Contact Transformations, Contact Algebras and Lifts on a Jet Bundle (*).

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Summary. – *It is shown that any contact preserving map germ on a jet bundle J is the lift of any map germ $J \to J^0$, satisfying some rank conditions. The different contact preserving homeomorphisms are given as well as the different lifts derived from these constructions. The relation with the infinitesimal and formal case is given.*

Introduction.

Infinitesimal contact transformations on a jet bundle of maps of manifolds have been considered mainly in relationship with partial differential equations and variational problems. The interest of such maps has increased as a consequence of the appearance of complete integrable evolution equations, which have an infinite set of symmetries. There is a rather lengthy list of publications on this subject, but some of the most important recent results to which the present work relates, are found in B. A. KUPERSHMIDT [7], [8], A. M. VINOGRADOV, V. V. LYCHAGIN [11], [13], [14] and also in R. L. ANDERSON and N. H. IBRAGIMOV [1].

It is the aim of this paper to consider the finite case of germs of contact preserving maps under certain differentiability conditions. The relation with the infinitesimal and formal case is given. The special case of contact maps preserving a partial differential equation is not treated in this paper but will appear later.

The contact structure is studied by means of the total vector bundle, introduced by J. M. BOARDMAN [2], which is the tangent bundle to the integrable sections of the jet bundle. Associated to this involutive vector bundle there is a de Rham complex defining a differential operator d_H, called the total differential. This differential operator enables one to construct the universal elements, introduced by A. M. VINOGRADOV [13] and hence is the natural operator in the analysis of contact preserving maps.

The main theorem then states that any contact preserving map germ on the infinite jet bundle is determined by its image in the 0-jet space. The converse construction gives rise to a lifting rule of germs of maps. The results obtained in the infinitesimal case generalize to contact preserving homeomorphism germs. The

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local contact homeomorphisms have the structure of a pseudo semi group. The obtained lifts are then extended to the case of formal vector fields.

The projective jet bundle [10] is not considered but the obtained results generalize in a natural way to it.

The main theorem of this paper has been announced in [4], where it was incorrectly formulated. Also the definitions have been modified as was suggested by Prof. A. M. VINOGRADOV. I like to express my gratitude to him for his interesting remarks on this subject.

1. – The jet bundle and its integrable sections.

a. The jet bundle J.

Let M and N be smooth real manifolds, $C(M, N)$ the set of smooth maps of M into N and $\mathbf{C}(M, N)$ the set of germs of elements of $C(M, N)$. Let $\mathscr{F}(M)$ denote $C(M, \mathbb{R})$ and $\mathscr{F}(M)$ the corresponding sheaf of germs. The stalk at $x \in M$ of $\mathscr{F}(M)$ is the local ring $\mathscr{F}(x)$.

The set of k-jets of elements of $\mathbf{C}(M, N)$, $J^k \equiv J^k(M, N)$, is a fibre bundle over $J^0 = M \times N$. One has the source map $\pi^k_M: J^k \to M$ and the target map $\pi^k_N: J^k \to N$. The space J^k inherits a topological and differentiable structure from the topological and differentiable structure of M and N. The set of spaces J^k, $\forall k \in \mathbb{N}$, is a projective family, with projection maps $\pi^k_l: J^k \to J^l$, $k \geqslant l$. The final object of this projective limiting system is the topological space $J = \varprojlim J^k$, which is equipped with the projective limit topology. One has the projections $\pi_k: J \to J^k$, $\pi_M: J \to M$ and $\pi_N: J \to N$.

Let $\mathscr{F}^{(k)}(p)$, $p \in J$, be the ring of germs of real functions on J, of the form $\boldsymbol{\varphi} \circ \pi_k$, with $\varphi \in \mathscr{F}(\pi_k(p))$. Elements of $\mathscr{F}^{(k)}(p)$ are said to factor through J^k. One has $\mathscr{F}^{(k)}(p) \subset \mathscr{F}^{(k+l)}(p)$, for all $l > 0$, which defines $\mathscr{F}(p) = \varinjlim \mathscr{F}^{(k)}(p)$ as a direct limit space. A real function germ on J is called smooth if it belongs to $\mathscr{F}(p)$, for some $p \in J$.

The space J, equipped with this direct limit differentiable structure, will be given a tangent bundle and a cotangent bundle which are defined by: $TJ = \varprojlim \pi^*_k TJ^k$ and $T^*J = \varinjlim \pi^*_k T^*J^k$. These bundles are equipped with the projective limit topology and the direct limit differentiable structure which makes them into manifolds, modelled on $\varprojlim \mathbb{R}^k$ [2], [3].

A map germ $\varphi: \varprojlim \mathbb{R}^k \to \varprojlim \mathbb{R}^k$ is smooth if $\forall l$, $\pi_l \circ \varphi: \varprojlim \mathbb{R}^k \to \mathbb{R}^l$ has smooth components. The use of local charts on $\varprojlim \mathbb{R}^k$-modelled spaces allows one to define smooth maps between $\varprojlim \mathbb{R}^k$-modelled spaces. Smooth vector field (resp. smooth 1-forms) on J will be smooth sections of TJ (resp. T^*J) [3].

A map germ $\boldsymbol{\varphi}: J \to J$ is a germ of smooth weak homeomorphisms at $p \in J$ if $\boldsymbol{\varphi}$ is a continuous open map on a neighbourhood of p. Because J is an infinite dimensional inverse limit manifold, such a map has not always a smooth inverse; if it has a smooth inverse we will call it a strong homeomorphism.

One also has the following operators:

(*a*) the germ operator $C(M, N) \to \mathbf{C}(M, N)\colon f \mapsto \mathbf{f}$.

(*b*) the k-jet operator $j^k\colon \mathbf{C}(M, N) \to J^k\colon (\mathbf{f}, x) \mapsto j_x f$;

(*c*) the k-section operator $J^k\colon C(M, N) \to \Gamma(J^k)$, where $\Gamma(J^k)$ is the space of sections of π^k_M, by $f \mapsto J^k f\colon x \mapsto j^k_x f$;

(*d*) the operator $\alpha^{l,k}\colon \Gamma(J^k) \to \Gamma(J^l)$, $k \geqslant l$, namely $\alpha^{l,k} = J^l \circ \pi^k_0$.

The space $C(M, N)$ will be identified with $\Gamma(J^0)$, and $\mathbf{C}(M, N)$ with $\mathbf{\Gamma}(J^0)$. It will be understood that if $\boldsymbol{\varphi}$ is a germ, φ will be a representant of $\boldsymbol{\varphi}$.

DEFINITION 1.1. – A section $\sigma \in \Gamma(J^l)$ is k-integrable at $x \in M$, $k \leqslant l$, if $\alpha^{k,l}(\sigma)(x) = (\pi^l_k \circ \sigma)(x)$.

If $\sigma \in \Gamma(J)$ is k-integrable at $x \in M$, $\forall k \in \mathrm{N}$, then σ is called integrable at $x \in M$.

It is a classical result of *E.* Borel that any point $p \in J$ is the infinite jet of an element of $\mathbf{C}(M, N)$. Hence through any $p \in J$ there is at least on integrable section.

