Annali di Matematica Pura ed Applicata (1923 -) (2019) 198:1481–1494
https://doi.org/10.1007/s10231-019-00827-y

Contact real hypersurfaces in the complex hyperbolic quadric

Sebastian Klein[1] · Young Jin Suh[2]

Received: 8 June 2018 / Accepted: 18 January 2019 / Published online: 1 February 2019

Abstract

We discuss the geometry of contact real hypersurfaces with constant mean curvature in the complex hyperbolic quadric $Q^{m*} = SO^o_{m,2}/SO_m SO_2$, where $m \geq 3$. These hypersurfaces were classified in Berndt and Suh (Proc Am Math Soc 143:2637–2649, 2015), and we study the individual types (two types of tubes around totally geodesic submanifolds of Q^{m*} and one type of horosphere) that have been found in that classification.

Keywords Contact hypersurface · Kähler structure · Complex conjugation · Complex hyperbolic quadric

Mathematics Subject Classification Primary 53C40 · Secondary 53C55

1 Introduction

Following Sasaki [19] and Okumura [13], an odd-dimensional, smooth manifold M^{2m-1} is called an *almost contact manifold* if the structure group of its tangent bundle can be reduced to $U_{m-1} \times 1$ (where U_{m-1} refers to the natural real representation of the unitary group in $m-1$ complex variables). M^{2m-1} is called a *contact manifold* if there exists a smooth 1-form η on M^{2m-1} so that $\eta \wedge d\eta^{m-1} \neq 0$; such an η is then called a *contact form* on M^{2m-1}.

It was shown by Sasaki [19, Theorem 5] that M^{2m-1} is an almost contact manifold if and only if there exists an *almost contact metric structure* (ϕ, ξ, η, g) on M^{2m-1}. Here ϕ is an

Young Jin Suh was supported by Grant Project No. NRF-2018-R1D1A1B-05040381 from National Research Foundation of Korea.

✉ Sebastian Klein
s.klein@math.uni-mannheim.de

Young Jin Suh
yjsuh@knu.ac.kr

1 Fakultät für Wirtschaftsinformatik und Wirtschaftsmathematik, Universität Mannheim, Seminargebäude A5, 68131 Mannheim, Germany

2 Department of Mathematics & RIRCM, College of Natural Sciences, Kyungpook National University, Daegu 41566, Republic of Korea

endomorphism field on M^{2m-1}, ξ is a vector field on M^{2m-1}, η is a 1-form on M^{2m-1} and g is a Riemannian metric on M^{2m-1}, and these data are related to each other in the following way: First we have

$$\phi^2 X = -X + \eta(X)\xi\ , \quad \phi(\xi) = 0\ , \quad \eta(\phi X) = 0\ , \quad \eta(\xi) = 1$$

for all vector fields X on M^{2m-1}, meaning that (ϕ, ξ, η) is an *almost contact structure*, and moreover this structure is adapted to the Riemannian metric g by

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y) \quad \text{and} \quad \eta(X) = g(X, \xi)$$

for all vector fields X, Y on M^{2m-1}.

Clearly, if M^{2m-1} has an almost contact metric structure (ϕ, ξ, η, g) so that $\eta \wedge d\eta^{m-1} \neq 0$ holds, then M^{2m-1} is a contact manifold. Conversely, if M^{2m-1} is a contact manifold, then for any contact form η on M^{2m-1} there exists an almost contact metric structure (ϕ, ξ, η, g) with this η by a result due to Sasaki [19, Theorem 4].

Let us now consider a real hypersurface M of a Kähler manifold $\bar{M}$ of complex dimension m. Then M has real dimension $2m - 1$, and the complex structure J and the Riemannian metric g of $\bar{M}$ induce an almost contact metric structure (ϕ, ξ, η, g) on M: Let N be a unit normal vector field of M in $\bar{M}$, then we choose ϕ as the *structure tensor field* defined by letting ϕX be the M-tangential part of JX for any $X \in TM$, choose ξ as the *Reeb vector field* $\xi = -JN$, and choose $\eta = g(\,\cdot\,, \xi)$. If there exists a smooth, everywhere nonzero function ρ on M in this setting so that

$$\mathrm{d}\eta(X, Y) = \rho \cdot g(\phi X, Y) \tag{1.1}$$

holds for all vector fields X, Y on M, then M is a contact manifold, and η a contact form on M. In this case M is called a *contact hypersurface* of $\bar{M}$, see also Blair [6], Dragomir and Perrone [7]. It was noted by Okumura [13, Eq. (2.13)] that the condition (1.1) is equivalent to

$$S\phi + \phi S = k \cdot \phi\ , \tag{1.2}$$

where S denotes the shape operator of the hypersurface M in $\bar{M}$ with respect to the unit normal vector field N, and $k = 2\rho$. If the complex dimension of $\bar{M}$ is at least 3 in this setting, then the function ρ resp. k is necessarily constant, see [4, Proposition 2.5].

Pursuant to these ideas, the contact hypersurfaces have been classified in the Hermitian symmetric spaces of rank 1, namely in the complex projective space $\mathbb{C}P^m$ and its non-compact dual, the complex hyperbolic space $\mathbb{C}H^m$. Yano and Kon showed in [24, Theorem VI.1.5] that a connected contact hypersurface with constant mean curvature of the complex projective space $\mathbb{C}P^m$ with $m \geq 3$ is locally congruent either to a geodesic hypersphere, or to a tube over a real projective space $\mathbb{R}P^n$, $m = 2n$, embedded in $\mathbb{C}P^m$ as a totally real, totally geodesic submanifold. Vernon proved in [23] that a complete, connected contact real hypersurface in $\mathbb{C}H^m$ with $m \geq 3$ is congruent to a tube around a totally geodesic $\mathbb{C}H^{m-1}$ in $\mathbb{C}H^m$, a tube around a real form $\mathbb{R}H^m$ in $\mathbb{C}H^m$, a geodesic hypersphere in $\mathbb{C}H^m$, or a horosphere in $\mathbb{C}H^m$. Note that all the contact hypersurfaces in $\mathbb{C}P^m$ or $\mathbb{C}H^m$ are homogeneous and therefore have constant principal curvatures, in particular constant mean curvature. We would like to mention that Pérez has carried out a nice investigation of certain real hypersurfaces in $\mathbb{C}P^m$ in [14].

When we consider more complicated Hermitian symmetric spaces as ambient space $\bar{M}$, there can be contact hypersurfaces which do not have constant mean curvature. The class of all contact hypersurfaces M in $\bar{M}$ is very complicated, and a full classification does not appear to be feasible at the present time. However, if one considers only contact hypersurfaces M with constant mean curvature, the classification problem becomes tractable at least when $\bar{M}$ is a Hermitian symmetric space of rank 2.

