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# Contact Hypersurfaces Of A Bochner-Kaehler Manifold

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**Abstract:** We have studied contact metric hypersurfaces of a Bochner-Kaehler manifold and obtained the following two results: (1) A contact metric constant mean curvature ($CMC$) hypersurface of a Bochner-Kaehler manifold is a $(k, \mu)$-contact manifold, and (2) If $M$ is a compact contact metric $CMC$ hypersurface of a Bochner-Kaehler manifold with a conformal vector field $V$ that is neither tangential nor normal anywhere, then it is totally umbilical and Sasakian, and under certain conditions on $V$, is isometric to a unit sphere.

*Keywords*: Bochner-Kaehler manifold, Contact metric hypersurface, Constant mean curvature, Conformal vector field.

*MS Classification*: 53B25, 53C55, 53 C15.

## 1 Introduction

Bochner curvature tensor was introduced in 1948 by S. Bochner [4] as a Kaehlerian analogue of the Weyl conformal tensor. It was shown by S.M. Webster [17] that the fourth order Chern-Moser curvature tensor of $CR$-manifolds coincides with the Bochner tensor. A Kaehler manifold with vanishing Bochner curvature tensor is known as Bochner-Kaehler manifold. Bochner-Kaehler surface is nothing but a self-dual Kaehler surface in Penrose's twistor theory. Some topological obstructions to

Bochner-Kaehler metrics were studied by Chen in [6]. Just as a real space-form is conformally flat, a complex space-form is Bochner flat, i.e. Bochner-Kaehler (the converse does not need to hold). The product of two complex space-forms of constant holomorphic sectional curvatures $c$ and $-c$ is non-Einstein Bochner-Kaehler. Though Bochner-Kaehler manifolds have been studied by quite a few geometers, nevertheless have received considerably less attention, compared to Kaehler metrics with vanishing scalar curvature and Kaehler-Einstein metrics. For details we refer to Bryant [5]. It is well known that a hypersurface $M$ of a Kaehler manifold $\bar{M}$ admits an almost contact metric structure induced from the Hermitian structure of $\bar{M}$. Okumura [13] studied and classified such hypersurfaces, mainly when the ambient space is a complex space-form. Generalizing the following result of Sharma [15] "The contact metric hypersurface of a complex space-form is a $(k, \mu)$-contact manifold", we prove the following main result of this paper.

**Theorem 1** *A contact metric constant mean curvature hypersurface of a Bochner-Kaehler manifold is a $(\kappa, \mu)$-contact manifold.*

Finally, we consider the case when the ambient space admits a conformal vector field and provide the following extrinsic characterization of a Sasakian manifold.

**Theorem 2** *Let $M$ be a compact contact metric constant mean curvature hypersurface of a Bochner-Kaehler manifold $\bar{M}$ admitting a conformal vector field $V$ which is neither tangential nor normal anywhere on $M$. Then $M$ is Sasakian and totally umbilical in $\bar{M}$, and the component $U$ of $V$, tangential to $M$ is conformal on $M$. Further, (i) if $U$ is non-Killing and dim.$M > 3$, then $M$ is isometric to the unit sphere $S^{2n+1}$, and (ii) if $V$ is closed, then for any dimension, $M$ is isometric to $S^{2n+1}$.*

# 2 Contact Metric Hypersurfaces Of A Kaehler Manifold

A $(2n + 1)$-dimensional smooth manifold $M$ is said to be a contact manifold if it carries a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M$. Given a contact 1-form $\eta$, there exists a unique vector field $\xi$ such that $(d\eta)(\xi, X)= 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle

$D$ ($\eta = 0$), one obtains a Riemannian metric $g$ and a (1,1)-tensor field $\varphi$ such that

$$(d\eta)(X,Y) = g(X, \varphi Y), \eta(X) = g(X, \xi), \varphi^2 = -I + \eta \otimes \xi \qquad (1)$$

$g$ is called an associated metric of $\eta$ and $(\varphi, \eta, \xi, g)$ a contact metric structure. The operators $h = \frac{1}{2}\pounds_\xi \varphi$ and $l = R(.,\xi)\xi$ are self-adjoint and satisfy: $h\xi = 0$ and $h\varphi = -\varphi h$. Furthermore, $h$, $h\varphi$ are trace-free. Following formulas hold on a contact metric manifold.

