Contact properties of surfaces in $\mathbb{R}^3$ with corank 1 singularities

Luciana F. Martins* and Juan J. Nuño-Ballesteros†

February 22, 2013

Abstract

We study the geometry of surfaces in $\mathbb{R}^3$ with corank 1 singularities. At a singular point we define the curvature parabola using the first and second fundamental forms of the surface, which contains all the local second order geometrical information about the surface. The curvature parabola is used to introduce the concepts of asymptotic directions and umbilic curvature, which are related to contact properties of the surface with planes and spheres.

1 Introduction

The aim of this paper is to introduce a new tool for the study of the differential geometry of second order of surfaces in Euclidean 3-space with corank 1 singularities: the *curvature parabola*. It induces the definitions of the *asymptotic directions* and of the *umbilic curvature function* on the singular set of the surface. With these tools we recover some of the results in [4, 12, 15] and obtain some other geometrical properties of the surface.

Surfaces in $\mathbb{R}^3$ are often defined explicitly as the image of a smooth mapping $f : U \to \mathbb{R}^3$, possibly with singularities, where U is an open subset of $\mathbb{R}^2$. Two maps germs $f, g : (\mathbb{R}^2, q) \to (\mathbb{R}^3, p)$ are said to be $\mathcal{A}$-equivalent, denoted by $f \sim g$, if $g = \phi \circ f \circ \psi^{-1}$ for some germs of diffeomorphisms ψ and ϕ of the source and target, respectively. D. Mond gave in [10] a list of normal forms of germs of surfaces under the $\mathcal{A}$-equivalence. If we consider surfaces in the same $\mathcal{A}$-orbit, clearly these surfaces have diffeomorphic image but may not have the same local differential geometry. So, if we are interested in the study of the *geometry* of the image, and no merely in its diffeomorphism type, we can not take a normal form given in [10] as a parametrisation for the surface.

*Work partially supported by FAPESP and CAPES.

†Work partially supported by DGICYT Grant MTM2012-33073 and MEC Grant PHB2009-0005-PC.

2010 Mathematics Subject classification. Primary 58K05; Secondary 57R45, 53A05.

Keywords and Phrase. Singular surfaces, umbilic curvature, curvature parabola.

The cross-cap is a singular surface image of any map germ $g : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ which is $\mathcal{A}$-equivalent to the normal form $f(x, y) = (x, y^2, xy)$ given in [10]. The geometry of the cross-cap was carried out, for instance, in [3, 5, 6, 7, 11, 12, 15]. It is shown in [3] that there are generically two types of cross-caps, one labeled hyperbolic cross-cap where all non-singular points of the immersed surface are hyperbolic, and the other labeled elliptic cross-cap where the parabolic set consists of two smooth curves meeting tangentially at the singularity and partitions the surface into hyperbolic and elliptic regions. The passage from one type to another is realized at a parabolic cross-cap whose parabolic set has a cusp. This classification turned out to be very useful when seeking to understand the projections of smooth two dimensional surfaces in $\mathbb{R}^4$ to 3-spaces and to obtain geometric information about the surface (see [11]).

The differential geometry of cross-caps is also investigated in [4], where the authors obtain several criteria of the singularity types of fronts of cross-caps in terms of differential geometric language. They also study singularities of the distance squared unfolding and investigate the focal set of cross-caps.

In [12] the authors study the flat geometry and the singularities of the parabolic set in the source as well as those of the height functions on surfaces parametrised by map-germs $\mathcal{A}$-equivalent to one of the normal forms given in [10]. More specifically, they consider the contact of theses surfaces with planes and then they apply their results to the flat geometry of surfaces in $\mathbb{R}^4$.

Let $M \subset \mathbb{R}^3$ be a surface with corank 1 singularities non necessarily isolated). This means that M is the image of a smooth map $g : \widetilde{M} \to \mathbb{R}^3$ from a smooth regular surface $\widetilde{M}$ whose differential has rank ≥ 1 at any point. Hence, the tangent space T_pM at a singularity p degenerates to a line and so there is a plane N_pM of directions orthogonal to T_pM. In Section 2 we consider the *first* and *second fundamental forms* of M at p and, using them, we define the *curvature parabola* Δ_p as a subset of N_pM (see Definition 2.2). The curvature parabola is in fact a parabola, which can degenerate in a half-line, line or a point. It is worth observing that the definitions of the fundamental forms do not depend on the choice of local coordinates on $\widetilde{M}$ and so neither the curvature parabola (although they may depend on the map g which parametrises M, see Example 2.3).

D. Mond gives in [10] a partition in four orbits of the set of all corank 1 map germs $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ according to their 2-jets under the action of $\mathcal{A}^2$, which is the space of 2-jets of diffeomorphisms in the source and target (see Proposition 2.4). We show that the curvature parabola can be easily used to distinguish between the four types of corank 1 singularities just by looking at the type of degeneracy of the parabola (Theorem 2.5). Furthermore, we show that two corank 1 2-jets $(\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ are equivalent under the action of the subgroup $\mathcal{R}^2 \times \mathcal{O}(3)$ of 2-jets of diffeomorphisms in the source and linear isometries of $\mathbb{R}^3$ if and only if there exists an isometry between the normal planes preserving the respective curvature parabolas (Theorem 2.7). In this sense we claim that the curvature parabola contains all the local second order geometrical information about the surface.

In Section 3 we look at the geometry of the surface M at a singular point p by means of the analysis of the singularities of the height and the distance-squared functions on the surface. The second order invariants are given in terms of the curvature parabola and its position with relation to the origin in the normal plane of the surface at p. Using the curvature parabola we define the *asymptotic directions* of M at p, the *binormal directions* in N_pM and the *osculating planes*, which are those with a degenerate contact with M at p. Moreover, if the singularity p is a cross-cap, then the asymptotic directions coincide with the limiting tangent directions at p of the parabolic set. When the singularity is not a cross-cap, we also use the curvature parabola to introduce the concept of *umbilic curvature* $\kappa_u(p)$ of M at p. Geometrically, $\kappa_u(p)$ either is the distance between p and the line containing Δ_p, when the curvature parabola is a half-line or a line, or $\kappa_u(p)$ is the distance between p and Δ_p, when it is a point.

The umbilic curvature $\kappa_u(p)$ turns out to be an important invariant: if $\kappa_u(p) \neq 0$, then $1/\kappa_u(p)$ is the radius of the unique sphere with umbilical contact (that is, contact of type $\Sigma^{2,2}$ in Thom-Boardman terminology) with M at p and, if $\kappa_u(p) = 0$, then there is a plane with umbilical contact with M at p (see Theorems 3.11 and 3.15). As an immediate consequence, the singular point is a non-flat 2-rounding of M or a 2-flattening of M respectively, according with the definitions given in [6] (Corollary 3.17).

Our definition of umbilic curvature $\kappa_u(p)$ generalizes the concept of limiting normal curvature defined in [13] for cuspidal edges to corank 1 singularities. This fact follows from the results of the first named author and K. Saji in [8]. In fact, they show that

$$\kappa^2 = \kappa_u^2 + \kappa_s^2,$$

where κ is the curvature of the singular curve of the cuspidal edge and κ_s is the singular curvature in the sense of [13]. This equality can be seen as a singular counterpart of the well known relation $\kappa^2 = \kappa_n^2 + \kappa_g^2$, between the normal and the geodesic curvatures κ_n and κ_g respectively, in the case of a regular surface. Thanks are due to Saji, Umehara and Yamada for pointing out the coincidence between our umbilic curvature and the normal curvature of cuspidal edges which appears in [13].

2 The second fundamental form at a corank 1 singularity

Given a smooth regular surface M in $\mathbb{R}^3$ parametrised by $f(x, y)$, where (x, y) is the usual Cartesian coordinate system of $\mathbb{R}^2$, consider the Gauss map $N : M \to S^2$ given by $N = (f_x \times f_y)/\|f_x \times f_y\|$. At a point $p \in M$, the map $-dN(p) : T_pM \to T_{N(p)}S^2$ can be thought of as an automorphism of T_pM, which is the classical shape operator S_p, or S by simplicity. The first fundamental form $I_p : T_pM \times T_pM \to \mathbb{R}$ is defined as $I_p(u, v) = \langle u, v\rangle$, where $\langle\ ,\ \rangle$ is the Euclidean metric in T_pM induced by that of $\mathbb{R}^3$. The coefficients of I_p are given by

$$E(p) = \langle f_x, f_x\rangle (q)\,, \qquad F(p) = \langle f_x, f_y\rangle (q)\,, \qquad G(p) = \langle f_y, f_y\rangle (q)\,,$$

where $f(q) = p$. The second fundamental form $II_p : T_pM \times T_pM \to \mathbb{R}$ is $II_p(u, v) = \langle S(u), v\rangle$, whose coefficients are given by

$$
\begin{aligned}
l(p) &= \langle S(f_x), f_x\rangle = \langle N, f_{xx}\rangle = \det(f_x, f_y, f_{xx}) / \sqrt{EG - F^2}\,, \\
m(p) &= \langle S(f_x), f_y\rangle = \langle N, f_{xy}\rangle = \det(f_x, f_y, f_{xy}) / \sqrt{EG - F^2}\,, \\
n(p) &= \langle S(f_y), f_y\rangle = \langle N, f_{yy}\rangle = \det(f_x, f_y, f_{yy}) / \sqrt{EG - F^2}\,,
\end{aligned}
$$

with derivatives calculated at q.

In the case of surfaces with singularities, we run into a problem as there is no a well defined normal vector to the surface at singular points.

