



Contact real hypersurfaces in the complex hyperbolic quadric

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Abstract

We discuss the geometry of contact real hypersurfaces with constant mean curvature in the complex hyperbolic quadric $Q^{m*} = SO_{m,2}^o/SO_m SO_2$, where $m \geq 3$. These hypersurfaces were classified in Berndt and Suh (Proc Am Math Soc 143:2637–2649, 2015), and we study the individual types (two types of tubes around totally geodesic submanifolds of Q^{m*} and one type of horosphere) that have been found in that classification.

Keywords Contact hypersurface · Kähler structure · Complex conjugation · Complex hyperbolic quadric

Mathematics Subject Classification Primary 53C40 · Secondary 53C55

1 Introduction

Following Sasaki [19] and Okumura [13], an odd-dimensional, smooth manifold M^{2m-1} is called an *almost contact manifold* if the structure group of its tangent bundle can be reduced to $U_{m-1} \times 1$ (where U_{m-1} refers to the natural real representation of the unitary group in $m-1$ complex variables). M^{2m-1} is called a *contact manifold* if there exists a smooth 1-form η on M^{2m-1} so that $\eta \wedge d\eta^{m-1} \neq 0$; such an η is then called a *contact form* on M^{2m-1} .

It was shown by Sasaki [19, Theorem 5] that M^{2m-1} is an almost contact manifold if and only if there exists an *almost contact metric structure* (ϕ, ξ, η, g) on M^{2m-1} . Here ϕ is an

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endomorphism field on M^{2m-1} , ξ is a vector field on M^{2m-1} , η is a 1-form on M^{2m-1} and g is a Riemannian metric on M^{2m-1} , and these data are related to each other in the following way: First we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1$$

for all vector fields X on M^{2m-1} , meaning that (ϕ, ξ, η) is an *almost contact structure*, and moreover this structure is adapted to the Riemannian metric g by

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad \text{and} \quad \eta(X) = g(X, \xi)$$

for all vector fields X, Y on M^{2m-1} .

Clearly, if M^{2m-1} has an almost contact metric structure (ϕ, ξ, η, g) so that $\eta \wedge d\eta^{m-1} \neq 0$ holds, then M^{2m-1} is a contact manifold. Conversely, if M^{2m-1} is a contact manifold, then for any contact form η on M^{2m-1} there exists an almost contact metric structure (ϕ, ξ, η, g) with this η by a result due to Sasaki [19, Theorem 4].

Let us now consider a real hypersurface M of a Kähler manifold \bar{M} of complex dimension m . Then M has real dimension $2m - 1$, and the complex structure J and the Riemannian metric g of \bar{M} induce an almost contact metric structure (ϕ, ξ, η, g) on M : Let N be a unit normal vector field of M in \bar{M} , then we choose ϕ as the *structure tensor field* defined by letting ϕX be the M -tangential part of JX for any $X \in TM$, choose ξ as the *Reeb vector field* $\xi = -JN$, and choose $\eta = g(\cdot, \xi)$. If there exists a smooth, everywhere nonzero function ρ on M in this setting so that

$$d\eta(X, Y) = \rho \cdot g(\phi X, Y) \tag{1.1}$$

holds for all vector fields X, Y on M , then M is a contact manifold, and η a contact form on M . In this case M is called a *contact hypersurface* of \bar{M} , see also Blair [6], Dragomir and Perrone [7]. It was noted by Okumura [13, Eq. (2.13)] that the condition (1.1) is equivalent to

$$S\phi + \phi S = k \cdot \phi, \tag{1.2}$$

where S denotes the shape operator of the hypersurface M in \bar{M} with respect to the unit normal vector field N , and $k = 2\rho$. If the complex dimension of \bar{M} is at least 3 in this setting, then the function ρ resp. k is necessarily constant, see [4, Proposition 2.5].

Pursuant to these ideas, the contact hypersurfaces have been classified in the Hermitian symmetric spaces of rank 1, namely in the complex projective space $\mathbb{C}P^m$ and its non-compact dual, the complex hyperbolic space $\mathbb{C}H^m$. Yano and Kon showed in [24, Theorem VI.1.5] that a connected contact hypersurface with constant mean curvature of the complex projective space $\mathbb{C}P^m$ with $m \geq 3$ is locally congruent either to a geodesic hypersphere, or to a tube over a real projective space $\mathbb{R}P^n$, $m = 2n$, embedded in $\mathbb{C}P^m$ as a totally real, totally geodesic submanifold. Vernon proved in [23] that a complete, connected contact real hypersurface in $\mathbb{C}H^m$ with $m \geq 3$ is congruent to a tube around a totally geodesic $\mathbb{C}H^{m-1}$ in $\mathbb{C}H^m$, a tube around a real form $\mathbb{R}H^m$ in $\mathbb{C}H^m$, a geodesic hypersphere in $\mathbb{C}H^m$, or a horosphere in $\mathbb{C}H^m$. Note that all the contact hypersurfaces in $\mathbb{C}P^m$ or $\mathbb{C}H^m$ are homogeneous and therefore have constant principal curvatures, in particular constant mean curvature. We would like to mention that Pérez has carried out a nice investigation of certain real hypersurfaces in $\mathbb{C}P^m$ in [14].

When we consider more complicated Hermitian symmetric spaces as ambient space \bar{M} , there can be contact hypersurfaces which do not have constant mean curvature. The class of all contact hypersurfaces M in \bar{M} is very complicated, and a full classification does not appear to be feasible at the present time. However, if one considers only contact

hypersurfaces M with constant mean curvature, the classification problem becomes tractable at least when \bar{M} is a Hermitian symmetric space of rank 2.

The series of irreducible Hermitian symmetric spaces of rank 2 comprise the complex quadrics $Q^m = SO_{m+2}/SO_2SO_m$ (isomorphic to the real 2-Grassmannians $G_2^+(\mathbb{R}^{m+2})$ of oriented planes in \mathbb{R}^{m+2}), the complex 2-Grassmannians $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$, and their non-compact duals, the complex hyperbolic quadrics $Q^{m*} = SO_{2,m}^o/SO_2SO_m$ and the duals of the complex 2-Grassmannians $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$. We would like to mention the very nice investigations by Pérez et al. of specific types of real hypersurfaces in Q^m in [15] and in $G_2(\mathbb{C}^{m+2})$ in [16] and [17].