$$\nabla_X \xi = -\varphi X - \varphi h X \qquad (2)$$

$$l - \varphi l \varphi = -2(h^2 + \varphi^2) \qquad (3)$$

If the associated CR-structure on $M$ is integrable, then $M$ is called a contact strongly pseudo-convex integrable CR manifold. This CR integrability condition was shown by Tanno [16] to be equivalent to

$$(\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \qquad (4)$$

and holds on a 3-dimensional contact metric manifold. A contact metric manifold $(M, g)$ is said to be $K$-contact if $\xi$ is Killing (equivalently, $h = 0$), and Sasakian if the almost Kaehler structure on the cone $M \times R$ with metric $dr^2 + r^2 g$ is Kaehler. Sasakian manifolds are $K$-contact and 3-dimensional $K$-contact manifolds are Sasakian. For details we refer to Blair [1]. In [2] Blair, Koufogiorgos and Papantoniou introduced a class of contact metric manifolds $M^{2n+1}(\eta, \xi, g, \varphi)$ satisfying the nullity condition:

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \qquad (5)$$

for real constants $k$ and $\mu$. Such manifolds are known as $(k, \mu)$-contact manifolds, and satisfy: $k \leq 1$, equality holding when $M$ is Sasakian.
Let $M$ be an isometrically embedded orientable hypersurface of a Kaehler manifold $\bar{M}$ of real dimension $2n + 2$ and with complex structure tensor $J : J^2 = -I$ and the Hermitian metric $g$. The induced metric on $M$ will also be denoted by $g$. If $N$ denotes the unit normal vector field to $M$, we set

$$JN = \xi \qquad (6)$$

$$JX = \varphi X - \eta(X)N, \qquad (7)$$

where $\varphi$ and $\eta$ denote a $(1,1)$-tensor field and a 1-form respectively, and $X$ an arbitrary vector field tangent to $M$. The Gauss and Weingarten formulas are

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \bar{\nabla}_X N = -AX$$

where $X, Y$ denote arbitrary vector fields tangent to $M$, $\nabla$ and $\bar{\nabla}$ the Riemannian connections of $M$ and $\bar{M}$ respectively, and $A$ the Weingarten operator. Differentiating (1) along an arbitrary vector field $X$ tangent to $M$, using the Weingarten formula, and comparing tangential parts gives

$$\nabla_X \xi = -\varphi AX. \tag{8}$$

One can easily verify using (6) and (7) that $(\varphi, \xi, \eta, g)$ defines the almost contact metric structure. We now assume that the almost contact metric structure induced on $M$ is a contact metric structure. Using the formula (2) in (8) yields

$$A\xi = (Tr.A - 2n)\xi. \tag{9}$$

$$AX = X + hX + (Tr.A - 2n - 1)\eta(X)\xi. \tag{10}$$

which were derived in [15]. Next, differentiating (7) along $M$, and using (10) gives equation (4). Hence $M$ is contact strongly pseudo-convex integrable CR manifold. We denote the Ricci tensor of $M$, of types $(0,2)$ and $(1,1)$ by $S$ and $Q$ respectively, and the scalar curvature by $r$ of $M$. Corresponding objects of $\bar{M}$ are denoted by the same letters with overbars. Recall the Gauss equation

$$\begin{aligned} g(\bar{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) \\ +g(AX,Z)g(AY,W) &- g(AY,Z)g(AX,W). \end{aligned} \tag{11}$$

and contract it as

$$\begin{aligned} \bar{S}(Y,Z) - g(\bar{R}(N,Y)Z,N) &= S(Y,Z) \\ +g(AX,AZ) &- (Tr.A)g(AY,Z). \end{aligned} \tag{12}$$

