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CONTEXTUAL GRAMMARS VS. CONTEXT-FREE ALGEBRAS*)

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1. INTRODUCTION

Contextual grammars and some of their modifications were introduced by S. Marcus [13]. Further modifications were defined by other authors [14], [15], [18], [20], [11]. These variants of contextual grammars were studied in several papers [16], [19], [10], [21].

In the present paper, we describe the greater part of these grammars in terms of context-free algebras (cf. [12], [6] p. 299). Since context-free algebras are equivalent with context-free grammars, we obtain the description of various types of contextual grammars in terms of context-free and, particularly, linear grammars. In two cases, we obtain linear grammars with regulated derivations where this regulation is different from the regulation studied in the literature (cf. [22] Chapter V). In two other cases, special types of linear grammars are obtained. In this way, the greater part of the theory of contextual grammars becomes part of the theory of linear grammars. Procedures transforming contextual grammars into the corresponding linear ones are described in the paper.

This approach requires some complements of the theory of context-free algebras. Since we try to formulate our theorems in the most general way, these complements are formulated for partial and heterogeneous algebras (cf. [2], [9]). The investigations of subalgebras and of so called indexing mappings in heterogeneous algebras prove to be useful. Two theorems concerning subalgebras generated by sets and by families of sets are used; the structure of such subalgebras is described in terms of derivations (cf. [4] p. 9, [5] p. 29, [22] p. 6). As a particular case, we obtain the equivalence of context-free algebras and context-free grammars (cf. [12], [6] p. 299) and also the assertion that the system of subalgebras of an algebra is algebraic (cf. [1], [23], [3] p. 81).

In what follows, we need some fundamental notions of the theory of formal

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languages. Especially, V^* is the free monoid over a set V. The elements in V^* are said to be *strings* over V. The binary operation of V^* is called *catenation*; the catenation of strings x, y in V^* is denoted by xy, the catenation of subsets X, Y, of V^* is written as XY. The length of $x \in V^*$ is denoted by $|x|$.

If V is a set and $L \subseteq V^*$, then L is said to be a *language*.

Let U be a set. An ordered pair (t, z) of strings over U is said to be a *production over U*. Let R be a set of productions over U. For x, y in U^*, we put $y \Rightarrow x\ (R)$ if there exist u, v in U^* and (t, z) in R such that $y = utv$, $uzv = x$. If x, y are in U^* and $(z_i)_{i=0}^p$ is a sequence of strings in U^* where $p \geqq 0$, then $(z_i)_{i=0}^p$ is called a *y-derivation of x in R of length p* whenever $y = z_0$, $z_p = x$, and $z_{i-1} \Rightarrow z_i\ (R)$ for $i = 1, 2, \ldots, p$. We write $y \Rightarrow^* x\ (R)$ if there exists at least one y-derivation of x in R.

An ordered triple $G = (V, S, R)$ where V, S are mutually disjoint sets and R is a set of productions over $V \cup S$ is called a *generalized grammar*. Elements in V are called *terminal symbols* and elements in S *nonterminal symbols* of G. A y-derivation of x in R is called *terminal with respect to G* if $x \in V^*$. We put $\mathrm{L}(G, s) = \{w;\ w \in V^*$ and $s \Rightarrow^* w\ (R)\}$ for every $s \in S$; the language $\mathrm{L}(G, s)$ is said to be *s-generated by G*. A language is said to be *generated by a generalized grammar $G = (V, S, R)$* if it is s-generated by G for some s in S.

A generalized grammar (V, S, R) is said to be a *grammar* if the sets V, S, R are finite. A grammar (V, S, R) is called *context-free* if $y \in S$ for every $(y, x) \in R$. A context-free grammar (V, S, R) is said to be *linear* if either $x \in V^*$ or $x \in V^*SV^*$ for every $(y, x) \in R$.

2. PARTIAL ALGEBRAS

Let A be a set, $a \geqq 0$ a nonnegative integer, and f a partial mapping of $\underbrace{A \times A \times \ldots \times A}_{a \text{ times}}$ into A. This means that there exists a subset D_f of A^a and f is a mapping of D_f into A. Then f is said to be a *partial a-ary operation on A*; the number a is called the *arity* of f. An a-ary operation on A is said to be *complete* if $D_f = A^a$.

Elements of A^a are a-tuples formed of elements in A. We note that there exists exactly one 0-tuple. Thus, a complete nullary operation on A contains exactly one ordered pair $(0, c)$ where 0 is the only 0-tuple and c is an element in A. Hence a complete nullary operation on A defines a fixed element of A or a constant in A; we identify complete nullary operations on A with elements in A. A noncomplete nullary operation is the empty set.

A *partial algebra* is an ordered pair (A, F) where A is a set and F a family of partial operations on A. We put $F = (f_t)_{t \in T}$ and denote by $a(t)$ the arity and by D_t the domain of f_t. Partial algebras will be denoted by Gothic types. Hence, a partial algebra $\mathfrak{A}$ can be written as $(A, (f_t)_{t \in T})$. A partial algebra is called *complete* if all its operations are complete.

Let $\mathfrak{A} = (A, (f_t)_{t\in T})$ be a partial algebra, B a subset of A. The set B is said to be *closed* in A if it has the following property. If $t \in T$ is arbitrary, $x_1, \ldots, x_{a(t)}$ are in B, and $(x_1, \ldots, x_{a(t)}) \in D_t$, then $f_t(x_1, \ldots, x_{a(t)}) \in B$. If B is a closed set in A, then we put, for every $t \in T$, $g_t = f_t \cap B^{a(t)+1}$. Clearly, $(B, (g_t)_{t\in T})$ is a partial algebra; it is said to be a *subalgebra* of $\mathfrak{A}$. Since g_t is completely determined by f_t and B, we shall not distinguish between f_t and g_t. Furthermore, if B is closed, the partial algebra $(B, (g_t)_{t\in T})$ is completely determined by B. Hence, we shall not distinguish between closed sets in $\mathfrak{A}$ and subalgebras of $\mathfrak{A}$.

If $\mathfrak{A} = (A, (f_t)_{t\in T})$ is a partial algebra, then $\mathfrak{A}$ is closed in $\mathfrak{A}$ and the intersection of a nonempty family of sets closed in $\mathfrak{A}$ is closed in $\mathfrak{A}$. Especially, if $B \subseteq A$ is arbitrary, there exists the least closed set in $\mathfrak{A}$ including B; it is the intersection of the nonempty family consisting of all closed sets including B. This least closed subset of $\mathfrak{A}$ including B and, by abuse of language, the corresponding subalgebra will be denoted by $[B]^{\mathfrak{A}}$.

We shall characterize the set $[B]^{\mathfrak{A}}$ by means of a generalized grammar.

1. Definition. Let $\mathfrak{A} = (A, (f_t)_{t\in T})$ be a partial algebra, B a subset of A. We put

$Q = B \cup \{(,),,\} \cup \{f_t;\ t \in T\}$, $S = A - B$,

$P = \{(x, f_t);\ t \in T,\ a(t) = 0,\ x \in A,\ x = f_t\} \cup \{(x, f_t(x_1, \ldots, x_{a(t)});\ t \in T,\ a(t) > 0,$
$x \in A,\ (x_1, \ldots, x_{a(t)}) \in D_t,\ x = f_t(x_1, \ldots, x_{a(t)}))\}$,

$\mathscr{G}(\mathfrak{A}, B) = (Q, S, P)$.

Clearly, $\mathscr{G}$ is an operator assigning a generalized grammar to any partial algebra $\mathfrak{A} = (A, (f_t)_{t\in T})$ and to any subset B of A. The left side member of a production of this generalized grammar is a symbol in A, the right side member is a string formed of elements in A, commas, parentheses, and symbols f_t.

2. Proposition. *Let $\mathfrak{A} = (A, (f_t)_{t\in T})$ be a partial algebra, B a subset of A. Then $[B]^{\mathfrak{A}}$ is the set of all elements x in A such that a terminal x-derivation with respect to $\mathscr{G}(\mathfrak{A}, B)$ exists.*

Proof. We put $\mathscr{G}(\mathfrak{A}, B) = (Q, S, P)$, $C = \{x;\ x \in A,\ x \Rightarrow^* w\ (P)$ for some $w \in Q^*\}$. It is easy to see that $B \subseteq C$ and that C is closed in A. This implies that $[B]^{\mathfrak{A}} \subseteq C$.