The classification of contact hypersurfaces with constant mean curvature in the complex quadric Q^m and in its non-compact dual, the complex hyperbolic quadric Q^{m*} has been carried out by Berndt and the second author of the present paper in [4]. The result for Q^{m*} is stated as Theorem A. For the case of the complex quadric Q^m , a different classification proof has been given in [22] by the second author of the present paper.

In the complex 2-Grassmannians $G_2(\mathbb{C}^{m+2})$, the contact hypersurfaces with constant mean curvature have also been classified by the second author of the present paper in [21]. He shows that such a hypersurface is congruent to an open part of a tube around a totally geodesic quaternionic projective space $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$. For the non-compact dual $G_2^*(\mathbb{C}^{m+2})$ of these Grassmannians, as far as we know there does not exist a classification of contact hypersurfaces with constant mean curvature. However, Berndt et al. [2] classified contact hypersurfaces of $G_2^*(\mathbb{C}^{m+2})$ which satisfy another curvature condition, namely that the principal curvature function α corresponding to the Reeb vector field of the hypersurface is constant. The result of the classification is that any such hypersurface of $G_2^*(\mathbb{C}^{m+2})$ is congruent either to an open part of a tube around a totally geodesic quaternionic hyperbolic space $\mathbb{H}H^n$ in $G_2(\mathbb{C}^{m+2})$ (only if $m = 2n$ is even), or to an open part of a horosphere in a certain position in $G_2^*(\mathbb{C}^{m+2})$. Note that all these hypersurfaces have constant mean curvature (this follows from [2, Propositions 4.1, 4.2(ii)]).

The purpose of the present paper is to study the local geometry of the contact real hypersurfaces with constant mean curvature in the complex hyperbolic quadric Q^{m*} . As mentioned above, those hypersurfaces were classified by Berndt and Suh. The result is as follows:

Theorem A (Berndt and Suh [4]) Let M be a connected orientable real hypersurface with constant mean curvature in the complex hyperbolic quadric Q^{m*} , $m \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of one of the following contact hypersurfaces in Q^{m*} :

- (i) the tube of radius $r > 0$ around the complex hyperbolic quadric Q^{m-1*} which is embedded in Q^{m*} as a totally geodesic complex hypersurface;
- (ii) a horosphere in Q^{m*} whose center at infinity is the equivalence class of an \mathfrak{A} -principal geodesic in Q^{m*} ;
- (iii) the tube of radius $r > 0$ around the n -dimensional real hyperbolic space $\mathbb{R}H^n$ which is embedded in Q^{m*} as a real space form of Q^{m*} .

We want to describe the local geometry of the three types (i)–(iii) of real hypersurfaces in Q^{m*} given in the above theorem. For this purpose we first need to study the geometry of Q^{m*} itself. In particular we need to describe the “fundamental geometric structures” of Q^{m*} ; they are its Riemannian metric g , its Hermitian structure J and a certain S^1 -subbundle

\mathfrak{A} of $\text{End}(TQ^{m*})$ which can be used to characterize the orbits of the isotropy action on the tangent space of Q^{m*} . The terms “ \mathfrak{A} -principal” which occurs in Theorem A refers to one of the two singular orbits of this action. For the complex quadric Q^m the corresponding S^1 -bundle \mathfrak{A} is obtained from the shape operator of the embedding $Q^m \hookrightarrow \mathbb{C}P^{m+1}$; it has first been introduced by Reckziegel in [18] and has many times been shown to be very useful for the investigation of the submanifold geometry of Q^m , for example for the classification of the totally geodesic submanifolds of Q^m by the first author of the present paper in [9], and in several classifications of real hypersurfaces in Q^m satisfying certain curvature conditions by the second author of the present paper and his coauthors. We would like to base our investigation of real hypersurfaces in Q^{m*} on the analogous S^1 -subbundle \mathfrak{A} for Q^{m*} , which has, as far as we know, not before been described in full detail in the literature. The situation is more complicated for Q^{m*} than for Q^m here, because there does not exist a complex hypersurface embedding of Q^{m*} into $\mathbb{C}H^{m+1}$, as was shown by Smyth [20], therefore \mathfrak{A} for Q^{m*} cannot be obtained in the analogous manner as for Q^m . Instead it needs to be obtained from the representation of Q^{m*} as the quotient manifold $SO_{2,m}/SO_2SO_m$ regarded as a non-compact Hermitian symmetric space. We thus carry out the description of the symmetric space structure of Q^{m*} and of the mentioned fundamental geometric structures in Sect. 2.

The remainder of the paper is concerned with the construction of the contact hypersurfaces in Q^{m*} that are given in Theorem A. In Sect. 3 we construct the tubes around totally geodesic submanifolds that occur in Theorem A(i),(iii). Section 4 recalls how horospheres in non-compact Riemannian symmetric spaces of rank 2 are constructed, and then a description of the horosphere of Theorem A(ii) is given.

2 The complex hyperbolic quadric

The m -dimensional complex hyperbolic quadric Q^{m*} is the non-compact dual of the m -dimensional complex quadric Q^m , i.e. the simply connected Riemannian symmetric space whose curvature tensor is the negative of the curvature tensor of Q^m .

The complex hyperbolic quadric Q^{m*} cannot be realized as a homogeneous complex hypersurface of the complex hyperbolic space $\mathbb{C}H^{m+1}$. In fact, Smyth [20, Theorem 3(ii)] has shown that every homogeneous complex hypersurface in $\mathbb{C}H^{m+1}$ is totally geodesic. This is in marked contrast to the situation for the complex quadric Q^m , which can be realized as a homogeneous complex hypersurface of the complex projective space $\mathbb{C}P^{m+1}$ in such a way that the shape operator for any unit normal vector to Q^m is a real structure on the corresponding tangent space of Q^m , see [18] and [9]. Another related result by Smyth, [20, Theorem 1], which states that any complex hypersurface of $\mathbb{C}H^{m+1}$ for which the square of the shape operator has constant eigenvalues (counted with multiplicity) is totally geodesic, also precludes the possibility of a model of Q^{m*} as a complex hypersurface of $\mathbb{C}H^{m+1}$ with the analogous property for the shape operator.