For a Bochner-Kaehler manifold $\bar{M}$, the Bochner curvature tensor $B$ (see [19]) vanishes, i.e. for arbitrarty vector fields $\bar{X}, \bar{Y}, \bar{Z}, \bar{W}$ on $\bar{M}$, we have

$$\begin{aligned}
0 &= g(B(\bar{X},\bar{Y})\bar{Z},\bar{W}) = g(\bar{R}(\bar{X},\bar{Y})\bar{Z},\bar{W}) - \frac{1}{2n+6}[g(\bar{Y},\bar{Z})g(\bar{Q}\bar{X},\bar{W}) \\
&- g(\bar{Q}\bar{X},\bar{Z})g(\bar{Y},\bar{W}) + g(J\bar{Y},\bar{Z})g(\bar{Q}J\bar{X},\bar{W}) - g(\bar{Q}J\bar{X},\bar{Z})g(J\bar{Y},\bar{W}) \\
&+ g(\bar{Q}\bar{Y},\bar{Z})g(\bar{X},\bar{W}) - g(\bar{X},\bar{Z})g(\bar{Q}\bar{Y},\bar{W}) + g(\bar{Q}J\bar{Y},\bar{Z})g(J\bar{X},\bar{W}) \\
&- g(J\bar{X},\bar{Z})g(\bar{Q}J\bar{Y},\bar{W}) - 2g(J\bar{X},\bar{Q}\bar{Y})g(J\bar{Z},\bar{W}) \\
&- 2g(J\bar{X},\bar{Y})g(\bar{Q}J\bar{Z},\bar{W}) + \frac{\bar{r}}{(2n+4)(2n+6)}[g(\bar{Y},\bar{Z})g(\bar{X},\bar{W}) \\
&- g(\bar{X},\bar{Z})g(\bar{Y},\bar{W}) + g(J\bar{Y},\bar{Z})g(J\bar{X},\bar{W}) \\
&- g(J\bar{X},\bar{Z})g(J\bar{Y},\bar{W}) - 2g(J\bar{X},\bar{Y})g(J\bar{Z},\bar{W})] \qquad (13)
\end{aligned}$$

# 3 Proofs Of The Results

**Lemma 1** *For a contact metric hypersurface of a Kaehler manifold,*

$$(a)\bar{S}(\varphi Y,Z) + \bar{S}(Y,\varphi Z) = \eta(Y)g(\bar{Q}N,Z) + \eta(Z)g(\bar{Q}N,Y)$$

$$(b)g(\xi,\bar{Q}N) = 0.$$

**Proof:** Since $\overline{M}$ is Kaehler, we have $\bar{S}(JY,Z) + \bar{S}(Y,JZ) = 0$. The use of (7) in this gives (a). Substituting $\xi$ for $Y$ and $Z$ in (a) yields (b).

**Lemma 2** *If $f$ is a smooth function on a contact metric manifold $M$ such that $df = (\xi f)\eta$ ($d$ denoting exterior derivation), then $f$ is constant on $M$.*

**Proof:** Taking the exterior derivative of the differential condition mentioned in the hypothesis gives $d(\xi f)\wedge\eta + (\xi f)d\eta = 0$. Taking its wedge product with $\eta$ we find $(\xi f)(d\eta)\wedge\eta = 0$. As $(d\eta)\wedge\eta$ is nowhere vanishing on $M$ (otherwise the definition of contact structure would break down), we conclude that $\xi f = 0$ on $M$. Consequently, $df = 0$ on $M$, and hence $f$ is constant on $M$, completing the proof.

**Lemma 3** *For a contact metric hypersurface $M$ of a Bochner-Kaehler manifold, the following conditions are equivalent:*
*(a) For any vector field $X$ tangent to $M$, $g(\bar{Q}N,X) = 0$*
*(b) $\xi$ is an eigenvector of the Ricci operator $Q$ at each point of $M$*
*(c) The mean curvature of $M$ is constant.*

**Proof :** Using equations (12), (13), and part (b) of Lemma 1 gives

$$\begin{aligned}(2n+5)\bar{S}(Y,Z) &= [\bar{S}(N,N)-\frac{\bar{r}}{2n+4}]g(Y,Z)-\frac{3\bar{r}}{2n+4}\eta(Y)\eta(Z)\\ &+ 3g(\bar{Q}Z,\xi)\eta(Y)+3g(\bar{Q}Y,\xi)\eta(Z)+(2n+6)[S(Y,Z)\\ &+ g(AY,AZ)-(Tr.A)g(AY,Z)]\end{aligned}$$

