_\lambda \in \bar{E}_\Lambda)$.

Arguing as before at each point y, we construct a restriction of $w(\chi)$ to an $m-1$ dimensional idealistic space $\bar{\chi}^{(0)}$ (Def. 1.18.3). Theorem 1.12 asserts that $\bar{\chi}^{(0)}$ is quasi-compact (Def. 1.19.1). And $\operatorname{Sing}\bar{\chi}^{(0)} = (\operatorname{Sing} w(\chi^{(0)})) - R(\tau)(w(\chi^0))$ which consists of "connected components" of $\operatorname{Sing} w(\chi^{(0)})$ (Remark 1.19.2).

In this case we define the restriction of $w(\chi^0)$ to be $\bar{\chi}^{(0)}$.

2.6 (*b*) Case $T(2)(y)=0$ *i.e.* $y \in R(\tau)(w(\chi^{(-r)}))$.

After a convenient restriction we may assume that $R(\tau)(w(\chi^{(-r)})) = \operatorname{Sing}(w(\chi^{(-r)}))$ (Def. 1.19.3).

Let α and $H \subset W^{(-r)}(\alpha)$ be as before. Since H has maximal contact with $w(\mathscr{A}^{(-r)}(\alpha))$, apply Theorem 1.15 case (*b*) if possible (*see* Remark I below) and let $(H, \mathscr{B}, E_\varnothing, A_\varnothing)$ induce a strong immersion with $(W^{(-r)}(\alpha), w(\mathscr{A}^{(-r)}(\alpha)), E_\varnothing, A_\varnothing)$ (we do not assume that $E_\Lambda^{(-r)} = \varnothing$ at $\chi^{(-r)}(\alpha)$).

One can check that, by this procedure an $m-1$ dimensional idealistic space $\bar{\chi}^{(-r)}$ has been defined such that:

(i) $\bar{\chi}^{(-r)}$ is quasi-compact

(ii) $\operatorname{Sing}\bar{\chi}^{(-r)} = \operatorname{Sing} w(\chi^{(-r)} = w\text{-}\operatorname{Sing}(\chi^{(-r)})$

(iii) The permissible sequence $\beta : \chi^{(-r)} \leftarrow \ldots \leftarrow \chi$ induces a permissible sequence

$$\bar{\beta} : \quad \bar{\chi}^{(-r)} \leftarrow \ldots \leftarrow \bar{\chi}^{(0)}.$$

(iv) $\operatorname{Sing}\bar{\chi}^{(j)} = \operatorname{Sing} w(\chi^{(j)})$, $j = -r, \ldots, 0$.

(v) $w(\chi^0)$ is restrictive to $\bar{\chi}^0$ (Def. 1.18.3).

In this case we define the restriction of $w(\chi^0)$ to be $\bar{\chi}^0$ (with birth $\bar{\chi}^{(-r)}$).

Remark 2.6.1. — Let $\bar{\chi}^0$ be the restriction of $w(\chi^0)$ as in 2.6 (*a*) or 2.6 (*b*), then:

(i) $\operatorname{Sing}(\bar{\chi}^0) = w\text{-}\operatorname{Sing}(\chi)$ (disregarding eventually connected components of the second term).

(ii) the function $T : w\text{-}\operatorname{Sing}(\chi) \to \mathbb{Z}^3 \times \Lambda^m$; is constant along $\operatorname{Sing}(\bar{\chi}^{(0)})$

Remark I. — The procedure of 2.6 is not defined at x if and only if

(i) $\tau(\chi^{(-r)}) = 1$

(ii) $T(2)(y)\,(= T(2)(x)) = 0$

since, in that case and only in that case Theorem 1.15 *b*) does not apply.

2.7

2.7.1. Before going into the development of this section we sketch the strategy to follow in a simplified form.

So we start with a pair (J, b) and $E = \{E_1, \ldots, E_n\}$ hypersurfaces with only normal crossings in a regular scheme W of dimension m (as in § 1). Recall that if χ is the induced idealistic space, then permissible transformations over χ correspond to w-permissible transformations over (J, b), E (Def. 1.18.2). Say

$$\begin{array}{ccccc} \chi & \chi_1 & \cdots\cdots & \chi_r \\ (J, b) \leftarrow & (J_1, b) & \ldots \leftarrow & (J_r, b) \\ E & E_1 & & E_r \end{array}$$

where: (i) (J_i, b) is the transform of (J_{i-1}, b) (Def. 1.3).

(ii) $J_i = MJ^{(i)}$, M a monomial (Def. 1.17.1).

(iii) $w\text{-ord}(J) \geqq \ldots \geqq w\text{-ord}(J_r)$ (Remark 1.17.6 (iii)).

(iv) $w\text{-}\operatorname{Sing}\chi_i = \operatorname{Sing}(w - \chi_i) = \operatorname{Sing} w(J_i, b)$ [$w(J_i, b)$ as in 1.17.6].

The notion of birth of χ_r (and of $E_r = E_r^- \cup E_r^+$) of Def. 2.1 corresponding to the smallest index k for which w-ord$((J_k, b)) = w$-ord$((J_r, b))$.

If the weighted order of (J_r, b) is zero *i. e.* if J_r is locally a monomial, the resolution of (J_r, b) will follow easily. So assume that w-order$(J_r, b) > 0$ (as in 2.3.2).

For further simplification we restrict our attention to the functions on w-Sing χ_r defined by T(1) [constantly equal to w-order(J_r, b)] and T(3), $T(3)(x) = n(x)$ (as in 2.3.2).

These two functions turn out to be substantial for this procedure of resolution.

In 2.7.2 we study the maximal value of this function (in a lexicographic sense) along w-Sing(χ_r), say $\operatorname{Max} T_r = (\omega, n)$. We set

$$\underline{\operatorname{Max}}\, T_r = \{ x \in w\text{-Sing}(\chi_r) / T(x) = (\omega, n) \}.$$

Fix $x \in \underline{\operatorname{Max}}(T_r)$, then $n(x) = n$, and say $\{E_1, \ldots, E_n\} = \{E_i \in E_r^- / x \in E_i\}$, E_i locally defined by $x_i = 0$.