Therefore we realize the complex hyperbolic quadric Q^{m*} as the quotient manifold $SO_{2,m}/SO_2SO_m$. As Q^1 is isomorphic to the real hyperbolic space $\mathbb{R}H^2 = SO_{1,2}/SO_2$, and Q^{2*} is isomorphic to the Hermitian product of complex hyperbolic spaces $\mathbb{C}H^1 \times \mathbb{C}H^1$, we suppose $m \geq 3$ in the sequel and throughout this paper. Let $G := SO_{2,m}$ be the transvection group of Q^{m*} and $K := SO_2SO_m$ be the isotropy group of Q^{m*} at the “origin” $p_0 := eK \in Q^{m*}$. Then

$$\sigma : G \rightarrow G, g \mapsto sgs^{-1} \quad \text{with} \quad s := \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

is an involutive Lie group automorphism of G with $\text{Fix}(\sigma)_0 = K$, and therefore $Q^{m*} = G/K$ is a Riemannian symmetric space. The center of the isotropy group K is isomorphic to SO_2 , and therefore Q^{m*} is in fact a Hermitian symmetric space.

The Lie algebra $\mathfrak{g} := \mathfrak{so}_{2,m}$ of G is given by

$$\mathfrak{g} = \{X \in \mathfrak{gl}(m + 2, \mathbb{R}) \mid X^t \cdot s = -s \cdot X\}$$

(see [10, p. 59]). In the sequel we will write members of \mathfrak{g} as block matrices with respect to the decomposition $\mathbb{R}^{m+2} = \mathbb{R}^2 \oplus \mathbb{R}^m$, i.e. in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where $X_{11}, X_{12}, X_{21}, X_{22}$ are real matrices of the dimension $2 \times 2, 2 \times m, m \times 2$ and $m \times m$, respectively. Then

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \mid X_{11}^t = -X_{11}, X_{12}^t = X_{21}, X_{22}^t = -X_{22} \right\}.$$

The linearization $\sigma_L = \text{Ad}(s) : \mathfrak{g} \rightarrow \mathfrak{g}$ of the involutive Lie group automorphism σ induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where the Lie subalgebra

$$\begin{aligned} \mathfrak{k} &= \text{Eig}(\sigma_*, 1) = \{X \in \mathfrak{g} \mid sXs^{-1} = X\} \\ &= \left\{ \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \mid X_{11}^t = -X_{11}, X_{22}^t = -X_{22} \right\} \cong \mathfrak{so}_2 \oplus \mathfrak{so}_m \end{aligned}$$

is the Lie algebra of the isotropy group K , and the $2m$ -dimensional linear subspace

$$\mathfrak{m} = \text{Eig}(\sigma_*, -1) = \{X \in \mathfrak{g} \mid sXs^{-1} = -X\} = \left\{ \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \mid X_{12}^t = X_{21} \right\}$$

is canonically isomorphic to the tangent space $T_{p_0}Q^{m*}$. Under the identification $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the Riemannian metric g of Q^{m*} (where the constant factor of the metric is chosen so that the formulae become as simple as possible) is given by

$$g(X, Y) = \frac{1}{2} \text{tr}(Y^t \cdot X) = \text{tr}(Y_{12} \cdot X_{21}) \quad \text{for} \quad X, Y \in \mathfrak{m}.$$

g is clearly $\text{Ad}(K)$ -invariant and therefore corresponds to an $\text{Ad}(G)$ -invariant Riemannian metric on Q^{m*} . The complex structure J of the Hermitian symmetric space is given by

$$JX = \text{Ad}(j)X \quad \text{for } X \in \mathfrak{m}, \quad \text{where } j := \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \in K.$$

Because j is in the center of K , the orthogonal linear map J is $\text{Ad}(K)$ -invariant and thus defines an $\text{Ad}(G)$ -invariant Hermitian structure on Q^{m*} . By identifying the multiplication with the unit complex number i with the application of the linear map J , the tangent spaces of Q^{m*} thus become m -dimensional complex linear spaces, and we will adopt this point of view in the sequel.

Like for the complex quadric (again compare [18] and [9]), there is another important structure on the tangent bundle of the complex quadric besides the Riemannian metric and the complex structure, namely an S^1 -bundle \mathfrak{A} of real structures (conjugations). The situation here differs from that of the complex quadric in that for Q^{m*} , the real structures in \mathfrak{A} cannot be interpreted as the shape operator of a complex hypersurface in a complex space form, but as the following considerations will show, \mathfrak{A} still plays a fundamental role in the description of the geometry of Q^{m*} .

Let

$$a_0 := \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}.$$

Note that we have $a_0 \notin K$, but only $a_0 \in O_2 SO_m$. However, $\text{Ad}(a_0)$ still leaves \mathfrak{m} invariant and therefore defines an \mathbb{R} -linear map A_0 on the tangent space $\mathfrak{m} \cong T_{p_0} Q^{m*}$. A_0 turns out to be an involutive orthogonal map with $A_0 \circ J = -J \circ A_0$ (i.e. A_0 is anti-linear with respect to the complex structure of $T_{p_0} Q^{m*}$), and hence a real structure on $T_{p_0} Q^{m*}$. But A_0 commutes with $\text{Ad}(g)$ not for all $g \in K$, but only for $g \in SO_m \subset K$. More specifically, for $g = (g_1, g_2) \in K$ with $g_1 \in SO_2$ and $g_2 \in SO_m$, say $g_1 = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ with $t \in \mathbb{R}$ (so that $\text{Ad}(g_1)$ corresponds to multiplication with the complex number $\mu := e^{it}$), we have

$$A_0 \circ \text{Ad}(g) = \mu^{-2} \cdot \text{Ad}(g) \circ A_0.$$

This equation shows that the object which is $\text{Ad}(K)$ -invariant and therefore geometrically relevant is not the real structure A_0 by itself, but rather the ‘‘circle of real structures’’

$$\mathfrak{A}_{p_0} := \{ \lambda A_0 \mid \lambda \in S^1 \}.$$

\mathfrak{A}_{p_0} is $\text{Ad}(K)$ -invariant and therefore generates an $\text{Ad}(G)$ -invariant S^1 -subbundle \mathfrak{A} of the endomorphism bundle $\text{End}(TQ^{m*})$, consisting of real structures (conjugations) on the tangent spaces of Q^{m*} . For any $A \in \mathfrak{A}$, the tangent line to the fibre of \mathfrak{A} through A is spanned by JA .