Now we replace $Y$ by $\varphi Y$ in the above equation to get one equation, and replace $Z$ by $\varphi Z$ to get another equation. Adding these two equations, and using part (a) of Lemma 1 we obtain

$$\begin{aligned}&(2n+5)\{\eta(Y)g(\bar{Q}N,Z)+\eta(Z)g(\bar{Q}N,Y)\}=3g(\bar{Q}\varphi Z,\xi)\eta(Y)\\ &+3g(\bar{Q}\varphi Y,\xi)\eta(Z)+(2n+6)[S(\varphi Y,Z)+S(Y,\varphi Z)+g(A\varphi Y,AZ)\\ &+g(AY,A\varphi Z)-(Tr.A)g(A\varphi Y,Z)-(Tr.A)g(AY,\varphi Z).\end{aligned} \tag{14}$$

Substituting $\xi$ for $Z$, using (9) and part (b) of Lemma 1 yields

$$(n+1)\varphi\bar{Q}\xi=(n+3)\varphi Q\xi. \tag{15}$$

We also note

$$g(\bar{Q}N,X)=-g(J\bar{Q}\xi,X)=-g(\varphi\bar{Q}\xi,X), \tag{16}$$

The equations (15) and (16) show that (a) is equivalent to (b). Contracting the Codazzi equation: $\bar{R}(X,Y)N=(\nabla_Y A)X-(\nabla_X A)Y$ at $X$ provides $\bar{S}(Y,N)=Y(Tr.A)-(divA)Y$. Equation (10) transforms the preceding equation into

$$\bar{S}(Y,N)=Y(Tr.A)-(\xi Tr.A)\eta(Y)-(divh)Y \tag{17}$$

Using equation (4) and the formula $(div.h)\xi=0$ (easy to verify) for a contact metric shows

$$(divh\varphi)\varphi Y=-(divh)Y \tag{18}$$

Let us assume (b), i.e. $Q\xi=(Tr.l)\xi$. Applying the formula: $(divh\varphi)Y=S(Y,\xi)-2n\eta(Y)$ (see [3]), we have $(divh\varphi)\varphi Y=S(\varphi Y,\xi)=0$. Hence equation (18) shows that $divh=0$. As $Q\xi=(Tr.l)\xi$ is equivalent to (a) [proven earlier], appealing to equation (17) we obtain $d(Tr.A)=(\xi Tr.A)\eta$. Application of Lemma 2 shows that $Tr.A$ is constant on $M$, proving $(b)\Rightarrow(c)$. For the converse, assume (c), i.e. $Tr.A$ constant. Then, we go back to equations (17) and (18) and use the formula

$(divh\varphi)Y = S(Y,\xi) - 2n\eta(Y)$ once again, getting $\bar{S}(X,N) = S(\varphi X,\xi) = -g(\varphi Q\xi, X)$. Using this in (16) we find $\varphi\bar{Q}\xi = \varphi Q\xi$. Finally, using this in (15) we conclude that $\varphi Q\xi = 0$ which implies (b), and complete the proof.

**Lemma 4** *For a contact metric hypersurface $M$ of a Bochner-Kaehler manifold $\bar{M}$,*

$$(a)(Q\varphi - \varphi Q) - (\eta \circ Q\varphi) \otimes \xi + \eta \otimes \varphi Q\xi = 2(Tr.A - 2)h\varphi.$$

$$(b) l\varphi - \varphi l = 2(Tr.A - 2n)h\varphi.$$

**Proof** Replacing $Y, Z$ by $\varphi Y, \varphi Z$ respectively, in (14) and then using (10) we get (a). Using the formula:

$$Q\varphi - \varphi Q = l\varphi - \varphi l + 4(n-1)h\varphi + (\eta \circ Q\varphi) \otimes \xi - \eta \otimes \varphi Q\xi = 0,$$

for a contact pseudo-convex integrable CR-manifold (see [10]), and using it in (a) we obtain (b).

