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CONTACT METRIC (κ, μ)-SPACES AS BI-LEGENDRIAN MANIFOLDS

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Abstract

We describe a contact metric manifold whose Reeb vector field belongs to the (κ, μ)-nullity distribution as a bi-Legendrian manifold and we study its canonical bi-Legendrian structure. Then we characterize contact metric (κ, μ)-spaces in terms of a canonical connection which can be naturally defined on them.

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1. Introduction

Contact metric (κ, μ)-spaces, introduced in [2] by Blair *et al.*, are those contact metric manifolds (M, ϕ, ξ, η, g) for which the Reeb vector field ξ belongs to the (κ, μ)-nullity distribution, that is, satisfies, for all vector fields V and W on M,

$$R_{VW}\xi = \kappa(\eta(W)V - \eta(V)W) + \mu(\eta(W)hV - \eta(V)hW), \tag{1.1}$$

for some real numbers κ and μ, where $2h$ is the Lie derivative of ϕ in the direction of ξ. This definition can be regarded as a generalization both of the Sasakian condition $R_{VW}\xi = \eta(W)V - \eta(V)W$ and of those contact metric manifolds satisfying $R_{VW}\xi = 0$ which were studied by Blair in [1].

Recently contact metric (κ, μ)-spaces have been studied by various authors (see, for example, [4–6, 11, 14]) and several important properties of these manifolds have been discovered. In fact there are many reasons for studying (κ, μ)-spaces. The first is that, in the non-Sasakian case (that is, for $\kappa \neq 1$), condition (1.1) determines the curvature completely. Moreover, while the values of κ and μ change, the form of (1.1) is invariant under $\mathcal{D}$-homothetic deformations. Finally, there are nontrivial examples of these manifolds, the most important being the unit tangent sphere bundle of a Riemannian manifold of constant sectional curvature with the usual contact metric structure.

A complete classification of contact metric (κ, μ)-spaces has been given in [5] by Boeckx, who proved also that any non-Sasakian contact metric (κ, μ)-space is locally homogeneous and strongly ϕ-symmetric.

One of the peculiarities of these manifolds is that they give rise to three mutually orthogonal distributions $\mathcal{D}_\lambda$, $\mathcal{D}_{-\lambda}$ and $\mathbb{R}\xi$, corresponding to the eigenspaces of the operator h. In particular, $\mathcal{D}_\lambda$ and $\mathcal{D}_{-\lambda}$ define two transverse Legendrian foliations of M so that these manifolds are endowed with a bi-Legendrian structure.

In the same period that contact metric (κ, μ)-spaces were introduced, the theory of Legendrian foliations was developed by Pang, Libermann and Jayne (see [13, 15, 16]), so it is tempting to use the techniques and language of Legendrian foliations for the study of contact metric (κ, μ)-spaces and to begin the investigation of the interactions between these two areas of the contact geometry. This is what we set out to do in this paper.

The paper is organized as follows. After some preliminaries on contact metric manifolds and Legendrian foliations, in Section 3 we study the Legendrian foliations canonically defined in any contact metric (κ, μ)-space. We find, for both the foliations, an explicit formula of the invariant Π introduced by Pang for classifying Legendrian foliations (see [16]) and we see that the Legendrian foliations in question are, according to this classification, either nondegenerate or flat. Then we relate these invariants to the invariant I_M used by Boeckx in [5] to classify contact metric (κ, μ)-spaces. In Section 4 we attach to any contact metric (κ, μ)-space a linear connection in a canonical way. We study the properties of this connection and, using it, we give an interpretation of the notion of contact metric (κ, μ)-space in terms of bi-Legendrian structures. In particular, we prove the following characterization of contact metric (κ, μ)-spaces.

THEOREM 1.1. *A contact metric manifold (M, ϕ, ξ, η, g) is a contact metric (κ, μ)-space if and only if M admits an orthogonal bi-Legendrian structure $(\mathcal{F}, \mathcal{G})$ such that the corresponding bi-Legendrian connection $\overline{\nabla}$ satisfies $\overline{\nabla}\phi = 0$ and $\overline{\nabla}h = 0$. Furthermore, the bi-Legendrian structure $(\mathcal{F}, \mathcal{G})$ coincides with that determined by the eigenspaces of h.*

This theorem should be compared with the well-known results obtained by Tanaka [17] and, independently, Webster [22]. They proved that any strongly pseudo-convex CR-manifold admits a unique linear connection $\widetilde{\nabla}$ such that the tensors ϕ, η, g are all $\widetilde{\nabla}$-parallel and whose torsion satisfies $\widetilde{T}(Z, Z') = 2\Phi(Z, Z')\xi$ for all $Z, Z' \in \Gamma(\mathcal{D})$ and $\widetilde{T}(\xi, \phi V) = -\phi\widetilde{T}(\xi, V)$ for all $V \in \Gamma(TM)$. In view of this remark and the fact that any contact metric (κ, μ)-space is a strongly pseudo-convex CR-manifold, one can see that the connection mentioned in Theorem 1.1 plays the same role for contact metric (κ, μ)-spaces that the Tanaka–Webster connection has for CR-manifolds. As we shall see, the connection $\overline{\nabla}$ uniquely determines a contact metric (κ, μ)-space modulo $\mathcal{D}$-homothetic deformations and it turns out to be very useful in the study of this kind of contact metric manifolds.

2. Preliminaries

2.1. Contact manifolds An *almost contact metric manifold* is a $(2n+1)$-dimensional Riemannian manifold (M, g) which admits a tensor field ϕ of type $(1, 1)$, a global 1-form η and a global vector field ξ, called a *Reeb vector field*, satisfying

$$\eta(\xi) = 1, \quad \phi^2 V = -V + \eta(V)\xi, \quad g(\phi V, \phi W) = g(V, W) - \eta(V)\eta(W), \tag{2.1}$$

for all vector fields V and W on M. Given an almost contact metric manifold, a 2-form Φ, called the *fundamental* 2*-form* of the structure, can be defined by $\Phi(V, W) = g(V, \phi W)$. Then we say that (M, ϕ, ξ, η, g) is a *contact metric manifold* if the additional property $d\eta = \Phi$ holds. From (2.1) it can be proved (see [3]) that:

(i) $\phi\xi = 0$, $\eta \circ \phi = 0$;
(ii) $\nabla_\xi \phi = 0$ and $\nabla_\xi \xi = 0$;
(iii) $\phi|_{\mathcal{D}}$ is an isomorphism.