For any $p \in Q^{m*}$ and $A \in \mathfrak{A}_p$, the real structure A induces a splitting

$$T_p Q^{m*} = V(A) \oplus JV(A)$$

into two orthogonal, maximal totally real subspaces of the tangent space $T_p Q^{m*}$. Here $V(A)$ resp. $JV(A)$ are the (+1)-eigenspace resp. the (-1)-eigenspace of A . For every unit vector $Z \in T_p Q^{m*}$ there exist $t \in [0, \frac{\pi}{4}]$, $A \in \mathfrak{A}_p$ and orthonormal vectors $X, Y \in V(A)$ so that

$$Z = \cos(t) \cdot X + \sin(t) \cdot JY$$

holds; see [18, Proposition 3]. Here t is uniquely determined by Z . The vector Z is singular, i.e. contained in more than one Cartan subalgebra of \mathfrak{m} , if and only if either $t = 0$ or $t = \frac{\pi}{4}$ holds. The vectors with $t = 0$ are called \mathfrak{A} -principal, whereas the vectors with $t = \frac{\pi}{4}$ are called \mathfrak{A} -isotropic. If Z is regular, i.e. $0 < t < \frac{\pi}{4}$ holds, then also A and X, Y are uniquely determined by Z .

Like for the complex quadric, the Riemannian curvature tensor R of Q^{m*} can be fully described in terms of the “fundamental geometric structures” g, J and \mathfrak{A} . In fact, under the correspondence $T_{p_0} Q^{m*} \cong \mathfrak{m}$, the Riemannian curvature tensor $\bar{R}(X, Y)Z$ corresponds to $-[[X, Y], Z]$ for $X, Y, Z \in \mathfrak{m}$, see [11, Chapter XI, Theorem 3.2(1)]. By evaluating the latter expression explicitly, one can show that one has

$$\begin{aligned} \bar{R}(X, Y)Z = & -g(Y, Z)X + g(X, Z)Y \\ & -g(JY, Z)JX + g(JX, Z)JY + 2g(JX, Y)JZ \\ & -g(AY, Z)AX + g(AX, Z)AY \\ & -g(JAY, Z)JAX + g(JAX, Z)JAY \end{aligned} \tag{2.1}$$

for arbitrary $A \in \mathfrak{A}_{p_0}$. Therefore the curvature of Q^{m*} is the negative of that of the complex quadric Q^m , compare [18, Theorem 1]. This confirms that the symmetric space Q^{m*} which we have constructed here is indeed the non-compact dual of the complex quadric.

As Nomizu [12, Theorem 15.3] has shown, there exists one and only one torsion-free covariant derivative $\bar{\nabla}$ on Q^{m*} so that the symmetric involutions $s_p : Q^{m*} \rightarrow Q^{m*}$ at $p \in Q^{m*}$ are all affine. $\bar{\nabla}$ is the *canonical covariant derivative* of Q^{m*} . With respect to $\bar{\nabla}$, the action of any member of G on Q^{m*} is also affine. Moreover, $\bar{\nabla}$ is the Levi-Civita connection corresponding to the Riemannian metric g , and therefore g is parallel with respect to $\bar{\nabla}$. Moreover, it is well-known that Q^{m*} becomes a Kähler manifold in this way, i.e. the complex structure J is also parallel. Finally, because the S^1 -subbundle \mathfrak{A} of the endomorphism bundle $\text{End}(TQ^{m*})$ is $\text{Ad}(G)$ -invariant, it is also parallel with respect to the covariant derivative $\bar{\nabla}^{\text{End}}$ induced by $\bar{\nabla}$ on $\text{End}(TQ^{m*})$. Because the tangent line of the fiber of \mathfrak{A} through some $A_p \in \mathfrak{A}$ is spanned by JA_p , this means precisely that for any section A of \mathfrak{A} there exists a real-valued 1-form $q : TQ^{m*} \rightarrow \mathbb{R}$ so that

$$\bar{\nabla}_v^{\text{End}} A = q(v) \cdot JA_p \quad \text{holds for } p \in Q^{m*}, v \in T_p Q^{m*}.$$

From the presentation (2.1) of the curvature tensor it follows analogously as for Q^m in [18, Sects. 5 and 6] that Q^{m*} has rank 2, that a linear subspace $\mathfrak{a} \subset \mathfrak{m}$ is a Cartan subalgebra if and only if there exist $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ so that $\mathfrak{a} = \mathbb{R}X \oplus \mathbb{R}JY$ holds, and that the positive root system $\Sigma^+ = \{\lambda_1, \dots, \lambda_4\}$ (in terms of the vectors $\lambda_k^\# \in \mathfrak{a}$ dual to the roots $\lambda_k \in \mathfrak{a}^*$) and the corresponding root spaces \mathfrak{m}_{λ_k} are given by

k	λ_k^\sharp	\mathfrak{m}_{λ_k}
1	$\sqrt{2} \cdot JY$	$J(V(A) \ominus \mathbb{R}X \ominus \mathbb{R}Y)$
2	$\sqrt{2} \cdot X$	$V(A) \ominus \mathbb{R}X \ominus \mathbb{R}Y$
3	$\sqrt{2} \cdot (X - JY)$	$\mathbb{R}(JX + Y)$
4	$\sqrt{2} \cdot (X + JY)$	$\mathbb{R}(JX - Y)$