**Proof Of Theorem 1.** First, we use equation (13) to obtain

$$\begin{aligned} g(\bar{R}(X,Y)\xi, W) = \frac{1}{2n+6}[\eta(Y)g(\bar{Q}X,W) - g(\bar{Q}X,\xi)g(Y,W) \\ -g(\bar{Q}X,N)g(JY,W) + g(\bar{Q}Y,\xi)g(X,W) - \eta(X)g(\bar{Q}Y,W) \\ +g(\bar{Q}Y,N)g(JY,W) - 2g(JX,Y)g(\bar{Q}\xi,W)] \\ -\frac{\bar{r}}{(2n+6)(2n+4)}[\eta(Y)g(X,W) - \eta(X)g(Y,W)] \end{aligned} \tag{19}$$

for arbitrary vector fields $X, Y, W$ tangent to $M$. By Lemma 3, the constant mean curvature hypothesis is equivalent to $g(\bar{Q}N, X) = 0$, i.e. $\bar{Q}N = fN$ for some function $f$ on $M$. We also have $\bar{Q}\xi = J\bar{Q}N = f\xi$. Hence equation (19) reduces to

$$\begin{aligned} (2n+6)g(\bar{R}(X,Y)\xi,W) &= \eta(Y)g(\bar{Q}X,W) - \eta(X)g(\bar{Q}Y,W) \\ &+ \sigma[\eta(Y)g(X,W) - \eta(X)g(Y,W)] \end{aligned} \tag{20}$$

where $\sigma = f - \frac{\bar{r}}{2n+4}$. This shows that $g(\overline{R}(X,Y)\xi, W) = 0$, for any vector fields $X$, $Y$ tangent to $M$ and orthogonal to $\xi$. Next, substituting $\xi$ for $Z$ in (11), and using equations (9) and (20) we obtain $R(X,Y)\xi = 0$,

for any vector fields $X$, $Y$ tangent to $M$ and orthogonal to $\xi$. Hence by the result of Koufogiorgos-Stamatiou ([11]) we conclude that $M$ is a $(\kappa, \mu)$-space provided the dimension of $M$ is $\geq 5$.
It remains to consider the 3-dimensional case for which we know that

$$\begin{aligned} R(X,Y)Z &= g(QY,Z)X - g(QX,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{21}$$

Making use of the formula (3) and the formula $h^2 = (k-1)\varphi^2$ ( for any 3-dimensional contact metric manifold [9], where $k$ is a function $= \frac{Tr.l}{2}$) in part (b) of Lemma 4, we obtain

$$lY = -k\varphi^2 Y + (Tr.A - 2)hY \tag{22}$$

Differentiating this along an arbitrary vector field $X$, using (2) and then contracting the resulting equation at $X$ with respect to a local orthonormal frame $e_i$, we find

$$\begin{aligned} g((\nabla_Y Q)\xi,\xi) - g((\nabla_\xi Q)Y,\xi) - g(R(Y,\varphi e_i + \varphi h e_i)\xi, e_i) \\ -g(R(Y,\xi)(\varphi e_i + \varphi h e_i), e_i) = -(\varphi^2 Y)\kappa + (Tr.A - 2)(divh)Y \end{aligned} \tag{23}$$

Using $Q\xi = (Tr.l)\xi$ and (2) we have $g((\nabla_Y Q)\xi,\xi) = 2(Yk)$, $g((\nabla_\xi Q)Y,\xi) = 2(\xi k)\eta(Y)$. We also had found during the proof of Lemma 3 that $div.h = 0$. Moreover, using (21) and $Q\xi = (Tr.l)\xi$ we compute