Then $\underline{\operatorname{Max}}\, T_r$ is the singular locus of a new pair of order 1 (Def. 1.2), say $T_r(J_r, b)$, where:

$$T_r(J_r, b) \sim w(J_r, b) \cap (\langle x_1 \rangle, 1) \cap \ldots \cap (\langle x_n \rangle, 1)$$

or equivalently, if $w(J_r, b) = (\mathscr{A}, d)$

$$T_r(J_r, b) \sim (\mathscr{A} + (x_1^d) + \ldots + (x_n^d), d)$$

[$\sim$: isomorphic in the sense of idealistic situations (Def. 1.9)].

If $n = 0$, in 2.6 we showed that the problem of resolution of $\omega(J_r, b)$ (the problem of "lowering" the weighted order), is a problem of resolution of an idealistic space of dimension smaller then m.

n is to be thought of as an obstruction in this sense.

The main results in this section are: [*see* conditions (1), (2), (3) and (4) of 2.7.3 for precise statements].

(*a*) The lowering of n [or of ω = weighted order of (J_r, b)], is equivalent to the resolution of the pair $T_r(J_r, b)$.

(*b*) The problem of resolution of $T_r(J_r, b)$ is a problem of resolution of idealistic spaces of dimension smaller then m.

Of course the number n, or any $n(x)$ is bounded by m. There cannot be more then m-hypersurfaces with normal crossings at $x \in W$.

2.7.2. Consider a sequence

$$\beta : \chi^{(-r)} \overset{\pi_{-r}}{\leftarrow} \chi^{(-r+1)} \leftarrow \ldots \chi^{(-1)} \overset{\pi_{-1}}{\leftarrow} \chi^0 = \chi$$

of permissible transformations over an m-dimensional idealistic space $\chi^{(-r)} : I \to C(m, \Lambda)$ such that

$$w\text{-ord}(\chi^{(-r)}) = w\text{-ord}(\chi) > 0.$$

We assume, inductively on r, that each π_j is a permissible transformation with center C_j, uniquely determined by an upper semicontinuous function on the "closed" sets w-Sing(χ^j).

In 2.3.2 we have constructed a function T on each w-Sing$(\chi^{(j)})$ which is upper semicontinuous. Now define for each such T : Max$(T(\chi^{(j)}))$ or simply.

Max(T) = maximum value of T at w-Sing$(\chi^{(j)})$, and

$\underline{\text{Max}}(T) = \{x \in w\text{-Sing}(\chi^{(j)}) \mid T(x) = \text{Max}\, T\}$

(*see* Remark 2.3.4).

Assume that the following conditions hold:

(i) $C_j \subset \underline{\text{Max}}\, T \subset w\text{-Sing}\, \chi^{(j)}$

(ii) for any $x \in w\text{-Sing}(\chi^{(j+1)})$; $T(\pi_j(x)) \geqq T(x)$.

Definition 2.7.2. — When these conditions hold then for each $x \in \underline{\text{Max}}(T) \subset w$-Sing$(\chi)$ we define:

1. m-Sing$(x) = T(x)\ (= \text{Max}(T))$.

2. the *m-birth* of x as the image y of x by the natural map $\pi : \chi \to \chi^{(-j)}$ where $-j$ is the smallest index for which $T(x) = \text{Max}(T(\chi^{(-j)})$.

Remark. – Given x as before, let y be the m-Sing birth of x and z the birth of x (Def. 2.5). Then z is also the birth of y.

2.7.3. In 2.6 we studied a sequence β (as before) such that w-ord$(\chi^{(-r)}) = w$-ord$(\chi) > 0$ and the additional hypothesis that $T(3)(x) = 0$, $\forall x \in w$-Sing(χ). In this section we consider the case that Max $T = (d_1, d_2, d_3, \infty)$ ($T : w\text{-Sing}(\chi) \to \mathbb{Z}^3 \times \Lambda^m$) where $d_3 > 0$ and we want to construct now a preparation procedure (Def. 2.4).

Let $-j$ and y be as before and $F^{(-j)} = \underline{\text{Max}}(T) \subset w\text{-Sing}(\chi^{(-j)})$, let z denote the birth of y and let $\alpha \in I$ be such that $z \in w\text{-Sing}(\mathscr{A}^{(-r)}(\alpha)) \subset W^{(-r)}(\alpha)$ where $\chi^{(-r)}(\alpha) = (W^{(-r)}(\alpha), \mathscr{A}^{(-r)}(\alpha), E_{\Lambda^{(-r)}(\alpha)}, A_{\Lambda^{(-r)}(\alpha)})$.

Now w-Sing$(\mathscr{A}^{(-r)}(\alpha)) = \text{Sing}(w(\mathscr{A}^{(-r)}(\alpha))$ and ord$(w(\mathscr{A}^{(-r)}(\alpha))) = 1$ (Remark 1.17.1). Again by theorem 1.16.1 there is a smooth hypersurface $H^{(-r)} \subset W^{(-r)}(\alpha)$ such that $z \in H^{(-r)}$ and $H^{(-r)}$ has maximal contact with $w(\mathscr{A}^{(-r)}(\alpha))$ [after shrinking $W^{(-r)}(\alpha)$].