Let $M \subset \mathbb{R}^3$ be a surface with a singularity of corank 1 at $p \in M$. We assume that M is the image of a C^∞ map $g : \widetilde{M} \to \mathbb{R}^3$, where $\widetilde{M}$ is a smooth regular surface and $q \in \widetilde{M}$ is a singular point of g of corank 1 such that $g(q) = p$. Given a coordinate system $\phi : U \to \mathbb{R}^2$ defined on some open neighbourhood U of q in $\widetilde{M}$, we say that $f = g \circ \phi^{-1}$ is a *local parametrisation of* M *at* p.

We define the *tangent line* to M at p as $T_pM = \operatorname{Im} g_*$, where $g_* : T_q\widetilde{M} \to T_p\mathbb{R}^3$ is the differential of g at q. We also have the *normal plane* N_pM at p, in such a way that $T_p\mathbb{R}^3 = T_pM \oplus N_pM$ and we denote the corresponding orthogonal projections by:

$$
\begin{array}{rcl}
\top : T_p\mathbb{R}^3 & \to & T_pM \\
w & \to & w^\top
\end{array}
\qquad\qquad
\begin{array}{rcl}
\perp : T_p\mathbb{R}^3 & \to & N_pM \\
w & \to & w^\perp
\end{array}
$$

The Euclidean metric of $\mathbb{R}^3$ induces the *first fundamental form* $I : T_q\widetilde{M} \times T_q\widetilde{M} \to \mathbb{R}$ in the obvious way:

$$
I(X, Y) = \langle g_*X, g_*Y\rangle\,, \quad \forall\, X, Y \in T_q\widetilde{M}\,.
$$

However, I is not a Riemannian metric on $T_q\widetilde{M}$, but a pseudometric. Taking a local parametrisation $f = g \circ \phi^{-1}$ of M at p, since $\{\partial_x, \partial_y\}$ provides a basis of $T_q\widetilde{M}$, the coefficients of the first fundamental form with respect to ϕ are:

$$
E(q) = I(\partial_x, \partial_x) = \langle f_x, f_x\rangle\,(\phi(q)), \;\; F(q) = I(\partial_x, \partial_y) = \langle f_x, f_y\rangle\,(\phi(q)),
$$
$$
G(q) = I(\partial_y, \partial_y) = \langle f_y, f_y\rangle\,(\phi(q)).
$$

Notice that if $X = a\partial_x + b\partial_y$ then $I(X, X) = a^2E(q) + 2abF(q) + b^2G(q)$.

Now we define the *second fundamental form* $II : T_q\widetilde{M} \times T_q\widetilde{M} \to N_pM$ *of* M *at* p. Given local coordinates of M at p as before, we define

$$
II(\partial_x, \partial_x) = f_{xx}^\perp(\phi(q))\,, \qquad II(\partial_x, \partial_y) = f_{xy}^\perp(\phi(q))\,, \qquad II(\partial_y, \partial_y) = f_{yy}^\perp(\phi(q))\,,
$$

and we extend II to $T_q\widetilde{M} \times T_q\widetilde{M}$ in a unique way as a symmetric bilinear map. Notice that if $X = a\partial_x + b\partial_y$ then $II(X, X) = a^2 f_{xx}^\perp(\phi(q)) + 2ab f_{xy}^\perp(\phi(q)) + b^2 f_{yy}^\perp(\phi(q))$.

Lemma 2.1 *The definition of the second fundamental form does not depend on the choice of local coordinates on* $\widetilde{M}$*.*

Proof Let $\overline{\phi}$ be another coordinate system of $\widetilde{M}$ at q with coordinates u, v and let us denote $\overline{f} = g \circ \overline{\phi}^{-1}$. We have

$$\overline{f}_u = f_x x_u + f_y y_u, \quad \overline{f}_v = f_x x_v + f_y y_v.$$

Now we compute the second order derivatives:

$$\begin{aligned}\overline{f}_{uu} &= f_x x_{uu} + f_y y_{uu} + f_{xx} x_u^2 + 2 f_{xy} x_u y_u + f_{yy} y_u^2, \\ \overline{f}_{uv} &= f_x x_{uv} + f_y y_{uv} + f_{xx} x_u x_v + f_{xy}(x_u y_v + x_v y_u) + f_{yy} y_u y_v, \\ \overline{f}_{vv} &= f_x x_{vv} + f_y y_{vv} + f_{xx} x_v^2 + 2 f_{xy} x_v y_v + f_{yy} y_v^2.\end{aligned}$$

Finally, we take the orthogonal projection to N_pM:

$$\begin{aligned}\overline{f}_{uu}^{\perp} &= f_{xx}^{\perp} x_u^2 + 2 f_{xy}^{\perp} x_u y_u + f_{yy}^{\perp} y_u^2, \\ \overline{f}_{uv}^{\perp} &= f_{xx}^{\perp} x_u x_v + f_{xy}^{\perp}(x_u y_v + x_v y_u) + f_{yy}^{\perp} y_u y_v, \\ \overline{f}_{vv}^{\perp} &= f_{xx}^{\perp} x_v^2 + 2 f_{xy}^{\perp} x_v y_v + f_{yy}^{\perp} y_v^2.\end{aligned}$$

We see that $\overline{f}_{uu}^{\perp}, \overline{f}_{uv}^{\perp}, \overline{f}_{vv}^{\perp}$ and $f_{xx}^{\perp}, f_{xy}^{\perp}, f_{yy}^{\perp}$ are related by the equations of basis change in a symmetric bilinear map with respect to the matrix

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}.$$

But this is the matrix of basis change from $\{\partial_x, \partial_y\}$ to $\{\partial_u, \partial_v\}$ in $T_q\widetilde{M}$. Hence, both coordinate systems define the same symmetric bilinear map. $\square$

For each normal vector $\nu \in N_pM$, we can consider the form $II_\nu : T_q\widetilde{M} \times T_q\widetilde{M} \to \mathbb{R}$ which we shall call *the second fundamental form of M at p along ν*, given by

$$II_\nu(X, Y) = \langle II(X, Y), \nu \rangle .$$

The coefficients of II_ν in terms of local coordinates (x, y) are:

$$l_\nu(q) = \left\langle f_{xx}^{\perp}, \nu \right\rangle (\phi(q)), \qquad m_\nu(q) = \left\langle f_{xy}^{\perp}, \nu \right\rangle (\phi(q)), \qquad n_\nu(q) = \left\langle f_{yy}^{\perp}, \nu \right\rangle (\phi(q)).$$

Given $X = a\partial_x + b\partial_y$ then it holds that $II_\nu(X, X) = a^2 l_\nu(q) + 2ab m_\nu(q) + b^2 n_\nu(q)$ and, if we fix an orthonormal frame $\{\nu_1, \nu_2\}$ of N_pM then

$$\begin{aligned} II(X, X) &= II_{\nu_1}(X, X)\, \nu_1 + II_{\nu_2}(X, X)\, \nu_2 \\ &= (a^2 l_{\nu_1} + 2ab m_{\nu_1} + b^2 n_{\nu_1})\, \nu_1 + (a^2 l_{\nu_2} + 2ab m_{\nu_2} + b^2 n_{\nu_2})\, \nu_2 \,, \end{aligned} \tag{1}$$

with the above coefficients calculated at q. Furthermore, the second fundamental form can be represented by the following matrix of coefficients:

$$\begin{pmatrix} l_{\nu_1} & m_{\nu_1} & n_{\nu_1} \\ l_{\nu_2} & m_{\nu_2} & n_{\nu_2} \end{pmatrix}.$$

We should remark that this matrix of coefficients will depend on the choices of coordinates on $\widetilde{M}$ and of the orthonormal frame of N_pM, although the second fundamental form does not depend.

2.1 The curvature parabola

Definition 2.2 Let C_q be the subset of unit vectors of $T_q\widetilde{M}$ and let $\eta : C_q \to N_pM$ be the map given by $\eta(X) = II(X,X)$. We define the *curvature parabola* of M at p, which we shall denote by Δ_p, as the image of this map, that is, $\Delta_p = \eta(C_q)$.

It is worth observing that $\Delta_p = \{II(X,X) \,|\, I(X,X)^{1/2} = 1\}$ and that it follows from Lemma 2.1 that the above definition does not depend on the choice of the local coordinates on $\widetilde{M}$.

Example 2.3 Let M be the surface image of $\widetilde{M} = \mathbb{R}^2$ by the map $g(x,y) = (x, y^2, xy)$. So M is the *cross-cap* surface given in Figure 2.3.

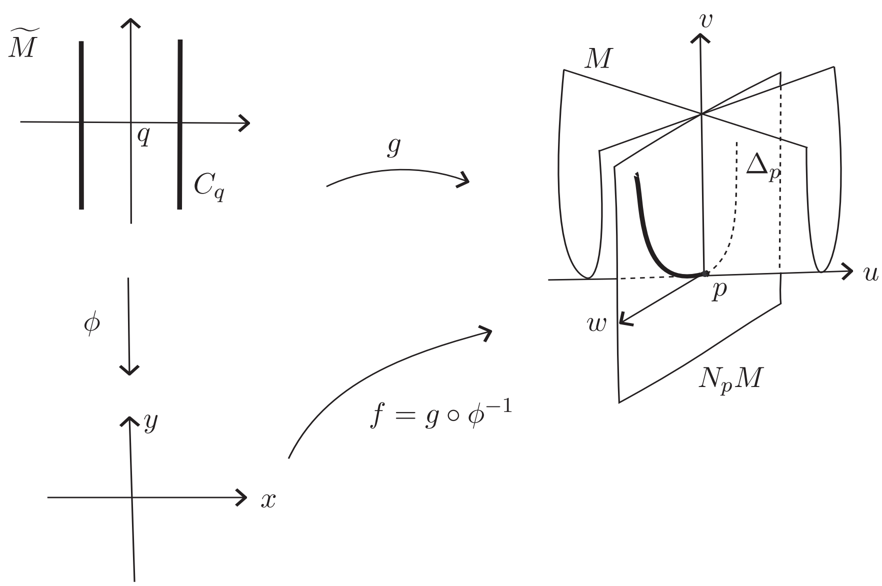

Figure 1: The cross-cap surface of Example 2.3.