Let $V(n)$ denote the following assertion. If $x \in A$ and a terminal x-derivation of length $\leqq n$ with respect to $\mathscr{G}(\mathfrak{A}, B)$ exists, then $x \in [B]^{\mathfrak{A}}$. By an easy induction, we prove that $V(n)$ holds for any nonnegative integer n. This means that $C \subseteq [B]^{\mathfrak{A}}$. $\square$

Since algebraic tools are elaborated for partial operations but not for relations, we master relations by decomposing them into partial operations.

3. Definition. Let A be a set, ϱ a binary relation on A. A family of mappings $(g_t)_{t\in T}$ from A to A is said to be a *ϱ-presenting family of partial unary operations* if $\bigcup_{t\in T} g_t = \varrho$.

It is easy to prove

4. Proposition. *Let A be a set, ϱ a binary relation on A, $(f_t)_{t\in T}$ an arbitrary ϱ-presenting family of partial unary operations, $(f_t)_{t\in T_0}$ a family of complete nullary operations on A, B a subset of A. Then the following assertions are equivalent.*

(i) *B is closed in $(A, (f_t)_{t\in T_0 \cup T})$.*

(ii) *The following conditions are satisfied.*

(a) *$f_t \in B$ for any $t \in T_0$.*

(b) *If $x \in B$, $(x, y) \in \varrho$, then $y \in B$.* □

5. Corollary. *Let A be a set, ϱ a binary relation on A, $(f_t)_{t\in T}$ an arbitrary ϱ-presenting family of partial unary operations, $(f_t)_{t\in T_0}$ a family of complete nullary operations on A, B a subset of A. Let D_{f_t} be the domain of f_t for every $t \in T$. Then the following assertions are equivalent.*

(i) *B is the least subset of A such that $f_t \in B$ for every $t \in T_0$ and that $x \in B$, $(x, y) \in \varrho$ imply $y \in B$.*

(ii) *B is the subalgebra of $(A, (f_t)_{t\in T})$ generated by the set $\{f_t;\ t \in T_0\}$.*

(iii) *B is the set of all elements $x \in A$ that are of the form $x = f_{t_1}(f_{t_2}(\dots(f_{t_p}(x_0))\dots))$ where $p \geqq 0$ is an integer, $x_0 = f_t$ for some $t \in T_0$, $t_i \in T$ for $1 \leqq i \leqq p$, $x_0 \in D_{f_{t_p}}$, and $f_{t_{i+1}}(\dots(f_{t_p}(x_0))\dots) \in D_{f_{t_i}}$ for $1 \leqq i \leqq p - 1$.*

Proof. The equivalence of (i) and (ii) is a consequence of 4, the equivalence of (ii) and (iii) is a consequence of 2 regarding the fact that the left side member and the right side member of any production of $\mathcal{G}(\mathfrak{A}, B)$ are two expressions for the same element in A. □

3. HETEROGENEOUS ALGEBRAS

Let S be a nonempty set, $\boldsymbol{A} = (A_s)_{s\in S}$ an indexed family of sets. Then the set A_s is said to be the *s-th component of* $\boldsymbol{A}$. A family $\boldsymbol{A} = (A_s)_{s\in S}$ is called *finite* if $\bigcup_{s\in S} A_s$ is a finite set. We denote by $\boldsymbol{O}$ the family $(O_s)_{s\in S}$ where $O_s = \emptyset$ for every $s \in S$.

Suppose that $\boldsymbol{A} = (A_s)_{s\in S}$, $\boldsymbol{B} = (B_s)_{s\in S}$ are indexed families of sets. If $B_s \subseteq A_s$ for every $s \in S$, then $\boldsymbol{B}$ is said to be a *subfamily of* $\boldsymbol{A}$; we also say that $\boldsymbol{A}$ "*includes*" $\boldsymbol{B}$.

Let S be a set, $\boldsymbol{K}$ a family of indexed families $\boldsymbol{A} = (A_s)_{s\in S}$. We put $\boldsymbol{A}(s) = A_s$ for any $\boldsymbol{A} \in \boldsymbol{K}$ and any $s \in S$ and we define the family $\boldsymbol{A}^0 = (A_s^0)_{s\in S}$ by putting $A_s^0 = \bigcap_{\boldsymbol{A}\in\boldsymbol{K}} \boldsymbol{A}(s)$ for every $s \in S$. By abuse of language, $\boldsymbol{A}^0$ is called the "*intersection*" of the family $\boldsymbol{K}$.

Let $(A_s)_{s\in S}$ be an indexed family of sets. If a is a nonnegative integer, $s = s(0)\,s(1)\ldots s(a)$ a string over S of length $a + 1$, and f a mapping of $A_{s(1)} \times A_{s(2)} \times \ldots \times A_{s(a)}$ into $A_{s(0)}$, then f is said to be an *operation on* $(A_s)_{s\in S}$ *of arity* a *with the scheme* s. The arity $a = 0$ is also admitted; if $a = 0$, then $s = s(0)$ and $f \in A_{s(0)}$ is a fixed element of $A_{s(0)}$ or a constant.

A *heterogeneous algebra* is an ordered pair $\mathfrak{A} = (A, F)$ where $A = (A_s)_{s\in S}$ is an indexed family of sets and $F = (f_t)_{t\in T}$ is an indexed family of operations with schemes on $(A_s)_{s\in S}$. For the sake of brevity, $a(t)$ will denote the arity, $s(t) = s(0, t)\,s(1, t)\ldots s(a(t), t)$ the scheme, and D_t the domain of f_t. Clearly, $D_t = A_{s(1,t)} \times \ldots \times A_{s(a(t),t)}$.

Let $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ be a heterogeneous algebra, Z a set, b a bijection of S onto Z. Let us put $B_z = A_{b^{-1}(z)}$ for any z in Z, $z(i, t) = b(s(i, t))$ for any $t \in T$ and any i, $0 \leqq i \leqq a(t)$. Then $\mathfrak{B} = ((B_z)_{z\in Z}, (f_t)_{t\in T})$ is a heterogeneous algebra where $z(t) = z(0, t)\,z(1, t)\ldots z(a(t), t)$ is the scheme of f_t for any $t \in T$. This algebra $\mathfrak{B}$ is obtained from $\mathfrak{A}$ by renaming the indices; it is said to be *equivalent to* $\mathfrak{A}$. The operations of $\mathfrak{B}$ are identical with those of $\mathfrak{A}$; only the schemes and components are expressed in a different way. Clearly, for any heterogeneous algebra $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$, we may find an equivalent algebra $\mathfrak{B} = ((B_z)_{z\in Z}, (f_t)_{t\in T})$ such that $Z \cap \bigcup_{z\in Z} B_z = \emptyset$; such a heterogeneous algebra is said to be *disjoint*.

Heterogeneous algebras can be considered as special cases of partial algebras. Indeed, let $((A_s)_{s\in S}, (f_t)_{t\in T})$ be a heterogeneous algebra. Then $(\bigcup_{s\in S} A_s, (f_t)_{t\in T})$ is a partial algebra.

Let $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ be a heterogeneous algebra, $(B_s)_{s\in S}$ a family of sets "included" in $(A_s)_{s\in S}$. The family $(B_s)_{s\in S}$ is said to be *closed in* $\mathfrak{A}$ if it has the following two properties.

1° If $t \in T$, $a(t) = 0$, $s(t) = s(0, t)$, then $f_t \in B_{s(0,t)}$.

2° If $t \in T$, $a(t) > 0$, $s(t) = s(0, t)\,s(1, t)\ldots s(a(t), t)$, $x_i \in B_{s(i,t)}$ for $i = 1, 2, \ldots, a(t)$, then $f_t(x_1, \ldots, x_{a(t)}) \in B_{s(0,t)}$.

Let $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ be a heterogeneous algebra, $(B_s)_{s\in S}$ a closed family of sets in $\mathfrak{A}$. For every $t \in T$, the set $g_t = f_t \cap (B_{s(0,t)} \times B_{s(1,t)} \times \ldots \times B_{s(a(t),t)})$ is an operation on $(B_s)_{s\in S}$ with the scheme $s(t)$. Hence, $((B_s)_{s\in S}, (g_t)_{t\in T})$ is a heterogeneous algebra; it is said to be a *subalgebra of* $\mathfrak{A}$. Since g_t is completely determined by f_t and $(B_s)_{s\in S}$, we shall not distinguish between f_t and g_t. Furthermore, if $(B_s)_{s\in S}$ is closed in $\mathfrak{A}$, the heterogeneous algebra $((B_s)_{s\in S}, (g_t)_{t\in T})$ is completely determined by $(B_s)_{s\in S}$. Hence, we shall not distinguish between closed families in $\mathfrak{A}$ and subalgebras of $\mathfrak{A}$. We shall denote the set of all subalgebras of $\mathfrak{A}$ by Sub $\mathfrak{A}$.