3 Tubes around the totally geodesic submanifolds $Q^{m-1*} \subset Q^{m*}$ and $\mathbb{R}H^m \subset Q^{m*}$

At first we let P be any submanifold of a Riemannian symmetric space \bar{M} , and for $p \in P$ we let $\perp_p P$ be the normal space of P in \bar{M} at p , and let $\perp_p^1 P := \{v \in \perp_p P \mid \|v\| = r\}$ be the unit sphere in $\perp_p P$. We let $\perp P$ resp. $\perp^r P$ be the vector bundle resp. the sphere bundle of normal vectors resp. of unit length normal vectors over P , and let $\tau : \perp P \rightarrow P$ be the bundle projection map. Moreover, we let $K^\perp : T(\perp P) \rightarrow \perp P$ be the normal connection map of $\perp P$, i.e. $K^\perp u = \nabla_u \text{id}_{\perp P}$. For any $\eta \in \perp P$,

$$T_\eta(\perp P) \rightarrow T_{\tau(\eta)}P \oplus \perp_{\tau(\eta)} P, u \mapsto (\tau_* u, K^\perp u)$$

is then an isomorphism of vector spaces.

The *tube map* of radius $r > 0$ is the map

$$\Phi : \perp^1 P \rightarrow \bar{M}, \eta \mapsto \exp_p^{\bar{M}}(r\eta),$$

where $\exp^{\bar{M}}$ denotes the (geodesic) exponential map of \bar{M} . If Φ is a diffeomorphism into \bar{M} , we call its image M the *tube* around P of radius r , it is then a real hypersurface of \bar{M} . Berndt [1, Corollary 4.4] has described how to calculate the eigenvalues and eigenvectors of the shape operator of the map Φ , and hence the principal curvatures, their multiplicities and the corresponding principal curvature directions of the tube M . We describe his results here only for the case where the submanifold P is totally geodesic.

Therefore suppose that P is a totally geodesic submanifold of \bar{M} . Let \bar{R} be the Riemannian curvature tensor of \bar{M} . For any $p \in P, \eta \in \perp_p P$ we let $\bar{R}_\eta := \bar{R}(\cdot, \eta)\eta$ be the corresponding Jacobi operator of \bar{M} and define $R_\eta := \bar{R}_\eta|_{T_p P}$. Let γ_η be the unit speed geodesic of \bar{M} with $\gamma_\eta(0) = p$ and $\dot{\gamma}_\eta(0) = \eta$; then $\gamma_\eta(r) = \Phi(p) \in M$ and $\dot{\gamma}_\eta(r)$ is a unit normal vector of M ; we say that $\dot{\gamma}_\eta(r)$ is *pointing outward* and that $-\dot{\gamma}_\eta(r)$ is *pointing inward*. For the description of the principal curvatures, we follow [1, Sect. 4.2] by considering for any $\kappa \in \mathbb{R}$ the solution functions \sin_κ and \cos_κ of the second order differential equation $y'' + \kappa y = 0$ with $\sin_\kappa(0) = 0, \sin'_\kappa(0) = 1$ and $\cos_\kappa(0) = 1, \cos'_\kappa(0) = 0$. Explicitly one has

$$\sin_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} t) & \text{for } \kappa > 0 \\ t & \text{for } \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} t) & \text{for } \kappa < 0 \end{cases} \quad \text{and} \quad \cos_\kappa(t) = \begin{cases} \cos(\sqrt{\kappa} t) & \text{for } \kappa > 0 \\ 1 & \text{for } \kappa = 0 \\ \cosh(\sqrt{-\kappa} t) & \text{for } \kappa < 0 \end{cases} .$$

According to [1, Corollary 4.4], the shape operator S^Φ with respect to the outward-pointing unit normal vector $\dot{\gamma}(r)$ has two types of eigenvalues, and any eigenvalue or eigenvector of S^Φ is obtained by either of these two methods:

- (I) For any eigenvalue κ of R_η , the number $\kappa \cdot \frac{\sin_\kappa(r)}{\cos_\kappa(r)}$ is an eigenvalue of S^Φ . The corresponding eigenspace of S^Φ is given by

$$\{u \in T_\eta(\perp^1 P) \mid \tau_*u \in E_\kappa, K^\perp u = 0\},$$

where E_κ denotes the eigenspace of R_η for the eigenvalue κ .

- (II) For any eigenvalue κ of $\bar{R}_\eta|(\perp_{\tau(\eta)} P \ominus \mathbb{R}\eta)$, the number $-\frac{\cos_\kappa(r)}{\sin_\kappa(r)}$ is an eigenvalue of S^Φ . The corresponding eigenspace of S^Φ is given by

$$\{u \in T_\eta(\perp^1 P) \mid \tau_*u = 0, K^\perp u \in \bar{E}_\kappa\},$$

where \bar{E}_κ denotes the eigenspace of $\bar{R}_\eta|(\perp_{\tau(\eta)} P \ominus \mathbb{R}\eta)$ for the eigenvalue κ .

The principal curvatures of M at $\Phi(\eta)$ are precisely the eigenvalues of S^Φ described above. The corresponding principal curvature spaces are the image under Φ_* of the eigenspaces of S^Φ ; hence, they are obtained from the eigenspaces E_κ (in the case (I)) resp. \bar{E}_κ (in the case (II)) by parallel transport along the geodesic γ_η .

We will now apply the preceding results to tubes in the complex hyperbolic quadric $\bar{M} = Q^{m*}$ around the two types of totally geodesic submanifolds that occur in Theorem A(i),(iii). It follows from Eq. (2.1) that if the unit normal vector $\eta \in \perp^1 Q^{m*}$ is \mathfrak{A} -principal, the corresponding Jacobi operator is given by

$$\bar{R}_\eta(X) = -X - AX + 2g(X, \eta)\eta - 2g(X, J\eta)J\eta.$$

It is easy to see that \bar{R}_η then has the two eigenvalues 0 and -2 . If $A \in \mathfrak{A}$ is such that $\eta \in V(A)$ holds, then the corresponding eigenspaces are $\mathbb{R}\eta \oplus J(V(A) \ominus \mathbb{R}\eta)$ and $(V(A) \ominus \mathbb{R}\eta) \oplus \mathbb{R}J\eta$, respectively.

