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Contact CR-Warped Product Submanifolds of Kenmotsu Manifolds

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Abstract : Recently B.Y. Chen [4] studied warped products which are CR-submanifolds in Kaehler manifolds. B. Sahin [8] extended the above result for warped product semi-slant submanifolds $N_\theta \times {}_f N_T$ of a Kaehler manifold. In the present paper we have investigated the existence of doubly as well as CR-warped product submanifolds of Kenmotsu manifolds.

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1. Preliminaries

A $(2m+1)$-dimensional Riemannian manifolds $(\bar{M}, g)$ is said to be a *Kenmotsu manifold* if it admits an endomorphism ϕ of its tangent bundle $T\bar{M}$, a vector field ξ and a 1-form η satisfying:

$$\left.\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \phi(\xi) = 0, \ \eta \circ \phi = 0, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi), \\ &(\bar{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \ \bar{\nabla}_X \xi = X - \eta(X)\xi \end{aligned}\right\} \quad (1.1)$$

for any vector fields X, Y on $\bar{M}$, where $\bar{\nabla}$ denotes the Riemannian connection with respect to g.

Now, let M be a submanifold immersed in $\bar{M}$. We also denote by g the induced metric on M. Let TM be the Lie algebra of vector fields in

M and $T^{\perp}M$ the set of all vector fields normal to M. Denote by ∇ the Levi-Civita connection on M. Then Gauss-Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{1.2}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{1.3}$$

for any $X\ Y \in TM$ and any $N \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection in the normal bundle, h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A are related by

$$g(A_N X, Y) = g(h(X,Y), N) \tag{1.4}$$

A submanifolds M tangent to ξ is called *contact CR-submanifold* if it admits an invariant distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant i.e., $TM = D \oplus D^{\perp} \oplus \langle\xi\rangle$ with $\phi(D_p) \subseteq D_p$ and $\phi(D_p^{\perp}) \subset T_p^{\perp}M$, for every $p \in M$. It is clear that contact CR-submanifold is a special case of semi-slant submanifolds.

For any $X \in TM$, we write

$$\phi X = PX + FX \tag{1.5}$$

where PX is the tangential component of ϕX and FX is the normal component of ϕX respectively. Similarly, for any vector field N, normal to M, we put

$$\phi N = tN + fN \tag{1.6}$$

where tN and fN are tangential and normal components of ϕN, respectively.

2. Doubly Warped Products

Doubly warped products can be considered as a generalization of warped product (M, g) is a product manifold of the form $M =_f B \times_b F$ with the metric $g = f^2 g_B + b^2 g_F$, where $b : B \to (0, \infty)$ and $f : F \to (0, \infty)$ are smooth maps and g_B, g_F are the metrics on the Riemannian manifolds B and F respectively [9]. If either $b = 1$ or $f = 1$, but not both, then we obtain a *(single)* warped product. If both $b = 1$ and $f = 1$, then we have a *product manifold*. If neither b nor f is constant, then we have a non *trivial doubly warped product*.

If $X \in T(B)$ and $Z \in T(F)$, then the Levi-Civita connection is

$$\nabla_X Z = Z(\ln f)X + X(\ln b)Z \tag{2.1}$$

Let $M = {}_{f_2}N_\perp \times {}_{f_1}N_T$ be doubly warped product CR-submanifolds of Kenmotsu manifold $\bar{M}$. Such submanifolds are always tangent to the structure vector field ξ.

Theorem 2.1. *There is no proper doubly warped product contact CR-submanifolds of Kenmotsu manifold $\bar{M}$.*

Proof. Let $M = {}_{f_2}N_\perp \times {}_{f_1}N_T$ be doubly warped product contact CR-submanifolds of Kenmotsu manifold $\bar{M}$. There are two possible case arise here.

Case (i) : When ξ tangent to N_T, then let $Z \in TN_\perp$

$$\nabla_Z \xi = Z(\ln f_1)\xi + \xi(\ln f_2)Z \tag{2.2}$$

Also, by equations (1.1) and (1.2)

$$\bar{\nabla}_Z \xi = Z - \eta(Z)\xi = Z$$

or

$$\nabla_Z \xi + h(Z, \xi) = Z$$

this means that

$$\left.\begin{aligned} \nabla_Z \xi &= Z, \\ h(Z, \xi) &= 0. \end{aligned}\right\} \tag{2.3}$$

Using equations (2.2) and (2.3), we get

$$Z = Z(\ln f_1)\xi + \xi(\ln f_2)Z.$$

By the orthogonality of two distribution, we get

$$\left.\begin{aligned} Z(\ln f_1) &= 0, \\ \xi(\ln f_2) &= 1. \end{aligned}\right\} \tag{2.4}$$

First part of equation (2.4) shows that f_2 is constant for all $Z \in TN_\perp$. In this case doubly warped product does not exists.