Here ∇ denotes the Levi-Civita connection and $\mathcal{D} = \ker(\eta)$ is the $2n$-dimensional distribution orthogonal to ξ and called the *contact distribution*. It is also easy to prove that, for any $X \in \Gamma(\mathcal{D})$, the bracket $[X, \xi]$ still belongs to $\mathcal{D}$.

In any contact metric manifold the 1-form η satisfies the relation

$$\eta \wedge (d\eta)^n \neq 0, \tag{2.2}$$

everywhere on M. Any $(2n+1)$-dimensional smooth manifold which carries a global 1-form satisfying (2.2) is called a *contact manifold*. Thus any contact metric manifold is a contact manifold. Conversely, it is well known that any contact manifold admits a compatible contact metric structure (ϕ, ξ, η, g). It should be remarked that (2.2) implies that the contact distribution $\mathcal{D}$ is never integrable.

Given a contact metric manifold, we can define a tensor field h by $h = (1/2)\mathcal{L}_\xi \phi$, $\mathcal{L}$ denoting the Lie differentiation. It can be shown (see [3]) that h is a trace-free, symmetric operator satisfying $h\xi = 0$, $\phi h = -h\phi$ and

$$\nabla_V \xi = -\phi h V - \phi V, \tag{2.3}$$

for all $V \in \Gamma(TM)$. Moreover, ξ is Killing if and only if h vanishes identically; in this case we say that (M, ϕ, ξ, η, g) is a K*-contact manifold*.

On a contact metric manifold M an almost complex structure J on the product manifold $M \times \mathbb{R}$ can be defined by setting $J(V, f\, d/dt) = (\phi V - f\xi, \eta(V)\, d/dt)$, where V is a vector field tangent to M and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable then (M, ϕ, ξ, η, g) is said to be *Sasakian*. It is well known that each of the following conditions characterizes Sasakian manifolds:

$$(\nabla_V \phi)W = g(V, W)\xi - \eta(W)V, \tag{2.4}$$

$$R_{VW}\xi = \eta(W)V - \eta(V)W, \tag{2.5}$$

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for all vector fields V and W on M. A generalization of condition (2.5) leads to the notion of the (κ, μ)-manifold. If the curvature tensor field of a contact metric manifold satisfies (1.1) for some real numbers κ and μ we say that ξ belongs to the (κ, μ)-nullity distribution or, simply, that (M, ϕ, ξ, η, g) is a contact metric (κ, μ)-space. This type of contact metric manifolds were introduced and studied in depth in [2]. Among other things, the authors proved the following results.

THEOREM 2.1 [2]. *Let (M, ϕ, ξ, η, g) be a contact metric manifold with ξ belonging to the (κ, μ)-nullity distribution. Then $\kappa \leq 1$. Moreover, if $\kappa = 1$ then $h = 0$ and (M, ϕ, ξ, η, g) is a Sasakian manifold; if $\kappa < 1$, the contact metric structure is not Sasakian and M admits three mutually orthogonal integrable distributions $\mathcal{D}_0 = \mathbb{R}\xi$, $\mathcal{D}_\lambda$ and $\mathcal{D}_{-\lambda}$ corresponding to the eigenspaces of h, where $\lambda = \sqrt{1-\kappa}$.*

THEOREM 2.2 [2]. *Let (M, ϕ, ξ, η, g) be a contact metric manifold with ξ belonging to the (κ, μ)-nullity distribution. Then the following relations hold, for any $X, Y \in \Gamma(TM)$:*

$$(\nabla_X \phi)Y = g(X, Y + hY)\xi - \eta(Y)\,(X + hX),$$
$$(\nabla_X h)Y = ((1-\kappa)g(X, \phi Y) + g(X, \phi hY))\xi + \eta(Y)\,h(\phi X + \phi hX) - \mu\phi hY.$$

Blair *et al.* proved also that the (κ, μ)-nullity condition remains unchanged under $\mathcal{D}$-homothetic deformations. The concept of $\mathcal{D}$-homothetic deformation for a contact metric manifold (M, ϕ, ξ, η, g) was introduced by Tanno in [18] and then intensively studied by many other authors. We recall that, given a real positive number a, by a $\mathcal{D}$-homothetic deformation of constant a we mean a change of the structure tensors in the following way:

$$\widetilde{\phi} = \phi, \quad \widetilde{\eta} = a\eta, \quad \widetilde{\xi} = \frac{1}{a}\xi, \quad \widetilde{g} = ag + a(a-1)\eta \otimes \eta. \tag{2.6}$$

In [2] the authors proved that if M is a contact metric manifold whose Reeb vector field belongs to the (κ, μ)-nullity distibution, then for the contact metric manifold $(M, \widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ the same property holds. Specifically, $\widetilde{\xi}$ belongs to the $(\widetilde{\kappa}, \widetilde{\mu})$-nullity distribution where

$$\widetilde{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \quad \widetilde{\mu} = \frac{\mu + 2a - 2}{a}.$$

2.2. Legendrian foliations A *Legendrian distribution* on a contact manifold (M^{2n+1}, η) is defined by an n-dimensional subbundle L of the contact distribution such that $d\eta(X, X') = 0$ for all $X, X' \in \Gamma(L)$. When L is integrable, it defines a *Legendrian foliation* of (M^{2n+1}, η). Legendrian foliations have been extensively investigated in recent years from various points of view (see, for example, [8, 13, 15, 16]). In particular, Pang provided a classification of Legendrian foliations by means of a bilinear symmetric form $\Pi_{\mathcal{F}}$ on the tangent bundle of the foliation, defined by $\Pi_{\mathcal{F}}(X, X') = -(\mathcal{L}_X \mathcal{L}_{X'} \eta)\,(\xi) = -\eta([X', [X, \xi]])$. He called a Legendrian foliation $\mathcal{F}$ *nondegenerate*, *degenerate* or *flat* depending respectively on whether the bilinear form $\Pi_{\mathcal{F}}$ is nondegenerate, degenerate or vanishes identically. In terms of an associated metric g, $\Pi_{\mathcal{F}}$ is given by

$$\Pi_{\mathcal{F}}(X, X') = 2g([\xi, X], \phi X'). \tag{2.7}$$

The above formula provides a geometrical interpretation of this classification.