DEFINITION 1.2. – The holonomic lift of a k-integrable section on $\mathscr{U} \subset M$, $\sigma \in \Gamma(\mathscr{U}, J^k)$ to $\Gamma(\mathscr{U}, J^{k+l})$, is the unique section on $\mathscr{U}$, defined by

$$J^{k+l} \circ \pi^k_0(\sigma) \equiv \sigma^{(l)} \ [10]\,.$$

It is a consequence of the definition that $J^{k+l} \circ \pi^k_0(\sigma)$ is $(k+l)$-integrable on $\mathscr{U} \subset M$, because $\alpha^{k,k}(\sigma) = \sigma$.

Definition 1.1 carries over to section germs under the germ operator.

I. *The total vector bundle* $H(J)$.

Two section germs, $\sigma_1, \sigma_2 \in \mathbf{\Gamma}(J^k)$, are P^k_{k-1}-equivalent at $x \in \mathscr{U}$, if

(1) $\sigma_1(x) = \sigma_2(x) = p$

(2) $\pi^k_{k-1*} T_p \sigma_1 = \pi^k_{k-1*} T_p \sigma_2$,

where $T_p \sigma$ is the tangent space to the submanifold σ at $p \in \sigma$.

PROPOSITION 1.3. – If one of two P^k_{k-1}-equivalent section germs, belonging to $\mathbf{\Gamma}(J^k)$, is k-integrable then so is the other.

PROOF. – Let $\boldsymbol{\sigma}, \boldsymbol{\mu} \in \mathbf{\Gamma}_x(J^k)$, $x \in M$, be P^k_{k-1}-equivalent and let μ be k-integrable. Then $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are 1-jet equivalent modulo the kernel of π^k_{k-1*}. But this implies that $\alpha^{k,k}(\sigma) = \alpha^{k,k}(\mu) = \mu = \sigma$ with σ and μ representatives of $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$. ∎

This equivalence suggests the consideration of the vector bundles

$$T^{(k)}(J^k)\colon \pi_{k-1}^{k*}\, TJ^{k-1} \to J^k \, .$$

From proposition (1.3) follows

PROPOSITION 1.4. – Let $f \in C(M, N)$; then $\pi^k_{k-1*} J^k f_*(x)\colon T_x M \to T^{(k)}_p(J^k)$, $p = J^k f(x)$, is a canonical monomorphism.

DEFINITION 1.5. – The k-th order total vector bundle is

$$H^{(k)}(J^k) = U_{J^k f(x)}\, \pi^k_{k-1*} J^k f_* T_x M \subset T^{(k)}(J^k) \, .$$

The total vector bundle is

$$H(J) = \varprojlim H^{(k)}(J^k) \subset TJ \, .$$

PROPOSITION 1.6. – The germ $\sigma \in \mathbf{\Gamma}(J^k)$ is a k-integrable germ at $x \in M$ iff

$$\pi^k_{k-1*}\sigma_*(T_x M) = H^{(k)}_{\sigma(x)}(J^k) \, .$$

This vector bundle was first introduced in a different way by J. M. BOARDMAN [2]. Its elements are called total vectors. One has the canonical isomorphisms

$$\#_p = \pi^k_{k-1*} J^k f_*(x)\colon T_x M \to H^{(k)}_p(J^k) \, , \qquad p = J^k f(x) \, ,$$

for each k and hence also $\#_p\colon T_x M \to H_p(J)$, $\pi_M(p) = x$. This isomorphism allows one to define the total lift $\#$ of any vector field $X \in \mathscr{X}(M)$ into a vector field $X^\#$ on J, called the total lift of X. One sets $X^\#_p = \#_p X$; then $X^\#$ is a section of the bundle $H(J)$.

PROPOSITION 1.7.

(1) For any $X \in \mathscr{X}(M)$ and $f \in C(M, N)$,

$$X \circ Jf^* = Jf^* \circ X^\# \, ,$$

where the vector fields are considered as derivations in the ring of real functions.

(2) The bundle $H(J)$ is involutive.

PROOF. – It is sufficient to prove relation (1) as a derivation on elements of $\mathscr{F}^{(k)}(p)$. But then (1) is a direct consequence of the chain rule for partial differentiation. Part (2) follows from (1).

The set of sections of $H(J)$ obtained by the total lift of $\mathscr{X}(M)$ will be denoted by $\mathscr{L}^\#(J)$. $\mathscr{L}^\#(J)$ will be the set of germs of elements of $\mathscr{L}^\#(J)$.

The lift $\#$ will be extended to a 1-form on J, defined by

$$\#\colon T_pJ \to H_p(J)\,, \quad p \in J\,,$$
$$Y \mapsto \#_p(\pi_{M*}Y)\,.$$

Hence $\operatorname{Ker}_p \pi_{M*} = \operatorname{Ker}_p \#$, $p \in J$, and $\#(\varphi\cdot X) = \varphi\cdot\#(X)$, for $X \in \mathscr{X}(J)$, $\varphi \in \mathscr{F}(p)$.

The bundle $H(J)$ is an involutive distribution, in the sense of the projective limit, on J. Hence $H(J)$ defines the total complex $(d_H, \pi_M^* \Lambda(M))$, where:

$$d_H\colon \mathscr{F}(p) \to \mathbf{\Lambda}^1(J)$$
$$\varphi \mapsto d_H\varphi\colon X^\# \mapsto X^\# \lrcorner\, d\varphi\,, \quad X^\# \in \mathscr{L}^\#(J)$$
$$X^V \mapsto 0\,, \quad X^V \in \operatorname{Ker}\#\,,$$

$\pi_M^*\Lambda(M)$ is considered as a module over $\mathscr{F}(J)$ and

$$d_H\colon \pi_M^*\Lambda^k(M) \to \pi_M^*\Lambda^{k+1}(M)\,, \quad \text{for } k \in \mathbf{N}\,.$$

(Small d always stands for exterior differentiation while capital D stands for the derivative.)

PROPOSITION 1.8. – $d_H \circ d_H = 0$.

PROOF. – For any $\varphi \in \mathscr{F}(J)$ and $X, Y \in \mathscr{X}(M)$

$$\begin{aligned}\big(d_H(d_H\varphi)\big((X^\#, Y^\#\big) &= \big(d(d_H\varphi)\big)(X^\#, Y^\#)\\ &= \tfrac{1}{2}\{X\big(d_H\varphi(Y^\#)\big) - Y^\#\big(d_H\varphi(X^\#)\big) - d_H\varphi\big([X^\#, Y^\#]\big)\}\\ &= 0\end{aligned}$$

by the involutiveness of $H(J)$. ■

PROPOSITION 1.9. – Let $\varphi \in \mathscr{F}^{(l)}(J)$, then $d_H\varphi \in \pi_M^*\Lambda(M)$ factors through J^{l+1} and is linear in the natural fibre coordinates of π_l^{l+1}.

Let $\boldsymbol{\varphi}$ be a map germ $J \to P$, at $p \in J$, with P a smooth manifold.