The obvious embedding of Lie groups $SO_{2,m-1}^o \rightarrow SO_{2,m}^o$ induces a totally geodesic embedding of $Q^{m-1*} = SO_{2,m-1}/SO_2SO_{m-1}$ into $Q^{m*} = SO_{2,m}/SO_2SO_m$. We will view Q^{m-1*} as a totally geodesic complex hypersurface of Q^{m*} by means of this embedding.

Proposition 3.1 The tube M around the totally geodesic Q^{m-1*} in Q^{m*} exists for every radius $r > 0$. For M the following statements hold:

- (1) Every normal vector N of M is \mathfrak{A} -principal.
- (2) M has constant principal curvatures, and in particular constant mean curvature. Let N be the outward-pointing unit normal vector of M at $q \in M$. By (1) we have $N \in V(A)$ for some $A \in \mathfrak{A}_q$. Then the principal curvatures of M with respect to N and the corresponding principal curvature spaces are

Principal curvature	Principal curvature space	Multiplicity
$\lambda = 0$	$J(V(A) \ominus \mathbb{R}N)$	$m - 1$
$\mu = -\sqrt{2} \tanh(\sqrt{2}r)$	$V(A) \ominus \mathbb{R}N$	$m - 1$
$\alpha = -\sqrt{2} \coth(\sqrt{2}r)$	$\mathbb{R}JN$	1

- (3) M is a Hopf hypersurface.
- (4) The shape operator S and the structure tensor field ϕ ($\phi = \text{pr}_{TM} \circ J$, where $\text{pr}_{TM} : TQ^{m*} \rightarrow TM$ denotes the orthogonal projection) satisfy

$$S\phi + \phi S = \mu \cdot \phi.$$

In particular M is a contact submanifold.

Proof It follows from the construction of the S^1 -subbundle \mathfrak{A} of $\text{End}(TQ^{m*})$ of real structures that for any $p \in Q^{m-1*}$, both complex subspaces $T_p Q^{m-1*}$ and $\perp_p Q^{m-1*}$ are invariant under every $A_p \in \mathfrak{A}_p$. Because $\perp_p Q^{m-1*}$ thus is a complex one-dimensional subspace that is invariant under $A_p \in \mathfrak{A}_p$, it follows that the vectors in this space are \mathfrak{A} -principal.

Now let $q \in M$ be given. It follows from the construction of the tube M that there exists $p \in Q^{m-1*}$ and a unit normal vector $\eta \in \perp_p Q^{m-1*}$ so that the normal space $\perp_q M$ of M at q is spanned by $\dot{\gamma}_\eta(r)$. Because $\dot{\gamma}_\eta(0) \in \perp_p Q^{m-1*}$ is \mathfrak{A} -principal by the preceding observation, and the S^1 -bundle \mathfrak{A} is parallel, it follows that also $\dot{\gamma}_\eta(r)$ and hence $\perp_q M$ is \mathfrak{A} -principal.

Moreover the outward-pointing unit normal vector N of M at q equals $\dot{\gamma}_\eta(r)$ and is contained in $V(A_q)$, where $A_q \in \mathfrak{A}_q$ is the parallel transport of A_p along the geodesic γ_η . Note that $T_p Q^{m-1*} = (V(A_p) \ominus \mathbb{R}\eta) \oplus J(V(A_p) \ominus \mathbb{R}\eta)$ and $\perp_p Q^{m-1*} = \mathbb{R}\eta \oplus \mathbb{R}J\eta$ holds. We now apply the two cases (I) and (II) given above to obtain the principal curvatures of M with respect to N . For case (I), we note that $R_\eta := \bar{R}_\eta|_{T_p Q^{m-1*}}$ has the two eigenvalues 0 and -2 , with eigenspaces $J(V(A_p) \ominus \mathbb{R}\eta)$ and $V(A_p) \ominus \mathbb{R}\eta$, respectively. From the eigenvalue $\kappa = 0$ we obtain the principal curvature $\kappa \cdot \frac{\sin_\kappa(r)}{\cos_\kappa(r)} = 0 = \lambda$, and from the eigenvalue $\kappa = -2$ we obtain the principal curvature $\kappa \cdot \frac{\sin_\kappa(r)}{\cos_\kappa(r)} = -\sqrt{2} \tanh(\sqrt{2}r) = \mu$. In case (II), we have $\perp_p Q^{m-1*} \ominus \mathbb{R}\eta = \mathbb{R}J\eta$, and therefore the only eigenvalue of $\bar{R}_\eta|_{(\perp_p Q^{m-1*} \ominus \mathbb{R}\eta)}$ is $\kappa = -2$ with the eigenspace $\mathbb{R}J\eta$. This yields the principal curvature $-\frac{\cos_\kappa(r)}{\sin_\kappa(r)} = -\frac{\cos_\kappa(r)}{\sin_\kappa(r)} = -\sqrt{2} \coth(\sqrt{2}r) = \alpha$. The principal curvature spaces are obtained from the corresponding eigenspaces of \bar{R}_η by parallel transport along γ_η and are therefore as stated in the proposition.

It follows from the calculation of the principal curvatures that M has constant principal curvatures, in particular constant mean curvature. The claim that M is Hopf means by definition that the Hopf vector field $\xi = -JN$ is a principal vector field; this is the case by our previous calculation. The corresponding principal curvature is $\alpha = -\sqrt{2} \coth(\sqrt{2}r)$. Moreover, it is easily seen that $\phi(JN) = 0$ holds, and that ϕ acts as J on $V(A_q) \ominus \mathbb{R}N$ and on $J(V(A_q) \ominus \mathbb{R}N)$. Therefore the equation in (4) is easily verified on the principal curvature spaces described in (2). Because Eq. (1.2) thus holds for M with $k = \mu$, it follows that M is a contact submanifold. □

A similar discussion applies to tubes around the totally geodesic $\mathbb{R}H^m$ in Q^{m*} . The obvious embedding of Lie groups $SO_{1,m} \rightarrow SO_{2,m}$ induces a totally geodesic embedding of $\mathbb{R}H^m = SO_{1,m}/SO_m$ into $Q^{m*} = SO_{2,m}/SO_2SO_m$. We will view $\mathbb{R}H^m$ as a real form, i.e. a totally geodesic, totally real, real- m -dimensional submanifold of Q^{m*} by means of this embedding.