LEMMA 2.3 [13]. *Let (M, ϕ, ξ, η, g) be a contact metric manifold and let $\mathcal{F}$ be a foliation on it. Then:*

(i) *$\mathcal{F}$ is flat if and only if $[\xi, X] \in \Gamma(T\mathcal{F})$ for all $X \in \Gamma(T\mathcal{F})$;*
(ii) *$\mathcal{F}$ is degenerate if and only if there exist $X \in \Gamma(T\mathcal{F})$ such that $[\xi, X] \in \Gamma(T\mathcal{F})$;*
(iii) *$\mathcal{F}$ is nondegenerate if and only if $[\xi, X] \notin \Gamma(T\mathcal{F})$ for all $X \in \Gamma(T\mathcal{F})$.*

Given a compatible contact metric structure (ϕ, ξ, η, g) and a Legendrian distribution L on M, we may consider the distribution $Q = \phi L$. It can be proved (see [13]) that Q is a Legendrian distribution on M which in general is not integrable, even if L is; it is called the *conjugate Legendrian distribution* of L, and the tangent bundle of M splits as the orthogonal sum $TM = L \oplus Q \oplus \mathbb{R}\xi$. When both L and Q are integrable, they define two orthogonal Legendrian foliations $\mathcal{F}$ and $\mathcal{G}$ on M, and the pair $(\mathcal{F}, \mathcal{G})$ is an example of a *bi-Legendrian structure* on M. More generally, a bi-Legendrian structure is a pair of two complementary, not necessarily orthogonal, Legendrian foliations on M.

To any contact manifold (M^{2n+1}, η) endowed with a pair of complementary Legendrian distributions (L, Q), it was attached [7] a linear connection $\overline{\nabla}$ uniquely determined by the following properties:

$$\begin{aligned} &\text{(i)} && \overline{\nabla}L \subset L,\ \overline{\nabla}Q \subset Q,\ \overline{\nabla}(\mathbb{R}\xi) \subset \mathbb{R}\xi;\\ &\text{(ii)} && \overline{\nabla}d\eta = 0;\\ &\text{(iii)} && \overline{T}(X, Y) = 2d\eta(X, Y)\xi,\ \text{ for all } X \in \Gamma(L), Y \in \Gamma(Q),\\ & && \overline{T}(V, \xi) = [\xi, V_L]_Q + [\xi, V_Q]_L,\ \text{ for all } V \in \Gamma(TM). \end{aligned} \tag{2.8}$$

Here $\overline{T}$ denotes the torsion tensor of $\overline{\nabla}$ and V_L and V_Q the projections of V onto the subbundles L and Q of TM, respectively. Such a connection is called the *bi-Legendrian connection* associated with the pair (L, Q) and it is defined as follows (see [7]). For all $V \in \Gamma(TM)$, $X \in \Gamma(L)$ and $Y \in \Gamma(Q)$, $\overline{\nabla}_V X := H(V_L, X)_L + [V_Q, X]_L + [V_{\mathbb{R}\xi}, X]_L$, $\overline{\nabla}_V Y := H(V_Q, Y)_Q + [V_L, Y]_Q + [V_{\mathbb{R}\xi}, Y]_Q$ and $\overline{\nabla}\xi = 0$, where H denotes the operator such that, for all $V, W \in \Gamma(TM)$, $H(V, W)$ is the unique section of $\mathcal{D}$ satisfying $i_{H(V,W)}d\eta|_{\mathcal{D}} = (\mathcal{L}_V i_W d\eta)|_{\mathcal{D}}$. Further properties of this connection are gathered together in the following proposition.

PROPOSITION 2.4 [7]. *Let (M, η) be a contact manifold endowed with two complementary Legendrian distributions L and Q and let $\overline{\nabla}$ denote the corresponding bi-Legendrian connection. Then the 1-form η and the vector field ξ are $\overline{\nabla}$-parallel and the torsion tensor field satisfies $\overline{T}(X, X') = -[X, X']_Q$ for all $X, X' \in \Gamma(L)$ and $\overline{T}(Y, Y') = -[Y, Y']_L$ for all $Y, Y' \in \Gamma(Q)$.*

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Now consider a contact metric manifold (M, ϕ, ξ, η, g) endowed with two complementary Legendrian distributions L and Q. The definition of the corresponding bi-Legendrian connection does not involve the compatible metric g; however, it makes sense to find conditions which ensure that $\overline{\nabla}$ is a metric connection at least when Q is orthogonal to L. This problem was solved in [9] where the author proves the following result.

PROPOSITION 2.5. *Let (M, ϕ, ξ, η, g) be a contact metric manifold and L be a Legendrian distribution on M. Let $Q = \phi L$ be the conjugate Legendrian distribution of L and $\overline{\nabla}$ the associated bi-Legendrian connection. Then the following statements are equivalent:*

(i) $\overline{\nabla} g = 0$;
(ii) $\overline{\nabla} \phi = 0$;
(iii) *g is a bundle-like metric with respect both to the distribution $L \oplus \mathbb{R}\xi$ and to $Q \oplus \mathbb{R}\xi$;*
(iv) *$\overline{\nabla}_X X' = -(\phi[X, \phi X'])_L$ for all $X, X' \in \Gamma(L)$, $\overline{\nabla}_Y Y' = -(\phi[Y, \phi Y'])_Q$ for all $Y, Y' \in \Gamma(Q)$ and the operator h maps the subbundle L onto L and the subbundle Q onto Q.*

Furthermore, assuming that L and Q are integrable, (i)–(iv) *are equivalent to the total geodesicity of the Legendrian foliations defined by L and Q.*

By a *bi-Legendrian manifold* we mean a contact manifold endowed with two transversal Legendrian foliations. In particular, in this paper we deal with contact metric manifolds foliated by two mutually orthogonal Legendrian foliations. In this regard, it will be useful to prove the following lemma, which states essentially that in a bi-Legendrian manifold the operator h is deeply linked to the given bi-Legendrian structure. This is the starting point for our work.