DEFINITION 1.10. – The total k-jet of $\boldsymbol{\varphi}$, $j^k_{T,p}(\varphi)$, $p \in J$, is defined by

$$Jf^* j^k_{T,p}(\varphi) = j^k_x(Jf^*\varphi)\,, \quad Jf(x) = p,\ f \in \mathbf{C}(M, N)\,.$$

$J^k_T(J, P)$ will denote the space of total k-jets of smooth map germs of J into P and $J_T(J, P) = \varprojlim J^k_T(J, P)$.

c. The universal elements.

For each $k \in \mathrm{N}$, there exists a unique universal element ϱ^k, section of $J^k_T(J^k, N)$ such that:

$$\forall f \in C(M, N)\,, \qquad J^k f^* \varrho^k = J^k f \;[13]\,.$$

The local expression of ϱ^k is given by

$$\varrho^k = (\pi^k_N, D_H \pi,^k_N D^2_H \pi^k_N, \ldots, D^k_H \pi^k_N)\,,$$

where D_H denotes the total derivative.

Because $\pi_N\colon J \to N$, one has with $x = \pi_M(p)$, $y = \pi_N(p)$:

$$D_H \pi_N\colon p \in J \to \mathrm{Hom}\,(T_x M, T_y N)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D^k_H \pi_N\colon p \in J \to \mathrm{Hom}\,(0^k T_x M, T_y N)\,.$$

0^k is the k-fold symmetric tensor product.

Let

$$F^k_{(x,y)} = \mathrm{Hom}\,(T_x M, T_y N) \oplus \mathrm{Hom}\,(0^2 T_x M, T_y N) \oplus \ldots \oplus \mathrm{Hom}\,(0^k T_x M, T_y N)\,,$$

and

$$\bar{F}^k_{(x,y)} = T_y N \oplus F^{k-1}_{(x,y)}\,.$$

The space $\bar{F}^k_{(x,y)}$ will also be identified with the corresponding subspace of $\mathrm{Hom}\,(T_x M, \bar{F}^{k-1}_{(x,y)})$.

From this follows that

(1) $\varrho^k\colon p \in J \to (y, F^k_{(y,x)})$;

(2) $d_H \varrho^k\colon p \in J \to \mathrm{Hom}\,(T_x M, \bar{F}^k_{(x,y)})$;

(3) $d\varrho^k\colon p \in J \to \mathrm{Hom}\,(\mathrm{Ker}_p\, \pi^k_{M*}, \bar{F}^k_{(x,y)})$.

and $\varrho^{k+1} = (\pi_N, D_H \varrho^k)$, by identification.

PROPOSITION 1.11. – The map $d\varrho^k\colon \varrho \in J \to \mathrm{Hom}\,(\mathrm{Ker}_p\, \pi^k_{M*}, \bar{F}^k_{(x,y)})$ is constant along the fibre $(\pi^0_k)^{-1}(\pi^k_0(p))$. ■

From this if follows that (*a*) $d\varrho^k$ restricted to $\mathrm{Ker}\,\pi^k_{k-1*}$;

$$d\varrho^k\colon \mathrm{Ker}\,\pi^k_{k-1*} \to 0^k T^*_x M \otimes T_y N$$

is the fundamental identification defined by M. KURANISHI [9] [12] and (*b*) all bundles $J^k \to J^l$, $k > l > 0$ are affine fibre bundles [5].

The forms $d_H \varrho^k$ and $d\varrho^k$ will also be considered as elements of $T^*J \oplus \bar{F}^k$ over J^k. This enables one to introduce the operator δ:

$$\delta\varrho^k = d\varrho^k - d_H \varrho^k \equiv \Phi^{k+1} \ [13] .$$

The elements Φ^k will be called the structure elements.

PROPOSITION 1.12. – Let $\sigma \in \Gamma J^k$; then σ is k-integrable at $x \in M$, iff

$$\sigma^* \Phi^k(x) = 0 .$$

The operator δ defines an operator $\mathfrak{D}$ on the space of sections by means of the pull back of $\delta\varrho^{k-1}$. In other words, if $\sigma \in \Gamma(J^k)$, then $\mathfrak{D}\sigma = \sigma^* \delta\varrho^{k-1}$. There is an exact sequence of sheafs of germs:

$$0 \to \mathbf{C}(M, N) \xrightarrow{j^k} \mathbf{\Gamma}(J^k) \xrightarrow{\mathfrak{D}} \mathbf{\Gamma}_M T^* M \otimes \bar{F}^{k-1} \to 0$$

which is the first resolvent of Spencer [5].

d. Integrability of sections $J^l \to J^k$.

Let $\Gamma_l(J^k)$ denote the space of smooth sections $J^l \to J^k$.

DEFINITION 1.13. – An element $\nu \in \Gamma_0(J^1)$ is integrable at $z \in J^0$ if $\nu^* H^{(1)}(J^1)$ is involutive at z.

Because $H^{(1)}(J^1) \subset \pi_0^{1*} TJ^0$, the pull-back bundle $\nu^* H^{(1)}(J^1)$ is a subbundle of TJ^0 and hence a distribution on J^0. Furthermore

$$\nu^* H^{(1)}(J^1) = \operatorname{Ker} \nu^* \Phi^{(1)} .$$

PROPOSITION 1.14. – If $\nu \in \Gamma_0(J^1)$ is integrable at $z \in J^0$, then there exists an element $\mu \in \Gamma(J^0)$, such that $\mu(x) = z$, $x = \pi_M^0(z)$, and $\mu^{(1)}(x) = \nu(z)$.

PROOF. – $\nu^* H^{(1)}(J^1)$ is a distribution on J^0, transversal to the fibre of π_M^0. Because $\nu^* H^{(1)}(J^1)$ is involutive at z, by Frobenius's theorem there exists an integral submanifold of $\nu^* H^{(1)}(J^1)$ at z, transversal to the fibre of π_M^0. $\operatorname{Dim} \nu^* H^{(1)}(J^1)(z) = \dim T_x M$ and the integral manifold is a local section, namely an element of $\Gamma(J^0)$. Let μ be such an element, then $\mu^* T_y M = \nu_z^* H^{(1)}(J^1)$ and $\nu \in \mu \in (J^1)$ is 1-in tegrable at $x \in M$. Hence $\mu^{(1)}(x) = \nu(z)$.

DEFINITION 1.15. – An element $\nu \in \Gamma_k(J^{k+1})$ is integrable at $z \in J^k$, if $\nu^* H^{(k+1)}(J^{k+1})$ is involutive at z.

The same remarks as before can be made because $\nu^* H^{(k+1)}(J^{k+1})$ defines a distribution germ on J^k at z.

PROPOSITION 1.16. – If $\nu \in \Gamma_k(J^{k+1})$ is integrable at $z \in J^k$, then there exists a $\mu \in \Gamma(J^k)$ such that $\mu(x) = z$, $\pi^k_M(z) = x$, μ is k-integrable at x and $\mu^{(1)}(x) = \nu(z)$.