Taking coordinates (u,v,w) in $\mathbb{R}^3$ and $q = p = \mathbf{0}$ then the tangent line T_pM to M at p is the u-axis and so, the normal plane N_pM is the vw-plane. Then, for any $X = a\partial_x + b\partial_y \in T_q\widetilde{M}$, it holds that: $E = 1$, $F = G = 0$, $I(X,X) = a^2$ and $II(X,X) = (0, 2b^2, 2ab)$. Therefore $C_q = \{(\pm 1, y);\, y \in \mathbb{R}\}$ and the curvature parabola Δ_p is a non-degenerate parabola which can be parametrised by $\eta(y) = (0, 2y^2, 2y)$.

Other interesting examples are the following:

1. The *cuspidal edge* is the surface M given by the image of $g(x,y) = (x, y^2, y^3)$ and it holds that Δ_p is the half-line parametrised by $\eta(y) = (0, 2y^2, 0)$, for all $p = (x,0,0)$ with $x \in \mathbb{R}$.

2. We can consider now the surface parametrized by $g(x,y) = (x,(y^3+x)^2,(y^3+x)^3)$, whose image set is in fact the same cuspidal edge of the above example. Then for $p = \mathbf{0}$ we have that Δ_p is just the point $\{(0,2,0)\}$. Thus, although Δ_p does not depend on the choice of local coordinates in $\widetilde{M}$, it depends on the map g which parametrises M.

3. The *swallowtail* is the surface M defined as the image of $g(x,y) = (x, 2xy+4y^3, 3y^4+xy^2)$. Then $p = \mathbf{0}$ is a corank 1 singularity of M and it is easy to check that $\eta(y) = (0,4y,0)$ is a parametrisation for Δ_p, which is a line in N_pM through the origin.

In general, since g has corank 1 at q, we can choose the coordinate system ϕ and make rotations in $\mathbb{R}^3$ in such a way that $f(x,y) = g\circ\phi^{-1}(x,y) = (x, f_2(x,y), f_3(x,y))$, with $(f_i)_x = (f_i)_y = 0$ at $\phi(q)$, for $i = 2,3$. In that case, we get $E = 1$ and $F = G = 0$. With these coordinates, given $X \in C_q$ and writing $X = x\partial_x + y\partial_y$, since $x^2E(q)+2xyF(q)+y^2G(q) = 1$, then we have $x = \pm 1$. Hence, fixing an orthonormal frame $\{\nu_1,\nu_2\}$ of N_pM and using (1), it holds that

$$y \mapsto (l_{\nu_1} + 2m_{\nu_1}y + n_{\nu_1}y^2)\,\nu_1 + (l_{\nu_2} + 2m_{\nu_2}y + n_{\nu_2}y^2)\,\nu_2 \qquad (2)$$

is a parametrisation for Δ_p in the normal plane. In particular, we deduce that Δ_p is a parabola in N_pM, which can degenerate.

We recall the following result due to Mond [10], which gives a partition of all corank 1 map germs $f:(\mathbb{R}^2,\mathbf{0})\to(\mathbb{R}^3,\mathbf{0})$ according to its 2-jet under the action of $\mathcal{A}^2$, which denotes the space of 2-jets of diffeomorphisms in the source and target. We will denote by $J^2(2,3)$ the space of 2-jets $j^2f(\mathbf{0})$ of map germs $f:(\mathbb{R}^2,\mathbf{0})\to(\mathbb{R}^3,\mathbf{0})$ and by $\Sigma^1J^2(2,3)$ the subset of 2-jets of corank 1.

Proposition 2.4 (Classification of 2-jets, [10]) *There exist four orbits in $\Sigma^1J^2(2,3)$ under the action of $\mathcal{A}^2$, which are*

$$(x,y^2,xy),\ (x,y^2,0),\ (x,xy,0),\ (x,0,0).$$

We should remark that $j^2f(\mathbf{0})$ has type (x,y^2,xy) if and only if the germ f is $\mathcal{A}$-equivalent to the cross-cap (Whitney umbrella), since it is 2-determined with respect to the $\mathcal{A}$-classification.

In the next theorem, we show that the curvature parabola can be easily used to distinguish between the four types of corank 1 singularities, just by looking at the type of degeneracy of the parabola.

Theorem 2.5 *Let $M \subset \mathbb{R}^3$ be a surface with a singularity of corank 1 at $p \in M$. We assume for simplicity that p is the origin of $\mathbb{R}^3$ and denote by $j^2f(\mathbf{0})$ be the 2-jet of a local parametrisation $f:(\mathbb{R}^2,\mathbf{0})\to(\mathbb{R}^3,\mathbf{0})$ of M. Then the following holds:*

a) *Δ_p is a non-degenerate parabola if and only if $j^2f(\mathbf{0}) \sim (x,y^2,xy)$;*

b) Δ_p *is a half-line if and only if* $j^2f(\mathbf{0}) \sim (x, y^2, 0)$*;*

c) Δ_p *is a line if and only if* $j^2f(\mathbf{0}) \sim (x, xy, 0)$*;*

d) Δ_p *is a point if and only if* $j^2f(\mathbf{0}) \sim (x, 0, 0)$*.*

Proof Without loss of generality we can assume that

$$j^2f(\mathbf{0}) = (x, \frac{1}{2}(a_{20}x^2 + 2a_{11}xy + a_{02}y^2), \frac{1}{2}(b_{20}x^2 + 2b_{11}xy + b_{02}y^2)).$$

We denote the standard basis of $\mathbb{R}^3$ by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. So, T_pM is generated by $\mathbf{e}_1$ and $\{\mathbf{e}_2, \mathbf{e}_3\}$ gives an orthonormal frame of N_pM. Then, the matrix of coefficients of the second fundamental form with respect to these coordinates is

$$\begin{pmatrix} a_{20} & a_{11} & a_{02} \\ b_{20} & b_{11} & b_{02} \end{pmatrix}.$$

According to [10], the classification of $j^2f(\mathbf{0})$ follows from the analysis of the coefficients $a_{02}, b_{02}, a_{11}, b_{11}$:

a) $j^2f(\mathbf{0}) \sim (x, y^2, xy)$ if and only if $a_{11}b_{02} - a_{02}b_{11} \neq 0$;

b) $j^2f(\mathbf{0}) \sim (x, y^2, 0)$ if and only if $a_{11}b_{02} - a_{02}b_{11} = 0$ and $a_{02}^2 + b_{02}^2 > 0$;

c) $j^2f(\mathbf{0}) \sim (x, xy, 0)$ if and only if $a_{02} = b_{02} = 0$ and $a_{11}^2 + b_{11}^2 > 0$;

d) $j^2f(\mathbf{0}) \sim (x, 0, 0)$ if and only if $a_{02} = b_{02} = a_{11} = b_{11} = 0$.

On the other hand, according to (2), the curvature parabola Δ_p is parametrised by

$$\eta(y) = (0, a_{20} + 2a_{11}y + a_{02}y^2, b_{20} + 2b_{11}y + b_{02}y^2)\,. \tag{3}$$

We have that

$$\det(\mathbf{e}_1, \eta', \eta'') = \begin{vmatrix} 2a_{11} + 2a_{02}y & 2a_{02} \\ 2b_{11} + 2b_{02}y & 2b_{02} \end{vmatrix} = 4(a_{11}b_{02} - a_{02}b_{11}).$$

Thus, the parabola is degenerate if and only if $a_{11}b_{02} - a_{02}b_{11} = 0$, which gives (a). The remaining cases (b), (c) and (d) follow immediately from the parametrisation of the parabola. □

We denote by $\mathcal{R}^2$ the group of 2-jets of diffeomorphisms from $(\mathbb{R}^2, \mathbf{0})$ to $(\mathbb{R}^2, \mathbf{0})$ and by $\mathcal{O}(3)$ the group of linear isometries of $\mathbb{R}^3$. Then, $\mathcal{R}^2 \times \mathcal{O}(3)$ is a subgroup of $\mathcal{A}^2$ which also acts on $\Sigma^1 J^2(2, 3)$, the subspace of 2-jets of corank 1.

We shall show that 2-jets of corank 1 map germs $(\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ are equivalent under the action $\mathcal{R}^2 \times \mathcal{O}(3)$ if and only if there exists an isometry preserving the respective curvature parabolas. For the proof of this, we first find "good parametrisations" for corank 1 surfaces in $\mathbb{R}^3$ (that is, parametrisations obtained with changes of coordinates at source and target which preserve the geometry of the image) according to the classification given in Proposition 2.4. The cross-cap case is done in [4, 15] and we include here for completeness.