LEMMA 2.6. *Let $\mathcal{F}$ and $\mathcal{G}$ be two mutually orthogonal Legendrian foliations on the contact metric manifold (M, ϕ, ξ, η, g). Then, for all $X, X' \in \Gamma(T\mathcal{F})$,*

$$\Pi_{\mathcal{F}}(X, X') - \Pi_{\mathcal{G}}(\phi X, \phi X') = 4g(hX, X'). \tag{2.9}$$

PROOF. By the orthogonality between $\mathcal{F}$ and $\mathcal{G}$, $\phi(T\mathcal{F}) = T\mathcal{G}$. Using (2.7), therefore,

$$\begin{aligned}\Pi_{\mathcal{F}}(X, X') - \Pi_{\mathcal{G}}(\phi X, \phi X') &= 2g([\xi, X], \phi X') - 2g([\xi, \phi X], \phi^2 X') \\ &= 2g([\xi, \phi X], X') - 2g(\phi[\xi, X], X') \\ &= 4g(hX, X'). \qquad \square\end{aligned}$$

COROLLARY 2.7. *If M is K-contact then $\mathcal{F}$ and $\mathcal{G}$ belong to the same class according to Pang's classification (see above).*

COROLLARY 2.8. *If $\mathcal{F}$ and $\mathcal{G}$ are both flat then M is K-contact.*

3. On the bi-Legendrian structure associated with a contact metric (κ, μ)-space

Let (M, ϕ, ξ, η, g) be a contact metric manifold such that ξ belongs to the (κ, μ)-nullity distribution. By Theorem 2.1 the orthogonal distributions $\mathcal{D}_\lambda$ and $\mathcal{D}_{-\lambda}$ defined by the eigenspaces of h are involutive and define on M two orthogonal Legendrian foliations which we denote by $\mathcal{F}_\lambda$ and $\mathcal{F}_{-\lambda}$, respectively. In this section we begin the study of the bi-Legendrian manifold $(M, \mathcal{F}_\lambda, \mathcal{F}_{-\lambda})$.

PROPOSITION 3.1. *Let (M, ϕ, ξ, η, g) be a contact metric (κ, μ)-space which is not K-contact. Then the Legendrian foliations $\mathcal{F}_\lambda$ and $\mathcal{F}_{-\lambda}$ are either nondegenerate or flat. Specifically, $\mathcal{F}_\lambda$ ($\mathcal{F}_{-\lambda}$) is flat if and only if $\kappa + \mu\lambda - (\lambda + 1)^2 = 0$ ($\kappa - \mu\lambda - (\lambda - 1)^2 = 0$), and nondegenerate otherwise.*

PROOF. Let $X \in \Gamma(\mathcal{D}_\lambda)$. Then, by (1.1),

$$R_{X\xi}\xi = \kappa X + \mu h X = (\kappa + \mu\lambda)X.$$

On the other hand, using (2.3),

$$\begin{aligned} R_{X\xi}\xi &= -\nabla_\xi \nabla_X \xi - \nabla_{[X,\xi]}\xi \\ &= \nabla_\xi \phi X + \lambda \nabla_\xi \phi X + \phi[X, \xi] + \phi h[X, \xi] \\ &= X - \lambda X - [\phi X, \xi] + \lambda X - \lambda^2 X - \lambda[\phi X, \xi] + \phi[X, \xi] + \phi h[X, \xi] \\ &= (\lambda + 1)^2 X - \lambda\phi[X, \xi] + \phi h[X, \xi], \end{aligned}$$

so that

$$\phi h[X, \xi] = \lambda\phi[X, \xi] + (\kappa + \mu\lambda - (\lambda + 1)^2)X.$$

Hence, applying ϕ and taking into account that $[X, \xi] \in \Gamma(\mathcal{D})$,

$$-h[X, \xi] = -\lambda[X, \xi] + (\kappa + \mu\lambda - (\lambda + 1)^2)\phi X.$$

Decomposing $[X, \xi]$ in the directions of $\mathcal{D}_\lambda$ and $\mathcal{D}_{-\lambda}$ gives

$$-h([X, \xi]_{\mathcal{D}_\lambda} + [X, \xi]_{\mathcal{D}_{-\lambda}}) = -\lambda[X, \xi] + (\kappa + \mu\lambda - (\lambda + 1)^2)\phi X,$$

from which it follows that

$$2\lambda[X, \xi]_{\mathcal{D}_{-\lambda}} = (\kappa + \mu\lambda - (\lambda + 1)^2)\phi X, \tag{3.1}$$

and we conclude, by Lemma 2.3, that $\mathcal{F}_\lambda$ is either flat or nondegenerate. The first case occurs if and only if $\kappa + \mu\lambda - (\lambda + 1)^2 = 0$ and the second if and only if $\kappa + \mu\lambda - (\lambda + 1)^2 \neq 0$. In a similar way one can prove the analogous results for $\mathcal{F}_{-\lambda}$. □

REMARK 3.1. From Corollary 2.7 it follows that the bi-Legendrian structure $(\mathcal{F}_\lambda, \mathcal{F}_{-\lambda})$ is flat if and only if $\kappa = 1$ and hence M is Sasakian. This can also be proved directly by observing that, by Proposition 3.1, the functions

$$f(\kappa, \mu) = \kappa + \mu\lambda - \lambda(\lambda + 1)^2 = 2(\kappa - 1) + (\mu - 2)\sqrt{1 - \kappa}$$

and

$$g(\kappa, \mu) = \kappa - \mu\lambda - \lambda(\lambda - 1)^2 = 2(\kappa - 1) + (2 - \mu)\sqrt{1 - \kappa}$$

both vanish if and only if $\kappa = 1$.