PROOF. – The distribution $\nu^* H^{(k+1)}(J^{k+1})$ is involutive at z and transversal to the fibres of π^k_M. Hence the integral manifold is an element of $\Gamma(J^k)$ at z. The section will be k-integrable at $\pi^k_M(z)$ and ν defines the first prolongation of this section at z. ■

From this approach it becomes now clear what an integrable element of $\Gamma_k(J^l)$ should be. Let $\mu \in \Gamma_k(\mathscr{U}, J^{k+1})$ be integrable on $\mathscr{U} \subset J^k$. Then there exists a unique section $\mu^{(l)} \in \Gamma_k(\mathscr{U}, J^{k+l+1})$, called the l-th prolongation of μ and defined by: $\forall \nu \in \in \Gamma(\pi^k_M(\mathscr{U}), J^k)$, integrable sections with image in $\mathscr{U}$ and such that $\nu^{(1)} = \mu \circ \nu$, one has $\nu^{(l+1)} = \mu^{(l)} \circ \nu$.

DEFINITION 1.17. – An element $\nu \in \Gamma_k(J^l)$, $l > k+1$, is integrable at $z \in J^k$, if (1) $\pi^l_{k+1} \circ \nu$ is integrable at $z \in J^k$ and (2) $\nu = (\pi^l_{k+1} \circ \nu)^{(l-1)}$.

PROPOSITION 1.18. – Let $\nu \in \Gamma_0(J)$ be integrable on $\mathscr{U} \subset J^0$; then on $\mathscr{U}$

$$\nu^* \circ d_H = (\nu^* d_H) \circ \nu^* .$$

PROOF. – For all integrable $\mu \in \Gamma(J^0)$ with image in $\mathscr{U} \subset J^0$ and such that $\pi_1 \circ \circ \nu \circ \mu = \mu^{(1)}$ on $\pi^1_M(\mathscr{U})$,

$$\begin{aligned} \mu^* \circ \nu^* \circ d_H &= d \circ \mu^* \circ \nu^* \\ &= \mu^*\big((\nu^* d_H) \circ \nu^*\big) . \quad ■ \end{aligned}$$

It is a consequence of the definition that if $\nu \in \Gamma_k(J)$ is integrable then $\nu^* d_H \circ \circ \nu^* d_H = 0$. In this case $\operatorname{Ker} \nu^* \Phi^{(k+1)}$ is an involutive distribution on J^k. Let $H(X)$ be the lift of a vector $X \in T_x M$ into this distribution. Then $H(X) \lrcorner \, \nu^* d_H = H(X) \lrcorner \, d$, which proves:

PROPOSITION 1.19. – Any $\nu \in \Gamma_k(\mathscr{U}, J^{k+1})$ is integrable iff $\nu^* d_H(\nu^* \varrho^k) = 0$, where $\nu^* \varrho^k$ is considered as an element of $\Lambda^1 J^k \otimes \bar{F}^{k-1}$ [10].

2. – Contact transformations.

DEFINITION 2.1. – Let $\boldsymbol{\varphi}$: $(J, p) \to (J, q)$ be a smooth map germ, then

(a) $\boldsymbol{\varphi}$ is a contact map germ if $\varphi_* H_p = H_q$;

(b) $\boldsymbol{\varphi}$ is a contact weak homeomorphism germ if $\boldsymbol{\varphi}$ satisfies (a) and φ is a weak homeomorphism;

(c) $\boldsymbol{\varphi}$ is a regular weak contact germ if φ satisfies (b) and if $\pi_k \circ \varphi$ has maximal rank for all $k \geqslant 0$;

(d) $\boldsymbol{\varphi}$ is a regular strong contact germ if φ satisfies (b) and if for some local fibre bundle coordinates $\psi^{(k)}: J \to J^k(\mathbb{R}^m, \mathbb{R}^n)$, containing p, the map $\pi_k \circ \circ \varphi \circ (\psi^{(k)})^{-1}$ is a local diffeomorphism at $\psi^{(k)}(p)$, for all $k \geqslant 0$;

(e) $\boldsymbol{\varphi}$ is a contact diffeomorphism germ if φ is a regular invertable contact homeomorphism.

For following examples show that one has strict inclusions

$$(a) \supset (b) \supset (c) \supset (d) .$$

Let $(x, u, p, r, \ldots)$ be coordinates on $J(\mathbb{R}, \mathbb{R})$, then at $(0, 0, 0, \ldots)$ the map germ

(1) $x' = x,\ u' = p,\ p' = r,$ '.. is (a) but not (b);

(2) $x' = x,\ u' = u^3 + p^2,\ p' = 3u^2p + 2pr,\ \ldots$ is (b) but not (c);

(3) $x' = x,\ u' = u^3 + p,\ p' = 3u^2p + r,\ \ldots$ is (c) but not (d);

(4) $x' = x,\ u' = u + p,\ p' = p + r,\ \ldots$ is (d), but not (e);

(5) $x' = x,\ u' = u + x^2,\ \ldots$ is (e).

Furthermore it follows from the definition that for any contact map $\varphi: \mathcal{O} \subset J \to J$ and any section $Jf: Jf^{-1}(\mathcal{O}) \to \mathcal{O} \subset J$, $f \in C(M, N)$, the map $\pi_M \circ \varphi \circ Jf$ is a diffeomorphism of $Jf^{-1}(\mathcal{O}) \to \pi_M(\varphi(Jf \cap \mathcal{O}))$.

THEOREM 2.2. – There exists a bijection between the contact map germs $\boldsymbol{\phi}: (J, p) \to (J, q)$ and the smooth map germs $\boldsymbol{\varphi}: (J, p) \to (J^0, \pi_0(q))$, such that $d_H(\pi_M \circ \circ \varphi)(p): T_x M \to T_y M$, $x = \pi_M(p)$, $y = \pi_M(q)$, has maximal rank.

PROOF. – The map germ $\boldsymbol{\phi}$ is a contact map germ if for any $\phi \in \Phi$ on some $\mathcal{O} \subset J$, $p \in \mathcal{O}$, and for all Jf, $f \in C(M, N)$ in $\mathcal{O}$, the image of $\Phi \circ Jf$ is an integrable section. This is equivalent with the requirements

(a) $\pi_0 \circ \phi \circ Jf \circ (\pi_M \circ \phi \circ Jf)^{-1}: \mathcal{U} \subset M \to J^0$ is a section for some $\mathcal{U}$ containing $y \in M$;

(b) $\phi(j_x f) = j_y(\pi_0 \circ \phi \circ Jf \circ (\pi_M \circ \phi \circ Jf)^{-1})$.

This last equation can be rewritten by means of

$$(\phi_M, \phi_N) \equiv (\pi_M \circ \phi, \pi_N \circ \phi) = \pi_0 \circ \phi$$

as

$$\phi(j_x f) = j_x(\phi_N \circ Jf)\cdot j_y(\phi_M \circ Jf)^{-1}$$

or

$$\phi(p) = j_{T,p}(\phi_N)\cdot j_{T,p}^{-1}(\phi_M) .$$

It suffices to put $\varphi = (\phi_M, \phi_N)$, which defines ϕ iff $d_H\phi_M(p)$ has maximal rank. ■

Because any map germ $\boldsymbol{\phi}\colon J \to J^0$, such that $d_H(\pi_M \circ \phi)$ has maximal rank, defines a unique contact map germ on J, one has a canonical lift, denoted $\natural$, of such map germs into the set of contact map germs on J.