The set Sub $\mathfrak{A}$ is nonempty, ordered by "inclusion", and closed with respect to "intersections" in the above defined sense.

Let $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ be a heterogeneous algebra, $\boldsymbol{B} = (B_s)_{s\in S}$ an arbitrary subfamily of $(A_s)_{s\in S}$. There exists the least closed family $(C_s)_{s\in S}$ in $\mathfrak{A}$ "including" $\boldsymbol{B}$; it is the "intersection" of the nonempty family consisting of all closed families "including" $\boldsymbol{B}$. The subalgebra $((C_s)_{s\in S}, (f_t)_{t\in T})$ of $\mathfrak{A}$ is called the *subalgebra generated by the family* $\boldsymbol{B}$. We put $C_s = [\boldsymbol{B}]_s^{\mathfrak{A}}$ for every $s \in S$.

In what follows, we shall deal with mappings assigning sets of indices to elements of heterogeneous algebras. More exactly:

1. Definition. Let $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ be a heterogeneous algebra. A mapping Φ of $\bigcup_{s\in S} A_s$ into 2^S is said to be *indexing in* $\mathfrak{A}$ if it has the following properties.

(o) If $s_0 \in S$, $x \in \bigcup_{s\in S} A_s$, and $s_0 \in \Phi(x)$, then $x \in A_{s_0}$.

(a) If $t \in T$, $a(t) = 0$, $s(t) = s(0, t)$, then $s(0, t) \in \Phi(f_t)$.

(b) If $t \in T$, $a(t) > 0$, $s(t) = s(0, t)\, s(1, t) \ldots s(a(t), t)$, $s(i, t) \in \Phi(x_i)$ for some $x_i \in \bigcup_{s\in S} A_s$ and $i = 1, 2, \ldots, a(t)$, then $s(0, t) \in \Phi(f_t(x_1, \ldots, x_{a(t)}))$.

We denote by Ind $\mathfrak{A}$ the set of all indexing mappings in $\mathfrak{A}$. If $\Phi, \Psi \in \operatorname{Ind} \mathfrak{A}$ and $\Phi(x) \subseteq \Psi(x)$ for every $x \in \bigcup_{s\in S} A_s$, we put $\Phi \leqq \Psi$.

2. Definition. Let $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ be a heterogeneous algebra, $\boldsymbol{B} = (B_s)_{s\in S}$ a subfamily of $(A_s)_{s\in S}$, Φ a mapping of $\bigcup_{s\in S} A_s$ into 2^S. We put $(\boldsymbol{B}, \Phi) \in \sigma_{\mathfrak{A}}$ if (and only if), for every $x \in \bigcup_{s\in S} A_s$ and every $s_0 \in S$, the conditions $x \in B_{s_0}$, $s_0 \in \Phi(x)$ are equivalent.

3. Proposition. *Let* $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ *be a heterogeneous algebra,* $\boldsymbol{B} = (B_s)_{s\in S}$ *a subfamily of* $(A_s)_{s\in S}$, Φ *a mapping of* $\bigcup_{s\in S} A_s$ *into* 2^S *such that* $(\boldsymbol{B}, \Phi) \in \sigma_{\mathfrak{A}}$.

Then $\Phi \in \operatorname{Ind} \mathfrak{A}$ *iff* $((B_s)_{s\in S}, (f_t)_{t\in T}) \in \operatorname{Sub} \mathfrak{A}$.

Proof. Clearly, (a) is equivalent to 1° and (b) to 2°. □

By 3, $\sigma_{\mathfrak{A}}$ is a bijection of Sub $\mathfrak{A}$ onto Ind $\mathfrak{A}$. Suppose $(\boldsymbol{B}, (f_t)_{t\in T}) \in \operatorname{Sub} \mathfrak{A}$, $(\boldsymbol{C}, (f_t)_{t\in T}) \in \operatorname{Sub} \mathfrak{A}$, $\Phi \in \operatorname{Ind} \mathfrak{A}$, $\Psi \in \operatorname{Ind} \mathfrak{A}$, $(\boldsymbol{B}, \Phi) \in \sigma_{\mathfrak{A}}$, $(\boldsymbol{C}, \Psi) \in \sigma_{\mathfrak{A}}$. It is easy to see that $\boldsymbol{B}$ is "included" in $\boldsymbol{C}$ iff $\Phi \leqq \Psi$. Thus we have

4. Proposition. *If* $\mathfrak{A}$ *is a heterogeneous algebra, then* $\sigma_{\mathfrak{A}}$ *is an isomorphism of* Sub $\mathfrak{A}$ *onto* Ind $\mathfrak{A}$. □

We shall characterize the subalgebra generated by a family of sets by means of a generalized grammar. It is necessary to mention that $f_t(x_1, x_2, \ldots, x_{a(t)})$ is not the only possible way of notation for the result of applying the operation f_t to the $a(t)$-tuple $(x_1, x_2, \ldots, x_{a(t)})$; for another way of notation, see for example [3] p. 48. We shall use a little more general way.

5. Definition. Let $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ be a heterogeneous algebra, W a set disjoint from S. Let u be a mapping of T into $\bigcup_{s\in S} A_s \cup W^*$ with the following properties:

(1) If $t \in T$ and $a(t) = 0$, then $u(t) = f_t$; we put $u(0, t) = u(t)$.

(2) If $t \in T$ and $a(t) > 0$, then $u(t)$ is a string in W^* of length $a(t) + 1$, $u(t) = u(0, t) \dots u(a(t), t)$ where $u(i, t) \in W$ for $0 \leqq i \leqq a(t)$.

(3) If $t \in T$, $t' \in T$, and $f_t \neq f_{t'}$, then $u(t) \neq u(t')$.

Then u is said to be a *standard mapping of* $\mathfrak{A}$ *into* $\bigcup_{s\in S} A_s \cup W^*$.

To every standard mapping of a heterogeneous algebra $\mathfrak{A}$, we assign a standard way of notation. For every $t \in T$ and any $x_i \in A_{s(i,t)}$ with $1 \leqq i \leqq a(t)$, we write $u(0, t)\, x_1\, u(1, t) \dots x_{a(t)}\, u(a(t), t)$ for $f_t(x_1, \dots, x_{a(t)})$ whenever $a(t) > 0$.

Clearly, putting $u(0, t) = f_t($, which may be considered as an indivisible symbol, $u(i, t) =$, for $1 \leqq i < a(t)$, $u(a(t), t) =)$, we obtain our usual way of notation.

We may form composites of the operations f_t and apply them to elements in $\bigcup_{s\in S} A_s$. For example, if $t \in T$, $t' \in T$, $x'_i \in A_{s(i,t')}$ for $1 \leqq i \leqq a(t')$, $x_i \in A_{s(i,t)}$ for $i \neq 2$, $1 \leqq i \leqq a(t)$, and $s(0, t') = s(2, t)$, then $f_t(x_1, f_{t'}(x'_1, \dots, x'_{a(t')}), \dots, x_{a(t)}) \in A_{s(0,t)}$. Since f_t are partial operations in $\bigcup_{s\in S} A_s$, the composites cannot be formed mechanically respecting only the arities of the operations; also the schemes must be respected. This leads us to the definition of terms and their values. Terms are strings of symbols and their values are elements in $\bigcup_{s\in S} A_s$.

6. Definition. Let $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ be a heterogeneous algebra and u its standard mapping.

1° If $x \in \bigcup_{s\in S} A_s$, then x is a term and $v(x) = x$.

2° If $t \in T$, $a(t) = 0$, then $u(0, t)$ is a term and $v(u(0, t)) = u(0, t)$.

3° If $t \in T$, $a(t) > 0$, and if x_i is a term such that $v(x_i) \in A_{s(i,t)}$ for $1 \leqq i \leqq a(t)$, then $u(0, t)\, x_1 u(1, t) \dots x_{a(t)} u(a(t), t)$ is a term and $v(u(0, t)\, x_1 u(1, t) \dots \dots x_{a(t)} u(a(t), t)) = u(0, t)\, v(x_1)\, u(1, t) \dots v(x_{a(t)})\, u(a(t), t)$.

4° There exist no other terms than those defined either by 1° or by 2° or by 3°.

If n is a term, then $v(n)$ is said to be its value; it is an element in $\bigcup_{s\in S} A_s$ while terms are formal expressions or strings. For a set N of terms, we put $v\langle N\rangle = \{v(n); n \in N\}$.