Proposition 3.2 The tube M around the totally geodesic $\mathbb{R}H^m$ in Q^{m*} exists for every radius $r > 0$. For M the following statements hold:

- (1) Every normal vector N of M is \mathfrak{A} -principal.
- (2) M has constant principal curvatures, and in particular constant mean curvature. Let N be the outward-pointing unit normal vector of M at $q \in M$. By (1) we have $N \in V(A)$ for some $A \in \mathfrak{A}_q$. Then the principal curvatures of M with respect to N and the corresponding principal curvature spaces are

Principal curvature	Principal curvature space	Multiplicity
$\lambda = 0$	$J(V(A) \ominus \mathbb{R}N)$	$m - 1$
$\mu = -\sqrt{2} \coth(\sqrt{2}r)$	$V(A) \ominus \mathbb{R}N$	$m - 1$
$\alpha = -\sqrt{2} \tanh(\sqrt{2}r)$	$\mathbb{R}JN$	1

- (3) M is a Hopf hypersurface.
- (4) The shape operator S and the structure tensor field ϕ ($\phi = \text{pr}_{TM} \circ J$, where $\text{pr}_{TM} : TQ^{m*} \rightarrow TM$ denotes the orthogonal projection) satisfy

$$S\phi + \phi S = \mu \cdot \phi .$$

In particular M is a contact submanifold.

Proof The proof of this proposition follows the same pattern as that of Proposition 3.1.

For any $p \in \mathbb{R}H^m$, there exists $A_p \in \mathfrak{A}_p$ so that $T_p \mathbb{R}H^m = V(-A_p)$ and therefore $\perp_p \mathbb{R}H^m = JV(-A_p) = V(A_p)$ holds. In particular, both $T_p \mathbb{R}H^m$ and $\perp_p \mathbb{R}H^m$ are \mathfrak{A} -principal. Now let $q \in M$ be given. Again there exists $p \in \mathbb{R}H^m$ and $\eta \in \perp_p^1 \mathbb{R}H^m$ so that $\perp_q M$ is spanned by $\dot{\gamma}_\eta(r)$. Because $\dot{\gamma}_\eta(0) \in \perp_p \mathbb{R}H^m = V(A_p)$ is \mathfrak{A} -principal, it again follows that also $\dot{\gamma}_\eta(r)$ and hence $\perp_q M$ is \mathfrak{A} -principal. Hence the outward-pointing unit normal vector $N = \gamma_\eta(r)$ is contained in $V(A_q)$ where $A_q \in \mathfrak{A}_q$ is the parallel transport of A_p along γ_η .

We again apply the two cases (I) and (II) given above to obtain the principal curvatures of M with respect to N . For case (I), we note that $R_\eta := \bar{R}_\eta|_{T_p \mathbb{R}H^m}$ has the two eigenvalues 0 and -2 with eigenspaces $J(V(A_p) \ominus \mathbb{R}\eta)$ and $\mathbb{R}J\eta$, respectively. From the eigenvalue $\kappa = 0$ we obtain $\kappa \cdot \frac{\sin_\kappa(r)}{\cos_\kappa(r)} = 0 = \lambda$, and from the eigenvalue $\kappa = -2$ we obtain the principal curvature $\kappa \cdot \frac{\sin_\kappa(r)}{\cos_\kappa(r)} = -\sqrt{2} \tanh(\sqrt{2}r) = \alpha$. In case (II), we have $\perp_p \mathbb{R}H^m \ominus \mathbb{R}\eta = V(A_p) \ominus \mathbb{R}\eta$, and therefore the only eigenvalue of $\bar{R}_\eta|_{(\perp_p Q^{m-1*} \ominus \mathbb{R}\eta)}$ is $\kappa = -2$ with the eigenspace $V(A_p) \ominus \mathbb{R}\eta$. This yields the principal curvature $-\frac{\cos_\kappa(r)}{\sin_\kappa(r)} = -\frac{\cos_\kappa(r)}{\sin_\kappa(r)} = -\sqrt{2} \coth(\sqrt{2}r) = \mu$.

The remaining parts of the proposition now follow in the same way as in the proof of Proposition 3.1.

4 Horospheres in Q^{m*}

Suppose that \bar{M} is a Hadamard manifold, i.e. a simply connected, complete Riemannian manifold with sectional curvature ≤ 0 . We denote by d the Riemannian distance function on \bar{M} . Two unit speed geodesics $\gamma_1, \gamma_2 : \mathbb{R} \rightarrow \bar{M}$ are said to be *asymptotic* to each other, if the function $t \mapsto d(\gamma_1(t), \gamma_2(t))$ remains bounded for $t \rightarrow \infty$. Asymptoticness defines an equivalence relation on the space of unit speed geodesics on \bar{M} . The equivalence classes are called *points at infinity*, and their set is denoted by $\bar{M}(\infty)$. For any unit speed geodesic $\gamma : \mathbb{R} \rightarrow \bar{M}$, the corresponding point at infinity is denoted by $\gamma(\infty) \in \bar{M}(\infty)$. The *horosphere* with center at infinity $\gamma(\infty)$ through some point $p \in \bar{M}$ is defined as

$$C(p, \gamma(\infty)) = \left\{ q \in \bar{M} \mid \lim_{t \rightarrow \infty} (d(q, \gamma(t)) - d(p, \gamma(t))) = 0 \right\} .$$

It can be shown that $C(p, \gamma(\infty))$ indeed does not depend on the choice of the representant γ within the equivalence class $\gamma(\infty)$ and that it is a real hypersurface of \bar{M} , see [8, Sect. 1.10].