Theorem 2.2 permits to write any contact map locally as

$$\phi(p) = \left(\phi_M(p), \phi_N(p), \tilde{D}_H\phi_N(p), \tilde{D}_H^2\phi_N(p), \ldots, \tilde{D}_H^k\phi_N(p), \ldots\right) ,$$

where $\tilde{D}_H = (D_H\phi_M)^{-1}D_H$ and $(\phi_M, \phi_N)^\natural = \phi$, at $p \in J$.

Let $\mathscr{M}_J(J^0)$ be the sheaf of map germs $\boldsymbol{\phi}\colon J \to J^0$ such that $d_H(\pi_M \circ \phi)$ has maximal rank and $\phi^\natural$ is a weak homeomorphism germ on J.

It is a direct consequence of this expression that if $(\boldsymbol{\phi}_M, \boldsymbol{\phi}_N) \in \mathscr{M}_J(J^0)$ has maximal rank as well as $D_H\phi_M$ then $(\boldsymbol{\phi}_M, \boldsymbol{\phi}_N)^\natural$ is a regular weak contact germ.

Let $\mathscr{N}_J(J^0)$ be the subset of $\mathscr{M}_J(J^0)$ of elements of maximal rank, then any regular weak contact germ is the $\natural$-lift of an element of $\mathscr{N}_J(J^0)$. The subset of $\mathscr{N}_J(J^0)$ of those elements factoring through J^k will be denoted by $\mathscr{N}_{J^k}(J^0)$, but will be considered as map germs on J, which avoid unnecessary shifts of base points of germs. One obviously has

$$\mathscr{M}_J(J^0) = \left(\mathscr{M}_J(M), \mathscr{M}_J(N)\right)$$

but both factors don't have to factor through the same J^k.

The following condition is introduced to simplify the next theorem.

CONDITION (C). – The pair $(\boldsymbol{\varphi}_M, \boldsymbol{\varphi}_N) \in \mathscr{M}_J(J^0)$ satisfies condition (C) if

$$\exists \boldsymbol{\theta} \in \boldsymbol{\Gamma}(J^1, \pi_M^* TM \otimes_J \pi_N^* TN) ,$$

such that

$$d_H \boldsymbol{\varphi}_N = d_H \boldsymbol{\varphi}_M \lrcorner\, \boldsymbol{\theta} .$$

THEOREM 2.3. – Let $\boldsymbol{\varphi}$ be a regular weak contact germ on J.

(*a*) If $\dim N = 1$, then $\varphi^*\mathscr{F}^l(J) \subseteq \mathscr{F}^l(J)$ for some $l \geq 1$, iff $\boldsymbol{\varphi} \in \left(\mathscr{N}_{J^1}(J^0)\right)^\natural$ and $\pi_0 \circ \varphi$ satisfies (C).
If $\dim N \neq 1$, then $\varphi^*\mathscr{F}^l(J) \subseteq \mathscr{F}^l(J)$ for some $l \geq 1$, iff $\boldsymbol{\varphi} \in \left(\mathscr{N}_{J_0}(J)\right)^\natural$.

(*b*) $\boldsymbol{\varphi}$ preserves π_M, iff $\boldsymbol{\varphi} \in \big(\mathcal{N}_M(M), \mathcal{N}_J(N)\big)^{\natural}$.

(*c*) $\boldsymbol{\varphi}$ preserves π_M and $\varphi^* \mathcal{F}^l(J) \subseteq \mathcal{F}^l(J)$ for some $l \geqslant 1$, iff $\boldsymbol{\varphi} \in \big(\mathcal{N}_M(M), \mathcal{N}_{J^0}(N)\big)^{\natural}$.

This theorem generalizes a result obtained by B. KUPERSHMIDT [7] [8], which has been reformulated later by R. L. ANDERSON and N. N. IBRAGIMOV [1].

The following lemmas will simplify the proof of this theorem.

LEMMA 2.4. – Let $(\varphi_M, \varphi_N) \in \mathcal{N}_{J^1}(J^0)$ satisfy condition (C), then if $\dim N > 1$, $(\varphi_M, \varphi_N) \in \mathcal{N}_{J^0}(J^0)$.

PROOF. – It is sufficient to prove the lemma on $J^1(\mathbb{R}, \mathbb{R}^2)$. Let (x, y^a, p^a), with $a = (1, 2)$, be coordinates on J^1. Then condition (C) becomes:

$$(\partial_x + p^a \partial_{y^a})\varphi_N^b = (\partial_x + p^a \partial_{y^a})\varphi_M \cdot \theta^b \tag{1}$$

$$\partial_{p^a}\varphi_N^b = \partial_{p^a}\varphi_M \cdot \theta^b\,. \tag{2}$$

Let $\partial_{p^a}\varphi_M$ be nonzero on an open subset $\mathcal{U} \subset J^1$. Equation (2) implies that $\partial_{p^a}\varphi_N^b \neq 0$ on an open dense subset of $\mathcal{U}$. Hence there exists a nonzero function C on $\mathcal{U}$ such that

$$(\partial_{p^1} - C\partial_{p^2})\varphi_N^b = 0 \tag{3}$$

$$(\partial_{p^1} - C\partial_{p^2})\varphi_M = 0\,. \tag{4}$$

The integrability conditions of equations (1) and (2), together with (3) and (4) give the conditions

$$(\partial_{y^1} - C\partial_{y^2})\varphi_N^b = (\partial_{y^1} - C\partial_{y^2})\varphi_M \cdot \theta^b + (\partial_x + p^c \partial^{y^c})\varphi_M \cdot (\partial_{y^1} - C\partial_{y^2})\theta^b\,. \tag{5}$$

If $(\partial_x + p^a \partial_{y^a})\varphi_M \neq 0$, then from equations (1) and (2) it follows that there exist on $\mathcal{U}$ nonzero functions A_a such that with $\Delta_a \equiv \partial_{p^a} - A_a(\partial_x + p^c \partial_{y^c})$

$$\Delta_a \varphi_N^b = 0 \tag{6}$$

$$\Delta_a \varphi_M = 0 \tag{7}$$

and $C = A_1/A_2$.

But this implies that $[\Delta_1, \Delta_2]\varphi_N^b = [\Delta_1, \Delta_2]\varphi_M = 0$, with

$$[\Delta_1, \Delta_2] = (\Delta_2 A_1 - \Delta_1 A_2)(\partial_x + p^c \partial_{y^c}) - A_2 \partial_{y^1} + A_1 \partial_{y^2}\,.$$

If $\Delta_2 A_1 - \Delta_1 A_2 \neq 0$, then equation (1) becomes

$$(A_2 \partial_{y^1} - A_1 \partial_{y^2})\varphi_N^b = (A_2 \partial_{y^1} - A_1 \partial_{y}\ )\varphi_M \cdot \theta^b ,$$

which together with (5) implies $(\partial_{p^1} - C\partial_{p^2})\theta^b = 0$.

If $\Delta_2 A_1 - \Delta_1 A_2 = 0$, then $[\Delta_1, \Delta_2]\varphi_M = 0$ together with yields (5)

$$(\partial_{p^1} - C\partial_{p^2})\theta^b = 0 .$$

Hence in both cases, the map

$$(\varphi_M, \varphi_N, \theta) \colon \mathscr{U} \subset J^1 \to J^1$$

is not of maximal rank.