Proposition 3.1 extends and improves the results obtained in [12] for contact metric manifolds for which ξ belongs to the κ-nullity distribution (see [19]), that is, the Levi-Civita connection of g satisfies $R_{VW}\xi = \kappa(\eta(W)V - \eta(V)W)$. Jayne proved in [12] that the bi-Legendrian structure associated with such contact metric manifolds is nondegenerate; we recall that in his proof he used the fact that the nondegenerate plane sections containing ξ have constant sectional curvature and this last property does not hold for contact metric (κ, μ)-spaces, as proved in [2].

We remark also that an explicit expression for the invariants $\Pi_{\mathcal{F}_\lambda}$ and $\Pi_{\mathcal{F}_{-\lambda}}$ of the Legendrian foliations $\mathcal{F}_\lambda$ and $\mathcal{F}_{-\lambda}$ follows from the proof of Proposition 3.1. Specifically, from (3.1) and (2.7) one can prove the following proposition.

Proposition 3.2. *Let (M, ϕ, ξ, η, g) be a contact metric (κ, μ)-space which is not K-contact. Then the canonical invariants associated with the Legendrian foliations $\mathcal{F}_\lambda$ and $\mathcal{F}_{-\lambda}$ are given by*

$$\Pi_{\mathcal{F}_\lambda} = \frac{(\lambda + 1)^2 - \kappa - \mu\lambda}{\lambda} g|_{\mathcal{F}_\lambda \times \mathcal{F}_\lambda} \quad \text{and} \tag{3.2}$$
$$\Pi_{\mathcal{F}_{-\lambda}} = \frac{-(\lambda - 1)^2 + \kappa - \mu\lambda}{\lambda} g|_{\mathcal{F}_{-\lambda} \times \mathcal{F}_{-\lambda}},$$

respectively.

It should be remarked that the pair $(\Pi_{\mathcal{F}_\lambda}, \Pi_{\mathcal{F}_{-\lambda}})$ is an invariant of the contact metric (κ, μ)-space in question up to $\mathcal{D}$-homothetic deformations. Indeed, let $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ be a $\mathcal{D}$-homothetic deformation of (ϕ, ξ, η, g). Then, first of all, since $\widetilde{h} = (1/2)\mathcal{L}_{\widetilde{\xi}}\widetilde{\phi} = (1/a)h$ (see [2]), the eigenvalues of $\widetilde{h}$ are $\pm\widetilde{\lambda} = \pm(1/a)\lambda$, in addition to 0. It follows that the eigenspaces $\mathcal{D}_{\widetilde{\lambda}}$ and $\mathcal{D}_{-\widetilde{\lambda}}$ coincide with $\mathcal{D}_\lambda$ and $\mathcal{D}_{-\lambda}$, respectively. Next, for all $X, X' \in \Gamma(\mathcal{D}_{\widetilde{\lambda}}) = \Gamma(\mathcal{D}_\lambda)$, the equalities $\Pi_{\mathcal{F}_{\widetilde{\lambda}}}(X, X') = -\widetilde{\eta}([X', [X, \widetilde{\xi}]]) = -a\eta((1/a)[X', [X, \xi]]) = \Pi_{\mathcal{F}_\lambda}(X, X')$ hold. Analogously one can prove that $\Pi_{\mathcal{F}_{-\widetilde{\lambda}}} = \Pi_{\mathcal{F}_{-\lambda}}$. Moreover, it should be observed that the invariant $\Pi_\mathcal{F}$ of any Legendrian foliation $\mathcal{F}$ depends only on the Legendrian foliation and on the contact form η and not on the associated metric g. In particular, the function

$$\frac{\Pi_{\mathcal{F}_\lambda}(X, X') + \Pi_{\mathcal{F}_{-\lambda}}(\phi X, \phi X')}{\Pi_{\mathcal{F}_\lambda}(X, X') - \Pi_{\mathcal{F}_{-\lambda}}(\phi X, \phi X')}, \tag{3.3}$$

for all $X, X' \in \Gamma(\mathcal{D}_\lambda)$ such that $\Pi_{\mathcal{F}_\lambda}(X, X') \neq 0$ (or, equivalently, $g(X, X') \neq 0$), is an invariant of the bi-Legendrian manifold M up to $\mathcal{D}$-homothetic deformations and does not depend on the vector fields $X, X' \in \Gamma(\mathcal{D}_\lambda)$. Indeed, a straightforward computation, taking into account Lemma 2.6, (3.2) and (2.1), shows that (3.3) is a constant,

$$\frac{\Pi_{\mathcal{F}_\lambda}(X, X') + \Pi_{\mathcal{F}_{-\lambda}}(\phi X, \phi X')}{\Pi_{\mathcal{F}_\lambda}(X, X') - \Pi_{\mathcal{F}_{-\lambda}}(\phi X, \phi X')} = \frac{1 - (\mu/2)}{4\sqrt{1 - \kappa}} = \frac{1}{4} I_M,$$

where I_M is the invariant introduced by Boeckx in [5] for classifying contact metric (κ, μ)-spaces. In particular, if $\mathcal{F}_\lambda$ ($\mathcal{F}_{-\lambda}$) is flat then I_M attains the value 4 (-4). Moreover, we can also give an explicit formula for the constant μ in terms of Legendrian foliations

$$\mu = \frac{\Pi_{\mathcal{F}_\lambda}(X, X')}{g(hX, X')} = \frac{\Pi_{\mathcal{F}_\lambda}(X, X')}{\lambda g(X, X')}, \tag{3.4}$$

for all $X, X' \in \Gamma(\mathcal{D}_\lambda)$ such that $g(X, X') \neq 0$.

4. An interpretation of contact metric (κ, μ)-spaces

Let (M, ϕ, ξ, η, g) be a contact metric (κ, μ)-space. We can attach to the bi-Legendrian structure $(\mathcal{F}_\lambda, \mathcal{F}_{-\lambda})$ the corresponding bi-Legendrian connection $\overline{\nabla}$, that is, the unique linear connection on M such that properties (2.8) hold. Furthermore, we have the following result.