7. Definition. Let $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ be a disjoint heterogeneous algebra, u its standard mapping into $\bigcup_{s\in S} A_s \cup W^*$, and $\boldsymbol{B} = (B_s)_{s\in S}$ a subfamily of $(A_s)_{s\in S}$. We put

$Q = \bigcup_{s\in S} B_s \cup W$,

$R_1 = \{(s, x);\ x \in B_s,\ s \in S\}$, $R_2 = \{(s(0, t), u(0, t));\ t \in T,\ a(t) = 0\}$,

$R_3 = \{(s(0, t), u(0, t)\, s(1, t) \ldots s(a(t), t)\, u(a(t), t));\ t \in T,\ a(t) > 0\}$,
$R \ = R_1 \cup R_2 \cup R_3$,
$\mathscr{K}(\mathfrak{A}, \boldsymbol{B}) = (Q, S, R)$.

Clearly, $\mathscr{K}(\mathfrak{A}, \boldsymbol{B})$ is a generalized grammar and $\mathbf{L}(\mathscr{K}(\mathfrak{A}, \boldsymbol{B}), s)$ is a set of strings for every $s \in S$. We prove that these strings are terms whose values form the set $[\boldsymbol{B}]_s^{\mathfrak{A}}$.

8. Proposition. *Let* $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ *be a disjoint heterogeneous algebra,* $\boldsymbol{B}$ *a subfamily of* $(A_s)_{s\in S}$. *Then* $v\langle \mathbf{L}(\mathscr{K}(\mathfrak{A}, \boldsymbol{B}), s)\rangle = [\boldsymbol{B}]_s^{\mathfrak{A}}$ *for every* $s \in S$ *where* v *and* $\mathscr{K}(\mathfrak{A}, \boldsymbol{B})$ *are constructed by means of the same standard mapping.*

Proof. Let $V(n)$ denote the following assertion. If $s \in S$, $x \in \mathbf{L}(\mathscr{K}(\mathfrak{A}, \boldsymbol{B}), s)$ and there exists an s-derivation of x of length $\leqq n$ with respect to $\mathscr{K}(\mathfrak{A}, \boldsymbol{B})$, then $v(x) \in [\boldsymbol{B}]_s^{\mathfrak{A}}$.

It is easy to see that $V(1)$ holds.

Let $m > 1$ be an integer and suppose that $V(n)$ holds for every n, $1 \leqq n < m$. Suppose $x \in \mathbf{L}(\mathscr{K}(\mathfrak{A}, \boldsymbol{B}), s)$, let $(z_i)_{i=0}^m$ be an s-derivation of x of length m with respect to $\mathscr{K}(\mathfrak{A}, \boldsymbol{B})$. Then $z_0 = s$ and there exists $t \in T$ such that $s = s(0, t)$, $(s(0, t), u(0, t)\, s(1, t)\, u(1, t) \ldots s(a(t), t)\, u(a(t), t)) \in R_3$, $z_1 = u(0, t)\, s(1, t)\, u(1, t) \ldots s(a(t), t)\, u(a(t), t)$. Then, for any $i = 1, 2, \ldots, a(t)$, there exists $x_i \in \mathbf{L}(\mathscr{K}(\mathfrak{A}, \boldsymbol{B}), s(i, t))$ and an $s(i, t)$-derivation of x_i of length less than m with respect to $\mathscr{K}(\mathfrak{A}, \boldsymbol{B})$ such that $x = u(0, t)\, x_1 u(1, t) \ldots x_{a(t)} u(a(t), t)$. This implies that $v(x_i) \in [\boldsymbol{B}]_{s(i,t)}^{\mathfrak{A}}$ for $i = 1, 2, \ldots, a(t)$. By definition, we obtain $v(x) = u(0, t)\, v(x_1)\, u(1, t) \ldots \ldots v(x_{a(t)})\, u(a(t), t) \in [\boldsymbol{B}]_{s(0,t)}^{\mathfrak{A}} = [\boldsymbol{B}]_s^{\mathfrak{A}}$.

Thus, $V(m)$ holds.

By induction, it follows that $v\langle \mathbf{L}(\mathscr{K}(\mathfrak{A}, \boldsymbol{B}), s)\rangle \subseteq [\boldsymbol{B}]_s^{\mathfrak{A}}$ for every $s \in S$.

We now easily prove that $(v\langle \mathbf{L}(\mathscr{K}(\mathfrak{A}, \boldsymbol{B}), s)\rangle)_{s\in S}$ is a closed family in $\mathfrak{A}$ "including" $\boldsymbol{B}$. This implies that $([\boldsymbol{B}]_s^{\mathfrak{A}})_{s\in S}$ is "included" in $(v\langle \mathbf{L}(\mathscr{K}(\mathfrak{A}, \boldsymbol{B}), s)\rangle)_{s\in S}$.

Thus, $v\langle \mathbf{L}(\mathscr{K}(\mathfrak{A}, \boldsymbol{B}), s)\rangle = [\boldsymbol{B}]_s^{\mathfrak{A}}$ for every $s \in S$. □

9. Corollary. *Let* $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ *be a disjoint heterogeneous algebra,* $\boldsymbol{B}$ *a subfamily of* $(A_s)_{s\in S}$, $s_0 \in S$, *and* $x \in [\boldsymbol{B}]_{s_0}^{\mathfrak{A}}$. *Then there exists a finite subfamily* $\boldsymbol{C}$ *of* $\boldsymbol{B}$ *such that* $x \in [\boldsymbol{C}]_{s_0}^{\mathfrak{A}}$.

If $v(t) = x$, there exists an s_0-derivation of t with respect to $\mathscr{K}(\mathfrak{A}, \boldsymbol{B}) = (Q, S, R)$ where only a finite number of productions of the form (s, r) with $s \in S$ and $r \in \bigcup_{s\in S} B_s$ has been used. Let C denote the finite set of all these elements $r \in \bigcup_{s\in S} B_s$. We put $C_s = C \cap B_s$ for every $s \in S$, $\boldsymbol{C} = (C_s)_{s\in S}$. Clearly, $x \in [\boldsymbol{C}]_{s_0}^{\mathfrak{A}}$. □

If $((A_s)_{s\in S}, (f_t)_{t\in T})$ is a heterogeneous algebra and S has exactly one element s, then we put $A = A_s$ and $(A, (f_t)_{t\in T})$ is a complete algebra. Hence, a complete algebra is a special case of a partial algebra and of a heterogeneous algebra; closed subsets of the partial algebra coincide with closed families of the heterogeneous algebra. Thus, we obtain

10. Corollary. *Let* $\mathfrak{A} = (A, (f_t)_{t\in T})$ *be a complete algebra,* $B \subseteq A$. *If* $x \in [B]^{\mathfrak{A}}$, *then* $x \in [C]^{\mathfrak{A}}$ *for a finite subset* C *of* B. □

See [1], [23], [3] p. 81.

11. Definition. A heterogeneous algebra $((A_s)_{s\in S}, (f_t)_{t\in T})$ is said to be *of simple structure if* $t \in T$, $t' \in T$, $t \neq t'$, $s(t) = s(t')$ imply $f_t \neq f_{t'}$.

12. Proposition. *For every heterogeneous algebra* $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ *there exists a heterogeneous algebra of simple structure* $\mathfrak{B} = ((A_s)_{s\in S}, (g_k)_{k\in K})$ *such that both have the same closed subfamilies.*

Proof. For any $t \in T$, $t' \in T$, we put $t \sim t'$ if $s(t) = s(t')$, $f_t = f_{t'}$. Clearly, $\sim$ is an equivalence on T. Let $K \subseteq T$ be a set such that $K \cap C$ has exactly one element for every $C \in T/\sim$. We put $g_k = f_k$, $z(k) = s(k)$ for every $k \in K$ where $z(k)$ is the scheme of g_k.

If $(C_s)_{s\in S}$ is closed in $\mathfrak{A}$, it is closed in $\mathfrak{B}$. Let, conversely, $(C_s)_{s\in S}$ be closed in $\mathfrak{B}$. Let $t \in T$ be arbitrary, $s(t) = s(0, t) \dots s(a(t), t)$, $x_i \in C_{s(i,t)}$ for $i = 1, 2, \dots, a(t)$. Suppose $k \in K$, $k \sim t$. Then $s(t) = z(k)$, $f_t = g_k$ which implies $f_t(x_1, \dots, x_{a(t)}) = g_k(x_1, \dots, x_{a(t)}) \in C_{z(0,k)} = C_{s(0,t)}$. Hence, $(C_s)_{s\in S}$ is closed in $\mathfrak{A}$. □

4. CONTEXT-FREE ALGEBRAS

1. Definition. A heterogeneous algebra $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ is said to be *context free* if the following conditions are satisfied.