Proposition 2.6 *Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a corank 1 map germ. Then after using smooth changes of coordinates in the source and isometries in the target, we can reduce f to the form*

$$(x, y) \;\mapsto\; (x\,,\, a\,x^2 + b\,xy + cy^2 + p(x, y),\; xy + q(x, y)\,)\;, \tag{4}$$

if $j^2 f(\mathbf{0}) \sim (x, y^2, xy)$, or

$$(x, y) \;\mapsto\; (x\,,\, a\,x^2 + y^2 + p(x, y)\,,\, d\,x^2 + q(x, y)\,)\;, \tag{5}$$

if $j^2 f(\mathbf{0}) \sim (x, y^2, 0)$, or

$$(x, y) \;\mapsto\; (x\,,\, xy + p(x, y)\,,\, d\,x^2 + q(x, y)\,)\;, \tag{6}$$

if $j^2 f(\mathbf{0}) \sim (x, xy, 0)$, or

$$(x, y) \;\mapsto\; (x\,, p(x, y)\,,\, d\,x^2 + q(x, y)\,)\;, \tag{7}$$

if $j^2 f(\mathbf{0}) \sim (x, 0, 0)$, where a, b, c, d are constants, $c > 0$, and $p, q \in \mathcal{M}_2^3$.

Proof Since f has rank 1 at the origin then there exist a rotation in $(\mathbb{R}^3, \mathbf{0})$ and a diffeomorphism in $(\mathbb{R}^2, \mathbf{0})$ that transform $j^2 f(\mathbf{0})$ into

$$(x\,,\; a_{20}\,x^2 + a_{11}\,xy + a_{02}\,y^2\,,\; b_{20}\,x^2 + b_{11}\,xy + b_{02}\,y^2)\;.$$

Now we use conditions given in the proof of Theorem 2.5 for each case of $j^2 f(\mathbf{0})$. For the case in that $j^2 f(\mathbf{0}) \sim (x, y^2, xy)$, see [4], Proposition 2.1.

Suppose then that $j^2 f(\mathbf{0}) \sim (x, y^2, 0)$. So we can take $a_{02} \neq 0$. Then a change of coordinates in the variable y (making $y = -(a_{11}/2a_{02})x + y'$) transforms $j^2 f(\mathbf{0})$ into

$$(x\,,\; a'_{20}\,x^2 + a_{02}\,y^2\,,\; b'_{20}\,x^2 + b'_{11}\,xy + b'_{02}\,y^2)\;.$$

From the hypothesis that f is not a cross-cap we have that $a_{02}\,b'_{11} = 0$, what implies that $b'_{11} = 0$. Next, considering coordinates (u, v, w) in $\mathbb{R}^3$, we choose a rotation in $(\mathbb{R}^3, \mathbf{0})$ through the angle $\theta = \arctan{(b'_{02}/a_{02})}$ about the u-axis what transforms $j^2 f(\mathbf{0})$ into

$$(x,\; \widetilde{a}_{20}\,x^2 + \widetilde{a}_{02}\,y^2\,,\; \widetilde{b}_{20}\,x^2)\;,$$

where $\widetilde{a}_{02} = \cos\theta((a_{02})^2 + (b'_{02})^2)/a_{02} \neq 0$. Finally, with a change in the variable y (and an isometry in the target if necessary), we can make $\widetilde{a}_{02} = 1$, getting (5).

If $j^2 f(\mathbf{0}) \sim (x, xy, 0)$ then $j^2 f(\mathbf{0}) \sim (x\,,\, a_{20}\,x^2 + a_{11}\,xy\,,\, b_{20}\,x^2 + b_{11}\,xy)$, and we can suppose that $a_{11} \neq 0$. So, with an analogous way that was done before, we get (6).

If $j^2 f(\mathbf{0}) \sim (x, 0, 0)$ then $j^2 f(\mathbf{0}) \sim (x\,,\, a_{20}\,x^2\,,\, b_{20}\,x^2)$, and just with isometries in the target we get (7). $\square$

Now we give the main result of this section, which tell us that the curvature parabola contains all the second order information of the surface M, i.e., we shall show that the geometry of second order of a surface at a singular point p is given by the curvature parabola and its position with relation to the origin in the normal plane of the surface at p.

Theorem 2.7 *Let $M_1, M_2 \subset \mathbb{R}^3$ be two surfaces with a corank 1 singularity at points $p_1 \in M_1$ and $p_2 \in M_2$, parametrised by f and g, respectively. The 2-jets $j^2f(\mathbf{0}), j^2g(\mathbf{0}) \in \Sigma^1 J^2(2,3)$ are equivalent under the action of $\mathcal{R}^2 \times \mathcal{O}(3)$ if and only if there is a linear isometry $\phi : N_{p_1}M_1 \to N_{p_2}M_2$ such that $\phi(\Delta_{p_1}(M_1)) = \Delta_{p_2}(M_2)$.*

Proof Assume that $j^2f(\mathbf{0}), j^2g(\mathbf{0})$ have both $\mathcal{A}^2$-type (x, y^2, xy). Then, by Proposition 2.6, they can be reduced to the form:

$$j^2f(\mathbf{0}) = (x, ax^2 + 2bxy + cy^2, xy), \quad j^2g(\mathbf{0}) = (x, \overline{a}x^2 + 2\overline{b}xy + \overline{c}y^2, xy),$$

with $c, \overline{c} > 0$. We denote the coordinates in $\mathbb{R}^3$ by (u, v, w). Then the normal planes $N_{p_1}M_1$, $N_{p_2}M_2$ are both equal to the vw-plane and the curvature parabolas $\Delta_{p_1}(M_1)$, $\Delta_{p_2}(M_2)$ are both non-degenerate and are given respectively by $v = a + 2bw + cw^2$ and $v = \overline{a} + 2\overline{b}w + \overline{z}w^2$.

Suppose there is $(\psi, \phi) \in \mathcal{R}^2 \times \mathcal{O}(3)$ such that $\phi \circ j^2f(\mathbf{0}) \circ \psi = j^2g(\mathbf{0})$. Since $j^2f(\mathbf{0}), j^2g(\mathbf{0})$ are both homogeneous we can assume that ψ is also a linear map. We denote the matrices of ψ and ϕ respectively as

$$P = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where $\det P \neq 0$ and $AA^t = I$.

By comparing coefficients in $\psi \circ j^2f(\mathbf{0}) \circ \phi$ and $j^2g(\mathbf{0})$ we get the following system of equations:

$$\begin{aligned}
&a_{11}p = 1, \quad a_{21}p = 0, \quad a_{31}p = 0, \quad a_{11}q = 0, \quad a_{21}q = 0, \quad a_{31}q = 0,\\
&aa_{12}p^2 + a_{13}rp + a_{12}brp + a_{12}cr^2 = 0,\\
&aa_{22}p^2 + a_{23}rp + a_{22}brp + a_{22}cr^2 = \overline{a},\\
&aa_{32}p^2 + a_{33}rp + a_{32}brp + a_{32}cr^2 = 0,\\
&2aa_{12}pq + a_{13}rq + a_{12}brq + a_{13}ps + a_{12}bps + 2a_{12}crs = 0,\\
&2aa_{22}pq + a_{23}rq + a_{22}brq + a_{23}ps + a_{22}bps + 2a_{22}crs = \overline{b},\\
&2aa_{32}pq + a_{33}rq + a_{32}brq + a_{33}ps + a_{32}bps + 2a_{32}crs = 1,\\
&aa_{12}q^2 + a_{13}sq + a_{12}bsq + a_{12}cs^2 = 0,\\
&aa_{22}q^2 + a_{23}sq + a_{22}bsq + a_{22}cs^2 = \overline{c},\\
&aa_{32}q^2 + a_{33}sq + a_{32}bsq + a_{32}cs^2 = 0.
\end{aligned}$$

From the analysis of this system we deduce easily that $p, s \neq 0$, $q = r = 0$, $a_{11}, a_{33} \neq 0$, $a_{22} > 0$, and $a_{21} = a_{31} = a_{12} = a_{32} = a_{13} = 0$. Hence the system reduces to:

$$a_{11}p = 1, \quad aa_{22}p^2 = \overline{a}, \quad a_{23}ps + a_{22}bps + 2a_{22}crs = \overline{b}, \quad a_{33}ps = 1, \quad a_{22}cs^2 = \overline{c}.$$

Now the fact that $AA^t = I$ implies $a_{23} = 0$, $a_{11} = \pm 1$, $a_{22} = 1$ and $a_{33} = \pm 1$. Thus, there are only four possible solutions to the system:

1. $p = s = 1$, $a_{11} = a_{33} = 1$ and $(\overline{a}, \overline{b}, \overline{c}) = (a, b, c)$;
2. $p = 1$, $s = -1$, $a_{11} = 1$, $a_{33} = -1$ and $(\overline{a}, \overline{b}, \overline{c}) = (a, -b, c)$;
3. $p = -1$, $s = 1$, $a_{11} = a_{33} = -1$ and $(\overline{a}, \overline{b}, \overline{c}) = (a, -b, c)$;
4. $p = s = -1$, $a_{11} = -1$, $a_{33} = 1$ and $(\overline{a}, \overline{b}, \overline{c}) = (a, b, c)$.

In any case, we conclude that ϕ preserves the vw-plane and that $\phi(\Delta_{p_1}(M_1)) = \Delta_{p_2}(M_2)$.

For the converse, suppose that there is a linear isometry ϕ in the vw-plane such that $\phi(\Delta_{p_1}(M_1)) = \Delta_{p_2}(M_2)$. We denote now by $\tilde{A}$ the matrix of ϕ. By comparing coefficients in the equations of the parabolas and using similar arguments, we find there are only two possibilities:

1. $\tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $(\overline{a}, \overline{b}, \overline{c}) = (a, b, c)$;
2. $\tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $(\overline{a}, \overline{b}, \overline{c}) = (a, -b, c)$.