The series of irreducible Hermitian symmetric spaces of rank 2 comprise the complex quadrics $Q^m = SO_{m+2}/SO_2SO_m$ (isomorphic to the real 2-Grassmannians $G_2^+(\mathbb{R}^{m+2})$ of oriented planes in $\mathbb{R}^{m+2}$), the complex 2-Grassmannians $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$, and their non-compact duals, the complex hyperbolic quadrics $Q^{m*} = SO^o_{2,m}/SO_2SO_m$ and the duals of the complex 2-Grassmannians $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$. We would like to mention the very nice investigations by Pérez et al. of specific types of real hypersurfaces in Q^m in [15] and in $G_2(\mathbb{C}^{m+2})$ in [16] and [17].

The classification of contact hypersurfaces with constant mean curvature in the complex quadric Q^m and in its non-compact dual, the complex hyperbolic quadric Q^{m*} has been carried out by Berndt and the second author of the present paper in [4]. The result for Q^{m*} is stated as Theorem A. For the case of the complex quadric Q^m, a different classification proof has been given in [22] by the second author of the present paper.

In the complex 2-Grassmannians $G_2(\mathbb{C}^{m+2})$, the contact hypersurfaces with constant mean curvature have also been classified by the second author of the present paper in [21]. He shows that such a hypersurface is congruent to an open part of a tube around a totally geodesic quaternionic projective space $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$. For the non-compact dual $G_2^*(\mathbb{C}^{m+2})$ of these Grassmannians, as far as we know there does not exist a classification of contact hypersurfaces with constant mean curvature. However, Berndt et al. [2] classified contact hypersurfaces of $G_2^*(\mathbb{C}^{m+2})$ which satisfy another curvature condition, namely that the principal curvature function α corresponding to the Reeb vector field of the hypersurface is constant. The result of the classification is that any such hypersurface of $G_2^*(\mathbb{C}^{m+2})$ is congruent either to an open part of a tube around a totally geodesic quaternionic hyperbolic space $\mathbb{H}H^n$ in $G_2(\mathbb{C}^{m+2})$ (only if $m = 2n$ is even), or to an open part of a horosphere in a certain position in $G_2^*(\mathbb{C}^{m+2})$. Note that all these hypersurfaces have constant mean curvature (this follows from [2, Propositions 4.1, 4.2(ii)]).

The purpose of the present paper is to study the local geometry of the contact real hypersurfaces with constant mean curvature in the complex hyperbolic quadric Q^{m*}. As mentioned above, those hypersurfaces were classified by Berndt and Suh. The result is as follows:

Theorem A (Berndt and Suh [4]) Let M be a connected orientable real hypersurface with constant mean curvature in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of one of the following contact hypersurfaces in Q^{m*}:

- (i) the tube of radius $r > 0$ around the complex hyperbolic quadric Q^{m-1*} which is embedded in Q^{m*} as a totally geodesic complex hypersurface;
- (ii) a horosphere in Q^{m*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m*};
- (iii) the tube of radius $r > 0$ around the n-dimensional real hyperbolic space $\mathbb{R}H^n$ which is embedded in Q^{m*} as a real space form of Q^{m*}.

We want to describe the local geometry of the three types (i)–(iii) of real hypersurfaces in Q^{m*} given in the above theorem. For this purpose we first need to study the geometry of Q^{m*} itself. In particular we need to describe the “fundamental geometric structures” of Q^{m*} ; they are its Riemannian metric g, its Hermitian structure J and a certain S^1-subbundle

$\mathfrak{A}$ of End(TQ^{m*}) which can be used to characterize the orbits of the isotropy action on the tangent space of Q^{m*}. The terms " $\mathfrak{A}$-principal" which occurs in Theorem A refers to one of the two singular orbits of this action. For the complex quadric Q^m the corresponding S^1 -bundle $\mathfrak{A}$ is obtained from the shape operator of the embedding $Q^m \hookrightarrow \mathbb{C}P^{m+1}$; it has first been introduced by Reckziegel in [18] and has many times been shown to be very useful for the investigation of the submanifold geometry of Q^m, for example for the classification of the totally geodesic submanifolds of Q^m by the first author of the present paper in [9], and in several classifications of real hypersurfaces in Q^m satisfying certain curvature conditions by the second author of the present paper and his coauthors. We would like to base our investigation of real hypersurfaces in Q^{m*} on the analogous S^1-subbundle $\mathfrak{A}$ for Q^{m*}, which has, as far as we know, not before been described in full detail in the literature. The situation is more complicated for Q^{m*} than for Q^m here, because there does not exist a complex hypersurface embedding of Q^{m*} into $\mathbb{C}H^{m+1}$, as was shown by Smyth [20], therefore $\mathfrak{A}$ for Q^{m*} cannot be obtained in the analogous manner as for Q^m. Instead it needs to be obtained from the representation of Q^{m*} as the quotient manifold $SO_{2,m}/SO_2SO_m$ regarded as a non-compact Hermitian symmetric space. We thus carry out the description of the symmetric space structure of Q^{m*} and of the mentioned fundamental geometric structures in Sect. 2.

The remainder of the paper is concerned with the construction of the contact hypersurfaces in Q^{m*} that are given in Theorem A. In Sect. 3 we construct the tubes around totally geodesic submanifolds that occur in Theorem A(i),(iii). Section 4 recalls how horospheres in non-compact Riemannian symmetric spaces of rank 2 are constructed, and then a description of the horosphere of Theorem A(ii) is given.

2 The complex hyperbolic quadric

The m-dimensional complex hyperbolic quadric Q^{m*} is the non-compact dual of the m-dimensional complex quadric Q^m, i.e. the simply connected Riemannian symmetric space whose curvature tensor is the negative of the curvature tensor of Q^m.

The complex hyperbolic quadric Q^{m*} cannot be realized as a homogeneous complex hypersurface of the complex hyperbolic space $\mathbb{C}H^{m+1}$. In fact, Smyth [20, Theorem 3(ii)] has shown that every homogeneous complex hypersurface in $\mathbb{C}H^{m+1}$ is totally geodesic. This is in marked contrast to the situation for the complex quadric Q^m, which can be realized as a homogeneous complex hypersurface of the complex projective space $\mathbb{C}P^{m+1}$ in such a way that the shape operator for any unit normal vector to Q^m is a real structure on the corresponding tangent space of Q^m, see [18] and [9]. Another related result by Smyth, [20, Theorem 1], which states that any complex hypersurface of $\mathbb{C}H^{m+1}$ for which the square of the shape operator has constant eigenvalues (counted with multiplicity) is totally geodesic, also precludes the possibility of a model of Q^{m*} as a complex hypersurface of $\mathbb{C}H^{m+1}$ with the analogous property for the shape operator.