PROPOSITION 4.1. *Let (M, ϕ, ξ, η, g) be a contact metric (κ, μ)-space and let $\overline{\nabla}$ be the bi-Legendrian connection associated with M. Then the tensors ϕ, h and g are $\overline{\nabla}$-parallel. Moreover, for the torsion tensor of $\overline{\nabla}$, one has $\overline{T}(Z, Z') = 2\Phi(Z, Z')\xi$ for all $Z, Z' \in \Gamma(\mathcal{D})$.*

PROOF. A well-known property of $\mathcal{F}_\lambda$ and $\mathcal{F}_{-\lambda}$ is that they are totally geodesic foliations (see [2]). Thus, applying Proposition 2.5 gives $\overline{\nabla} g = 0$ and $\overline{\nabla}\phi = 0$. Next, for all $V \in \Gamma(TM)$, $X \in \Gamma(\mathcal{D}_+)$, $Y \in \Gamma(\mathcal{D}_-)$,

$$(\overline{\nabla}_V h)X = \overline{\nabla}_V hX - h\overline{\nabla}_V X = \overline{\nabla}_V(\lambda X) - \lambda\overline{\nabla}_V X = 0,$$
$$(\overline{\nabla}_V h)Y = \overline{\nabla}_V hY - h\overline{\nabla}_V Y = \overline{\nabla}_V(-\lambda Y) + \lambda\overline{\nabla}_V Y = 0,$$

since $\overline{\nabla}$ preserves $\mathcal{F}_\lambda$ and $\mathcal{F}_{-\lambda}$. Finally, for any $f \in C^\infty(M)$,

$$(\overline{\nabla}_V h)f\xi = \overline{\nabla}_V(h(f\xi)) - h(\overline{\nabla}_V(f\xi)) = -h(f\overline{\nabla}_V\xi) - V(f)h\xi = 0,$$

since $\overline{\nabla}\xi = 0$ and $h\xi = 0$. It remains to prove the property for the torsion, but this follows easily from Proposition 2.4 and from the integrability of $\mathcal{D}_\lambda$ and $\mathcal{D}_{-\lambda}$. □

COROLLARY 4.2. *With the assumptions and notation of Proposition 4.1, the connection $\overline{\nabla}$ is related to the Levi-Civita connection of (M, ϕ, ξ, η, g) by the following formula, for all $X, Y \in \Gamma(\mathcal{D})$:*

$$\overline{\nabla}_X Y = \nabla_X Y - \eta(\nabla_X Y)\xi. \tag{4.1}$$

PROOF. Since $\overline{\nabla}$ is torsion-free along the leaves of the foliations $\mathcal{F}_\lambda$ and $\mathcal{F}_{-\lambda}$, and is metric by Proposition 4.1, it coincides with the Levi-Civita connection along the leaves of $\mathcal{F}_\lambda$ and $\mathcal{F}_{-\lambda}$. Hence (4.1) holds for all $X, Y \in \Gamma(\mathcal{D}_\lambda)$ or $X, Y \in \Gamma(\mathcal{D}_{-\lambda})$ because

$\mathcal{F}_\lambda$ and $\mathcal{F}_{-\lambda}$ are totally geodesic foliations. Now let $X \in \Gamma(\mathcal{D}_\lambda)$ and $Y \in \Gamma(\mathcal{D}_{-\lambda})$. It is well known (see [2]) that $\nabla_X Y \in \Gamma(\mathcal{D}_{-\lambda} \oplus \mathbb{R}\xi)$. For all $Y' \in \Gamma(\mathcal{D}_{-\lambda})$, using $\overline{\nabla} g = 0$,

$$\begin{aligned} 2g(\nabla_X Y, Y') &= X(g(Y, Y')) + Y(g(X, Y')) - Y'(g(X, Y)) + g([X, Y], Y') \\ &\quad + g([Y', X], Y) - g([Y, Y'], X) \\ &= X(g(Y, Y')) + g([X, Y], Y') + g([Y', X], Y) \\ &= X(g(Y, Y')) - g([X, Y']_{\mathcal{D}_{-\lambda}}, Y) + g([X, Y]_{\mathcal{D}_{-\lambda}}, Y') \\ &= 2g([X, Y]_{\mathcal{D}_{-\lambda}}, Y') \\ &= 2g(\overline{\nabla}_X Y, Y'), \end{aligned}$$

from which it follows that $\overline{\nabla}_X Y = (\nabla_X Y)_{\mathcal{D}_{-\lambda}}$ and hence (4.1). Analogously one can prove (4.1) for $X \in \Gamma(\mathcal{D}_{-\lambda})$ and $Y \in \Gamma(\mathcal{D}_\lambda)$. □

Now we examine an 'inverse' problem, in a certain sense. We start with a bi-Legendrian structure on an arbitrary contact metric manifold M and ask whether M is a contact metric (κ, μ)-space for some $\kappa, \mu \in \mathbb{R}$.

THEOREM 4.3. *Let (M, ϕ, ξ, η, g) be a contact metric manifold, non-K-contact, endowed with two orthogonal Legendrian foliations $\mathcal{F}$ and $\mathcal{G}$, and suppose that the bi-Legendrian connection corresponding to $(\mathcal{F}, \mathcal{G})$ satisfies $\overline{\nabla}\phi = 0$ and $\overline{\nabla} h = 0$. Then (M, ϕ, ξ, η, g) is a contact metric (κ, μ)-space. Furthermore, the bi-Legendrian structure $(\mathcal{F}, \mathcal{G})$ coincides with that determined by the eigenspaces of h.*