If on $(\partial_x + p^c \partial_{y^c})\varphi_M = 0$, then one also has $(\partial_x + p^c \partial_{y^c})\varphi_N^b = 0$ and equation (5) becomes

$$(\partial_{p^1} - C\partial_{p^2})\varphi_N^b = (\partial_{p^1} - C\partial_{p^2})\varphi_M \cdot \theta^b . \tag{8}$$

A similar argument as above leads to the equations $\delta_a \varphi_N^b = \delta_a \varphi_M = 0$, with $\delta_1 = B_1 \partial_{p^1} + (\partial_{y^1} - C\partial_{y^2})$ and $\delta_2 = B_2 \partial_{p^2} + (\partial_{y^1} - C\partial_{y^2})$, for some functions B_a on $\mathscr{U}$. Consideration of $[\partial_1, \partial_2]$ then yields the same conclusion.

Now if only $\partial_{p^1}\varphi_M \neq 0$ and $\partial_{p^2}\varphi_M = 0$ on $\mathscr{U}$ then also $\partial_{p^1}\varphi_N^b \neq 0$ and $\partial_{p^2}\varphi_M = 0$, which implies from (2) that $\partial_{p^2}\theta^b = 0$. Hence the map $(\varphi_M, \varphi_N, \theta)$ is not of maximal rank. ∎

The following example shows that if $\dim N = 1$ there exist regular weak contact germs satisfying condition (C). Let J^1 be $J^1(\mathbb{R}, \mathbb{R})$. The map germ at $(0, 0, 0)$ given by: $x' = x + p$, $u' = u + \frac{1}{2}p^2$, $p' = p$, is a strong contact diffeomorphism satisfying condition (C).

This example has the property that the higher order terms are no longer defined when $r = -1$ ($r = p_x$ on J^2), because $r' = r/(1 + r)$. Or equivalently $d_H \varphi_M$ is no longer of maximal rank. All theorems in this paper are formulated on the infinite jet bundle; a correct way of dealing with these infinities is given by A. VINOGRADOV [13] in terms of the completed or projective jetbundle. Inside that framework the condition of maximal rank of $d_H \varphi_M$ may be dropped.

LEMMA 2.5. – If $(\varphi_M, \varphi_N) \in \mathscr{M}_J(J^0)$ and $(\varphi_M, \varphi_N, \tilde{D}_H \varphi_N) \colon J \to J^1$ factors through J^l, $l \geqslant 1$; then $(\varphi_M, \varphi_N, \ldots, \tilde{D}_M^k \varphi_N) \colon J \to J^k$ factors through J^{l+k-1}.

PROOF. – This is a direct consequence of the chainrule for partial derivatives, proposition (1.9) and the maximal rank condition on $D_H \varphi_M$. ∎

PROOF OF THE THEOREM. – Let $\varphi^\natural \in (\mathcal{N}_J(J^0))^\natural$, then if $\varphi^* \mathcal{F}^l(J) \subseteq \mathcal{F}^l(J)$ for some $l \geq 1$, by lemma (2.5) the map $(\varphi_M, \varphi_N, \tilde{D}_H \varphi_N)$ has to factor through J^1. But from lemma (2.4) it follows that if $\dim N \neq 1$, $\boldsymbol{\varphi} \in \mathcal{N}_{J^0}(J^0)$ and if $\dim N = 1$, $\boldsymbol{\varphi} \in \mathcal{N}_{J^1}(J^0)$ and $\boldsymbol{\varphi}$ satisfies condition (C). In both cases if follows from the lifting rule, given by theorem (2.2), that $\varphi^* \mathcal{F}^l(J) \subseteq \mathcal{F}^l(J)$, for all $l \geq 1$.

Moreover, $\varphi^\natural \in (\mathcal{N}_J(J^0))^\natural$ preserves π_M iff the map germs $\pi_M \circ \varphi^\natural \circ Jf$ is independent of the section germ $\boldsymbol{Jf}$. Hence $\boldsymbol{\varphi}_M$ has to be independent of the fibre coordinates of π_M. This is the case iff $\boldsymbol{\varphi} \in (\mathcal{N}_M(M), \mathcal{N}_J(N))$.

To prove (*c*) it is now sufficient to restrict attention to those $\boldsymbol{\varphi}^\natural$ belonging to $(\mathcal{N}_M(M), \mathcal{N}_J(N))^\natural$. Then if $\dim N \neq 1$, it follows from (*a*) that $\boldsymbol{\varphi} \in (\mathcal{N}_M(M), \mathcal{N}_{J^0}(N))$. If $\dim N = 1$, $\boldsymbol{\varphi}$ has to satisfy condition (C). But then $X \lrcorner\, d\varphi_N = 0$, $\forall X$ in $\ker \pi^1_{0*}$. The converse is trivial. ■

REMARK. – Because lemma (2.4) does no longer hold for contact maps, the theorem is not valid for contact maps. This is shown by the following example on $J(\mathbb{R}, \mathbb{R}^2)$. Let $(x, u_1, u_2, p_1, p_2, \ldots)$ be natural coordinates on $J(\mathbb{R}, \mathbb{R}^2)$, then the map germ at $0 \in J(\mathbb{R}, \mathbb{R}^2)$ defined by $x' = x + p_1$, $u'_1 = u_2 + \frac{1}{2} p_1{}^2{}_1$, $u'_2 = p_1$, $p'_1 = p_1$, $p'_2 = 1, \ldots$, is contact preserving and preserves $\mathcal{F}^l(J)$, $l \geq 1$.

From now on we will restrict attention to regular strong contact germs. Let $\mathcal{D}_J(J^0)$ denote the subset of $\mathcal{N}_J(J^0)$ of those elements defining a regular strong contact homeomorphism.

PROPOSITION 2.6. – Let $\boldsymbol{\phi} \in \mathcal{D}_J(J^0)$, then

(*a*) $\phi^{\natural *} \circ d_H = (d_H \circ \phi^{\natural *}) \circ (d_H \varphi_M^{-1} \circ d\varphi_M)$;

(*b*) $\phi^{\natural *} \circ d_H = d_H \circ \phi^{\natural *}$ iff $\boldsymbol{\phi} \in (\mathcal{D}_M(M), \mathcal{D}_{J^0}(N))$.

PROOF. – Part (*a*) is a direct consequence of theorem (2.2). Part (*b*) follows from the fact that $\phi^\natural$ commutes with d_H iff $\phi^\natural$ is contact preserving and preserves π_M. This follows from theorem (2.3). ■

DEFINITION 2.7. – A contact germ $\boldsymbol{\varphi}$ on J will be called a special contact germ if $\varphi \in (\mathcal{D}_M(M), \mathcal{D}_J(N))^\natural$.

Let $\boldsymbol{SC}(J)$ be the sheaf of special contact germs on J. One has the filtration given by $\boldsymbol{SC}^{(k)}(J) = (\mathcal{D}_M(M), \mathcal{D}_{J^k}(N))^\natural$. It has to be remarked that, as a consequence of theorem (2.3), on J^k only $\boldsymbol{SC}^{(0)}(J^k)$ makes sense.