Case (ii) : When ξ tangent to $N_\perp$ and let $X \in TN_T$

$$\nabla_X \xi = \xi(\ln f_1)X + X(\ln f_2)\xi \tag{2.5}$$

From structure equation and Gauss formula, we get

$$X - \eta(X)\xi = \bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi)$$

i.e.,

$$\left.\begin{aligned} &\nabla_X \xi = X, \\ &h(X, \xi) = 0. \end{aligned}\right\} \tag{2.6}$$

Equations (2.5) and (2.6) gives

$$X = (\xi \ln f_1)X + (X \ln f_2)\xi \qquad (2.7).$$

By orthogonality of two distributions, equation (2.7) gives

$$\left.\begin{aligned} &\xi(\ln f_1) = 1, \\ &X(\ln f_2) = 0. \end{aligned}\right\} \tag{2.8}$$

Second part of equation (2.8) follows that f_2 is constant for all $X \in T(N_T)$. In this case also, doubly warped product does not exists. Hence the proof is complete.

As in both the cases doubly warped product submanifold $M = {}_{f_2}N_\perp \times {}_{f_1}N_T$, ξ tangent to N_T and ξ tangent to $N_\perp$ doubly warped products does not exist. Now, we study warped product submanifolds.

3. Warped Product Submanifolds

B.Y. Chen studied warped product CR-submanifolds in Kaehler manifolds and introduce the notion of CR-warped product [4]. Later I. Hasegawa and I. Mihai studied contact CR-warped product submanifolds in Sasakian manifolds [5]. In this section we study contact warped product CR-submanifolds of Kenmotsu manifolds.

Definition 3.1. *Let* (B, g_1) *and* (F, g_2) *be two Riemannian manifolds with Riemannian metric* g_1 *and* g_2 *respectively and* f *a positive differentiable* *function on B. The warped product of B and F is the Riemannian manifold $B \times_f F = (B \times F, g)$, where*

$$g = g_1 + f^2 g_2. \tag{3.1}$$

More explicitly, if U is tangent to $M = B \times_f F$ at (p, q), then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|^2$$

where $\pi_i (i = 1,\ 2)$ are the canonical projections of $B \times F$ onto B and F respectively.

We recall that on a warped product one has

$$\nabla_U V = \nabla_V U = (U \ln f)V \tag{3.2}$$

for any vector fields U tangent to B and V tangent to F [3].

Let $M = N_\perp \times_f N_T$ be a contact warped product CR-submanifolds of Kenmotsu manifold $\bar{M}$. Such submanifolds are always tangent to the structure vector field ξ. We distinguish 2 cases

(i) ξ tangent to N_T.

(ii) ξ tangent to $N_\perp$.

First we consider the case (i), when ξ tangent to N_T.

Theorem 3.1. *Let $\bar{M}$ be a $(2m+1)$-dimensional Kenmotsu manifold. Then there do not exist warped product submanifolds $M = N_\perp \times_f N_T$ such that N_T is an invariant submanifold tangent to ξ and $N_\perp$ is anti-invariant submanifold of $\bar{M}$.*

Proof. Assume $M = N_\perp \times_f N_T$ is a contact warped product CR-submanifold of a Kenmotsu manifold $\bar{M}$, such that N_T is an invariant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of $\bar{M}$.

By equation (3.2), we get

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X \tag{3.3}$$

for any vector fields Z and X tangent to $N_\perp$ and N_T respectively.

In particular,

$$\nabla_Z \xi = (Z \ln f)\xi \tag{3.4}$$

Also, by structure equation (1.1) and Gauss formula, we have

$$Z = \bar{\nabla}_Z \xi = \nabla_Z \xi + h(Z, \xi)$$

$$\left.\begin{aligned} &\nabla_Z \xi = Z, \\ &h(Z, \xi) = 0. \end{aligned}\right\} \tag{3.5}$$

Equations (3.4) and (3.5) follows that $Z \ln f = 0$, $\forall\ Z \in T(N_\perp)$ i.e., f is constant for all $Z \in T(N_\perp)$. This means that $N_\perp$ is trivial, i.e., M is invariant submanifold. This complete the proof of the theorem.