PROOF. Firstly, we prove that under our assumptions (4.1) holds. Since, by Proposition 2.5, $\overline{\nabla} g = 0$ and $\overline{T}(X, X') = 0$, $\overline{T}(Y, Y') = 0$ for all $X, X' \in \Gamma(T\mathcal{F})$ and $Y, Y' \in \Gamma(T\mathcal{G})$, it follows immediately that the bi-Legendrian connection and the Levi-Civita connection coincide along the leaves of $\mathcal{F}$ and $\mathcal{G}$. Moreover, for all $X \in \Gamma(T\mathcal{F})$ and $Y \in \Gamma(T\mathcal{G})$, $\nabla_X Y \in \Gamma(T\mathcal{G} \oplus \mathbb{R}\xi)$ because, for all $X' \in \Gamma(T\mathcal{F})$,

$$g(\nabla_X Y, X') = X(g(Y, X')) - g(Y, \nabla_X X') = 0,$$

since $\mathcal{F}$, as well as $\mathcal{G}$, is totally geodesic by Proposition 2.5. Then one can argue as in the proof of Corollary 4.2 and prove that

$$\nabla_Z Z' = \overline{\nabla}_Z Z' + \eta(\nabla_Z Z')\xi, \tag{4.2}$$

for all $Z, Z' \in \Gamma(\mathcal{D})$. Now for all $X, Y, Z \in \Gamma(\mathcal{D})$, applying (4.2),

$$\begin{aligned} g((\nabla_X h)Y, Z) &= g(\nabla_X hY - h\nabla_X Y, Z) \\ &= g(\overline{\nabla}_X hY + \eta(\nabla_X hY)\xi - h\overline{\nabla}_X Y - \eta(\nabla_X Y)h\xi, Z) \\ &= g((\overline{\nabla}_X h)Y, Z) + \eta(\nabla_X hY)\eta(Z) \\ &= g((\overline{\nabla}_X h)Y, Z) = 0, \end{aligned}$$

since, by assumption, $\overline{\nabla} h = 0$. Thus the tensor field h is η-parallel and so, by [6, Theorem 4], (M, ϕ, ξ, η, g) is a contact metric (κ, μ)-space. To prove the last part

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of the theorem, suppose for the sake of argument that $\mathcal{F}$ does not coincide with both $\mathcal{F}_\lambda$ and $\mathcal{F}_{-\lambda}$. Let X be a vector field tangent to $\mathcal{F}$ and decompose it as $X = X_+ + X_-$, with $X_+ \in \Gamma(\mathcal{D}_\lambda)$ and $X_- \in \Gamma(\mathcal{D}_{-\lambda})$. Then $hX = h(X_+) + h(X_-) = \lambda X_+ - \lambda X_- = \lambda(X_+ - X_-)$, from which, since by Proposition 2.5 h preserves $\mathcal{F}$, it follows that $X_+ - X_- \in \Gamma(T\mathcal{F})$. On the other hand, $X_+ + X_- = X \in \Gamma(T\mathcal{F})$, hence X_+ and X_- are both tangent to $\mathcal{F}$, and this is a contradiction. □

From Theorem 4.3 we get the following characterization of contact metric (κ, μ)-spaces. Here, in an abuse of terminology, we call any n-dimensional subbundle L of the distribution $\mathcal{D} = \ker(\eta)$ such that $d\eta(X, X') = 0$ for all $X, X' \in \Gamma(L)$ a Legendrian distribution of an almost contact manifold, and, as in contact metric geometry, define $2h$ as the Lie differentiation of the tensor ϕ along the Reeb vector field ξ.

THEOREM 4.4. *Let (M, ϕ, ξ, η, g) be an almost contact metric manifold with ξ non-Killing. Then (M, ϕ, ξ, η, g) is a contact metric (κ, μ)-space if and only if it admits two orthogonal conjugate Legendrian distributions L and Q and a linear connection $\widetilde{\nabla}$ satisfying the following properties:*

(i) $\widetilde{\nabla} L \subset L$, $\widetilde{\nabla} Q \subset Q$;
(ii) $\widetilde{\nabla}\eta = 0$, $\widetilde{\nabla} d\eta = 0$, $\widetilde{\nabla} g = 0$, $\widetilde{\nabla} h = 0$;
(iii) $\widetilde{T}(Z, Z') = 2\Phi(Z, Z')\xi$ *for all* $Z, Z' \in \Gamma(\mathcal{D})$,
$\widetilde{T}(V, \xi) = [\xi, V_L]_Q + [\xi, V_Q]_L$ *for all* $V \in \Gamma(TM)$,

where $\widetilde{T}$ denotes the torsion tensor field of $\widetilde{\nabla}$. Furthermore, $\widetilde{\nabla}$ is uniquely determined, and L and Q are integrable and coincide with the eigenspaces of the operator h.

PROOF. The proof is rather obvious in one direction: it is sufficient to take as $\widetilde{\nabla}$ the bi-Legendrian connection associated with the bi-Legendrian structure defined by the eigenspaces of h. We now prove the converse. Note that by (ii) it follows also that ξ is parallel with respect to $\widetilde{\nabla}$, since, for any $V \in \Gamma(TM)$, $(\widetilde{\nabla}_V \eta)\xi = -\eta(\widetilde{\nabla}_V \xi) = 0$, so $\widetilde{\nabla}_V \xi \in \Gamma(\mathcal{D})$. On the other hand, for any $Z \in \Gamma(\mathcal{D})$, since $\widetilde{\nabla}$ is a metric connection and preserves the subbundle $\mathcal{D} = L \oplus Q$,

$$g(\widetilde{\nabla}_V \xi, Z) = V(g(\xi, Z)) - g(\xi, \widetilde{\nabla}_V Z) = 0,$$

from which $\widetilde{\nabla}_V \xi$ is also orthogonal to $\mathcal{D}$ and hence vanishes. Now we can prove the result. We show first that $d\eta = \Phi$, so M is a contact metric manifold. For any $X, X' \in \Gamma(L)$ and $Y, Y' \in \Gamma(Q)$, $d\eta(X, X') = 0 = g(X, \phi X')$ and $d\eta(Y, Y') = 0 = g(Y, \phi Y')$. Moreover,

$$2\Phi(X, Y)\xi = \widetilde{T}(X, Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y],$$

from which

$$2\Phi(X, Y) = g(\widetilde{\nabla}_X Y, \xi) - g(\widetilde{\nabla}_Y X, \xi) - g([X, Y], \xi). \tag{4.3}$$