Therefore we realize the complex hyperbolic quadric Q^{m*} as the quotient manifold $SO_{2,m}/SO_2SO_m$. As Q^{1*} is isomorphic to the real hyperbolic space $\mathbb{R}H^2 = SO_{1,2}/SO_2$, and Q^{2*} is isomorphic to the Hermitian product of complex hyperbolic spaces $\mathbb{C}H^1 \times \mathbb{C}H^1$, we suppose $m \geq 3$ in the sequel and throughout this paper. Let $G := SO_{2,m}$ be the transvection group of Q^{m*} and $K := SO_2SO_m$ be the isotropy group of Q^{m*} at the "origin" $p_0 := eK \in Q^{m*}$. Then

$$\sigma : G \to G,\ g \mapsto sgs^{-1} \quad \text{with} \quad s := \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

is an involutive Lie group automorphism of G with $\mathrm{Fix}(\sigma)_0 = K$, and therefore $Q^{m*} = G/K$ is a Riemannian symmetric space. The center of the isotropy group K is isomorphic to SO_2, and therefore Q^{m*} is in fact a Hermitian symmetric space.

The Lie algebra $\mathfrak{g} := \mathfrak{so}_{2,m}$ of G is given by

$$\mathfrak{g} = \left\{ X \in \mathfrak{gl}(m+2, \mathbb{R}) \middle| X^t \cdot s = -s \cdot X \right\}$$

(see [10, p. 59]). In the sequel we will write members of $\mathfrak{g}$ as block matrices with respect to the decomposition $\mathbb{R}^{m+2} = \mathbb{R}^2 \oplus \mathbb{R}^m$, i.e. in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where X_{11}, X_{12}, X_{21}, X_{22} are real matrices of the dimension 2×2, $2 \times m$, $m \times 2$ and $m \times m$, respectively. Then

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \middle| X_{11}^t = -X_{11},\ X_{12}^t = X_{21},\ X_{22}^t = -X_{22} \right\}.$$

The linearization $\sigma_L = \mathrm{Ad}(s) : \mathfrak{g} \to \mathfrak{g}$ of the involutive Lie group automorphism σ induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where the Lie subalgebra

$$\begin{aligned} \mathfrak{k} &= \mathrm{Eig}(\sigma_*, 1) = \{X \in \mathfrak{g} | sXs^{-1} = X\} \\ &= \left\{ \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \middle| X_{11}^t = -X_{11},\ X_{22}^t = -X_{22} \right\} \cong \mathfrak{so}_2 \oplus \mathfrak{so}_m \end{aligned}$$

is the Lie algebra of the isotropy group K, and the $2m$-dimensional linear subspace

$$\mathfrak{m} = \mathrm{Eig}(\sigma_*, -1) = \{X \in \mathfrak{g} | sXs^{-1} = -X\} = \left\{ \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \middle| X_{12}^t = X_{21} \right\}$$

is canonically isomorphic to the tangent space $T_{p_0}Q^{m*}$. Under the identification $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the Riemannian metric g of Q^{m*} (where the constant factor of the metric is chosen so that the formulae become as simple as possible) is given by

$$g(X, Y) = \tfrac{1}{2}\,\mathrm{tr}(Y^t \cdot X) = \mathrm{tr}(Y_{12} \cdot X_{21}) \quad \text{for} \quad X, Y \in \mathfrak{m}.$$

g is clearly $\mathrm{Ad}(K)$-invariant and therefore corresponds to an $\mathrm{Ad}(G)$-invariant Riemannian metric on Q^{m*}. The complex structure J of the Hermitian symmetric space is given by

$$JX = \mathrm{Ad}(j)X \quad \text{for} \quad X \in \mathfrak{m}, \quad \text{where} \quad j := \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \in K .$$

Because j is in the center of K, the orthogonal linear map J is $\mathrm{Ad}(K)$-invariant and thus defines an $\mathrm{Ad}(G)$-invariant Hermitian structure on Q^{m*}. By identifying the multiplication with the unit complex number i with the application of the linear map J, the tangent spaces of Q^{m*} thus become m-dimensional complex linear spaces, and we will adopt this point of view in the sequel.

Like for the complex quadric (again compare [18] and [9]), there is another important structure on the tangent bundle of the complex quadric besides the Riemannian metric and the complex structure, namely an S^1-bundle $\mathfrak{A}$ of real structures (conjugations). The situation here differs from that of the complex quadric in that for Q^{m*}, the real structures in $\mathfrak{A}$ cannot be interpreted as the shape operator of a complex hypersurface in a complex space form, but as the following considerations will show, $\mathfrak{A}$ still plays a fundamental role in the description of the geometry of Q^{m*}.

Let

$$a_0 := \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} .$$

Note that we have $a_0 \notin K$, but only $a_0 \in O_2\, SO_m$. However, $\mathrm{Ad}(a_0)$ still leaves $\mathfrak{m}$ invariant and therefore defines an $\mathbb{R}$-linear map A_0 on the tangent space $\mathfrak{m} \cong T_{p_0}Q^{m*}$. A_0 turns out to be an involutive orthogonal map with $A_0 \circ J = -J \circ A_0$ (i.e. A_0 is anti-linear with respect to the complex structure of $T_{p_0}Q^{m*}$), and hence a real structure on $T_{p_0}Q^{m*}$. But A_0 commutes with $\mathrm{Ad}(g)$ not for all $g \in K$, but only for $g \in SO_m \subset K$. More specifically, for $g = (g_1, g_2) \in K$ with $g_1 \in SO_2$ and $g_2 \in SO_m$, say $g_1 = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ with $t \in \mathbb{R}$ (so that $\mathrm{Ad}(g_1)$ corresponds to multiplication with the complex number $\mu := e^{it}$), we have

$$A_0 \circ \mathrm{Ad}(g) = \mu^{-2} \cdot \mathrm{Ad}(g) \circ A_0 .$$

This equation shows that the object which is $\mathrm{Ad}(K)$-invariant and therefore geometrically relevant is not the real structure A_0 by itself, but rather the “circle of real structures”

$$\mathfrak{A}_{p_0} := \{\lambda A_0 | \lambda \in S^1\} .$$

$\mathfrak{A}_{p_0}$ is $\mathrm{Ad}(K)$-invariant and therefore generates an $\mathrm{Ad}(G)$-invariant S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{m*})$, consisting of real structures (conjugations) on the tangent spaces of Q^{m*}. For any $A \in \mathfrak{A}$, the tangent line to the fibre of $\mathfrak{A}$ through A is spanned by JA.