If $H^{(-j)}$ denotes the strict transform of $H^{(-r)}$ at $W^{(-j)}(\alpha)$ by the maps induced over $W^{(-r)}(\alpha)$, then $y \in H^{(-j)}$ and $H^{(-j)}$ has maximal contact with $w(\mathscr{A}^{(-j)}(\alpha))$ (which is the transform of the idealistic exponent $w(\mathscr{A}^{(-r)}(\alpha)))$ at $W^{(-j)}(\alpha)$ [Remark 1.17.6 (iii)]. Recall (as in 2.6) that $H^{(-j)}$ has normal crossings with $E^+_{\Lambda^{(j)}(\alpha)}$ (2.1). If $w(\mathscr{A}^{(-j)}(\alpha))$ *is defined locally at* y by a pair (J, b), then consider the idealistic exponent

$$K = ((J + \sum_{y \in E_s \in \Gamma} P_s^b, b)), \qquad \Gamma = (E_{\Lambda^{(j)}})^-$$

(2.1) where $P_s \subset O_{W^{(j)}(\alpha)}$ is the sheaf of ideals $O(-E_s)$.

One can check that:

(*a*) Sing $K = F^{(-j)}$ (locally at y).

(*b*) K is well defined independently of the election of (J, b).

Remark. — Assume that $T(2)(y)=(=T(2)(z))=0$ then

$$(J+\sum_{y\in E_s\in\Gamma} P_s^b,\ b)\sim(J+\sum_{y\in E_t\in\Gamma'} P_t^b,\ b)\quad \text{(Def. 1.1)}$$

where $\Gamma'=\{E_t\in(E_{\Lambda^{(j)}})^-\mid w\text{-Sing}(\chi^{(-j)})\notin E_t\}$ (locally at y).

Since $H^{(j)}$ has maximal contact with $w(\mathscr{A}^{(-j)}(\alpha))=((J, b))$, then it also has maximal contact with K.

Now consider at $W^{(-j)}(\alpha)$ the weighted idealistic situation $(W^{(-j)}(\alpha), K, (E_{\Lambda^{(-j)}(\alpha)})^+, \bar{A}_{\Lambda^{(-j)}(\alpha)})$ where $(E_{\Lambda^{(-j)}(\alpha)})^+$ is as before and $\bar{A}_{\Lambda^{(-j)}(\alpha)}$ consists of functions $\alpha(\lambda): E_\lambda\to\mathbb{Q}$, for each $E_\lambda\in(E_{\Lambda^{(-j)}(\alpha)})^+$ where $\alpha(\lambda)(x)=0$, $\forall x\in E_\lambda$.

Now for each $E_\lambda\in(E_{\Lambda^{(j)}(\alpha)})^+$ let $\bar{E}_\lambda=E_\lambda\cap H^{(-j)}$ and define $E_{\bar{\Lambda}}=\{\bar{E}_\lambda \text{ (as before)}\}$ and $A_{\bar{\Lambda}}=\{\alpha(\lambda): \bar{E}_\lambda\to\mathbb{Q}$ ($\bar{E}_\lambda$ as before) such that $\alpha(\lambda)(x)=0$, $\forall x\in\bar{E}_\lambda\}$.

$E_{\bar{\Lambda}}$ consists of hypersurfaces (at $H^{(-j)}$) with only normal crossings.

We claim that the conditions of Theorem 1.15 (*b*) are given (*see* Remark II below), so that there is an idealistic exponent $\mathscr{B}$ at $H^{(-j)}$ and a strong immersion

$$(H^{(-j)}, \mathscr{B}, E_{\bar{\Lambda}})\subsetneq(W^{(-j)}(\alpha), K, (E_{\Lambda^{(-j)}(\alpha)})^+).$$

Arguing in the same way for all points $x\in\underline{\mathrm{Max}}(T)\subset\chi^0=\chi$ and all election of hypersurfaces $H^{(-r)}$, we construct an $m-1$ dimensional idealistic space $\bar{\chi}^{(-j)}$ which is quasi-compact and satisfies the following conditions:

(1) $\mathrm{Sing}\,\bar{\chi}^{(-j)}=\underline{\mathrm{Max}}(T)\subset w-\mathrm{Sing}(\chi^{(-j)})$.

(2) The permissible sequence

$$\chi^{(-j)}\overset{\pi_{-j}}{\leftarrow}\chi^{(-j+1)}\leftarrow\ldots\overset{\pi_{-1}}{\leftarrow}\chi^{(0)}=\chi$$

induces a permissible sequence

$$(A):\quad \bar{\chi}^{(-j)}\leftarrow\bar{\chi}^{(-j+1)}\leftarrow\ldots\leftarrow\bar{\chi}^{(0)}$$

over $\bar{\chi}^{(-j)}$ such that $\mathrm{Sing}(\bar{\chi}^{(l)})=\underline{\mathrm{Max}}(T)\subset w\text{-Sing}(\chi^{(l)})$ for all $l=-j, -j+1, \ldots, 0$.

(3) If $\bar{\chi}^{(-j)}\leftarrow\bar{\chi}^{(-j+1)}\leftarrow\ldots\leftarrow\bar{\chi}^{(0)}\leftarrow\ldots\bar{\chi}^{(k)}$ is a permissible sequence [extending that of (2)] then it induces a permissible sequence

$$(\chi^{(-r)}\ldots\leftarrow)\,\chi^{(-j)}\leftarrow\ldots\leftarrow\chi^{(0)}\leftarrow\chi^{(1)}\leftarrow\ldots\leftarrow\chi^{(k)}$$

at permissible centers $C_l(-r\leqq l\leqq k)$ such that (i) and (ii) of 2.7 hold. Moreover $\mathrm{Sing}\,\bar{\chi}^{(l)}=\underline{\mathrm{Max}}(T)\subset w\text{-Sing}(\chi^{(l)})$ $0\leqq l\leqq k$ and

$$\mathrm{Max}(T: w\text{-Sing}(\chi)\to\mathbb{Z}^3\times\Lambda^m)>\mathrm{Max}(T: w\text{-Sing}(\chi^k)\to\mathbb{Z}^3\times\Lambda^m)$$

if and only if $\mathrm{Sing}\,\bar{\chi}^{(k)}=\varnothing$.