Hence, we extend ϕ to a linear isometry of $\mathbb{R}^3$ in the obvious way, so that $\phi \circ j^2 f(\mathbf{0}) = j^2 g(\mathbf{0})$.

The remaining cases where the 2-jets have type $(x, y^2, 0)$, $(x, xy, 0)$ or $(x, 0, 0)$ are treated in a similar way. Details are left to the reader. $\square$

3 Second order contact properties

An usual approach of getting information about the geometry of smooth surfaces is to analyse their generic contacts with planes and spheres. Such contacts are measured by composing the implicit equation of the plane or sphere with the parametrisation of the surface, and seeing what types of singularities arise.

Given a local parametrisation $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ of a surface M in $\mathbb{R}^3$, let $h_{\mathbf{v}}$ and $d_{\mathbf{u}}$ be the height and distance-squared functions, which are defined on M by $h_{\mathbf{v}}(x, y) = \langle \mathbf{v}, f(x, y) \rangle$ and $d_{\mathbf{u}}(x, y) = |\mathbf{u} - f(x, y)|^2$, respectively, where $\mathbf{v}, \mathbf{u} \in \mathbb{R}^3$ with $\mathbf{v}$ a unit vector. Then, for contact with a plane (resp. sphere), it is enough to study the singularities of the height function $h_{\mathbf{v}}$, with $\mathbf{v}$ an orthogonal vector to the plane (resp. of the distance-squared function

$d_{\mathbf{u}}$, with $\mathbf{u}$ being the centre of the sphere), and we label the contact with the type of this singularity.

Recall that the A_n and D_n singularities are singularities $(\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}, \mathbf{0})$ that are $\mathcal{R}$-equivalent to:

$$\begin{array}{rcrcll} A_n^{\pm} & : & (x,y) & \mapsto & \pm x^2 \pm y^{n+1} & n \geq 1, \\ D_n^{\pm} & : & (x,y) & \mapsto & x^2 y \pm y^{n-1} & n \geq 4 \ . \end{array}$$

For regular surfaces the following result is well known.

Proposition 3.1 [1, 2] *Suppose that M is a generic smooth surface in $\mathbb{R}^3$. The height function $h_{\mathbf{v}}$ can only have singularities of types A_n, $n = 1, 2, 3$. The distance squared function $d_{\mathbf{u}}$ can only have singularities of types A_n, $n = 1, 2, 3, 4$ and D_4.*

We shall now deal with similar approach for corank 1 surfaces in $\mathbb{R}^3$. Recall that $\Sigma^{2,2}$ is the Thom-Boardman submanifold in the jet space $J^r(\mathbb{R}^2, \mathbb{R})$, $r \geq 2$, given by the jets whose partial derivatives up to order 2 are equal to zero. First we shall define asymptotic and binormal directions, and umbilic curvature, which are directly related with degenerate singularities of the height and the distance squared functions.

In this section $M = g(\widetilde{M}) \subset \mathbb{R}^3$ denotes a surface with a singularity of corank 1 at $p = g(q)$ and Δ_p is its curvature parabola at p.

3.1 Asymptotic and binormal directions

The following definitions of asymptotic and binormal directions are inspired by those of a regular surface in $\mathbb{R}^4$, where we have the curvature ellipse in the normal plane (see [9]). We define both concepts in terms of the second fundamental form and then, we characterize them geometrically by means of the curvature parabola Δ_p.

Definition 3.2 We say that a non zero tangent direction $X \in T_q\widetilde{M}$ is *asymptotic* if there is a non zero normal vector $\mathbf{v} \in N_pM$ such that $II_{\mathbf{v}}(X, Y) = 0$, for any $Y \in T_q\widetilde{M}$. Moreover, in such case we say that $\mathbf{v}$ is a *binormal* direction.

Let (x, y) be local coordinates on $\widetilde{M}$ near q and take an orthonormal frame $\{\nu_1, \nu_2\}$ of N_pM such that the coefficient matrix of the second fundamental form is

$$\begin{pmatrix} l_{\nu_1} & m_{\nu_1} & n_{\nu_1} \\ l_{\nu_2} & m_{\nu_2} & n_{\nu_2} \end{pmatrix}.$$

Lemma 3.3 *A tangent direction $X = x\partial_x + y\partial_y \in T_q\widetilde{M}$ is asymptotic if and only if*

$$\begin{vmatrix} y^2 & -xy & x^2 \\ l_{\nu_1} & m_{\nu_1} & n_{\nu_1} \\ l_{\nu_2} & m_{\nu_2} & n_{\nu_2} \end{vmatrix} = 0.$$

Proof Given another tangent vector $Y = u\partial_x + v\partial_y \in T_q\widetilde{M}$ and a normal vector $\mathbf{v} = \alpha\nu_1 + \beta\nu_2 \in N_pM$ we have that

$$II_{\mathbf{v}}(X,Y) = \alpha(l_{\nu_1}xu + m_{\nu_1}(xv+yu) + n_{\nu_1}yv) + \beta(l_{\nu_2}xu + m_{\nu_2}(xv+yu) + n_{\nu_2}yv).$$

Then X is asymptotic if and only if there is $(\alpha,\beta) \neq (0,0)$ such that the above expression is 0, for any (u,v). By looking at the coefficients in u, v and after eliminating α, β we arrive to the desired quadratic equation:

$$x^2(l_{\nu_1}m_{\nu_2} - l_{\nu_2}m_{\nu_1}) + xy(l_{\nu_1}n_{\nu_2} - l_{\nu_2}n_{\nu_1}) + y^2(m_{\nu_1}n_{\nu_2} - m_{\nu_2}n_{\nu_1}) = 0.$$

□

Remark 3.4 Let A, B be the coefficient matrices of the quadratic forms II_{ν_1} and II_{ν_2} respectively, that is,

$$A = \begin{pmatrix} l_{\nu_1} & m_{\nu_1} \\ m_{\nu_1} & n_{\nu_1} \end{pmatrix}, \quad B = \begin{pmatrix} l_{\nu_2} & m_{\nu_2} \\ m_{\nu_2} & n_{\nu_2} \end{pmatrix}.$$

Then, the asymptotic and binormal directions are the solutions of the so called generalized eigenvalue problem of the matrix pair (A, B). This means that $X = x\partial_x + y\partial_y$ is asymptotic associated with the binormal direction $\mathbf{v} = \alpha\nu_1 + \beta\nu_2$ if and only if

$$\alpha A \begin{pmatrix} x \\ y \end{pmatrix} = \beta B \begin{pmatrix} x \\ y \end{pmatrix}.$$

We can choose local coordinates for $\widetilde{M}$ such that the curvature parabola is parametrised in the normal plane by (2). The parameter value $y \in \mathbb{R}$ corresponds to a unit tangent direction $X = \partial_x + y\partial_y \in C_q$. We also denote by y_∞ the parameter value corresponding to the null tangent direction $X = \partial_y$. In the case that Δ_p degenerates to a line or a half-line we define $\eta(y_\infty) = \eta'(y_\infty)$ as $\eta'(y)/|\eta'(y)|$ where $y > 0$ is any value such that $\eta'(y) \neq 0$. In the case that Δ_p degenerates to a point ν, then we define $\eta(y_\infty) = \nu$ and $\eta'(y_\infty) = 0$. In the case that Δ_p is a non-degenerate parabola, $\eta(y_\infty)$ and $\eta'(y_\infty)$ are not defined.

Lemma 3.5 *A tangent direction given by a parameter value $y \in \mathbb{R} \cup \{y_\infty\}$ is asymptotic if and only if $\eta(y)$ and $\eta'(y)$ are collinear (provided they are defined).*

Proof Given $y \in \mathbb{R}$, we compute the determinant of $\eta(y)$ and $\eta'(y)$ in the normal plane:

$$\det(\eta(y), \eta'(y)) = 2\left((l_{\nu_1}m_{\nu_2} - l_{\nu_2}m_{\nu_1}) + y(l_{\nu_1}n_{\nu_2} - l_{\nu_2}n_{\nu_1}) + y^2(m_{\nu_1}n_{\nu_2} - m_{\nu_2}n_{\nu_1})\right),$$

which coincides with the equation of asymptotic directions for $x = 1$.

When $y = y_\infty$, then $\eta(y)$ and $\eta'(y)$ are always collinear, provided they are defined. Thus, $\eta(y)$ and $\eta'(y)$ are collinear if and only if Δ_p is a degenerate parabola. But this happens if and only if

$$m_{\nu_1}n_{\nu_2} - m_{\nu_2}n_{\nu_1} = 0,$$

(see proof of Theorem 2.5) which is again the equation of asymptotic directions for $x = 0$ and $y = 1$. □

Now we analyse the different possibilities for the asymptotic directions for each type of curvature parabola (see Figure 2):

1. If Δ_p is a non-degenerate parabola then it can exist 0, 1 or 2 asymptotic directions, according to p lies "inside", on or "outside" Δ_p, respectively.

2. If Δ_p is a half-line, then either there exist two asymptotic directions $\{y_v, y_\infty\}$, with $\eta(y_v)$ being the vertex of Δ_p, or every $y \in \mathbb{R} \cup \{y_\infty\}$ is an asymptotic direction, according to the line containing Δ_p does not pass through p or it does, respectively.

3. IfΔ_p is a line, then either y_∞ is the only asymptotic direction or every $y \in \mathbb{R} \cup \{y_\infty\}$ is an asymptotic direction, according to the line does not contain p or it does, respectively.