For any $p \in Q^{m*}$ and $A \in \mathfrak{A}_p$, the real structure A induces a splitting

$$T_pQ^{m*} = V(A) \oplus JV(A)$$

into two orthogonal, maximal totally real subspaces of the tangent space T_pQ^{m*}. Here $V(A)$ resp. $JV(A)$ are the $(+1)$-eigenspace resp. the (-1)-eigenspace of A. For every unit vector $Z \in T_pQ^{m*}$ there exist $t \in [0, \frac{\pi}{4}]$, $A \in \mathfrak{A}_p$ and orthonormal vectors $X, Y \in V(A)$ so that

$$Z = \cos(t) \cdot X + \sin(t) \cdot JY$$

holds; see [18, Proposition 3]. Here t is uniquely determined by Z. The vector Z is singular, i.e. contained in more than one Cartan subalgebra of $\mathfrak{m}$, if and only if either $t = 0$ or $t = \frac{\pi}{4}$ holds. The vectors with $t = 0$ are called $\mathfrak{A}$-*principal*, whereas the vectors with $t = \frac{\pi}{4}$ are called $\mathfrak{A}$-*isotropic*. If Z is regular, i.e. $0 < t < \frac{\pi}{4}$ holds, then also A and X, Y are uniquely determined by Z.

Like for the complex quadric, the Riemannian curvature tensor R of Q^{m*} can be fully described in terms of the "fundamental geometric structures" g, J and $\mathfrak{A}$. In fact, under the correspondence $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the Riemannian curvature tensor $\bar{R}(X, Y)Z$ corresponds to $-[[X, Y], Z]$ for $X, Y, Z \in \mathfrak{m}$, see [11, Chapter XI, Theorem 3.2(1)]. By evaluating the latter expression explicitly, one can show that one has

$$\begin{aligned} \bar{R}(X, Y)Z = &- g(Y, Z)X + g(X, Z)Y \\ &- g(JY, Z)JX + g(JX, Z)JY + 2g(JX, Y)JZ \\ &- g(AY, Z)AX + g(AX, Z)AY \\ &- g(JAY, Z)JAX + g(JAX, Z)JAY \end{aligned} \tag{2.1}$$

for arbitrary $A \in \mathfrak{A}_{p_0}$. Therefore the curvature of Q^{m*} is the negative of that of the complex quadric Q^m, compare [18, Theorem 1]. This confirms that the symmetric space Q^{m*} which we have constructed here is indeed the non-compact dual of the complex quadric.

As Nomizu [12, Theorem 15.3] has shown, there exists one and only one torsion-free covariant derivative $\bar{\nabla}$ on Q^{m*} so that the symmetric involutions $s_p : Q^{m*} \to Q^{m*}$ at $p \in Q^{m*}$ are all affine. $\bar{\nabla}$ is the *canonical covariant derivative* of Q^{m*}. With respect to $\bar{\nabla}$, the action of any member of G on Q^{m*} is also affine. Moreover, $\bar{\nabla}$ is the Levi-Civita connection corresponding to the Riemannian metric g, and therefore g is parallel with respect to $\bar{\nabla}$. Moreover, it is well-known that Q^{m*} becomes a Kähler manifold in this way, i.e. the complex structure J is also parallel. Finally, because the S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{m*})$ is $\mathrm{Ad}(G)$-invariant, it is also parallel with respect to the covariant derivative $\bar{\nabla}^{\mathrm{End}}$ induced by $\bar{\nabla}$ on $\mathrm{End}(TQ^{m*})$. Because the tangent line of the fiber of $\mathfrak{A}$ through some $A_p \in \mathfrak{A}$ is spanned by JA_p, this means precisely that for any section A of $\mathfrak{A}$ there exists a real-valued 1-form $q : TQ^{m*} \to \mathbb{R}$ so that

$$\bar{\nabla}^{\mathrm{End}}_v A = q(v) \cdot JA_p \quad \text{holds for } p \in Q^{m*}, v \in T_pQ^{m*}.$$

From the presentation (2.1) of the curvature tensor it follows analogously as for Q^m in [18, Sects. 5 and 6] that Q^{m*} has rank 2, that a linear subspace $\mathfrak{a} \subset \mathfrak{m}$ is a Cartan subalgebra if and only if there exist $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ so that $\mathfrak{a} = \mathbb{R}X \oplus \mathbb{R}JY$ holds, and that the positive root system $\Sigma^+ = \{\lambda_1, \cdots, \lambda_4\}$ (in terms of the vectors $\lambda_k^\sharp \in \mathfrak{a}$ dual to the roots $\lambda_k \in \mathfrak{a}^*$) and the corresponding root spaces $\mathfrak{m}_{\lambda_k}$ are given by

k	$\lambda_k^\sharp$	$\mathfrak{m}_{\lambda_k}$
1	$\sqrt{2}\cdot JY$	$J(V(A)\ominus\mathbb{R}X\ominus\mathbb{R}Y)$
2	$\sqrt{2}\cdot X$	$V(A)\ominus\mathbb{R}X\ominus\mathbb{R}Y$
3	$\sqrt{2}\cdot(X-JY)$	$\mathbb{R}(JX+Y)$
4	$\sqrt{2}\cdot(X+JY)$	$\mathbb{R}(JX-Y)$

3 Tubes around the totally geodesic submanifolds $Q^{m-1^*}\subset Q^{m*}$ and $\mathbb{R}H^m\subset Q^{m*}$

At first we let P be any submanifold of a Riemannian symmetric space $\bar{M}$, and for $p\in P$ we let $\perp_p P$ be the normal space of P in $\bar{M}$ at p, and let $\perp_p^1 P := \{v\in\perp_p P \mid \|v\| = r\}$ be the unit sphere in $\perp_p P$. We let $\perp P$ resp. $\perp^r P$ be the vector bundle resp. the sphere bundle of normal vectors resp. of unit length normal vectors over P, and let $\tau :\perp P\to P$ be the bundle projection map. Moreover, we let $K^\perp : T(\perp P)\to\perp P$ be the normal connection map of $\perp P$, i.e. $K^\perp u = \nabla_u \mathrm{id}_{\perp P}$. For any $\eta\in\perp P$,

$$T_\eta(\perp P)\to T_{\tau(\eta)}P\oplus\perp_{\tau(\eta)}P,\; u\mapsto(\tau_* u, K^\perp u)$$

is then an isomorphism of vector spaces.

The *tube map* of radius $r>0$ is the map

$$\Phi :\perp^1 P\to\bar{M},\; \eta\mapsto\exp_p^{\bar{M}}(r\,\eta)\,,$$

where $\exp^{\bar{M}}$ denotes the (geodesic) exponential map of $\bar{M}$. If Φ is a diffeomorphism into $\bar{M}$, we call its image M the *tube* around P of radius r, it is then a real hypersurface of $\bar{M}$. Berndt [1, Corollary 4.4] has described how to calculate the eigenvalues and eigenvectors of the shape operator of the map Φ, and hence the principal curvatures, their multiplicities and the corresponding principal curvature directions of the tube M. We describe his results here only for the case where the submanifold P is totally geodesic.