$$g(R(Y,\varphi e_i + \varphi h e_i)\xi, e_i) = -\eta(Y)Tr.(Q\varphi h)$$

and

$$g(R(Y,\xi)(\varphi e_i + \varphi h e_i), e_i) = 0$$

Utilizing all these findings in (23), we obtain

$$Yk - (\xi k)\eta(Y) + \eta(Y)Tr.(Q\varphi h) = 0$$

Taking $Y = \xi$ it is easy to see that $Tr.Q\varphi h = 0$. Hence $Yk = (\xi k)\eta(Y)$, i.e. $dk = (\xi k)\eta$. Applying Lemma 2, we conclude that $k$ is constant. Thus, the hypothesis : $Q\xi = (Tr.l)\xi$ and (21) imply $R(X,Y)\xi = 0$, for any vector field $X$, $Y$ orthogonal to $\xi$. Replacing $X$ by $X - \eta(X)\xi$ and $Y$ by $Y - \eta(Y)\xi$ (as these vector fields are orthogonal to $\xi$) we obtain

$$R(X,Y)\xi = \eta(Y)lX - \eta(X)lY$$

The use of (22) in the foregoing equation shows that $M^3$ is a $(\kappa, \mu)$ space with $\mu = (Tr.A - 2)$. This completes the proof.

**Proof Of Theorem 2.** As $V$ is conformal on $\bar{M}$,

$$\pounds_V g = 2\rho g \tag{24}$$

We decompose the conformal vector field $V$ along $M$ orthogonally as

$$V = U + \alpha N \tag{25}$$

where $U$ is the tangential part of $V$ and $\alpha$ a smooth function on $M$. In view of the Lemma 3, the constant mean curvature hypothesis is equivalent to $\bar{S}(X, N) = 0$ for arbitrary vector field $X$ tangent to $M$. Following the procedure given on pages 101-104 of Yano [18], we have

$$2n \int_M \bar{S}(U, N) dM = \int_M \alpha \sum_{i \neq j}^{2n+1} (k_i - k_j)^2 dM \tag{26}$$

where $dM$ is the volume element of $M$, and $k_i$ are the principal curvatures of $M$. As the left hand side of the above equation is zero, and $\alpha$ is nowhere zero on $M$ (otherwise $V$ would become tangent to $M$ somewhere, contradicting our hypothesis), we conclude that $k_i = k_j$, i.e. $M$ is totally umbilical. Hence, using (10) provides $A = I$, and $h = 0$, i.e. $M$ is Sasakian. The conformal Killing equation (24), together with the Gauss and Weingarten formulas show that $\pounds_U g = 2(\rho + \alpha)g$, i.e. $U$ is conformal on $M$. If $U$ is homothetic, then $U$ reduces to Killing, since $M$ is compact. Hence, if $U$ is not Killing, and $dim > 3$, then by the following theorem of Okumura [14] "A complete Sasakian manifold of dimension $> 3$ and admitting a non-Killing conformal vector field is isometric to a unit sphere", $M$ is isometric to $S^{2n+1}$ which proves (i). For (ii), we know from the following result of Goldberg [8] "A closed conformal vector field on a non-flat Kaehler manifold is homothetic and holomorphic" that $V$ is homothetic. Hence $\nabla_X V = \rho X$ ($\rho$ constant). Using the decomposition (25), and bearing in mind that $X$ is arbitrary tangent vector on $M$, we immediately obtain $U = -D\alpha$ ($D$ is the gradient operator of $(M, g)$) and $\nabla_X U = (\alpha + \rho)X$. We note that $\alpha$ cannot be constant on $M$, otherwise $U$ would vanish on $M$ turning $V$ normal to $M$ and thus contradicting our hypothesis. Thus obtain

$$\nabla_X D(\alpha + \rho) = -(\alpha + \rho)X \tag{27}$$

Hence, by Obata's theorem [12]"A complete Riemannian manifold of dimension $> 2$ is isometric to a sphere of radius $\frac{1}{c}$ if and only if it admits a non-trivial solution $f$ of the differential equation $\nabla\nabla f = -c^2 fg$", we conclude that $M$ is isometric to $S^{2n+1}$, completing the proof.

**Remark.** As indicated in [7], the Kaehler cone manifold $(M \times R^+, d(r^2\eta)$ with metric $dr \otimes dr + r^2 g$ over a Sasakian manifold $(M, \eta, g)$ admits a conformal vector field $ar\partial_r - b\xi$ (for $a, b$ real constants) which is nowhere tangent and nowhere normal to $M$ and therefore serves as an example of the conformal vector field satisfying the hypothesis of Theorem 2.

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