(4) Conversely, if $\chi^{(-r)}\leftarrow\ldots\leftarrow\chi^{(0)}\leftarrow\chi^{(1)}\leftarrow\ldots\chi^{(k)}$ is an extension of $\chi^{(-r)}\leftarrow\ldots\leftarrow\chi^0=\chi$ by permissible transformations at centers

$$C_j\subset\underline{\mathrm{Max}}\,T\subset w\text{-Sing}(\chi^{(j)}),\qquad 0\leqq j\leqq k$$

such that (i) and (ii) of 2.7 hold, and if

$$\operatorname{Max}(\mathrm{T} : w\text{-Sing}(\chi^{(k)}) \to \mathbb{Z}^3 \times \Lambda^m) = \operatorname{Max}(\mathrm{T} : w\text{-Sing}(\chi) \to \mathbb{Z}^3 \times \Lambda^m)$$

then it induces a sequence of permissible transformations

$$\bar{\chi}^{(-j)} \leftarrow \ldots \leftarrow \bar{\chi}^{(0)} \leftarrow \bar{\chi}^{(1)} \leftarrow \ldots \leftarrow \bar{\chi}^{(k)}$$

and $\operatorname{Sing}(\bar{\chi}^{(l)}) = \underline{\operatorname{Max}}\, \mathrm{T} \subset w\text{-Sing}\, \chi^{(l)}\ l=0, \ldots, k$.

Remark II. — The construction of the restricted situation at y would not be possible if and only if:

(1) $\tau(\chi^{(-r)}) = 1$

(2) $\mathrm{T}(2)(y) = 0$

(3) $\mathrm{T}(3)(y) = 0$

(*see* Remark I) but we assumed in the construction of 1.7.2 that $\mathrm{T}(3)(y) \neq 0$.

2.8. Now let $\mathrm{D}_m = \mathrm{Z}^3 \times \Lambda^m$, $\mathrm{J}_m = \mathrm{D}_m \times \mathrm{D}_{m-1} \times \ldots \times \mathrm{D}_1$ and suppose that the theorem of constructive resolutions (2.2) holds in dimension smaller then m.

We assume that the sequence (A) is a constructive sequence, *i.e.* that there is a resolution

$$\chi^{(-j)} \leftarrow \chi^{(-j+1)} \leftarrow \ldots \leftarrow \chi^{(0)} \leftarrow \chi^{(1)} \leftarrow \ldots \leftarrow \chi^{(l)},\ \chi^{(0)} = \chi$$

together with functions $\psi_{m-1}^{(k)} : \operatorname{Sing} \bar{\chi}^{(k)} \to \mathrm{J}_{m-1}$, $-j \leqq k < l$ satisfying the conditions at 2.2 (*see* observation 2.3.3). Recall that $\operatorname{Sing}(\bar{\chi}^{(k)}) = \underline{\operatorname{Max}}(\mathrm{T}) \subset w\text{-Sing}(\chi^{(k)})$ where now:

$$\chi^{(-k)} \leftarrow \ldots \leftarrow \chi^{(0)} \leftarrow \chi^{(1)} \leftarrow \ldots \leftarrow \chi^{(l)},\ \chi^{(0)} = \chi$$

is the permissible sequence constructed with these centers.

Moreover this maximum value of T along $w\text{-Sing}(\chi^{(-s)})$ is the same, say c, for all $-j \leqq s \leqq l$.

So if c_1 is the maximum of T along $w\text{-Sing}(\chi^{(l)})$ (assuming that the birth of $\chi^{(l)}$ is still $\chi^{(-k)}$), then $c_1 < c$. But this simply means that

$$\operatorname{Max}\{\mathrm{T}(3)(x) \mid x \in w\text{-Sing}(\chi^{(l)})\} < \operatorname{Max}\{\mathrm{T}(3)(x) \mid x \in w\text{-Sing}(\chi^{(-k)})\}$$

But $\mathrm{T}(3)(x) \leqq m = \dim \chi^{(l)}$ (Def. 1.18.2). So repeating this argument we are left in the situation at which either $w\text{-ord}(\chi^{(l)}) < w\text{-ord}(\chi^{(-k)})$ or $w\text{-ord}(\chi^{(l)}) = w\text{-ord}(\chi^{(-k)})$ and $\mathrm{T}(3)(x) = 0$, $\forall x \in w\text{-Sing}(\chi^{(l)})$. In this way we have constructed a preparation procedure (Def. 2.4) and now the inductive procedure of 2.6 can be applied.

In either case at $\mathrm{F}^{(s)} = \{x \in w\text{-Sing}(\chi^{(s)}) \mid \mathrm{F}(x) \text{ is maximum}\} = \underline{\operatorname{Max}}(\mathrm{T})$ define $\psi_m^{(k)}(x) = (\mathrm{T}(x), \psi_{m-1}^{(k)}(x))$; this defines a map:

$$\psi_m^{(k)} : \mathrm{F}^{(k)} \to \mathrm{D}_m \times \mathrm{J}_{m-1} (= \mathrm{J}_m)$$

We are still left with the case (within w-ord$(\chi)>0$) where:

$$T(2)(x)=0, \qquad \forall x \in w\text{-Sing}(\chi)$$
$$T(3)(x)=0, \qquad \forall x \in w\text{-Sing}(\chi)$$

and $\tau(w(\chi))=1$.

In this case and only in this case, the procedure introduced before are of no use. But then w-Sing χ is regular at each point and w-Sing χ itself is a center of a permissible transformation and such transformation defines a resolution of $w(\chi)$. On the other hand the function $T: w\text{-Sing}(\chi) \to D_m$ is constant. So we define

$$\psi_m(x)=(T(x), \infty) \in D_m \times J_{m-1}=J_m$$

Finally, if w-ord$(\chi)=0$ define

$$\psi_m : \operatorname{Sing}\chi \to J_m$$

by

$$\psi_m(x)=(T(x), \infty)$$

(Remark 2.3.1 asserts that a resolution of χ can be "constructed").