Therefore suppose that P is a totally geodesic submanifold of $\bar{M}$. Let $\bar{R}$ be the Riemannian curvature tensor of $\bar{M}$. For any $p\in P$, $\eta\in\perp_p P$ we let $\bar{R}_\eta := \bar{R}(\,\cdot,\eta)\eta$ be the corresponding Jacobi operator of $\bar{M}$ and define $R_\eta := \bar{R}_\eta|T_pP$. Let γ_η be the unit speed geodesic of $\bar{M}$ with $\gamma_\eta(0)=p$ and $\dot\gamma_\eta(0)=\eta$; then $\gamma_\eta(r)=\Phi(p)\in M$ and $\dot\gamma_\eta(r)$ is a unit normal vector of M ; we say that $\dot\gamma_\eta(r)$ is *pointing outward* and that $-\dot\gamma_\eta(r)$ is *pointing inward*. For the description of the principal curvatures, we follow [1, Sect. 4.2] by considering for any $\kappa\in\mathbb{R}$ the solution functions $\sin_\kappa$ and $\cos_\kappa$ of the second order differential equation $y''+\kappa y=0$ with $\sin_\kappa(0)=0$, $\sin_\kappa'(0)=1$ and $\cos_\kappa(0)=1$, $\cos_\kappa'(0)=0$. Explicitly one has

$$\sin_\kappa(t)=\begin{cases}\frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}\,t) & \text{for } \kappa>0\\ t & \text{for } \kappa=0\\ \frac{1}{\sqrt{-\kappa}}\sinh(\sqrt{-\kappa}\,t) & \text{for } \kappa<0\end{cases}\quad\text{and}\quad\cos_\kappa(t)=\begin{cases}\cos(\sqrt{\kappa}\,t) & \text{for } \kappa>0\\ 1 & \text{for } \kappa=0\\ \cosh(\sqrt{-\kappa}\,t) & \text{for } \kappa<0\end{cases}.$$

According to [1, Corollary 4.4], the shape operator S^Φ with respect to the outward-pointing unit normal vector $\dot\gamma(r)$ has two types of eigenvalues, and any eigenvalue or eigenvector of S^Φ is obtained by either of these two methods:

(I) For any eigenvalue κ of R_η, the number $\kappa \cdot \frac{\sin_\kappa(r)}{\cos_\kappa(r)}$ is an eigenvalue of S^Φ. The corresponding eigenspace of S^Φ is given by

$$\{u \in T_\eta(\perp^1 P) \mid \tau_* u \in E_\kappa,\ K^\perp u = 0\}\ ,$$

where E_κ denotes the eigenspace of R_η for the eigenvalue κ.

(II) For any eigenvalue κ of $\bar{R}_\eta|(\perp_{\tau(\eta)} P \ominus \mathbb{R}\eta)$, the number $-\frac{\cos_\kappa(r)}{\sin_\kappa(r)}$ is an eigenvalue of S^Φ. The corresponding eigenspace of S^Φ is given by

$$\{u \in T_\eta(\perp^1 P) \mid \tau_* u = 0,\ K^\perp u \in \bar{E}_\kappa\}\ ,$$

where $\bar{E}_\kappa$ denotes the eigenspace of $\bar{R}_\eta|(\perp_{\tau(\eta)} P \ominus \mathbb{R}\eta)$ for the eigenvalue κ.

The principal curvatures of M at $\Phi(\eta)$ are precisely the eigenvalues of S^Φ described above. The corresponding principal curvature spaces are the image under Φ_* of the eigenspaces of S^Φ; hence, they are obtained from the eigenspaces E_κ (in the case (I)) resp. $\bar{E}_\kappa$ (in the case (II)) by parallel transport along the geodesic γ_η.

We will now apply the preceding results to tubes in the complex hyperbolic quadric $\bar{M} = Q^{m*}$ around the two types of totally geodesic submanifolds that occur in Theorem A(i),(iii). It follows from Eq. (2.1) that if the unit normal vector $\eta \in \perp^1 Q^{m*}$ is $\mathfrak{A}$-principal, the corresponding Jacobi operator is given by

$$\bar{R}_\eta(X) = -X - AX + 2\,g(X,\eta)\eta - 2\,g(X,J\eta)J\eta\ .$$

It is easy to see that $\bar{R}_\eta$ then has the two eigenvalues 0 and -2. If $A \in \mathfrak{A}$ is such that $\eta \in V(A)$ holds, then the corresponding eigenspaces are $\mathbb{R}\eta \oplus J(V(A) \ominus \mathbb{R}\eta)$ and $(V(A) \ominus \mathbb{R}\eta) \oplus \mathbb{R}J\eta$, respectively.

The obvious embedding of Lie groups $SO^o_{2,m-1} \to SO^o_{2,m}$ induces a totally geodesic embedding of $Q^{m-1^*} = SO_{2,m-1}/SO_2SO_{m-1}$ into $Q^{m*} = SO_{2,m}/SO_2SO_m$. We will view Q^{m-1^*} as a totally geodesic complex hypersurface of Q^{m*} by means of this embedding.

Proposition 3.1 The tube M around the totally geodesic Q^{m-1^*} in Q^{m*} exists for every radius $r > 0$. For M the following statements hold:

(1) Every normal vector N of M is $\mathfrak{A}$-principal.

(2) M has constant principal curvatures, and in particular constant mean curvature. Let N be the outward-pointing unit normal vector of M at $q \in M$. By (1) we have $N \in V(A)$ for some $A \in \mathfrak{A}_q$. Then the principal curvatures of M with respect to N and the corresponding principal curvature spaces are

Principal curvature	Principal curvature space	Multiplicity
$\lambda = 0$	$J(V(A) \ominus \mathbb{R}N)$	$m-1$
$\mu = -\sqrt{2}\tanh(\sqrt{2}r)$	$V(A) \ominus \mathbb{R}N$	$m-1$
$\alpha = -\sqrt{2}\coth(\sqrt{2}r)$	$\mathbb{R}JN$	1

(3) M is a Hopf hypersurface.

(4) The shape operator S and the structure tensor field ϕ ($\phi = \mathrm{pr}_{TM} \circ J$, where $\mathrm{pr}_{TM} : TQ^{m*} \to TM$ denotes the orthogonal projection) satisfy

$$S\phi + \phi S = \mu \cdot \phi.$$

In particular M is a contact submanifold.

Proof It follows from the construction of the S^1-subbundle $\mathfrak{A}$ of $\mathrm{End}(TQ^{m*})$ of real structures that for any $p \in Q^{m-1*}$, both complex subspaces T_pQ^{m-1*} and $\perp_p Q^{m-1*}$ are invariant under every $A_p \in \mathfrak{A}_p$. Because $\perp_p Q^{m-1*}$ thus is a complex one-dimensional subspace that is invariant under $A_p \in \mathfrak{A}_p$, it follows that the vectors in this space are $\mathfrak{A}$-principal.