We now suppose that \bar{M} is a Riemannian symmetric space of non-compact type and rank 2. The following construction principle for horospheres in \bar{M} was described by Berndt and Suh in [3, Sect. 2]. Let us consider $\bar{M} = G/K$ with the ‘‘origin’’ $o := eK \in \bar{M}$, the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{m}$. Further consider the root system $\Sigma \subset \mathfrak{a}^*$ and the corresponding root space decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{k}_{\mathfrak{a}} \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_{\lambda}$. For a positive root system $\Sigma^+ \subset \Sigma$, $\mathfrak{n} := \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_{\lambda}$ is a nilpotent subalgebra of \mathfrak{g} , and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is an Iwasawa decomposition of \mathfrak{g} . Let A and N be the connected Lie subgroups of G with Lie algebra \mathfrak{a} resp. \mathfrak{n} . Then $G = KAN$ is an Iwasawa decomposition of G , more precisely, A and N are simply connected, $K \cap AN = \{e\}$ holds, and the maps

$$K \times A \times N \rightarrow G, (k, a, n) \mapsto kan \quad \text{and} \quad A \times N \rightarrow \bar{M}, (a, n) \mapsto an \cdot o$$

are surjective diffeomorphisms, see [10, Theorem VI.6.46, p. 374].

Now suppose that a unit vector $H \in \mathfrak{a}$ is given. Then $\mathfrak{g}_H := (\mathfrak{a} \ominus \mathbb{R}H) \oplus \mathfrak{n}$ is a solvable Lie subalgebra of \mathfrak{g} . Let S_H be the connected subgroup of AN with Lie algebra \mathfrak{g}_H . Then the orbits of the action of S_H on \bar{M} are the horospheres of \bar{M} with the center at infinity $\gamma_H(\infty)$, where γ_H is the geodesic with $\gamma_H(0) = o$ and $\dot{\gamma}_H(0) = H$ (and where we identify \mathfrak{m} with $T_o\bar{M}$ in the usual manner). In particular we have $C(o, \gamma_H(\infty)) = S_H \cdot o$. It was shown by Berndt and Tamaru in [5, Proposition 3.1(2),(3)] that the shape operator of $C(o, \gamma_H(\infty))$ with respect to the unit normal vector H is given by $\text{ad}(H)|_{\mathfrak{g}_H}$. Therefore the principal curvatures are constant, and their values are given by 0 and by $\lambda(H)$ for every $\lambda \in \Sigma^+$. The corresponding principal curvature spaces are $\mathfrak{a} \ominus \mathbb{R}H$ and \mathfrak{g}_{λ} ; under the standard identification $T_e\bar{M} \cong \mathfrak{m}$, the latter space corresponds to $\mathfrak{m}_{\lambda} = \{X - \theta X \mid X \in \mathfrak{g}_{\lambda}\}$ (where $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ denotes the Cartan involution).

We will now apply this construction to $\bar{M} = Q^{m*}$ and an \mathfrak{A} -principal vector $H \in \mathfrak{a}$.

Proposition 4.1 Let M be a horosphere in Q^{m*} with its center at infinity being given by an \mathfrak{A} -principal geodesic γ . Then the following statements hold:

- (1) Every normal vector N of M is \mathfrak{A} -principal.
- (2) M has constant principal curvatures, and in particular constant mean curvature. Then the principal curvatures of M with respect to the unit normal vector¹ $N := -\dot{\gamma}(0)$ and the corresponding principal curvature spaces are

Principal curvature	Principal curvature space	Multiplicity
0	$J(V(A) \ominus \mathbb{R}N)$	$m - 1$
$-\sqrt{2}$	$(V(A) \ominus \mathbb{R}N) \oplus \mathbb{R}JN$	m

Here $A \in \mathfrak{A}$ is chosen such that $-\dot{\gamma}(0) \in V(A)$ holds.

- (3) M is a Hopf hypersurface.
- (4) The shape operator S and the structure tensor field ϕ ($\phi = \text{pr}_{TM} \circ J$, where $\text{pr}_{TM} : TQ^{m*} \rightarrow TM$ denotes the orthogonal projection) satisfy

¹ We choose the negative of $\dot{\gamma}(0)$ as normal vector here so that the orientation matches the one considered for the tubes in Sect. 3.

$$S\phi + \phi S = -\sqrt{2} \cdot \phi.$$

In particular M is a contact submanifold.

Proof We use the description of the Cartan subalgebras, the roots and the root spaces of Q^{m*} given at the end of Sect. 2. We may assume without loss of generality that $\gamma(0) = o$ and $\dot{\gamma}(0) \in \mathfrak{a}$ holds. Because the geodesic γ is \mathfrak{A} -principal, there exists $A \in \mathfrak{A}_o$ with $X := \dot{\gamma}(0) \in V(A)$, and we may further assume that the Cartan subalgebra \mathfrak{a} is given by $\mathfrak{a} = \mathbb{R}X \oplus \mathbb{R}JY$ with a unit vector $Y \in V(A) \ominus \mathbb{R}X$. Numbering the positive roots of Q^{m*} as in Sect. 2 we then have

$$\lambda_1(-X) = 0 \quad \text{and} \quad \lambda_2(-X) = \lambda_3(-X) = \lambda_4(-X) = -\sqrt{2}.$$

It follows that the horosphere M has the two principal curvatures 0 with principal curvature space $(\mathfrak{a} \ominus \mathbb{R}X) \oplus \mathfrak{m}_{\lambda_1} = J(V(A) \ominus \mathbb{R}X)$, and $-\sqrt{2}$ with the principal curvature space $\mathfrak{m}_{\lambda_2} \oplus \mathfrak{m}_{\lambda_3} \oplus \mathfrak{m}_{\lambda_4} = (V(A) \ominus \mathbb{R}X) \oplus \mathbb{R}JX$.

The remainder of the statements follows in the same way as for the proofs of Sect. 3.

Remark 4.2 Note that for both the family of tubes around Q^{m-1*} (Proposition 3.1) and the family of tubes around $\mathbb{R}H^m$ (Proposition 3.2), when one lets the radius $r > 0$ of the tube tend to infinity, the values of the principal curvatures and the corresponding principal curvature spaces tend to the values and spaces of the horosphere with \mathfrak{A} -principal center at infinity (Proposition 4.1). In this sense, this horosphere is the joint limit of both mentioned families of tubes as $r \rightarrow \infty$.

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