4. IfΔ_p is a point, then every $y \in \mathbb{R} \cup \{y_\infty\}$ is an asymptotic direction.

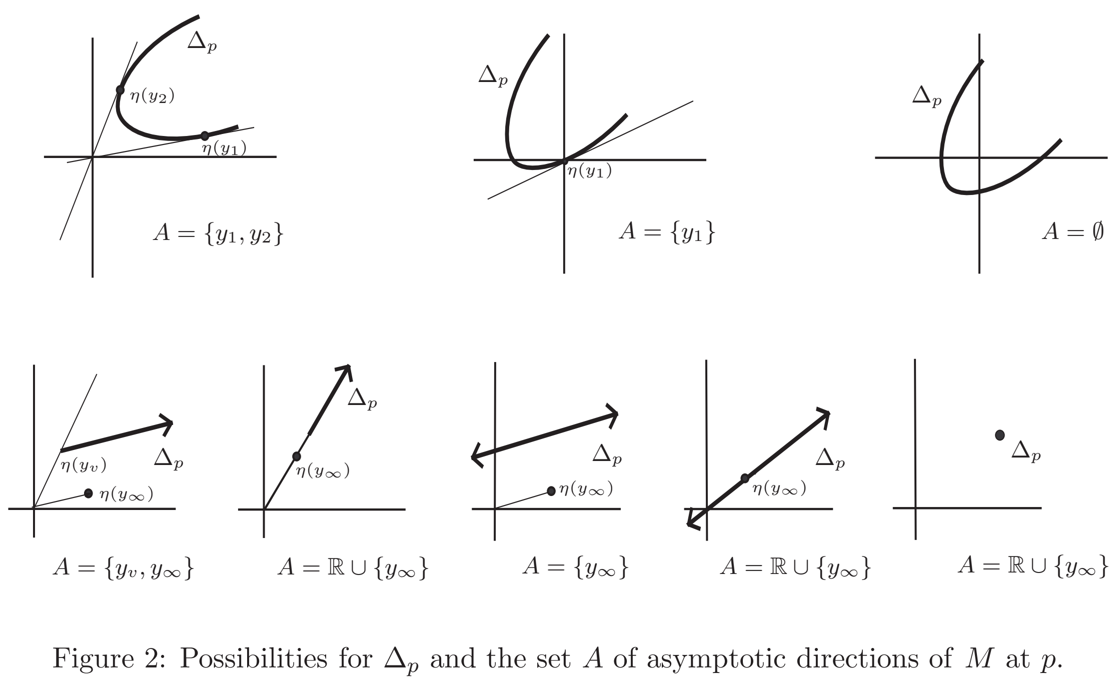

Figure 2: Possibilities for Δ_p and the set A of asymptotic directions of M at p.

The following lemma follows immediately from the computations in Lemmas 3.3 and 3.5.

Lemma 3.6 *A normal direction* $\mathbf{v} \in N_pM$ *is binormal if and only if there is an asymptotic direction* $y \in \mathbb{R} \cup \{y_\infty\}$ *such that* $\mathbf{v}$ *is orthogonal to the subspace spanned by* $\eta(y)$ *and* $\eta'(y)$*.*

Definition 3.7 Given a binormal direction $\mathbf{v} \in N_pM$, the plane through p orthogonal to $\mathbf{v}$ is called an *osculating plane* of M at p.

Again the number of binormal directions (and hence of osculating planes) depends on the different possibilities for the curvature parabola Δ_p. If Δ_p is non-degenerate, then we may have 0,1 or 2 binormal directions, one for each asymptotic direction. When Δ_p is a half-line, there are three possibilities: if the line containing Δ_p does not pass through the origin there are 2 binormal directions, if it does but the vertex of the parabola is not the origin there is 1 binormal direction, but if the vertex is the origin then all directions are binormal. If Δ_p is a line or a point different from the origin, then there is a unique binormal direction. Finally, if Δ_p is the origin, then any normal direction is binormal.

We finish this part by showing that the asymptotic directions, in the case of a singularity of cross-cap type, can be seen as the limiting tangent directions to the parabolic set in the source.

Proposition 3.8 *Assume that* Δ_p *is non-degenerate. Then* $X \in T_q\widetilde{M}$ *is asymptotic if and only if it is a limiting tangent direction to the set of points* $q' \in \widetilde{M}$*, in a neighbourhood of* q*, such that* $g(q')$ *is a parabolic regular point of* M*.*

Proof By Proposition 2.6, we choose coordinates such that M is locally parametrised by $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ given by

$$f(x, y) = (x, ax^2 + bxy + cy^2 + p(x, y), xy + q(x, y)),$$

where $a, b, c \in \mathbb{R}$, $c > 0$ and $p, q \in \mathcal{M}_2^3$. The equation of the parabolic set is:

$$\det(f_x, f_y, f_{xx}) \det(f_x, f_y, f_{yy}) - \det(f_x, f_y, f_{xy})^2 = 4c(ax^2 - cy^2) + h.o.t. = 0.$$

The limiting tangent directions are computed by the initial part $ax^2 - cy^2 = 0$, which coincides with the equation of asymptotic directions. □

3.2 Umbilic curvature

Suppose that M is not of cross-cap type at p, that is, Δ_p is a degenerate parabola. Given an orthonormal frame $\{\nu_1, \nu_2\}$ of N_pM and $X \in T_q\widetilde{M}$, recall that

$$II(X, X) = II_{\nu_1}(X, X)\, \nu_1 + II_{\nu_2}(X, X)\, \nu_2\,. \tag{8}$$

Let us consider a special frame of N_pM. Suppose first that Δ_p is not a point. Then the asymptotic direction y_∞ is well defined. Denote by $\mathbf{v}_\infty$ the infinite binormal direction such that $\{\eta(y_\infty), \mathbf{v}_\infty\}$ is an orthonormal positively oriented frame of N_pM. Suppose now that Δ_p is a point which is not the origin. Then $\eta(y)$ is a non null constant and so take the orthonormal positively oriented frame $\{\mathbf{v}, \frac{\eta(y)}{|\eta(y)|}\}$ of N_pM such that $\mathbf{v}$ is a binormal direction. We call the above frames of *adapted frames* of N_pM. We remark that when Δ_p is the origin then any orthonormal frame of N_pM can be taken as an adapted frame. See Figure 3.

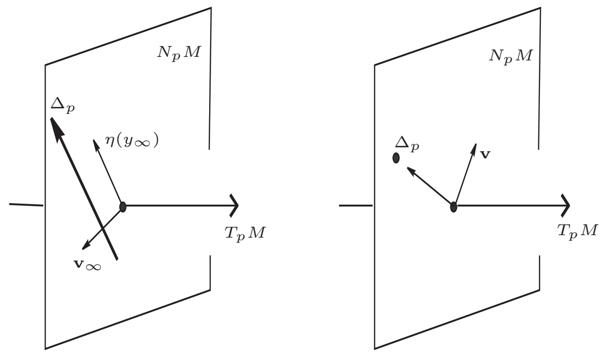

Figure 3: Adapted frame of N_pM.

Given $X \in C_q$, then $II(X,X) \in \Delta_p$. Thus, it follows from (8) that in an adapted frame $\{\nu_1, \nu_2\}$ of N_pM, $II_{\nu_2}(X,X)$ does not depend on $X \in C_q$, up to the sign. This suggest us the following definition.

Definition 3.9 Given a unit vector $X \in T_q\widetilde{M}$ and an adapted frame $\{\nu_1, \nu_2\}$ of N_pM, we call the positive number

$$\kappa_u(p) = |\langle II(X,X), \nu_2\rangle| = |II_{\nu_2}(X,X)|$$

the *umbilic curvature* of M at p.

Remark 3.10 (1) The definition of umbilic curvature does not depend on the choice of the adapted frame of N_pM, nor the parametrisation for Δ_p, nor the choice of local coordinates in $\widetilde{M}$. However, it may depend on the map $g : \widetilde{M} \to \mathbb{R}^3$ which parametrises M. In fact, the cuspidal edge parametrised by $g(x,y) = (x, y^2, y^3)$ has $\kappa_u(\mathbf{0}) = 0$, but the same cuspidal edge parameterised by $g(x,y) = (x, (y^3+x)^2, (y^3+x)^3)$ has $\kappa_u(\mathbf{0}) = 2$ (see Example 2.3).

(2) (*Geometric interpretation for the umbilic curvature*) If Δ_p is a point, then $\kappa_u(p)$ is the distance between Δ_p and p. Furthermore, $\kappa_u(p) = 0$ if and only if either $\Delta_p = \{p\}$ or Δ_p is contained in a line through p in the case that Δ_p is a half-line or a line.

If Δ_p is a half-line or a line then $\kappa_u(p)$ is the length of the projection of Δ_p on the direction given by an infinity binormal direction. Then it follows the formula:

$$\kappa_u(p) = \left|\left\langle \eta(y), \frac{\eta'(y) \times e}{|\eta'(y)|}\right\rangle\right| = \frac{1}{|\eta'(y)|}|\det(\eta(y), \eta'(y), e)|,$$

where η is a parametrisation for Δ_p, e is a unit vector in T_pM and y is any real number such that $\eta'(y) \neq 0$. Equivalently, since in this case $\kappa_u(p)$ also can be seen as the distance between p and the line containing Δ_p then we also can write

$$\kappa_u(p) = \frac{|\eta(y) \times \eta'(y)|}{|\eta'(y)|}, \tag{9}$$

which does not depend on $y \in \mathbb{R}$, with $\eta'(y) \neq 0$.