Now let $q \in M$ be given. It follows from the construction of the tube M that there exists $p \in Q^{m-1*}$ and a unit normal vector $\eta \in \perp_p Q^{m-1*}$ so that the normal space $\perp_q M$ of M at q is spanned by $\dot\gamma_\eta(r)$. Because $\dot\gamma_\eta(0) \in \perp_p Q^{m-1*}$ is $\mathfrak{A}$-principal by the preceding observation, and the S^1-bundle $\mathfrak{A}$ is parallel, it follows that also $\dot\gamma_\eta(r)$ and hence $\perp_q M$ is $\mathfrak{A}$-principal.

Moreover the outward-pointing unit normal vector N of M at q equals $\dot\gamma_\eta(r)$ and is contained in $V(A_q)$, where $A_q \in \mathfrak{A}_q$ is the parallel transport of A_p along the geodesic γ_η. Note that $T_pQ^{m-1*} = (V(A_p) \ominus \mathbb{R}\eta) \oplus J(V(A_p) \ominus \mathbb{R}\eta)$ and $\perp_p Q^{m-1*} = \mathbb{R}\eta \oplus \mathbb{R}J\eta$ holds. We now apply the two cases (I) and (II) given above to obtain the principal curvatures of M with respect to N. For case (I), we note that $R_\eta := \bar{R}_\eta | T_pQ^{m-1*}$ has the two eigenvalues 0 and -2, with eigenspaces $J(V(A_p) \ominus \mathbb{R}\eta)$ and $V(A_p) \ominus \mathbb{R}\eta$, respectively. From the eigenvalue $\kappa = 0$ we obtain the principal curvature $\kappa \cdot \frac{\sin_\kappa(r)}{\cos_\kappa(r)} = 0 = \lambda$, and from the eigenvalue $\kappa = -2$ we obtain the principal curvature $\kappa \cdot \frac{\sin_\kappa(r)}{\cos_\kappa(r)} = -\sqrt{2}\tanh(\sqrt{2}\,r) = \mu$. In case (II), we have $\perp_p Q^{m-1*} \ominus \mathbb{R}\eta = \mathbb{R}J\eta$, and therefore the only eigenvalue of $\bar{R}_\eta|(\perp_p Q^{m-1*} \ominus \mathbb{R}\eta)$ is $\kappa = -2$ with the eigenspace $\mathbb{R}J\eta$. This yields the principal curvature $-\frac{\cos_\kappa(r)}{\sin_\kappa(r)} = -\frac{\cos_\kappa(r)}{\sin_\kappa(r)} = -\sqrt{2}\coth(\sqrt{2}\,r) = \alpha$. The principal curvature spaces are obtained from the corresponding eigenspaces of $\bar{R}_\eta$ by parallel transport along γ_η and are therefore as stated in the proposition.

It follows from the calculation of the principal curvatures that M has constant principal curvatures, in particular constant mean curvature. The claim that M is Hopf means by definition that the Hopf vector field $\xi = -JN$ is a principal vector field; this is the case by our previous calculation. The corresponding principal curvature is $\alpha = -\sqrt{2}\coth(\sqrt{2}r)$. Moreover, it is easily seen that $\phi(JN) = 0$ holds, and that ϕ acts as J on $V(A_q) \ominus \mathbb{R}N$ and on $J(V(A_q) \ominus \mathbb{R}N)$. Therefore the equation in (4) is easily verified on the principal curvature spaces described in (2). Because Eq. (1.2) thus holds for M with $k = \mu$, it follows that M is a contact submanifold. □

A similar discussion applies to tubes around the totally geodesic $\mathbb{R}H^m$ in Q^{m*}. The obvious embedding of Lie groups $SO_{1,m} \to SO_{2,m}$ induces a totally geodesic embedding of $\mathbb{R}H^m = SO_{1,m}/SO_m$ into $Q^{m*} = SO_{2,m}/SO_2SO_m$. We will view $\mathbb{R}H^m$ as a real form, i.e. a totally geodesic, totally real, real-m-dimensional submanifold of Q^{m*} by means of this embedding.

Proposition 3.2 The tube M around the totally geodesic $\mathbb{R}H^m$ in Q^{m*} exists for every radius $r > 0$. For M the following statements hold:

(1) Every normal vector N of M is $\mathfrak{A}$-principal.
(2) M has constant principal curvatures, and in particular constant mean curvature. Let N be the outward-pointing unit normal vector of M at $q \in M$. By (1) we have $N \in V(A)$ for some $A \in \mathfrak{A}_q$. Then the principal curvatures of M with respect to N and the corresponding principal curvature spaces are

Principal curvature	Principal curvature space	Multiplicity
$\lambda = 0$	$J(V(A) \ominus \mathbb{R}N)$	$m-1$
$\mu = -\sqrt{2}\coth(\sqrt{2}r)$	$V(A) \ominus \mathbb{R}N$	$m-1$
$\alpha = -\sqrt{2}\tanh(\sqrt{2}r)$	$\mathbb{R}JN$	1

(3) M is a Hopf hypersurface.

(4) The shape operator S and the structure tensor field ϕ ($\phi = \mathrm{pr}_{TM} \circ J$, where $\mathrm{pr}_{TM} : TQ^{m*} \to TM$ denotes the orthogonal projection) satisfy

$$S\phi + \phi S = \mu \cdot \phi \,.$$

In particular M is a contact submanifold.

Proof The proof of this proposition follows the same pattern as that of Proposition 3.1.

For any $p \in \mathbb{R}H^m$, there exists $A_p \in \mathfrak{A}_p$ so that $T_p\mathbb{R}H^m = V(-A_p)$ and therefore $\perp_p \mathbb{R}H^m = JV(-A_p) = V(A_p)$ holds. In particular, both $T_p\mathbb{R}H^m$ and $\perp_p \mathbb{R}H^m$ are $\mathfrak{A}$-principal. Now let $q \in M$ be given. Again there exists $p \in \mathbb{R}H^m$ and $\eta \in \perp^1_p \mathbb{R}H^m$ so that $\perp_q M$ is spanned by $\dot{\gamma}_\eta(r)$. Because $\dot{\gamma}_\eta(0) \in \perp_p \mathbb{R}H^m = V(A_p)$ is $\mathfrak{A}$-principal, it again follows that also $\dot{\gamma}_\eta(r)$ and hence $\perp_q M$ is $\mathfrak{A}$-principal. Hence the outward-pointing unit normal vector $N = \gamma_\eta(r)$ is contained in $V(A_q)$ where $A_q \in \mathfrak{A}_q$ is the parallel transport of A_p along γ_η.