PROPOSITION 2.8. – Let $\boldsymbol{\varphi}^\natural \in \boldsymbol{SC}^{(0)}(J^k)$, then the l-jet of $\boldsymbol{\varphi}^\natural$ is defined by the $(k+l)$-jet of $\boldsymbol{\varphi}$.

It suffices to write $\boldsymbol{\varphi}^\natural$ in local coordinates to prove the proposition.

PROPOSITION 2.9. – Let $\mu \in \Gamma_0(J)$ be integrable; then for any $\boldsymbol{\varphi}^\natural \in \boldsymbol{SC}^{(k)}(J)$

$$\mu^*(\varphi^\natural) = \mu^*\big((\mu^* \varphi)^\natural\big) .$$

PROOF. – If $\mu \in \Gamma_0(J)$ is integrable, then $\mu^* \circ d_H = (\mu^* d_H) \circ \mu^*$, by proposition 1.18. Hence locally

$$\begin{aligned}\mu^*(D_H^2 \varphi_N) &= (\mu^* D)_H (\mu^* D_H \varphi_N) \\ &= (\mu^* D_H^2)(\mu^* \varphi_N) .\end{aligned}$$

By iteration and the use of the lifting procedure, the proposition follows. ■

3. – Contact Lie algebras and ♮-lifts.

a. The contact Lie algebras of vector fields.

The ♮-lift defined for germs of J-parametric diffeomorphisms will now be extended to J-parametric vector fields.

DEFINITION 3.1. – A contact vector field X on J is an element of $\mathscr{X}(J)$, such that $[X, H(J)] \subset H(J)$.

Let $\mathscr{L}_{J^k}(N)$ be the space of germs of elements of $\Gamma^k_{N*}(J^k, \pi_N^{k*} TN)$; then there is an unique lift

$$\natural : \mathscr{L}_{J^k}(N) \to \mathscr{X}(J) ,$$

defined by $[\natural(X), H(J)] \subset H(J)$ and $\pi_{0*}(\natural(X)) = X$. Let $\mathscr{L}^\natural(J^k, N)$ be the image of $\natural(\mathscr{L}_{J^k}(N))$ [8].

PROOF OF THE EXISTENCE AND UNIQUENESS OF $\overset{*}{\natural}$. – Let $(\mathscr{U} \subset M, x^i)$ and $(\mathscr{V} \subset N, x^a)$ be local coordinates; then in terms of the induced natural coordinates on $\pi_0^{-1}(\mathscr{U} \times \mathscr{V})$, one has

$$X = X^i \partial_i + X^a \partial_a + X^a_i \partial^i_a + X^a_{ij} \partial^{ij}_a + \ldots .$$

The condition $\pi_{0*}(X) \in \mathscr{L}_{J^k}(N)$ yields $X^i \partial_i = 0$ and $X^a \in \mathscr{F}^{(k)}(J)$. Now from $[X, H(J)] \subset H(J)$ it follows that $X^a_i = \partial^\#_i X^a$, $X^a_{ij} = \partial^\#_j \partial^\#_i X^a$, ..., which defines X uniquely in terms of $X^a \partial_a = \pi_{0*} X$. ■

THEOREM 3.2. – $X \in \mathscr{X}(J)$ is a contact vector field iff

$$X = X^\# + (\pi_{N*} X - \pi_{N*} X^\#)^\natural .$$

PROOF. – Any $X \in \mathscr{X}(J)$ may be decomposed as $X = X^\# + X^V$, with $X^A \in \operatorname{Ker} \pi_{M*}$. Then X is a contact vector field iff X^V is the ♮-lift of some $Y \in \mathscr{L}_J(N)$. From the $\pi_{N*} X$ components it follows that $Y = \pi_{N*} X - \pi_{N*} X^\#$. ■

It follows from this theorem that the set of contact vector fields can be written as the sum

$$\mathscr{L}^{\natural}(J, N) + \mathscr{L}^{\natural}(J, M) = \mathscr{L}^{\natural}(J) .$$

The sheaf $\mathscr{L}^{\natural}(J)$ is filtered by

$$\mathscr{L}^{\natural(k)}(J) = \mathscr{L}^{\natural}(J^k, N) + \mathscr{L}^{\natural}(J^k, M) .$$

For $1 < k < \infty$, the sheaf $\mathscr{L}^{\natural(k)}(J)$ is not closed for the Lie bracket of vector fields. The following properties are easily proven.

PROPERTIES 3.3.

(1) The following sub-sheaves are Lie sub-algebras over $\mathbb{R}$ of $\mathscr{L}^{\natural}(J)$:

$$\mathscr{L}^{\natural}(J, M),\ \mathscr{L}^{\natural}(J, N),\ \mathscr{L}^{\#}(J),\ \mathscr{L}^{\natural}(J^0, M),\ \mathscr{L}^{\natural}(J^0, N) .$$

(2) The following lifts are isomorphisms of $\mathbb{R}$-Lie algebras:

$$\begin{aligned}\natural &: \mathscr{L}_{J^0,y}(M) \to \mathscr{L}^{\natural}_{p}(J^0, M)\\ \natural &: \mathscr{L}_{J^0,y}(N) \to \mathscr{L}^{\natural}_{p}(J^0, N)\\ \# &: \mathscr{X}_{x}(M) \quad \to \mathscr{L}^{\#}_{p}(J)\end{aligned}$$

where $p \in J$, $\pi_0(p) = y$, $\pi_M(p) = x$.

(3) $[\mathscr{L}^{\natural}(J, M), \mathscr{L}^{\natural}(J, N)] \subset \mathscr{L}^{\natural}(J, N)$

$[\mathscr{L}^{\#}(J), \mathscr{L}^{\natural}(J, N)] = 0.$

REMARK ON THE SPECIAL CASE $\dim N = 1$. – It is a consequence of theorem 2.3 that there exists a Lie sub-algebra of contact vector fields of $\mathscr{L}^{\natural(1)}(J)$. Using the decomposition

$$X = X^{\#} + (\pi_{N*}X)^{\natural} - (\pi_{N*}X^{\#})^{\natural} ,$$

and the local expression

$$\pi_{1*}X = X^i\partial_i + A\partial_y + Y_i\partial^i ,$$

the condition that $\pi_{1*}X$ factors through J^1 becomes

$$\partial^j(X^i y_i - A) = X^j$$

and

$$(\partial_j + y_j \partial_y)(X^i y_i - A) + Y_j = 0 .$$

With $\varphi = X^i y_i - A$, $\varphi \in \mathscr{F}^{(1)}(J)$, any element of $\mathscr{L}^{\natural(1)}(J)$ preserving $\mathscr{F}^{(1)}(J)$ may be written as:

$$X = \partial^i \varphi(\partial_i + y_i \partial_y) - \varphi \partial_y - (\partial_j + y_j \partial_y)\varphi \cdot \partial^j .$$

Hence the vector field X is uniquely defined by the function $X \rfloor \delta\varrho^0$ on J^1. From this fact one derives the Jacobi, Poisson and Lagrange Brackets on $\mathscr{F}(J^1)$ [11].

b. The $\natural_p$ *lift of formal vector fields on* J^0.