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Now $g(\widetilde{\nabla}_X Y, \xi) = X(g(Y, \xi)) - g(Y, \widetilde{\nabla}_X \xi) = 0$ and, analogously, $g(\widetilde{\nabla}_Y X, \xi) = 0$, so that (4.3) becomes

$$2\Phi(X, Y) = -\eta([X, Y]),$$

from which it follows that $d\eta(X, Y) = \Phi(X, Y)$. To conclude that (M, ϕ, ξ, η, g) is a contact metric manifold it remains to check that $d\eta(Z, \xi) = \Phi(Z, \xi)$ for any $Z \in \Gamma(\mathcal{D})$. Indeed, $d\eta(Z, \xi) = -(1/2)\eta([Z, \xi]) = 0 = \Phi(Z, \xi)$ since

$$[Z, \xi] = \widetilde{\nabla}_Z \xi - \widetilde{\nabla}_\xi Z - \widetilde{T}(Z, \xi) = -\widetilde{\nabla}_\xi Z - [\xi, Z_L]_Q - [\xi, Z_Q]_L \in \Gamma(\mathcal{D}),$$

because of (i). Therefore (M, ϕ, ξ, η, g) is a contact metric manifold endowed with two complementary (in particular, orthogonal) Legendrian distributions L and Q, and since $\widetilde{\nabla}\xi = 0$ the connection $\widetilde{\nabla}$ coincides with the bi-Legendrian connection $\overline{\nabla}$ associated with (L, Q). This fact and (iii) imply the integrability of L and Q. Indeed, for any $X, X' \in \Gamma(L)$,

$$[X, X']_Q = -\overline{T}(X, X') = -\widetilde{T}(X, X') = -2d\eta(X, X')\xi = 0$$

and

$$g([X, X'], \xi) = \eta([X, X']) = -2d\eta(X, X') = 0,$$

hence $[X, X'] \in \Gamma(L)$, and in a similar manner one can prove the integrability of Q. Thus L and Q define two orthogonal Legendrian foliations on M and now the result follows from Theorem 4.3. □

The connection $\widetilde{\nabla}$ is, from certain points of view, an 'invariant' of the contact metric (κ, μ)-space modulo $\mathcal{D}$-homothetic deformations. Indeed, a direct computation leads to the following result.

PROPOSITION 4.5. *The bi-Legendrian connection associated with a contact metric (κ, μ)-space remains unchanged under a $\mathcal{D}$-homothetic deformation.*

The connection stated in Theorem 4.4 should be compared to the Tanaka–Webster connection of a nondegenerate integrable CR-manifold (see [17, 22]) and to the generalized Tanaka–Webster connection introduced by Tanno in [20]. This can be seen in the following theorem, where we prove, using Theorem 4.4, the already quoted result that any contact metric (κ, μ)-space is a strongly pseudo-convex CR-manifold.

COROLLARY 4.6. *Any contact metric (κ, μ)-space is a strongly pseudo-convex CR-manifold.*

PROOF. We define a connection on M as follows. Put

$$\widehat{\nabla}_V W = \begin{cases} \overline{\nabla}_V W & \text{if } V \in \Gamma(\mathcal{D}), \\ -\phi h W + [\xi, W] & \text{if } V = \xi. \end{cases}$$

Then it easy to check that $\widehat{\nabla}$ coincides with the Tanaka–Webster connection of M and so we get the assertion. □

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The above characterization may be also a tool for proving properties on (κ, μ)-spaces. As an application we show in a very simple way that an invariant submanifold of a contact metric (κ, μ)-space, that is, a submanifold N such that $\phi T_p N \subset T_p N$ for all $p \in N$, is in turn a contact metric (κ, μ)-space (see [21]).

COROLLARY 4.7. *Any invariant submanifold of a contact metric (κ, μ)-space is in turn a (κ, μ)-space.*

PROOF. It is well known (see [3]) that an invariant submanifold of a contact metric manifold inherits a contact metric structure by restriction. Now let N^{2m+1} be an invariant submanifold of M^{2n+1} and consider the distribution on N given by $L_x := T_x N \cap \mathcal{D}_{\lambda x}$ and $Q_x := T_x N \cap \mathcal{D}_{-\lambda x}$ for all $x \in N$. It is easy to check that L and Q define two mutually orthogonal Legendrian foliations of N^{2m+1} and that the bi-Legendrian connection corresponding to (L, Q) is just the connection induced on N by the bi-Legendrian connection associated with $(\mathcal{D}_\lambda, \mathcal{D}_{-\lambda})$. The result now follows from Theorem 4.4. □

We conclude by showing that the assumption in Theorem 4.4 that ξ must be non-Killing is essential. This can be seen in the following example.

EXAMPLE 4.1. Consider the sphere

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

with the following Sasakian structure:

$$\eta = x_3 dx_1 + x_4 dx_2 - x_1 dx_3 - x_2 dx_4, \quad \xi = x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} - x_2 \frac{\partial}{\partial x_4},$$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Set

$$X := x_2(\partial/\partial x_1) - x_1(\partial/\partial x_2) - x_4(\partial/\partial x_3) + x_3(\partial/\partial x_4)$$

and

$$Y := \phi X = x_4(\partial/\partial x_1) - x_3(\partial/\partial x_2) + x_2(\partial/\partial x_3) - x_1(\partial/\partial x_4),$$

and consider the one-dimensional distributions L and Q on S^3 generated by X and Y, respectively. An easy computation shows that $[X, \xi] = -2Y$, $[Y, \xi] = 2X$, $[X, Y] = 2\xi$. Thus L and Q define two nondegenerate, orthogonal Legendrian foliations on the Sasakian manifold $(S^3, \phi, \xi, \eta, g)$. For the bi-Legendrian connection corresponding to this bi-Legendrian structure, a straightforward computation leads to $\overline{\nabla}_X X = \overline{\nabla}_X Y = \overline{\nabla}_X \xi = 0$ and $\overline{\nabla}_Y X = \overline{\nabla}_Y Y = \overline{\nabla}_Y \xi = 0$. Therefore $\overline{\nabla}\phi = 0$ and so, by Proposition 2.5, also $\overline{\nabla} g = 0$. Moreover, as ξ is Killing, obviously $\overline{\nabla} h = 0$.

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