(1) The sets S, T are finite.

(2) There exists a finite set V such that $A_s = V^*$ for any $s \in S$.

(3) There exists a standard mapping u of $\mathfrak{A}$ into $\bigcup_{s\in S} A_s \cup (V^*)^* = (V^*)^*$ in such a way that for any $t \in T$, $f_t(x_1, \dots, x_{a(t)}) = u(0, t)\, x_1 u(1, t) \dots x_{a(t)} u(a(t), t)$ where the expression $u(0, t)\, x_1 u(1, t) \dots x_{a(t)} u(a(t), t)$ means the catenation of strings $u(0, t), x_1, u(1, t), \dots, x_{a(t)}, u(a(t), t)$ in the given order. This standard mapping u will be said to be *principal* and the symbol u will be reserved for the principal mapping of the algebra.

Terms and generalized grammars of a heterogeneous algebra (see 3.6 and 3.7) are defined by means of an arbitrary standard mapping. By a term and a generalized grammar of a context-free algebra, we always mean a term and a generalized grammar defined by means of the principal mapping.

By definition, a term of a context-free algebra is a string in $(V^*)^*$. A term that is a string of length 1 in $(V^*)^*$ coincides with its value by 3.6 1° and 2°. Using 3° of 3.6 and (3), we prove by an easy induction that the value of an arbitrary term which is a string $z_1 z_2 \dots z_p$ in $(V^*)^*$ is obtained by catenation of the elements $z_1, z_2, \dots, z_p$

where these elements are strings in V^*. Since our way of notation does not distinguish between a string of strings and the catenation of these strings, we obtain, by 3.8,

2. Proposition. *Let* $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ *be a disjoint context-free algebra. Then* $[\boldsymbol{O}]_s^{\mathfrak{A}} = \mathbf{L}(\mathscr{K}(\mathfrak{A}, \boldsymbol{O}), s)$ *for every* $s \in S$. $\square$

Regarding 3.7, we see that, for a context-free algebra $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ with $A_s = V^*$, we choose $W = V^*$, and, hence, $Q = V^*$. Further, if $\boldsymbol{B} = \boldsymbol{O}$, we obtain $R_1 = \emptyset$ and, hence, $R = R_2 \cup R_3 = \{(s(0, t), u(0, t)\, s(1, t)\, u(1, t) \ldots s(a(t), t)\,.\,u(a(t), t)); t \in T\}$. Thus, $\mathscr{K}(\mathfrak{A}, \boldsymbol{O}) = (V^*, S, R)$. Therefore, if we take V instead of V^* in the triple (V^*, S, R), we obtain a context-free grammar. To be precise, we have

3. Definition. Let $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ be a disjoint context-free algebra with $A_s = V^*$ for every $s \in S$. Let $\mathscr{K}(\mathfrak{A}, \boldsymbol{O}) = (V^*, S, R)$. We put $\mathscr{L}(\mathfrak{A}) = (V, S, R)$.

Hence, $\mathscr{L}$ is an operator assigning a context-free grammar to any disjoint context-free algebra.

As a consequence, we obtain

4. Corollary. *Let* $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ *be a disjoint context-free algebra. Then* $[\boldsymbol{O}]_s^{\mathfrak{A}} = \mathbf{L}(\mathscr{L}(\mathfrak{A}), s)$ *for any* $s \in S$. $\square$

A context-free algebra of simple structure may be always reconstructed from a context-free grammar as follows from the following

5. Definition. Let $G = (V, S, P)$ be a context-free grammar. We put $A_s = V^*$ for every $s \in S$. If $t \in P$, then $t = (s(0, t), u(0, t)\, s(1, t) \ldots s(a(t), t)\, u(a(t), t))$ for an integer $a(t) \geqq 0$, $s(0, t), s(1, t), \ldots, s(a(t), t)$ in S and $u(0, t), u(1, t), \ldots, u(a(t), t)$ in V^*. For every $t \in P$, we put $f_t(x_1, \ldots, x_{a(t)}) = u(0, t)\, x_1 u(1, t) \ldots x_{a(t)} u(a(t), t)$ where $x_1, \ldots, x_{a(t)}$ are arbitrarily chosen in V^* and we define the scheme $s(t)$ of f_t to be $s(0, t)\, s(1, t) \ldots s(a(t), t)$. Then $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in P})$ is a disjoint context-free algebra of simple structure. We put $\mathscr{A}(G) = \mathfrak{A}$.

6. Proposition. *If* G *is a context-free grammar, then* $\mathscr{L}(\mathscr{A}(G)) = G$. $\square$

7. Corollary. *Let* $G = (V, S, R)$ *be a context-free grammar. Then* $[\boldsymbol{O}]_s^{\mathscr{A}(G)} = \mathbf{L}(G, s)$ *for every* $s \in S$.

Indeed, by 4 and 6, we have $[\boldsymbol{O}]_s^{\mathscr{A}(G)} = \mathbf{L}(\mathscr{L}(\mathscr{A}(G)), s) = \mathbf{L}(G, s)$. $\square$

Regarding 4 and 7, we have

8. Corollary ([12], [6] p. 299). *A language is generated by a context-free grammar iff it is a component of the least subalgebra of a context-free algebra.* $\square$

Disjoint context-free algebras of simple structure may be described in the form of context-free grammars. There exists one more way of describing them.

9. Definition. Let $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ be a disjoint context-free algebra of simple structure with $A_s = V^*$ for every $s \in S$. We put

$N = \{u(t);\ t \in T\}$,
$\chi(u) = \{s(t);$ there exists $t \in T$ with $u = u(t)\}$ for every $u \in N$,
$L = \{u;\ u \in N,\ |u| = 1\}$, $M = \{u;\ u \in N,\ |u| > 1\}$,
$\varphi = \chi \upharpoonright L$, $\psi = \chi \upharpoonright M$ (where $f \upharpoonright P$ denotes the restriction of the mapping f to the subset P of its domain),
$\mathcal{H}(\mathfrak{A}) = (V, L, M, S, \varphi, \psi)$.

Hence, a disjoint context-free algebra of simple structure is expressed in the form of a 6-tuple of sets. This leads us to the following definition.

10. Definition. Let V, S be finite disjoint sets, L a finite subset of V^*, M a finite set of strings of length >1 formed of strings over V^*, φ a mapping assigning, to every $x \in L$, a subset of S, ψ a mapping assigning, to every string of length $n > 1$ over V^*, a finite set of strings of length n over S. Then the ordered 6-tuple $(V, L, M, S, \varphi, \psi)$ is called a *labelled multicontextual grammar*.

We have seen that the operator $\mathcal{H}$ assings a labelled multicontextual grammar to every disjoint context-free algebra of simple structure. We now define an operator assigning a disjoint context-free algebra of simple structure to every labelled multicontextual grammar.

11. Definition. Let $H = (V, L, M, S, \varphi, \psi)$ be a labelled multicontextual grammar. We put

$U = L \cup M$; any $u \in U$ is a string $u(0)\, u(1) \dots u(a)$ over V^*,
$\gamma = \varphi \cup \psi$,
$T = \{(u, s);\ u \in U,\ s \in \gamma(u)\}$.

If $t = (u, s) \in T$, then we put $a(t) = |u| - 1 = |s| - 1$, $s(t) = s$, $u(t) = u$, $f_t(x_1, \dots, x_{a(t)}) = u(0, t)\, x_1 u(1, t) \dots x_{a(t)} u(a(t), t)$ for any $x_1, x_2, \dots, x_{a(t)}$ in V^* where $u(0, t), u(1, t), \dots, u(a(t), t)$ are members of the string $u(t)$.

Furthermore, we put $A_s = V^*$ for any $s \in S$, $\mathcal{B}(H) = ((A_s)_{s\in S}, (f_t)_{t\in T})$.

Clearly, $\mathcal{B}$ is an operator assigning a disjoint context-free algebra of simple structure to every labelled multicontextual grammar.

Let $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ be a context-free algebra of simple structure with $A_s = V^*$ for any $s \in S$. Then any f_t is defined by means of $u(t)$. Thus, $t \in T$, $t' \in T$, $t \neq t'$, $s(t) = s(t')$ imply $u(t) \neq u(t')$. Therefore, the mapping assigning the ordered pair $(u(t), s(t))$ to any $t \in T$ is a bijection. Hence, without loss of generality, we may suppose that T is a finite set of ordered pairs (u, s) where u is a string over V^* and s a string over S, both of the same length. For any $t = (u, s) \in T$, we have $|u| = a(t) + 1 = |s|$, $u(t) = u$, $s(t) = s$, $f_t(x_1, \dots, x_{a(t)}) = u(0, t)\, x_1 u(1, t) \dots x_{a(t)} u(a(t), t)$ where $x_1, \dots, x_{a(t)}$ are arbitrary strings over V and $u(0, t)\, u(1, t) \dots u(a(t), t) = u(t)$.