2.9. With the assumption of constructive resolutions of singularities for idealistic spaces of dimension smaller then m, we have produced in 2.8, for any m-dimensional idealistic space χ a unique resolution:

$$\text{(A)} \qquad \begin{array}{l} \chi_0 \leftarrow \chi_1 \ldots \leftarrow \chi_r \overset{\Pi_r}{\leftarrow} \leftarrow \chi_n \\ Y_0 \quad Y_1 \qquad Y_r \end{array}$$

where each $\chi_r \overset{\Pi_r}{\leftarrow} \chi_{r+1}$ is a permissible transformation with center $Y_r \subset \operatorname{Sing}\chi_r$.

DEFINITION 2.9.1. — Given a point $x \in \operatorname{Sing}\chi_r$, if $x \notin Y_r$ we identified it with a point $x \in \operatorname{Sing}\chi_{r+1}$ in such a way that Π_r: $\operatorname{Sing}\chi_{r+1} \to \operatorname{Sing}\chi_r$ is locally an isomorphism (at x). Since (A) is finite there is a well defined number $r' \geqq r$ which is maximal with the condition that $\Pi_r^{r'}$: $\operatorname{Sing}\chi_{r'} \to \operatorname{Sing}\chi_r$ (the composition of all intermediate maps) is an isomorphism locally at x. We say that "$x \in \operatorname{Sing}\chi_{r'}$". In this case $x \in Y_{r'} \subset \operatorname{Sing}(\chi_{r'})$, because of the maximality of r', r' is called the *level of* x.

DEFINITION 2.9.2. — Given an upper semicontinuous function h: $F \to (D, \leqq)$, if $(D, \leqq)$ is totally ordered then set $\operatorname{Max} h=\{\text{maximal value of } h\}$ (a unique element) and $\underline{\operatorname{Max}}\, h=\{x \mid h(x)=\operatorname{Max}(h)\}$. If D is not totally ordered, then $\operatorname{Max} h$ might consist of more than one element. We will assume moreover that for each $x \in F$, there is a totally ordered subset $(D_x, <) \subset (D, <)$ and that $\operatorname{Im} g(h) \subset D_x$ locally at x.

Examples of these maps are given by

T: $\operatorname{Sing} w(\chi) \to D$ as pointed out in 2.3.1 and 2.3.2.

Now $\underline{\text{Max}}\, h$ becomes a disjoint union of closed sets

$$\underline{\text{Max}}\, h = \bigcup_{d \in \text{Max}\, h} \underline{\text{Max}}\,(h)(d), \qquad \underline{\text{Max}}\,(h)(d) = \{x \mid h(x) = d\}$$

LEMMA 2.9.3. — *Suppose we are given the following data:*

$$\text{(B)} \qquad \begin{array}{ccccccc} \chi & \overset{\Pi_0}{\leftarrow} & \chi_1 & \overset{\pi_1}{\leftarrow} & \chi_j & \overset{\Pi_j}{\leftarrow} \leftarrow \chi_n \\ Y_0 \subset F_0 & & Y_1 \subset F_1 & & Y_j \subset F_j & \end{array}$$

and upper semicontinuous functions h_r: $F_r \to (D, \leqq)$ such that:

(i) the data (B) is a resolution of χ.

(ii) $F_r \subset \text{Sing}\, \chi_r$ is closed, Y_r is the center of Π_r and $Y_r \subset \underline{\text{Max}}\,(h_r)$.

(iii) if $x \in F_{r+1}$ and $\Pi(x) \in F_r$ then $h_{r+1}(x) \leqq h_r(\Pi_r(x))$ and the equality holds if moreover $\Pi(x) \notin Y_r$.

(iv) ST $(F_r) \subset F_{r+1}$ [ST(F_r) strict transform of F_r], (ST$(F_r) = \varnothing$ if $Y_r = F_r$).

(v) If $x \in Y_s$ $(s > r)$ and $\Pi_r^s(x) \in Y_r$ than $h_s(x) \leqq h_r(\Pi_r^s(x)$.

(vi) If $s > r$, $\forall x \in F_s \exists d \in \text{Max}\, h_r$ such that $h_s(x) \leqq d$ and if equality holds then $\Pi_r^s(x) \in \underline{\text{Max}}\, h_r$ (Π_r^s: the composition of all intermediate maps).

Define now H_r: $\text{Sing}\, \chi_r \to (D, \leqq)$ as follows: given $x \in \text{Sing}\, \chi_r$ let r' be the level of x (Def. 2.9.1) then $x \in Y_{r'}$, and we define $H_r(x) = h_{r'}(x)$. We claim that

(*a*) If $x \in F_r$, $H_r(x) = h_r(x)$ *i. e.* H_r extends h_r.

(*b*) $H_r(x) \leqq H_{r-1}(\Pi(x))$ and equality holds if $\Pi(x) \notin Y_{r-1}$.

(*c*) H_r is upper semicontinuous, $\text{Max}\, H_r = \text{Max}\, h_r$ and $\underline{\text{Max}}\, H_r = \underline{\text{Max}}\, h_r$.

Remark 2.9.3.1. — In the conditions of (vi), if $h_s(x) = d$ then $x \in \underline{\text{Max}}\, h_s$.

Proof (of the Lemma). — (*a*) *Let* $x \in F_r$ and r' be the level of x. We must prove that $h_r(x) = h_{r'}(x)$, this follows from (iv) and (iii).

(*b*) If $\Pi(x) \notin Y_{r-1}$, then level of x and $\Pi(x)$ concide, so $H_{r-1}(\Pi(x)) = H_r(x)$. If $\Pi(x) \in Y_{r-1}$ then the level of $\Pi(x)$ is $r-1$ and $H_{r-1}(\Pi(x)) = h_{r-1}(\Pi(x))$. Let r' be the level of x, then $x \in Y_{r'}$ and clearly $\Pi_{r-1}^{r'}(x) = \Pi(x)$ so

$$H_r(x) = h_{r'}(x) \leqq h_{r-1}(\Pi(x)) = H_{r-1}(\Pi(x))$$

[inequality due to (v)].