Now, we consider the case (ii), when ξ tangent to $N_\perp$.

Assume that $\bar{M}$ be $(2m+1)-$dimensional Kenmotsu manifold and consider the warped product submanifold $M = N_\perp \times_f N_T$ such that $N_\perp$ is an anti-invariant submanifold of dimension p tangent to ξ and N_T is an invariant submanifold of $\bar{M}$. Then for any $X \in T(N_T)$

$$g(\nabla_X \xi, X) = g(\bar{\nabla}_X \xi, X) = g(X - \eta(X)\xi, X)$$

i.e.,

$$\xi \ln f \|X\|^2 = \|X\|^2$$

or,

$$\xi \ln f = 1$$

i.e.,

$$g(\nabla \ln f, \xi) = 1$$

i.e.,

$$\nabla \ln f = \xi, \tag{3.9}$$

where $\nabla \ln f$ denotes the gradient of $\ln f$ and defined as $g(\nabla \ln f, U) = U \ln f$, for all $U \in T(M)$. Equation (3.9) can also be written as

$$\sum_{i=1}^{p} \frac{\partial \ln f_i}{\partial x_i} = \xi, \qquad i = 1, 2 \cdots p. \tag{3.10}$$

Equation (3.10) is the first order partial differential equation and has a unique solution, i.e., warped product exist in this case.

Now, we will give an example of warped product contact CR-submanifold of type $N_\perp \times_f N_T$ in Kenmotsu manifold, with ξ tangent to $N_\perp$:

Example. Consider the complex space C^8 with the usual Kaehler structure and real global coordinates $(x^1, y^1, \cdots x^8, y^8)$. Let $\bar{M} = R\times\ _fC^8$ be the warped product between the real line R and C^8, where the warping function is $f = e^t$, t being the global coordinate on R. Then $\bar{M}$ is Kenmotsu manifold [1]. Consider the distribution $D = span\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \cdots \frac{\partial}{\partial x^4}, \frac{\partial}{\partial y^4}\}$ and $D^\perp = span\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x^5}, \cdots \frac{\partial}{\partial x^8},\}$ which are integrable and denote by N_T and $N_\perp$, the integral submanifolds, respectively. Let $g_{N_T} = \sum\limits_{i=1}^{4}((dx^i)^2 + (dy^i)^2)$ and $g_{N_\perp} = dt^2 + e^{2t}\sum\limits_{a=5}^{8}(dx^a)^2$ be Riemannian metrics on N_T and $N_\perp$, respectively. Then, $M = N_\perp \times\ _fN_T$ is a contact CR-submanifold, isometrically immersed in $\bar{M}$. Here, the warping function is $f = e^t$.

References

[1] D.E. Blair, Riemannian Geometry of Contact and Symplectic manifolds, Progress in Mathematics, Vol. 203, Birkhauser, Boston, Basel, Berlin, (2001).

[2] R.L. Bishop and B. O'Neill, Manifolds of Negative curvature, *Trans. Amer. Math. Soc.,* **145**, (1969) 1-49.

[3] B.Y. Chen, On isometric minimal immersions from warped products into real space forms, *Proc. Edinburg Math. Soc.* **45** (2002), 579-587.

[4] B.Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifold, *Monatsh. math.* **133** (2001), 177-195.

[5] I. Hasegawa, and I. Mihai, Contact CR-warped product submanifolds in Sasakian manifolds, *Geom. Dedicata* **102** (2003), 143-150.

[6] M.I. Munteanu, A note on doubly warped product contact CR-submanifolds in trans-sasakian manifolds, To appear in Acta Mathematica Hungaria (2007) (eprint: arXiv:math.DG/0604008 v1).

[7] C. Murathan, K. Arslan, R. Ezentas, I. Mihai, Contact CR-Warped product submanifolds in Kenmotsu space forms, *J. Korean Math. Soc.* **42** (2005), 1101-1110.

[8] B. Sahin, Nonexistence of warped product semi-slant submanifolds of Kaehler manifold, Geom. Dedicata **117** (2006) 195-202.

[9] B. Unal, Doubly warped products, *Differential geometry and Its Applications* **15** (3) (2001), 253-263.

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