We again apply the two cases (I) and (II) given above to obtain the principal curvatures of M with respect to N. For case (I), we note that $R_\eta := \bar{R}_\eta | T_p\mathbb{R}H^m$ has the two eigenvalues 0 and -2 with eigenspaces $J(V(A_p) \ominus \mathbb{R}\eta)$ and $\mathbb{R}J\eta$, respectively. From the eigenvalue $\kappa = 0$ we obtain $\kappa \cdot \frac{\sin_\kappa(r)}{\cos_\kappa(r)} = 0 = \lambda$, and from the eigenvalue $\kappa = -2$ we obtain the principal curvature $\kappa \cdot \frac{\sin_\kappa(r)}{\cos_\kappa(r)} = -\sqrt{2}\,\tanh(\sqrt{2}\,r) = \alpha$. In case (II), we have $\perp_p \mathbb{R}H^m \ominus \mathbb{R}\eta = V(A_p) \ominus \mathbb{R}\eta$, and therefore the only eigenvalue of $\bar{R}_\eta|(\perp_p Q^{m-1^*} \ominus \mathbb{R}\eta)$ is $\kappa = -2$ with the eigenspace $V(A_p) \ominus \mathbb{R}\eta$. This yields the principal curvature $-\frac{\cos_\kappa(r)}{\sin_\kappa(r)} = -\frac{\cos_\kappa(r)}{\sin_\kappa(r)} = -\sqrt{2}\,\coth(\sqrt{2}\,r) = \mu$.

The remaining parts of the proposition now follow in the same way as in the proof of Proposition 3.1.

4 Horospheres in Q^{m*}

Suppose that $\bar{M}$ is a Hadamard manifold, i.e. a simply connected, complete Riemannian manifold with sectional curvature ≤ 0. We denote by d the Riemannian distance function on $\bar{M}$. Two unit speed geodesics $\gamma_1, \gamma_2 : \mathbb{R} \to \bar{M}$ are said to be *asymptotic* to each other, if the function $t \mapsto d(\gamma_1(t), \gamma_2(t))$ remains bounded for $t \to \infty$. Asymptoticness defines an equivalence relation on the space of unit speed geodesics on $\bar{M}$. The equivalence classes are called *points at infinity*, and their set is denoted by $\bar{M}(\infty)$. For any unit speed geodesic $\gamma : \mathbb{R} \to \bar{M}$, the corresponding point at infinity is denoted by $\gamma(\infty) \in \bar{M}(\infty)$. The *horosphere* with *center at infinity* $\gamma(\infty)$ through some point $p \in \bar{M}$ is defined as

$$C(p, \gamma(\infty)) = \left\{ q \in \bar{M} \;\middle|\; \lim_{t\to\infty}\big(d(q,\gamma(t)) - d(p,\gamma(t))\big) = 0 \right\}.$$

It can be shown that $C(p, \gamma(\infty))$ indeed does not depend on the choice of the representant γ within the equivalence class $\gamma(\infty)$ and that it is a real hypersurface of $\bar{M}$, see [8, Sect. 1.10].

We now suppose that $\bar{M}$ is a Riemannian symmetric space of non-compact type and rank 2. The following construction principle for horospheres in $\bar{M}$ was described by Berndt and Suh in [3, Sect. 2]. Let us consider $\bar{M} = G/K$ with the "origin" $o := eK \in \bar{M}$, the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{m}$. Further consider the root system $\Sigma \subset \mathfrak{a}^*$ and the corresponding root space decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{k}_{\mathfrak{a}} \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$. For a positive root system $\Sigma^+ \subset \Sigma$, $\mathfrak{n} := \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$ is a nilpotent subalgebra of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$. Let A and N be the connected Lie subgroups of G with Lie algebra $\mathfrak{a}$ resp. $\mathfrak{n}$. Then $G = KAN$ is an Iwasawa decomposition of G, more precisely, A and N are simply connected, $K \cap AN = \{e\}$ holds, and the maps

$$K \times A \times N \to G,\ (k, a, n) \mapsto kan \quad \text{and} \quad A \times N \to \bar{M},\ (a, n) \mapsto an \cdot o$$

are surjective diffeomorphisms, see [10, Theorem VI.6.46, p. 374].

Now suppose that a unit vector $H \in \mathfrak{a}$ is given. Then $\mathfrak{s}_H := (\mathfrak{a} \ominus \mathbb{R}H) \oplus \mathfrak{n}$ is a solvable Lie subalgebra of $\mathfrak{g}$. Let S_H be the connected subgroup of AN with Lie algebra $\mathfrak{s}_H$. Then the orbits of the action of S_H on $\bar{M}$ are the horospheres of $\bar{M}$ with the center at infinity $\gamma_H(\infty)$, where γ_H is the geodesic with $\gamma_H(0) = o$ and $\dot{\gamma}_H(0) = H$ (and where we identify $\mathfrak{m}$ with $T_e\bar{M}$ in the usual manner). In particular we have $C(o, \gamma_H(\infty)) = S_H \cdot o$. It was shown by Berndt and Tamaru in [5, Proposition 3.1(2),(3)] that the shape operator of $C(o, \gamma_H(\infty))$ with respect to the unit normal vector H is given by $\mathrm{ad}(H)|\mathfrak{s}_H$. Therefore the principal curvatures are constant, and their values are given by 0 and by $\lambda(H)$ for every $\lambda \in \Sigma^+$. The corresponding principal curvature spaces are $\mathfrak{a} \ominus \mathbb{R}H$ and $\mathfrak{g}_\lambda$; under the standard identification $T_e\bar{M} \cong \mathfrak{m}$, the latter space corresponds to $\mathfrak{m}_\lambda = \{X - \theta X \mid X \in \mathfrak{g}_\lambda\}$ (where $\theta : \mathfrak{g} \to \mathfrak{g}$ denotes the Cartan involution).

We will now apply this construction to $\bar{M} = Q^{m*}$ and an $\mathfrak{A}$-principal vector $H \in \mathfrak{a}$.

Proposition 4.1 Let M be a horosphere in Q^{m*} with its center at infinity being given by an $\mathfrak{A}$-principal geodesic γ. Then the following statements hold:

(1) Every normal vector N of M is $\mathfrak{A}$-principal.
(2) M has constant principal curvatures, and in particular constant mean curvature. Then the principal curvatures of M with respect to the unit normal vector[1] $N := -\dot{\gamma}(0)$ and the corresponding principal curvature spaces are

Principal curvature	Principal curvature space	Multiplicity
0	$J(V(A) \ominus \mathbb{R}N)$	$m-1$
$-\sqrt{2}$	$(V(A) \ominus \mathbb{R}N) \oplus \mathbb{R}JN$	m

Here $A \in \mathfrak{A}$ is chosen such that $-\dot{\gamma}(0) \in V(A)$ holds.