(*c*) Given $d \in D$, we define

$$U = \{x \in \text{Sing}\, \chi_r / H_r(x) \geqq d\}$$

$$V = \bigcup_{(s, d') \in \Gamma} \Pi_r^s(F(s, d'))$$

$$\Gamma = \{(s, d')/d' \in \text{Max}\,(h_s)\, d' \geqq d \quad \text{and} \quad s \geqq r\},$$

$$F(s, d) = \underline{\text{Max}}\,(h_s)(d') = \{x \in \underline{\text{Max}}\,(h_s)/h_s(x) = d'\}.$$

We claim that $U=V$. In which case, since each Π_r^s is proper and the $F(s, d')$ are closed, U is a finite union of closed sets.

Fix $x \in U$, $H_r(x)=d' \geqq d$ and let r' be the level of x. Then $x \in Y_{r'} (\subseteq \underline{\operatorname{Max}} h_{r'})$ so $d' \in \operatorname{Max} h_{r'})$ and $d' \geqq d$ *i. e.* $(r', d') \in \Gamma$, so $x \in \Pi_r^{r'}(F(r', d'))$ *i. e.* $x \in V$.

If $x \in V$ there is $y \in \underline{\operatorname{Max}}(h_s)(d')((s, d') \in \Gamma)$ such that $\Pi_r^s(y)=x$, so $h_s(y)=d' \in \operatorname{Max}(h_s)$ and $d' \geqq d$.

Let s' be the level of y and r' the level of x. Clearly $s' \geqq r'$, $\Pi_{r'}^{s'}(y)=x \in Y_{r'}$ and $y \in Y_{s'}$, so

$$H_r(x)=h_{r'}(x) \geqq h_s(y)=h_{s'}(y)=d' \geqq d$$

[inequelity do to (v)] *i. e.* $x \in U$.

Let us show that $\operatorname{Max} h_r = \operatorname{Max} H_r$. First we prove that: $\forall d \in \operatorname{Max} H_r$, $\exists d' \in \operatorname{Max} h_r$ such that $d \leqq d'$. In fact if $H_r(x)=d$ for some point $x \in \operatorname{Sing} \chi_r$ of level r', then $x \in Y_{r'} \subset F_{r'}$ and $h_{r'}(x)=d$. By (vi) there is $d' \in \operatorname{Max}(h_r)$ such that $d \leqq d'$. Since (a) is proved it follows that $\operatorname{Max} h_r = \operatorname{Max} H_r$. Again because of (a), $\underline{\operatorname{Max}} h_r \subseteq \underline{\operatorname{Max}} H_r$ and the equality is clear from (vi).

Remark 2.9.4. — Suppose that the sets F_r are replaced by $F^{(r)}$ satisfying:

(a) $\underline{\operatorname{Max}}(h_r) \subset F^{(r)} \subset F_r$ and $F^{(r)}$ is closed

(b) Condition (iv) of Lemma 2.9.3.

and (c) h'_r: $F^{(r)} \to D$ are defined by restricting h_r to $F^{(r)}$.
With this conditions we assert that:

(1) the statement of the Lemma still holds.

(2) If H'_r is defined as in the Lemma then $H'_r = H_r$.

Proof of (1) is straightforwards [*see* Remark 2.9.3.1 for (vi)] and (2) is due to the fact that the construction of H_r depends only on $h_s|_{Y_s}$, $\forall s \geqq r$, and $Y_s \subset \underline{\operatorname{Max}} h_s \subset F^{(s)}$.

PROPOSITION 2.9.5. — *Given the resolution* (A) *of* 2.9, *let* F_r *be defined as:*

(A) $F_r = \operatorname{Sing} w(\chi_r)$ *if* w-ord$(\chi_r) > 0$.

(B) $F_r = \operatorname{Sing} \chi_r$ *if* w-ord$(\chi_r) = 0$

and set T_r: $F_r \to D$ *as in* 2.3.1 *and* 2.3.2, *then all the conditions of Lemma* 2.9.3 *are satisfied.*

Proof. — (i) and (ii) follow by construction.

(iv): If w-ord$(\chi_r) > 0$ and the strict transform of $F_r = \operatorname{Sing} \omega(\chi_r)$ is non-empty, then the w-ord$(\chi_{r+1}) = w$-ord(χ_r) and $w(\chi_{r+1})$ is the transform of $w(\chi_r)$ (2.7). Now (iv) is clear in this case.

If w-ord$(\chi_r)=0$ then $F_r = \operatorname{Sing} \chi_r$, w-ord $\chi_{r+1}=0$ and $F_{r+1} = \operatorname{Sing} \chi_{r+1}$, so also in this case (iv) is clear.

(iii) We prove it by considering different cases:

(a) w-ord$(\chi_{r+1}) < w$-ord(χ_r). In this case it is clear that w-ord$(\chi_r > 0$ and as discussed above [in the prove of (iv)], $F_r = \operatorname{Sing} w(\chi_r)$ must be Y_r, (iii) is now obvious from these remarks.

(*b*) $w\text{-ord}(\chi_{r+1}) = w\text{-ord}(\chi_r) = \omega > 0$. The first coordinate of T_r is constant alonog F_r (equal to ω) and the some holds at F_{r+1}. The second coordinate is $T(2)$, the good behavior of this function is given by Prop. 1.16.4 which states that $T(2)(x) = T(2)(\Pi(x))$, $\forall x \in \mathrm{Sing}(\chi_{r+1})$. So that we are left with proving (iii) by looking at the function $T(3)$, now the statement follows from the fact that E_{r+1}^- is the strict transform of E_r^- and by the construction of (A) in terms of T [condition (1) (2) (3) and (4) of 2.7.2].