We prove

12. Proposition. *Let* $\mathfrak{A}$ *be a disjoint context-free algebra of simple structure. Then* $\mathscr{B}(\mathscr{H}(\mathfrak{A})) = \mathfrak{A}$.

Proof. We put $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ where $A_s = V^*$ for every $s \in S$. We suppose that T is a finite set of ordered pairs (u, s) where u is a string over V^*, s is a string over S, and $|u| = |s|$. Hence, for any $t = (u, s) \in T$, we have $|u| = a(t) + 1 = |s|$, $u(t) = u$, $s(t) = s$, $f_t(x_1, \ldots, x_{a(t)}) = u(0, t)\, x_1 u(1, t) \ldots x_{a(t)} u(a(t), t)$ where $u(0, t) \ldots \ldots u(a(t), t) = u(t)$. Further, we put $\mathscr{H}(\mathfrak{A}) = (V', L, M, S', \varphi, \psi)$, $\mathscr{B}(\mathscr{H}(\mathfrak{A})) = ((A'_s)_{s\in S''}, (g_t)_{t\in T'})$. By 9, we have $V' = V$, $S' = S$, $N = \{u(t);\ t \in T\} = \{u;\ (u, s) \in T\}$, $L = \{u;\ (u, s) \in T,\ |u| = 1\}$, $M = \{u;\ (u, s) \in T,\ |u| > 1\}$, $\chi(u) = \{s(t);\ u = u(t)$ for some $t \in T\} = \{s;\ (u, s) \in T\}$, $\varphi(u) = \{s;\ (u, s) \in T\}$ for $|u| = 1$, $\psi(u) = \{s;\ (u, s) \in T\}$ for $|u| > 1$. By 11, this implies $S'' = S' = S$, $A'_s = V^* = A_s$ for every $s \in S$, $U = L \cup M = N = \{u(t);\ t \in T\}$, $\gamma = \varphi \cup \psi = \chi$, $T' = \{(u, s);\ u \in U,\ s \in \gamma(u)\} = \{(u, s);\ u \in N,\ s \in \chi(u)\} = T$. For every $(u, s) \in T' = T$, $|u| - 1 = |s| - 1$ is the arity of g_t, $s(t)$ is its scheme and $g_t(x_1, \ldots, x_{a(t)}) = u(0, t)\, x_1 u(1, t) \ldots \ldots x_{a(t)} u(a(t)) = f_t(x_1, \ldots, x_{a(t)})$. Hence, for every $t \in T' = T$, we have $g_t = f_t$ and they both have the same scheme. Thus, $\mathscr{B}(\mathscr{H}(\mathfrak{A})) = ((A'_s)_{s\in S''}, (g_t)_{t\in T'}) = ((A_s)_{s\in S}, (f_t)_{t\in T}) = \mathfrak{A}$. □

A labelled multicontextual grammar has been defined as an ordered 6-tuple of sets. The language generated by such a grammar is defined as follows.

13. Definition. Let $H = (V, L, M, S, \varphi, \psi)$ be a labelled multicontextual grammar, Φ the smallest indexing mapping on $\mathscr{B}(H)$. Then, for every $s \in S$, the language s-generated by H is defined to be $\{x;\ x \in V^*,\ s \in \Phi(x)\}$ and is denoted by $\mathbf{L}(H, s)$. A language is said to be *generated by H* if it is s-generated by H for some $s \in S$.

By 3.2 and 3.4, we obtain

14. Proposition. *Let* $H = (V, L, M, S, \varphi, \psi)$ *be a labelled multicontextual grammar. Then, for any* $s \in S$, *we have* $\mathbf{L}(H, s) = [\boldsymbol{O}]_s^{\mathscr{B}(H)}$. □

By 12 and 14, we have

15. Corollary. *Let* $\mathfrak{A} = ((A_s)_{s\in S}, (f_t)_{t\in T})$ *be a disjoint context-free algebra of simple structure,* $s \in S$. *Then* $[\boldsymbol{O}]_s^{\mathfrak{A}} = \mathbf{L}(\mathscr{H}(\mathfrak{A}), s)$. □

Regarding 14, 15, and 3.12, we obtain

16. Corollary. *A language is generated by a labelled multicontextual grammar iff it is a component of the least subalgebra of a context-free algebra.* □

1. Definition. A labelled multicontextual grammar $H = (V, L, M, S, \varphi, \psi)$ is called a *labelled contextual grammar* if $|u| = 2$ for every $u \in M$.

Let $H = (V, L, M, S, \varphi, \psi)$ be a labelled contextual grammar. By 4.11, $\mathscr{B}(H) = ((A_s)_{s \in S}, (f_t)_{t \in T})$ has the property that $t \in T$ implies either $a(t) = 0$, $u(t) = u(0, t)$, $s(t) = s(0, t)$, $f_t = u(0, t)$ or $a(t) = 1$, $u(t) = u(0, t)\, u(1, t)$, $s(t) = s(0, t)\, s(1, t)$, $f_t(x) = u(0, t)\, x u(1, t)$ for any $x \in V^*$. By 4.3, we obtain $\mathscr{L}(\mathscr{B}(H)) = (V, S, R)$ where $R = \{(s(0, t), u(0, t));\ t \in T,\ a(t) = 0\} \cup \{(s(0, t), u(0, t)\, s(1, t)\, u(1, t));\ t \in T,\ a(t) = 1\}$. Thus, $\mathscr{L}(\mathscr{B}(H))$ is a linear grammar.

On the other hand, if $G = (V, S, R)$ is a linear grammar, then, by 4.5, $\mathscr{A}(G) = ((A_s)_{s \in S}, (f_t)_{t \in T})$ has the property that $a(t) = 0$ or $a(t) = 1$ for any $t \in T$ and, hence, $\mathscr{H}(\mathscr{A}(G))$ is a labelled contextual grammar by 4.9. Thus

2. Proposition. (i) *If H is a labelled contextual grammar, $\mathscr{L}(\mathscr{B}(H))$ is a linear grammar.*

(ii) *If G is a linear grammar, $\mathscr{H}(\mathscr{A}(G))$ is a labelled contextual grammar.* □

3. Proposition. (i) *For any labelled contextual grammar $H = (V, L, M, S, \varphi, \psi)$ and any $s \in S$, the assertion $\mathbf{L}(H, s) = \mathbf{L}(\mathscr{L}(\mathscr{B}(H)), s)$ holds.*

(ii) *For any linear grammar $G = (V, S, R)$ and any $s \in S$, the assertion $\mathbf{L}(G, s) = \mathbf{L}(\mathscr{H}(\mathscr{A}(G)), s)$ holds.*

Proof. The first assertion follows by 4.14 and 4.4, the second by 4.7, 4.12, 4.14. □

By 2, we obtain

4. Corollary. *A language is generated by a linear grammar iff it is generated by a labelled contextual grammar.* □

5. Definition. Let $H = (V, L, M, S, \varphi, \psi)$ be a labelled multicontextual grammar with card $S = 1$. Then it is said to be a *multicontextual grammar*. By 1, we know what a contextual grammar is.

It is easy to see that φ, ψ mean no restrictions if card $S = 1$. Thus, they may be omitted as well as S.

6. Agreement. A multicontextual grammar is denoted by (V, L, M), a contextual grammar by (V, L, C).

By 3, we obtain

7. Proposition. (i) *A language is generated by a multicontextual grammar iff it is generated by a context-free grammar having exactly one nonterminal symbol.*

(ii) *A language is generated by a contextual grammar iff it is generated by a linear grammar having exactly one nonterminal symbol.* □

Hence, we have characterized the class of all context-free and of all linear languages L with the property Var $L = 1$ where Var is a complexity measure studied by several authors (see, for example, [7], [8], [17]).

8. Definition. A contextual grammar with choice is an ordered quadruple (V, L, C, φ) where (V, L, C) is a contextual grammar and φ is a mapping of V^* into 2^C. The language $\mathbf{L}(V, L, C, \varphi)$ generated by (V, L, C, φ) is defined to be the smallest set $K \subseteq V^*$ such that

1° $L \subseteq K$;

2° $x \in K$, $(u, v) \in \varphi(x)$ imply $uxv \in K$.