(3) For any $X \in T_q\widetilde{M}$ and $\{\nu_1, \nu_2\}$ adapted frame of N_pM, it holds that

$$\kappa_u(p) = \frac{|II_{\nu_2}(X, X)|}{I(X, X)}.$$

In fact,

$$|II_{\nu_2}(X, X)| = |X|^2 |II_{\nu_2}(\frac{X}{|X|}, \frac{X}{|X|})| = I(X, X)\kappa_u(p),$$

since we are considering in $T_q\widetilde{M}$ the pseudometric induced by the first fundamental form I.

(4) The umbilic curvature is not defined for singularities of cross-cap type.

(5) If M has a singularity of cuspidal edge type at p, then the umbilic curvature coincides with the limiting normal curvature defined in [13]. This follows from the results of [8].

3.3 Contact with planes

In this section we shall deal with contact of planes with M at p. So we consider the singularities of the height function $h_{\mathbf{v}}$, with $\mathbf{v}$ a unit vector in $\mathbb{R}^3$. Recall that a plane in $\mathbb{R}^3$ is said to be *transverse* to M at p if $h_{\mathbf{v}}$ is a submersion at p.

Theorem 3.11 *Let $M \subset \mathbb{R}^3$ be a surface with a singularity of corank 1 at $p \in M$. Then $h_{\mathbf{v}}$ is singular at p if and only if $\mathbf{v} \in N_pM$. Furthermore:*

(i) *Assume Δ_p is not a point. Then $h_{\mathbf{v}}$ has a degenerate singularity at p if and only if $\mathbf{v}$ is a binormal direction at p. Moreover, the singularity is of type $\Sigma^{2,2}$ if and only if Δ_p is degenerate, $\kappa_u(p) = 0$ and $\mathbf{v}$ is a infinite binormal direction at p.*

(ii) *Assume Δ_p is a point. Then $h_{\mathbf{v}}$ has a degenerate singularity at p for all direction $\mathbf{v} \in N_pM$. Moreover, the singularity is of type $\Sigma^{2,2}$ if and only if either $\kappa_u(p) = 0$ or $\kappa_u(p) \neq 0$ and $\mathbf{v}$ is a binormal direction at p.*

Proof The first assertion that $h_{\mathbf{v}}$ is singular at p if and only if $\mathbf{v} \in N_pM$ is obvious. We show (i) and (ii) by looking at the different types of curvature parabolas. We denote by (u, v, w) the coordinates in $\mathbb{R}^3$ and assume for simplicity that $p = \mathbf{0}$.

(1) Δ_p *is non-degenerate.* This case corresponds to a singularity of cross-cap type and the degenerate singularities of the height function can be found in [15]: if we take the

parametrisation given at (4), then the directions with a degenerate singularity are: $\mathbf{v}_1 = (0, 1, -b - 2c\sqrt{\frac{a}{c}})$ and $\mathbf{v}_2 = (0, 1, -b + 2c\sqrt{\frac{a}{c}})$ (for $a \geq 0$). On the other hand, Δ_p is parametrised by

$$\eta(y) = 2(0, a + by + cy^2, y)\,,$$

and a simple computation shows that the asymptotic directions occur at $y = \pm\sqrt{\frac{a}{c}}$ and the corresponding binormal directions are $\mathbf{v}_1$ and $\mathbf{v}_2$. Moreover, there are no $\Sigma^{2,2}$ singularities in this case.

(2) Δ_p *is a half-line.* By Theorem 2.5, $j^2f(\mathbf{0}) \sim (x, y^2, 0)$. Taking the parametrisation given at (5), the curvature parabola is parametrised by

$$\eta(y) = 2(0, a + y^2, d)\,,$$

with $a, d \in \mathbb{R}$ constants, such that $\kappa_u(p) = 2|d|$. We have 3 subcases:

- $d \neq 0$. In this case the line containing Δ_p does not pass through the origin and there are 2 asymptotic directions $y = 0$ and y_∞, whose binormal directions give $\mathbf{v}_1 = (0, -d, a)/(\sqrt{a^2 + d^2})$ and $\mathbf{v}_\infty = (0, 0, 1)$, respectively.

- $d = 0$ and $a \neq 0$. In this case the line containing Δ_p passes through the origin, but the origin is not the vertex of the parabola. Any direction $y \in \mathbb{R} \cup \{y_\infty\}$ is asymptotic, but there is only one binormal direction $\mathbf{v}_\infty$.

- $d = a = 0$. The vertex of Δ_p is the origin. Any direction is binormal.

The height function in the direction $\mathbf{v} = (u, v, w)$ is $h_{\mathbf{v}}(x, y) = ux + v(ax^2 + y^2) + wdx^2 + h.o.t.$. Then $\mathbf{0}$ is a degenerate singularity if and only if $u = 0$ and $v(av + dw) = 0$. We see that the solutions are exactly the binormal directions in the 3 subcases. Moreover, we have a $\Sigma^{2,2}$ singularity only in the case that $d = 0$ and $\mathbf{v} = \mathbf{v}_\infty$.

(3) Δ_p *is a line.* So, $j^2f(\mathbf{0}) \sim (x, xy, 0)$ and taking f as in (6), the curvature parabola is parametrised by

$$\eta(y) = 2(0, y, d)\,,$$

where $d \in \mathbb{R}$ and $\kappa_u(p) = 2|d|$. Any direction is asymptotic, but there is only one binormal direction $\mathbf{v}_\infty = (0, 0, 1)$. The height function $h_{\mathbf{v}}(x, y) = ux + vxy + wdx^2 + h.o.t.$, has a degenerate singularity at the origin if and only if $u = v = 0$ which gives $\mathbf{v} = \mathbf{v}_\infty$. Moreover, we have a $\Sigma^{2,2}$ singularity only in the case $d = 0$.

(4) Δ_p *is a point.* In this case $j^2f(\mathbf{0}) \sim (x, 0, 0)$ and taking f as in (7), then

$$\eta(y) = (0, 0, 2d).$$

where $d \in \mathbb{R}$ and $\kappa_u(p) = 2|d|$. If $d \neq 0$ there is only one binormal direction $\mathbf{v} = (0, 1, 0)$. Otherwise, if $d = 0$ any direction is binormal. On the other hand, the height function is given by $h_{\mathbf{v}}(x,y) = ux + wdx^2 + h.o.t.$ If $d \neq 0$, then any normal direction ($u = 0$) gives a degenerate singularity but only the binormal direction $\mathbf{v} = (0,1,0)$ gives a $\Sigma^{2,2}$ singularity. Otherwise, if $d \neq 0$, then any normal direction gives a $\Sigma^{2,2}$ singularity. □

Corollary 3.12 *Let $M \subset \mathbb{R}^3$ be a surface with a singularity of corank 1 at $p \in M$. If Δ_p is not a point, then a plane has degenerate contact with M at p if and only if it is an osculating plane of M at p.*

Example 3.13 Let M be the swallowtail surface given in Example 2.3. Since $\mathbf{v} = (0,0,1)$ is the unique binormal direction of M at $\mathbf{0}$, then the orthogonal plane to $\mathbf{v}$ is the unique osculating plane of M at p. It follows from Theorem 3.11 that this is the unique plane having degenerate contact with M at p. Moreover, since $\kappa_u(\mathbf{0}) = 0$, the contact is of type $\Sigma^{2,2}$.

Now, let M be the cuspidal edge from the same example. Since the vertex of Δ_p is the origin of N_pM then all unit directions in N_pM are binormal directions with $\mathbf{v} = (0,0,1)$ given the infinite binormal direction. So, it follows that every plane containing the u-axis is an osculating plane of M at p. Since $\kappa_u(p) = 0$ so, by Theorem 3.11, such planes have degenerate contact with M at p and the uv-plane is the only one having contact of type $\Sigma^{2,2}$.

3.4 Contact with spheres

We shall deal now with contact of the surface M with spheres.

Definition 3.14 The *focal set of M at p* is the locus of points of $\mathbb{R}^3$ which are centres of spheres with degenerate contact with M at p or, equivalently, it is the locus of points $\mathbf{u} \in \mathbb{R}^3$ such that $d_{\mathbf{u}}$ has a degenerate singularity at p, where $d_{\mathbf{u}}$ is the distance-squared function on M.

In the following result, the cross-cap case can be found in [4], Proposition 3.2 and 3.4 (see also [15]) but we include it in the statement for completeness. When M has not a cross-cap type singularity, let $\{\nu_1, \nu_2\}$ be an adapted frame of N_pM, $\varepsilon = \operatorname{sgn} II_{\nu_2}(X,X)$, for any $X \in C_q$, and let $\ell_p \subset N_pM$ be the line through p parallel to ν_2, that is, $\ell_p = \{p + t\nu_2\,;\, t \in \mathbb{R}\}$. Note that when Δ_p is a half-line or a line, then ℓ_p is the line in N_pM through p orthogonal to Δ_p.