(*c*) If $w\text{-ord}(\chi_{r+1}) = w\text{-ord}(\chi_r) = 0$ we refer to Remark 2.3.1.

(v) (*a*) $w\text{-ord}(\chi_s) < w\text{-ord}(\chi_r)$ there is nothing to prove. We must consider the cases.

(*b*) $w\text{-ord}(\chi_s) = w\text{-ord}(\chi_r) > 0$ and (*c*) $w\text{-ord}(\chi_s) = w\text{-ord}(\chi_s) = 0$ both undergo essentially the some proofs as those given above for (*b*) and (*c*) of (iii).

(vi): is clear from the construction of (A) in terms of T.

PROPOSITION 2.9.6. — *Let* (A), F_r, Y_r *be as in Prop.* 2.9.5, *if each* F_r *is replaced by* $F^{(r)} = \underline{\mathrm{Max}}\, T_r$, *then the conditions of Remark* 2.9.4 *hold.*

Proof. – the non trivial point is to show that condition (iv) of Lemma 2.9.3 still holds *i. e.* $ST(F'_r) \subset F'_{r+1}$.

If $w\text{-ord}(\chi_r) > 0$, there is an $n-1$ dimensional idealistic space $\bar{\chi}^{(l)}$ such that $\mathrm{Sing}(\bar{\chi}^{(l)}) = \underline{\mathrm{Max}}(T_r)$ $(= F^{(r)})$, and if $\mathrm{Max}(T_r) = d$ then the lowering of d is equivalent to the resolution of $\bar{\chi}^{(l)}$ [conditions (1), (2), (3) aond (4) of 2.7.2], so we look at $\chi_r \overset{\Pi}{\leftarrow} \chi_{r+1}$.

If $\mathrm{Max}\, T_{r+1} < d$, Y_r must be $\mathrm{Sing}\, \bar{\chi}^{(l)}$ $(= F_r)$ and there is nothing to prove. If $\mathrm{Max}\, T_{r+1} = d$ then $\underline{\mathrm{Max}}\, T_{r+1}$ is the singular locus of $\bar{\chi}^{l+1}$ which is the transform of $\bar{\chi}^{l}$ by a permissible map $\bar{\chi}^{l} \leftarrow \bar{\chi}^{l+1}$, but then the $ST(\mathrm{Sing}\, \bar{\chi}^{l}) \subset \mathrm{Sing}\, \bar{\chi}^{l+1}$ as was to be shown.

If $w\text{-ord}(\chi_r) = 0$ then $F_r^{(r)}$ is the center *i. e.* $F^{(r)} = Y_r$ and there is nothing to prove.

2.9.7. In 2.8 we defined at $F^{(s)} = \underline{\mathrm{Max}}\, T_s$ a function

$$\psi_m^s = F^{(s)} \to D = D_m \times J_m$$

in such a way that $p_1^r \circ \psi_m^s = T_s$ (p_1^r projection on D_m).

THEOREM 2.9.7. — *The data*

$$\text{(A)} \qquad \begin{array}{ccccccc} \chi_0 & \leftarrow & \chi_1 & \leftarrow & \chi_j & \leftarrow & \chi_n \\ Y_0 \subset F^{(0)} & & Y_1 \subset F^{(1)} & & Y_j \subset F^{(j)} & & \end{array}$$

together with the functions ψ_m^r: $F^{(r)} \to D$ *satisfies the conditions of Lemma* 2.9.3. *In particular there are, for each s, functions* ψ_m^s: $\mathrm{Sing}\, \chi_s \to D_m \times J_m$ *making of* (A) *a constructive resolution in the sense of* 2.2.

Proof. – After Prop 2.9.6, (i), (ii) and (iv) deserve no proof (vi) is clear from the construction of (A) [recall that $Y_s = \underline{\mathrm{Max}}\, \psi_m^s$, and for $s > r$, x and d as in (vi) then $h_s(x) < d$].

(iii) (*a*) If $w\text{-ord}(\chi_r) = 0$, then ψ_m^r is basically T_r and again this case is in Prop. 2.9.6.

(*b*) If $w\text{-ord}(\chi_r) > 0$ and $\mathrm{Max}\, T_r > \mathrm{Max}\, T_{r+1}$, then $Y_r = \underline{\mathrm{Max}}\, T_r (= F^{(r)})$ and the assertion is clear.

(*c*) If $\operatorname{Max} T_r = \operatorname{Max} T_{r+1}$, there is $\bar{\chi}^l$ (as in the proof of Prop 2.9.6) such that $F^{(r)} = \operatorname{Sing}(\bar{\chi}^l)$, $F^{(r+1)} = \operatorname{Sing}(\bar{\chi}^{l+1})$.

Now $T(x) = T(\Pi(x))$ so one must prove (iii) for ψ_{m-1} and now x and $\Pi(x)$ are singular points of an $m-1$ dimensional resolution.

But ψ_{m-1} is constructive and (iii) follows from (ii), of 2.2.

(v) Reduces immediatly to the case $T_s(x) = T_r(\Pi_r^s(x))$ and undergoes essentially the some argument of the proof of (*c*) given just above.

2.10

Remark 2.10.1. — Why T(2)?

As pointed out in 2.7, the role ot T(2) is not essential for our constructions *i. e.* we can define $T(2)(x) = 1$ whenever $T(1)(x) > 0$ without affecting the general strategy. However if we consider (J, 1), E, $J = \langle x, y \rangle \subset \mathbb{C}|x, y, z|$, $E = \{E_1\}$, $E_1 = \{z = 0\} \subset \mathbb{C}^3$, then one can check that the number of unnecessary quadratics transformations applied before solving the pair, will diminish if we do consider this function.

2.10.1. — At this point we give a punctual construction of the functions ψ_m defined at 2.8.

Let χ an idealistic space of dimension m, if w-ord $\chi = 0$ *i. e.* if χ is locally a monomial, ψ_m reduces to T (2.3.1).