Any integrable $\mu \in \Gamma_0(J)$ defines a local foliation of J^0 by the integral submanifolds of $(\pi_1 \circ \mu)^* H^{(1)}(J^1)$. Let $\nu \in \Gamma(J^0)$ be a local integral of μ, $p = \mu(y)$, $y = \nu(x)$, $x \in M$; then one sets for $\varphi \in \mathscr{F}^{(0)}(J)$

$$J^k_{\mu,\nu}(\varphi) = \mu^* J^k_{T,p}(\varphi) ,$$

and one has

$$\begin{aligned} \nu^* J^k_{\mu,\nu}(\varphi) &= \nu^* \mu^* J^k_{T,p}(\varphi) \\ &= J^k_x(\nu^* \varphi) . \end{aligned}$$

Locally one has

$$j^k_{\mu,\nu}(\varphi) = \big(\varphi(y), (\mu^* D_H)(\varphi), \ldots, (\mu^* D_H)^k(\varphi)\big) .$$

It is clear that $j^k_{\mu,\nu}\varphi$ depends only on $p = \mu(y)$ and hence one adopts the notation $j^k_{p,\nu}\varphi$, where $y = \pi_0(p)$.

Applying this to germs of vector fields on J^0 one sets:

$$\begin{aligned} J^k_p \mathscr{X}_y(J^0) &= \{ j^k_{p,\nu} X \mid X \in \mathscr{X}_y(J^0) \} . \\ J^k_\mu \mathscr{X}(J^0) &= U_p J^k_p \mathscr{X}_y(J^0) , \quad p \in \mu(J^0),\ \mu \in \Gamma_0(J^k) . \\ J^k_T \mathscr{X}(J^0) &= U_p J^k_p \mathscr{X}_y(J^0) , \quad p \in J^k . \end{aligned}$$

$J^k_T \mathscr{X}(J^0)$ is a vector bundle over J^k. Let $\mathscr{X}^V(J^0)$ be the subset of germs of vector fields in $\operatorname{Ker} \pi^0_{M*}$; then $J^k_T \mathscr{X}^V(J^0)$ is a subbundle of $J^k_T \mathscr{X}(J^0)$ over J^k.

Theorem 3.6.

(*a*) The vector bundles over J^k: $J^k_T \mathscr{X}^V(J^0)$ and $\operatorname{Ker} \pi^k_{M*}$ are isomorphic.

(*b*) The canonical isomorphism

$$\natural\colon J^k_p \mathscr{X}^V_y(J^0) \to \operatorname{Ker}_p \pi^k_{M*}$$

is given by: Let

$$X \in \mathscr{X}^V_y(J^0)\,, \qquad X^{(k)} \in J^k_p \mathscr{X}^V_k(J^0)$$

be such that $J^k_{p,y} X = X^{(k)}$; then

$$\natural_p(X^{(k)}) = \pi_{k*} \natural(X)(p) \in \operatorname{Ker}_p \pi^k_M\,.$$

(*c*) If $\mu \in \Gamma_0(J^k)$ is integrable, then for $X \in \mathscr{X}^V(J^0)$:

$$\mu^*(\pi_{k*} \natural(X))(y) = \natural_{\mu(y)}(J_{\mu(y),y} X)\,.$$

PROOF. – Let $X \in \mathscr{X}^V(J^0)$; then it is a consequence of theorem 2.3 that $\pi_{k*} \natural(X)$ is vector field on J^k. Hence $\natural_p\colon J^k_p \mathscr{X}^V_V(J^0) \to \operatorname{Ker}_p \pi^k_{M*}$ is a monomorphism.

An easy calculation shows that $\dim J^k_p \mathscr{X}^V_y(J^0) = \dim \operatorname{Ker}_p \pi^k_{M*}$, which proves (*a*) and (*b*). Let $\mu \in \Gamma_0(J^k)$ be integrable, then in terms of some natural coordinates one has for $X \in \mathscr{X}^V(\mathcal{O}^0)$:

$$\mu^* \natural(X) = X^a \partial_a + (\mu^* \partial_i)(X^a)\, \partial^i_a + (\mu^* \partial_i)(\mu^* \partial_j)(X^a)\, \partial^{ij}_a + \dots$$

from proposition 1.18. This proves (*c*). ∎

CONSEQUENCES. – Let $J^k \mathscr{X}^V(J^0)$ be the space of k-jets of germs of vertical vector fields on J^0 and let

$$\lambda^{(k)}\colon J^k \mathscr{X}^V(J^0) \times J^k \to J^k_T \mathscr{X}^V(J^0)$$

be the natural projection map. Then $\natural_p \circ \lambda^{(k)}$ is an epimorphism:

$$J^k \mathscr{X}^V_y(J^0) \times \{p\} \to \operatorname{Ker}_p \pi^k_M\,,$$

with $y = \pi^k_0(p)$.

Let $\mathscr{X}^V_M(J^0)$ be the subset of germs of vertical vector fields on J^0, factoring through M; then:

$$\natural_p \circ \lambda^{(k)}\colon J^k \mathscr{X}^V_{M,y}(J^0) \times \{p\} \to \operatorname{Ker}_p \pi^k_{M*}$$

is an isomorphism. The restriction of this map to $J^k \mathscr{X}^V_M(J^0) \times J^k$ will be denoted by $\beta^{(k)}$.

At any $p \in J^k$ one has the following exact sequence:

$$0 \to \operatorname{Ker} \pi^k_{k-1*} \to \operatorname{Ker} \pi^k_{M*} \to \operatorname{Ker} \pi^{k-1}_{M*} \to 0 ,$$

while at $\pi^k_0(p) = y$, the following exact sequence can be written $(x = \pi^0_M(y))$

$$0 \to 0^k(T^*_x M) \otimes T_y N \to J^k \mathscr{X}^V_{M,y}(J^0) \to J^{k-1} \mathscr{X}^V_{M,y}(J^0) \to 0 .$$

From both exact sequences and the isomorphism $\beta^{(k)}$ follows:

THEOREM 3.7.

$$d\varrho^k(\operatorname{Ker} \pi^k_{k-1*}) = (\beta^{(k)})^{-1}(\operatorname{Ker} \pi^k_{k-1*}) .$$

Let $\dim M = \dim N$ and let $S^{(k)} \subset J^k$ be the subset of k-jets of diffeomorphism germs. If $\mathscr{X}^V_N(J^0)$ is the subset of vertical vector fields germs on J^0 factoring through N, then follows:

THEOREM 3.8. – The morphism

$$\natural_p \circ \lambda^{(k)} \colon J^k \mathscr{X}^V_{N,y}(J^0) \times \{p\} \to \operatorname{Ker}_p \pi^k_{M*}$$

is an isomorphism on $S^{(k)}$.

Let α_p be the morphism $\natural_p \circ \lambda^{(k)}$ restricted to $J^k \mathscr{X}^V_{N,y}(J^0)$ on $S^{(k)}$; then in contrast to the morphism $\beta^{(k)}$, the morphism α_p depends explicitly on $p \in J^k$.

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