Theorem 3.15 *Let $M \subset \mathbb{R}^3$ be a surface with a singularity of corank 1 at $p \in M$. The function $d_{\mathbf{u}}$ is singular at p if and only if $\mathbf{u} \in N_pM$. Furthermore, the following possibilities hold:*

(i) *If Δ_p is a non-degenerate parabola then the focal set of M at p is a conic. More precisely, it is either an ellipse, parabola or hyperbola according to p is "inside", on or "outside" of Δ_p, respectively.*

(ii) *If Δ_p is a half-line then the focal set of M at p either is the union of two transverse lines intercepting at $\mathbf{u} = p + \varepsilon \dfrac{1}{\kappa_u(p)}\nu_2$ (if $\kappa_u(p) \neq 0$) or it is the union of two parallel lines in N_pM (if $\kappa_u(p) = 0$), which can be coincident. In both cases, ℓ_p is one of the lines of the focal set.*

(iii) *If Δ_p is a line then the focal set of M at p is the line ℓ_p.*

(iv) *If Δ_p is a a point then the focal set of M at p is the plane N_pM.*

Moreover, $d_{\mathbf{u}}$ has a singularity of type $\Sigma^{2,2}$ if and only if Δ_p is a degenerate parabola, $\kappa_u(p) \neq 0$ and $\mathbf{u} = p + \varepsilon \frac{1}{\kappa_u(p)}\nu_2$.

Proof The first assertion that $d_{\mathbf{u}}$ is singular if and only if $\mathbf{u} \in N_pM$ is obvious and the proof of case (i) can be found in [4, 15], so we consider only the other cases. In all the cases we assume that $p = \mathbf{0}$ and that M is given by the corresponding parametrisation in Proposition 2.6. If we denote the coordinates in $\mathbb{R}^3$ by (u, v, w), in all the three cases we have that $\{\nu_1, \nu_2\} = \{(0, 1, 0), (0, 0, 1)\}$ is the adapted frame of N_pM, ℓ_p is the w-axis, $\kappa_u(\mathbf{0}) = 2|d|$ and $\varepsilon = \mathrm{sgn}(d)$.

(ii) Suppose that Δ_p is a half-line. Given $\mathbf{u} = (u, v, w)$ we have

$$d_{\mathbf{u}}(x, y) = (u - x)^2 + (v - ax^2 - y^2)^2 + (w - d\,x^2)^2 + h.o.t.$$

Hence, $\mathbf{0}$ is a degenerate singularity of $d_{\mathbf{u}}$ if and only if $u = 0$ and $v(1 - 2av - 2dw) = 0$. If $d \neq 0$, we have two transverse lines, intercepting at $(0, 0, \frac{1}{2d})$. Otherwise, if $d = 0$, we have two parallel lines. In both cases, $v = 0$ is one of the lines. Moreover, there is a $\Sigma^{2,2}$ singularity only in the case that $d \neq 0$ and $\mathbf{u} = (0, 0, \frac{1}{2d})$.

(iii) Let us suppose now that Δ_p is a line. The distance-squared function is now given by

$$d_{\mathbf{u}}(x, y) = u^2 - 2ux + (1 - 2dw)x^2 - 2vxy + h.o.t.$$

Then $d_{\mathbf{u}}$ has a degenerate singularity at $\mathbf{0}$ if and only if $u = v = 0$. Furthermore, the singularity is of type $\Sigma^{2,2}$ if and only if $d \neq 0$ and $\mathbf{u} = (0, 0, \frac{1}{2d})$.

(iv) Finally, suppose now that Δ_p is a point, then

$$d_{\mathbf{u}}(x, y) = u^2 - 2ux + (1 - 2dw)x^2 + h.o.t.$$

We conclude that for any $u = 0$ we have a degenerate singularity at the origin. Moreover, it is a $\Sigma^{2,2}$ singularity if and only if $d \neq 0$ and $\mathbf{u} = (0, 0, \frac{1}{2d})$.

□

Example 3.16 Let M be the swallowtail surface given in Example 2.3. Since Δ_p is a line (the v-axis) and $\kappa_u(\mathbf{0}) = 0$ then, by Theorem 3.15, it follows that the only spheres with degenerate contact with M at $\mathbf{0}$ are those ones with center belonging to the w-axis. Moreover, there are no spheres with $\Sigma^{2,2}$ contact.

Let M be the cuspidal edge surface image of $f(x, y) = (x, ax^2 + y^2, by^2 + cy^3)$. So, the singular set of f is the x-axis and its image in $\mathbb{R}^3$ by f is $\{(x, ax^2, 0)\,;\, x \in \mathbb{R}\}$. Taking $p = \mathbf{0}$, a parametrisation for Δ_p is $\eta(y) = (0, 2a, 0) + (0, 2, 2b)y^2, y \in \mathbb{R}$, which is a half-line in N_pM contained in a line passing through the origin if and only if $ab = 0$. Then $\eta'(y) = 4(0, y, by)$ and so $\eta(y_\infty) = \frac{1}{\sqrt{1+b^2}}(0, 1, b)$. Therefore taking the infinite binormal vector $\mathbf{v}_\infty = (0, -b, 1)$ we get that $\{\eta(y_\infty), \mathbf{v}_\infty\}$ is an adapted frame of N_pM. By (9), we have

$$\kappa_u(\mathbf{0}) = \frac{2|ab|}{\sqrt{1 + b^2}} \,.$$

Since $\varepsilon = \operatorname{sgn} \langle \eta(y), \mathbf{v}_\infty \rangle = \operatorname{sgn}(-ab)$, we deduce the following conclusion: (a) if $ab = 0$ then the only spheres with degenerate contact with M at $\mathbf{0}$ are those ones with center belonging to the line $\ell_p = \{t(0, -b, 1); t \in \mathbb{R}\}$ and there are no spheres with $\Sigma^{2,2}$ contact; (b) if $ab \neq 0$ then the sphere with centre at $\mathbf{u} = \varepsilon \frac{1}{\kappa(p)} \mathbf{v}_\infty$ is the unique sphere having $\Sigma^{2,2}$ contact with M at $\mathbf{0}$.

Given a surface $M \subset \mathbb{R}^3$ and a local parametrisation $f : (\mathbb{R}^2, q) \to (\mathbb{R}^3, p)$, recall that p is a 2-*rounding* of M if p is either a 2-*flattening*, that is, there is $\mathbf{v} \in S^2$ such that $j^r h_{\mathbf{v}}(q) \in \Sigma^{2,2}$, $r \geq 2$, or a *non-flat* 2-*rounding*, that is, it is not a 2-flattening and there is $\mathbf{u} \in \mathbb{R}^3$ such that $j^r d_{\mathbf{u}}(q) \in \Sigma^{2,2}$. It is known that a regular (resp. singular) point of f is a 2-rounding if and only if it is an umbilic point (resp. it is not a cross-cap point). See [6] for details. One concludes from Theorems 3.11 and 3.15 the following corollary:

Corollary 3.17 *Let $M \subset \mathbb{R}^3$ be a surface with a singularity of corank 1 at p which is not a cross-cap. If the umbilic curvature of M at p is non zero then p is a non-flat 2-rounding of M; if the umbilic curvature is zero, then p is a 2-flattening of M.*

References

[1] T. Banchoff, T. Gaffney and C. McCrory, *Cusps of Gauss Mappings*, Research Notes in Mathematics, 55, Pitman, London (1982).

[2] J. W. Bruce and P. J. Giblin, Generic curves and surfaces, *J. London Math. Soc.* 2 (1981), 555-561.

[3] J. W. Bruce and J. M. West, Functions on cross-caps, *Math. Proc. Cambridge Philos. Soc.* 123 (1998), 19-39.

[4] T. Fukui and M. Hasegawa, Fronts of Whitney umbrella - a differential geometric approach via blowing up, *Journal of Singularities* 4 (2012), 35-67.

[5] T. Fukui and M. Hasegawa, Height functions on Whitney umbrellas, preprint.

[6] T. Fukui and J. J. Nuño-Ballesteros, Isolated roundings and flattenings of submanifolds in euclidean spaces, *Tohoku Math. J.* (2) 57 (2005), 469-503.

[7] R. Garcia, J. Sotomayor and C. Gutierrez, Lines of principal curvature around umbilics and Whitney umbrellas, *Tohoku Math. J.* 52 (2000), 163-172.

[8] L. F. Martins and K. Saji, Geometric invariants of cuspidal edges, preprint, 2012. Available from www.mat.ibilce.unesp.br/singularidades/docs/Martins-Saji.pdf.

[9] D. K. H. Mochida, M. C. Romero Fuster, and M. A. S. Ruas, The geometry of surfaces in 4-space from a contact viewpoint. *Geom. Dedicata* 54 (1995), no. 3, 323–332.

[10] D. Mond, On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, *Proc. London Math. Soc.*, (3), 50 (1985), 333-369.

[11] J. J. Nuño-Ballesteros and F. Tari. Surfaces in $\mathbb{R}^4$ and their projections to 3-spaces, *Proc. Royal Edinburgh Math. Soc.*, 137A (2007), 1313-1328.

[12] R. Oset Sinha and F. Tari, On the geometry of singular surfaces, preprint, 2012. Available from www2.icmc.usp.br/~faridtari/Papers/OsetTariSingularSurfaces.pdf.

[13] K. Saji, M. Umehara, and K. Yamada, The geometry of fronts, *Ann. of Math.* **169** (2009), 491–529.

[14] F. Tari, On pairs of geometric foliations on a cross-cap, *Tohoku Math.J.*, 59 (2007), 233-258.

[15] J. West, *The differential geometry of the cross-cap.* Ph.D. Thesis, Liverpool University, 1995.

Departamento de Matemática, IBILCE - UNESP
R. Cristóvão Colombo, 2265, CEP 15054-000
São José do Rio Preto, SP, Brazil
E-mail: `lmartins@ibilce.unesp.br`

Departament de Geometria i Topologia,
Universitat de València
Campus de Burjassot 46100, Spain
E-mail: `Juan.Nuno@uv.es`