We consider therefore the case w-ord $\chi > 0$. In order to simplify set (J, *b*) as in paragraph 1 and (J_r, b) aristing from $(J, b) \leftarrow (J_1, b) \ldots \leftarrow (J_r, b) \ldots \leftarrow (J_n, b)$ with the notations and assumptions of 2.7.1, where only the functions T(1) and T(3) where considered [*i. e.* $T(2)(x) = 1$ if $T(1)(x) > 0$].

So let (ω, n) be $\operatorname{Max} T_r$, and $k \leq r$ be the smallest number for which $\operatorname{Max} T_k = (\omega, n_0)$. Recall from 2.7.1 that $T_r(J_r, b)$ was an "$m-1$-dimension" idealistic pair such that $\underline{\operatorname{Max}}\, T_r = \operatorname{Sing} T_r(J_r, b)$ and that

$$(J_k, b) \overset{\Pi_k}{\leftarrow} \ldots \leftarrow (J_r, b)$$

induces a sequence of permissible maps:

$$T_k(J_k, b) \overset{\Pi_k}{\leftarrow} \ldots \leftarrow T_r(J_r, b),$$

each $T_i(J_i, b)$ being the transform of $T_{i-1}(J_{i-1}, b)$ (Def. 1.3), for $i > k$.

Given $x \in \operatorname{Sing}(J_p, b)$ we express $\psi_m^p(x)$ by three coordinates, the first two corresponding to T_p, the third to ψ_{m-1}^p. We begin by defining, inductively on p, sets $E_{x,p}^-$ as follows:

(i) if $\omega - \nu_x(J_p, b) < \omega - \nu_{\Pi(x)}(J_{p-1}, b)$ $(\Pi = \Pi_{p-1})$ (Def 1.17.1), or if $p = 0$:

$$E_{x,p}^- = \{E_i \in E_p / x \in E_i\}$$

(ii) if $\omega - \nu_x(J_p, b) = \omega - \nu_{\pi(x)}(J_{p-1}, b)$

$$E_{x,p}^- = \{ST(E_i) / E_i \in E_{p-1, \Pi(x)}^- \text{ and } x \in ST(E_i)\}$$

(as usual ST denotes the strict transform).

Now we claim that:

(*a*) $T_p(1)(x)=\omega-\nu_x(J_p, b)$

(*b*) $T_p(3)(x)=E^-_{p,x}$

(*c*) If $q(\leqq p)$ is the smallest index for which $T_q(\Pi^p_q(x))=T_p(x)$. Consider at a neighbourhood of $y=\Pi^p_q(x)$ the pair:

$$(\mathscr{A}, d)=w(J_{q,y}, b)\cap(x_1, 1)\cap(x_2, 1)\cap\ldots\cap(x_h, 1)$$

[notation as in 2.7.1, where $h=T_q(3)(y)$ and $x_i=0$ defines $E_i\in E^-_{q,y}$ locally at y]. Then the third coordinate is $\psi^t_{m-1}(x)$, $t=p-q$ and ψ^t_{m-1} arises from the constructive resolution of the $m-1$ dimensional pair $(\mathscr{A}, d)$.

Let r denote the level of $x\,(r\geqq p)$ (Def. 2.9.1) and recall the definition of $\psi^p_m(x)$ in terms of the level of x (2.9.3 and 2.9.7).

Point (*a*) is clear and (*b*) will follow by proving inductively on p, that:

(*d*) $E^-_{x,p}=\{E_i\in E^-_r/x\in E_i\}$.

In the case (i), either $p=0$ or the weighted order of (J_r, b) is smaller then that of (J_{p-1}, b) and (*d*) follows in this case from the definition of E^-_r in terms of the weighted orders of the pairs (2.1).

In the case (ii), if s is the level of $\Pi(x)$, clearly $s\leqq r$ and (with the identifications of Def. 2.9.1)

$$w\text{-ord}(J_s)=\omega-\nu_{\Pi(x)}(J_s, b)=\omega-\nu_x(J_r, b)=w\text{-ord}(J_r)$$

since $\Pi(x)\in Y_s\subset\underline{\text{Max}}\,\psi_m$ and $x\in Y_r\subset\underline{\text{Max}}\,\psi_m$. So (*d*) follows now from the relations between E^-_s and E^-_r given in 2.1.

Now that (*d*) is settled (for any p) we prove (*c*). So let $s(\geqq q)$ be the level of y and r as before that of x. Clearly $s\leqq r$. On the other hand $y\in Y_s\subset\underline{\text{Max}}\,T_s$ and $x\in Y_r\subset\underline{\text{Max}}\,T_r$ so:

$$\text{Max}\,T_s=T_s(y)=T_r(x)=\text{Max}\,T_r=(w, n_0).$$

In particular $k\leqq s$ (k defined as above).

Consider the composition of the intermediate maps: Π^s_k and the point $z=\Pi^s_k(y)$. If the level of z is the level of y, Π^s_k is the identity map locally at y and (*c*) follows from (*d*) and the construction of $T_k(J_k, b)$ (2.7.1).

If Π^s_k would not be an isomorphism at y, since $\Pi^s_q=\text{id}$, then $k<q$ contradicting the minimality of q.

So if x is considered as a point of $\text{Sing}(J_r, b)$, the point $\Pi^r_k(x)\in\text{Sing}(J_k, b)$ (which is the m-birth of x Def. 2.7.2) has the same level as y.

Suppose now that the function T_p is replaced by $T_p(1)$ and q by q_1 $(\leqq p)$: the smallest index for which $y_1=T_p(1)(\Pi^p_{q_1}(x))=T_p(1)(x)$. Then the same argument as above will show that the birth of $x\in\text{Sing}(J_r, b)$ (Def. 2.5) has the same level as y_1. Therefore in the construction of 2.7.3 the election of the hypersurface of maximal contact can be done locally at y_1.

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