(3) M is a Hopf hypersurface.
(4) The shape operator S and the structure tensor field ϕ ($\phi = \mathrm{pr}_{TM} \circ J$, where $\mathrm{pr}_{TM} : TQ^{m*} \to TM$ denotes the orthogonal projection) satisfy

[1] We choose the negative of $\dot{\gamma}(0)$ as normal vector here so that the orientation matches the one considered for the tubes in Sect. 3.

$$S\phi + \phi S = -\sqrt{2} \cdot \phi.$$

In particular M is a contact submanifold.

Proof We use the description of the Cartan subalgebras, the roots and the root spaces of Q^{m*} given at the end of Sect. 2. We may assume without loss of generality that $\gamma(0) = o$ and $\dot{\gamma}(0) \in \mathfrak{a}$ holds. Because the geodesic γ is $\mathfrak{A}$-principal, there exists $A \in \mathfrak{A}_o$ with $X := \dot{\gamma}(0) \in V(A)$, and we may further assume that the Cartan subalgebra $\mathfrak{a}$ is given by $\mathfrak{a} = \mathbb{R}X \oplus \mathbb{R}JY$ with a unit vector $Y \in V(A) \ominus \mathbb{R}X$. Numbering the positive roots of Q^{m*} as in Sect. 2 we then have

$$\lambda_1(-X) = 0 \quad \text{and} \quad \lambda_2(-X) = \lambda_3(-X) = \lambda_4(-X) = -\sqrt{2}\,.$$

It follows that the horosphere M has the two principal curvatures 0 with principal curvature space $(\mathfrak{a} \ominus \mathbb{R}X) \oplus \mathfrak{m}_{\lambda_1} = J(V(A) \ominus \mathbb{R}X)$, and $-\sqrt{2}$ with the principal curvature space $\mathfrak{m}_{\lambda_2} \oplus \mathfrak{m}_{\lambda_3} \oplus \mathfrak{m}_{\lambda_4} = (V(A) \ominus \mathbb{R}X) \oplus \mathbb{R}JX$.

The remainder of the statements follows in the same way as for the proofs of Sect. 3.

Remark 4.2 Note that for both the family of tubes around Q^{m-1*} (Proposition 3.1) and the family of tubes around $\mathbb{R}H^m$ (Proposition 3.2), when one lets the radius $r > 0$ of the tube tend to infinity, the values of the principal curvatures and the corresponding principal curvature spaces tend to the values and spaces of the horosphere with $\mathfrak{A}$-principal center at infinity (Proposition 4.1). In this sense, this horosphere is the joint limit of both mentioned families of tubes as $r \to \infty$.

Acknowledgements This work was done while Sebastian Klein was visiting professor at the Research Institute of Real and Complex Submanifolds in Kyungpook National University during October, 2017.

References

1. Berndt, J.: Über Untermannigfaltigkeiten von komplexen Raumformen, Ph.D. thesis, Universität zu Köln (1989)
2. Berndt, J., Lee, H., Suh, Y.J.: Contact hypersurfaces in noncompact complex Grassmannians of rank two. Int. J. Math. **24**(11), 1350089 (2013)
3. Berndt, J., Suh, Y.J.: Hypersurfaces in noncompact complex Grassmannians of rank two. Int. J. Math. **23**(35), 1250103 (2012)
4. Berndt, J., Suh, Y.J.: Contact hypersurfaces in Kähler manifolds. Proc. Am. Math. Soc. **143**, 2637–2649 (2015)
5. Berndt, J., Tamaru, H.: Homogeneous codimension one foliations on noncompact symmetric spaces. J. Differ. Geom. **63**, 1–40 (2003)
6. Blair, D.E.: Riemannian Geometry of Contact and Symplectic Manifolds, Progress in Mathematics. Springer, Berlin (2010)
7. Dragomir, S., Perrone, D.: Harmonic Vector Fields: Variational Principles and Differential Geometry. Elsevier, New York (2011)
8. Eberlein, P.: Geometry of Nonpositively Curved Manifolds. The University of Chicago Press, Chicago (1996)
9. Klein, S.: Totally geodesic submanifolds in the complex quadric. Differ. Geom. Appl. **26**, 79–96 (2008)
10. Knapp, A.W.: Lie Groups Beyond an Introduction, Progress in Mathematics. Birkhäuser, Basel (2002)

11. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. II, Wiley Classics Library edn. Wiley, New York (1996)
12. Nomizu, K.: Invariant affine connections on homogeneous spaces. Am. J. Math. **76**, 33–65 (1954)
13. Okumura, M.: Contact hypersurfaces in certain Kählerian manifolds. Tôhoku Math. J. **18**, 74–102 (1966)
14. Pérez, J.D.: Commutativity of Cho and structure Jacobi operators of a real hypersurface in a complex projective space. Ann. di Mat. Pure Appl. **194**, 1781–1794 (2015)
15. Pérez, J.D., Jeong, I., Ko, J., Suh, Y.J.: Real hypersurfaces with Killing shape operator in the complex quadric. Mediterr. J. Math. **15**(1), 15 (2018)
16. Pérez, J.D., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with parallel and commuting Ricci tensor. J. Korean Math. Soc. **44**, 211–235 (2007)
17. Pérez, J.D., Suh, Y.J., Watanabe, Y.: Generalized Einstein real hypersurfaces in complex two-plane Grassmannians. J. Geom. Phys. **60**, 1806–1818 (2010)
18. Reckziegel, H.: On the geometry of the complex quadric. In: Dillen F, Komrakov B, Simon U, Van de Woestyne I, Verstraelen L (eds.) Geometry and Topology of Submanifolds VIII (Brussels/Nordfjordeid 1995), World Scientific Publishing, River Edge, pp. 302–315 (1995)
19. Sasaki, S.: On differentiable manifolds with certain structures which are closely related to almost contact structure, I. Tôhoku Math. J. **12**, 459–476 (1960)
20. Smyth, B.: Homogeneous complex hypersurfaces. J. Math. Soc. Jpn. **20**, 643–647 (1968)
21. Suh, Y.J.: Real hypersurfaces of type B in complex two-plane Grassmannians. Monatsh. Math. **147**, 337–355 (2006)
22. Suh, Y.J.: Contact Real Hypersurfaces in the Complex Quadric, submitted for publication, p. 22 (2017)
23. Vernon, M.H.: Contact hypersurfaces of a complex hyperbolic space. Tôhoku Math. J. **39**, 215–222 (1987)
24. Yano, K., Kon, M.: CR submanifolds in Kählerian and Sasakian manifolds. Progress in Mathematics. Birkhäuser, Boston (1983)