9. Definition. A linear grammar with choice is an ordered quadruple $(V, \{s\}, P, \varphi)$ where $(V, \{s\}, P)$ is a linear grammar with exactly one nonterminal s and φ is a mapping of V^* into 2^C, $C = \{(u, v); (s, usv) \in P\}$.

Any s-derivation of $w \in V^*$ with respect to $(V, \{s\}, P)$ is of the form $(z_i)_{i=0}^p$ where $p \geqq 0$, $s = z_0$, $z_p = w$, $z_i = u_1 \dots u_i s v_i \dots v_1$ for $1 \leqq i \leqq p - 1$ with $(s, u_i s v_i) \in P$, $z_p = u_1 \dots u_{p-1} w_p v_{p-1} \dots v_1$ with $(s, w_p) \in P$. Putting $u_i \dots u_{p-1} w_p v_{p-1} \dots v_i = w_i$ for $i = 1, 2, \dots, p - 1$, we obtain $u_i w_{i+1} v_i = w_i$ for $i = 1, 2, \dots, p - 1$, $s \Rightarrow^* w_i$ for $i = 1, 2, \dots, p$.

An s-derivation $(z_i)_{i=0}^p$ with respect to $(V, \{s\}, P)$ is said to be *φ-restricted* if $(u_i, v_i) \in \varphi(w_{i+1})$ for $i = 1, 2, \dots, p - 1$. We define the language $\mathbf{L}(V, \{s\}, P, \varphi)$ generated by $(V, \{s\}, P, \varphi)$ to be the set $\{w; w \in V^*$ and there exists at least one φ-restricted s-derivation of w with respect to $(V, \{s\}, P)\}$.

Hence, a linear grammar with choice $(V, \{s\}, P, \varphi)$ may be considered as a linear grammar $(V, \{s\}, P)$ whose derivations are regulated by means of the mapping φ. This regulation is different from that described in [22] Chapter V because the production chosen at the i th step of the derivation depends – roughly speaking – on the subsequent steps of the derivation.

We prove

10. Proposition. *A language is generated by a contextual grammar with choice iff it is generated by a linear grammar with choice.*

Proof. Let $H = (V, L, C, \varphi)$ be a contextual grammar with choice. We put $T_0 = L$, $f_t = t$ for every $t \in T_0$, $\varrho = \{(x, y); x \in V^*, y = uxv, (u, v) \in \varphi(x)\}$. For every $(u, v) \in C$, we put $D_{(u,v)} = \{x; (u, v) \in \varphi(x)\}$, $f_{(u,v)}(x) = uxv$ for every $x \in D_{(u,v)}$. Clearly, $D_{f_{(u,v)}} = D_{(u,v)}$, where $D_{f_{(u,v)}}$ is the domain of $f_{(u,v)}$, and $(f_{(u,v)})_{(u,v) \in C}$ is a ϱ-presenting family of partial unary operations. By 8, the language B generated by H is defined by means of the condition (i) of 2.5 which is equivalent with (iii) of 2.5. Clearly, for any $(u, v) \in C$ and any $x \in V^*$, the condition $x \in D_{f_{(u,v)}}$ is equivalent with $(u, v) \in \varphi(x)$. Thus, (iii) of 2.5 is equivalent with the following condition.

(iv) B is the set of all $x \in V^*$ that are of the form $x = u_1 \ldots u_p x_0 v_p \ldots v_1$ where $x_0 \in L$, $p \geqq 0$, $(u_i, v_i) \in C$ for $1 \leqq i \leqq p$, $(u_p, v_p) \in \varphi(x_0)$, $(u_i, v_i) \in \varphi(u_{i+1} \ldots \ldots u_p x_0 v_p \ldots v_{p+1})$ for $1 \leqq i \leqq p - 1$.

Clearly, (iv) is equivalent to the following condition.

(v) B is the set of all $x \in V^*$ having a φ-restricted s-derivation with respect to $(V, \{s\}, P)$ where $P = \{(s, w);\ w \in L\} \cup \{(s, usv);\ (u, v) \in C\}$. Clearly, the correspondence of the class of all contextual grammars with choice to the class of all linear grammars with choice we have constructed is a bijection. This implies the assertion of 10. □

11. Definition (see [14]). Let (V, L, C, φ) be a contextual grammar with choice, ψ a mapping of V^* into 2^C such that $\psi(x) \subseteq \varphi(x)$ for any $x \in V^*$, $\mathbf{L}(V, L, C, \varphi)$ the language generated by (V, L, C, φ). Then the ordered quintuple (V, L, C, φ, ψ) is said to be a *contextual grammar with double choice*. The language $\mathbf{L}(V, L, C, \varphi, \psi)$ generated by (V, L, C, φ, ψ) is defined to be $\{uxv;\ x \in \mathbf{L}(V, L, C, \varphi),\ (u, v) \in \psi(x)\}$.

12. Definition. Let $(V, \{s\}, P, \varphi)$ be a linear grammar with choice, ψ a mapping of V^* into 2^C such that $\psi(x) \subseteq \varphi(x)$ for any $x \in V^*$, $\mathbf{L}(V, \{s\}, P, \varphi)$ the language generated by $(V, \{s\}, P, \varphi)$. Then the ordered quintuple $(V, \{s\}, P, \varphi, \psi)$ is said to be a *linear grammar with double choice*. The language $\mathbf{L}(V, \{s\}, P, \varphi, \psi)$ generated by $(V, \{s\}, P, \varphi, \psi)$ is defined to be $\{uxv;\ x \in \mathbf{L}(V, \{s\}, P, \varphi),\ (u, v) \in \psi(x)\}$.

Hence, a linear grammar with double choice $(V, \{s\}, P, \varphi, \psi)$ may be considered as a linear grammar $(V, \{s\}, P)$ whose derivations are regulated by means of the mappings φ, ψ, i.e., in a way different from the regulations described in [22] Chapter V.

It is easy to prove (see [18])

13. Proposition. *Any language over a finite vocabulary may be generated by a contextual grammar with double choice and by a linear grammar with double choice.* □

We have dealt with contextual grammars with choice and with double choice that are equivalent to linear grammars with choice and with double choice, respectively, i.e., to linear grammars with restricted derivations. The restriction of derivations increases the generative capacity of linear grammars; for linear grammars with choice, this follows from [15] 2.12, for linear grammars with double choice from 13.

In the literature there are two other types of contextual grammars that are equivalent to linear grammars of special kind: regular and programmed contextual grammars.

14. Definition (see [15]). Let (V, L, C, φ) be a contextual grammar with choice such that $\varphi(x) = \varphi(y)$, $(u, v) \in \varphi(x)$ imply $\varphi(uxv) = \varphi(uyv)$. Then (V, L, C, φ) is called *regular*.

15. Definition. Let (V, L, C, φ) be a regular contextual grammar. We put

$S = \{\varphi(x);\ x \in V^*\}$,

$A_s = V^*$ for any $s \in S$,

$T_0 = L$,

$a(t) = 0$, $f_t = t$, $s(0, t) = \varphi(t)$ for any $t \in T_0$,

$T_1 = \{(s, (u, v));\ s \in S,\ (u, v) \in s\}$.

If $t = (s, (u, v)) \in T_1$, we put $a(t) = 1$, $f_t(w) = uwv$ for any $w \in V^*$, $s(1, t) = s$, $s(0, t) = \varphi(uxv)$ where x is an arbitrary string with $\varphi(x) = s$.

Further, we set

$T = T_0 \cup T_1$,

$\mathscr{R}(V, L, C, \varphi) = ((A_s)_{s \in S}, (f_t)_{t \in T})$.

Then the heterogeneous algebra $\mathscr{R}(V, L, C, \varphi)$ (which need not be disjoint) is called *regular*.

Since $\varphi(x) \subseteq C$ for any $x \in V^*$ and since C is finite, S is finite as well. This implies that T_1 is also finite. Hence

16. Proposition. *If (V, L, C, φ) is a regular contextual grammar, then $\mathscr{R}(V, L, C, \varphi)$ is a context-free algebra.* □

17. Proposition. *Let $R = (V, L, C, \varphi)$ be a regular contextual grammar, $\mathscr{R}(R) = ((A_s)_{s \in S}, (f_t)_{t \in T})$. Then $\mathbf{L}(R) = \bigcup_{s \in S} [O]_s^{\mathscr{R}(R)}$.*

Proof. Let $\mathfrak{A}$ be a disjoint heterogeneous algebra equivalent with $\mathscr{R}(R)$. For the sake of simplicity, we denote object of $\mathfrak{A}$ by the same symbols as corresponding objects of $\mathscr{R}(R)$.

We put $\mathscr{L}(\mathfrak{A}) = (V, S, P)$. By 4.3 and 15, we have $P = \{(s(0, t), u(0, t));\ t \in T_0\} \cup \cup \{(s(0, t), u(0,t)\, s(1, t)\, u(1, t));\ t \in T_1\} = \{(\varphi(x), x);\ x \in L\} \cup \{(\varphi(uxv), u\, \varphi(x)\, v);\ x \in V^*,\ (u, v) \in \varphi(x)\}$.

Then, for an arbitrary $x \in V^*$, the following assertions are equivalent.

(α) $x \in L$.

(β) $(\varphi(x), x) \in P$.

Let x, x', u, v be in V^*, $x' = uxv$. Then the following assertions are equivalent.

(γ) $(u, v) \in \varphi(x)$.

(δ) $(\varphi(x'), u\, \varphi(x)\, v) \in P$.

Let $p \geqq 0$ be an integer, $x_0, x_1, \ldots, x_p$ strings in V^*. By the above equivalence, it follows that the following assertions (ε), (η) are equivalent.

(ε) $x_0 \in L$ and there exist $(u_i, v_i) \in V^* \times V^*$ such that $x_{i+1} = u_i x_i v_i$, $(u_i, v_i) \in \in \varphi(x_i)$ for $i = 0, 1, \ldots, p - 1$.

(η) $(\varphi(x_0), x_0) \in P$ and there exist $(u_i, v_i) \in V^* \times V^*$ such that $x_{i+1} = u_i x_i v_i$, $(\varphi(x_{i+1}), u_i\, \varphi(x_i)\, v_i) \in P$ for $i = 0, 1, \ldots, p - 1$.

Clearly, (ε) means that $x_p \in \mathbf{L}(V, L, C, \varphi)$ and (η) means that there exists a $\varphi(x_p)$-derivation of x_p in P, i.e., $x_p \in [O]_{\varphi(x_p)}^{\mathscr{R}(R)}$ by 4.4. □

18. Definition. Let $R = (V, L, C, \varphi)$ be a regular contextual grammar, $\mathfrak{A}$ a disjoint heterogeneous algebra equivalent to $\mathscr{R}(R)$. Then the linear grammar $\mathscr{L}(\mathfrak{A})$ is said to be *regular*.

19. Corollary. *Let $R = (V, L, C, \varphi)$ be a regular contextual grammar, $\mathfrak{A}$ a disjoint heterogeneous algebra equivalent to $\mathscr{R}(R)$, $\mathscr{L}(\mathfrak{A}) = (V, S, P)$. Then* $\mathbf{L}(R) = \bigcup_{s\in S} \mathbf{L}(\mathscr{L}(\mathfrak{A}), s)$.

Proof. This is a consequence of 17 and 4.4. □

Hence, we have proved that a language is generated by a regular contextual grammar iff it is constructed by means of a regular linear grammar in the sense of 19.

20. Definition (see [18]). Let (V, L, C) be a contextual grammar, φ a mapping of $L \cup C$ into 2^C. Then the ordered quadruple (V, L, C, φ) is said to be a *programmed contextual grammar*. We define the language $\mathbf{L}(V, L, C, \varphi)$ generated by (V, L, C, φ) to be the set $\{u_p u_{p-1} \dots u_1 x v_1 \dots v_{p-1} v_p;\ p \geqq 0,\ x \in L,\ (u_1, v_1) \in \varphi(x),\ (u_i, v_i) \in \varphi(u_{i-1}, v_{i-1}) \text{ for } i = 2, \dots, p\}$.

21. Definition. Let (V, L, C, φ) be a programmed contextual grammar. We put

$S = L \cup C$,

$T = L \cup \{(x, (u, v));\ x \in L,\ (u, v) \in \varphi(x)\} \cup \{((u_1, v_1), (u, v));\ (u_1, v_1) \in C,\ (u, v) \in \varphi(u_1, v_1)\}$.

For every $t \in L$, we put $a(t) = 0$, $f_t = t$, $s(0, t) = t$. For every $x \in L$ and every $(u, v) \in \varphi(x)$, we put $t = (x, (u, v))$, $a(t) = 1$, $f_t(w) = uwv$ where $w \in V^*$ is arbitrary, $s(0, t) = (u, v)$, $s(1, t) = x$.

For every $(u_1, v_1) \in C$ and every $(u, v) \in \varphi(u_1, v_1)$, we put $t = ((u_1, v_1), (u, v))$, $a(t) = 1$, $f_t(w) = uwv$ where $w \in V^*$ is arbitrary, $s(0, t) = (u, v)$, $s(1, t) = (u_1, v_1)$,

$A_s = V^*$ for any $s \in S$,

$\mathscr{P}(V, L, C, \varphi) = ((A_s)_{s\in S}, (f_t)_{t\in T})$.

Then the heterogeneous algebra $\mathscr{P}(V, L, C, \varphi)$ (which need not be disjoint) is called a *programmed heterogeneous algebra*.

Since the sets L, C are finite, the sets S, T are finite as well. This implies

22. Proposition. *If (V, L, C, φ) is a programmed contextual grammar, then $\mathscr{P}(V, L, C, \varphi)$ is a context-free algebra.* □

23. Proposition. *Let $P = (V, L, C, \varphi)$ be a programmed contextual grammar, $\mathscr{P}(P) = ((A_s)_{s\in S}, (f_t)_{t\in T})$. Then* $\mathbf{L}(P) = \bigcup_{s\in S} [O]_s^{\mathscr{P}(P)}$.

Proof. Let $w \in V^*$ be arbitrary. Then $w \in \bigcup_{s\in S} [O]_s^{\mathscr{P}(P)}$ iff $w = f_{t_p}(\dots (f_{t_1}(f_{t_0})) \dots)$ for some $p \geqq 0$ where f_{t_0} is nullary and f_{t_i} is unary for $i = 1, 2, \dots, p$. This means that $f_{t_0} \in L$, $f_{t_i}(w) = u(0, t_i)\, w v(1, t_i)$, $s(0, t_{i-1}) = s(1, t_i)$ for $i = 1, 2, \ldots, p$. This implies that $(u(0, t_1), u(1, t_1)) \in \varphi(f_{t_0})$, $(u(0, t_i), u(1, t_i)) \in \varphi(u(0, t_{i-1}), u(1, t_{i-1}))$ for $i = 1, 2, \ldots, p$. Hence $w \in \mathbf{L}(V, L, C, \varphi)$.

On the other hand, if $w \in \mathbf{L}(V, L, C, \varphi)$, there exist $x \in L$, $(u_1, v_1) \in \varphi(x), \ldots, (u_p, v_p) \in \in \varphi(u_{p-1}, v_{p-1})$ such that $w = u_p \ldots u_1 x v_1 \ldots v_p$. We put $x = f_{t_0} = t_0$, $t_1 = = (x, (u_1, v_1))$, $t_i = ((u_{i-1}, v_{i-1}), (u_i, v_i))$ for $i = 2, \ldots, p$. Then $w = = f_{t_p}(\ldots(f_{t_1}(f_{t_0}))\ldots) \in \bigcup_{s \in S} [O]_s^{\mathscr{P}(P)}$. □

24. Definition. Let $P = (V, L, C, \varphi)$ be a programmed contextual grammar, $\mathfrak{A}$ a disjoint heterogeneous algebra equivalent to $\mathscr{P}(P)$. Then the linear grammar $\mathscr{L}(\mathfrak{A})$ is said to be *programmed.*

The term "programmed" expresses the fact that the linear grammar has been constructed starting with a programmed contextual grammar; our programmed linear grammars have nothing to do with programmed grammars in the sense of [22] Chapter V.

25. Corollary. *Let $P = (V, L, C, \varphi)$ be a programmed contextual grammar, $\mathfrak{A}$ a disjoint heterogeneous algebra equivalent to $\mathscr{P}(P)$, $\mathscr{L}(\mathfrak{A}) = (V, S, P)$. Then* $\mathbf{L}(P) = \bigcup_{s \in S} \mathbf{L}(\mathscr{L}(\mathfrak{A}), s)$.

Proof. This is a consequence of 23 and 4.4. □

Hence, we have proved that a language is generated by a programmed contextual grammar iff it is constructed by means of a programmed linear grammar in the sense of 25.

On the basis of 19, we may construct, to any regular contextual grammar, a linear grammar that generates the same language; similarly, by 25, the construction of a linear grammar is described that generates the same language as a given programmed contextual grammar. Hence, regular and programmed contextual grammars lead to some special types of linear grammars, and, hence, to subclasses of the class of all linear languages.

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