Contingency-based equilibrium logic

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Abstract. We investigate an alternative language for equilibrium logic that is based on the concept of positive and negative contingency. Beyond these two concepts our language has the modal operators of necessity and impossibility and the Boolean operators of conjunction and disjunction. Neither negation nor implication are available. Our language is just as expressive as the standard language of equilibrium logic (that is based on conjunction and intuitionistic implication).

Introduction

Traditionally, modal logics are presented as extensions of classical propositional logic by modal operators of necessity L and possibility M (often written $\Box$ and $\Diamond$). These operators are interpreted in Kripke models: triples $M = \langle W, R, V \rangle$ where W is a nonempty set of possible worlds, $R : W \longrightarrow 2^W$ associates to every $w \in W$ the set of worlds $R(w) \subseteq W$ that are accessible from w, and $V : W \longrightarrow 2^{\mathbb{P}}$ associates to every $w \in W$ the subset of the set of propositional variables $\mathbb{P}$ that is true at w. The truth conditions are:

$$M, w \Vdash \mathsf{L}\varphi \text{ iff } M, v \Vdash \varphi \text{ for every } v \in R(w)$$
$$M, w \Vdash \mathsf{M}\varphi \text{ iff } M, v \Vdash \varphi \text{ for some } v \in R(w)$$

Let $\mathcal{L}_{\mathsf{L},\mathsf{M}}$ be the language built from L, M, and the Boolean operators $\neg$, $\vee$ and $\wedge$. Other languages to talk about Kripke models exist. One may e.g. formulate things in terms of strict implication $\varphi > \psi$, which has the same interpretation as $\mathsf{L}(\varphi \to \psi)$ [6]. In this paper we study yet another set of primitives that is based on the notion of *contingency*. Contingency is the opposite of what might be called 'being settled', i.e. being either necessary or impossible. Contingency of φ can be expressed in $\mathcal{L}_{\mathsf{L},\mathsf{M}}$ by the formula $\neg\mathsf{L}\varphi \wedge \neg\mathsf{L}\neg\varphi$. One may distinguish contingent truth $\varphi \wedge \neg\mathsf{L}\varphi \wedge \neg\mathsf{L}\neg\varphi$ from contingent falsehood $\neg\varphi \wedge \neg\mathsf{L}\varphi \wedge \neg\mathsf{L}\neg\varphi$. If the modal logic is at least **KT** (relation R is reflexive, characterised by the axiom $\mathsf{L}\varphi \to \varphi$) then contingent truth of φ reduces to $\varphi \wedge \neg\mathsf{L}\varphi$, and contingent falsehood of φ reduces to $\neg\varphi \wedge \neg\mathsf{L}\neg\varphi$. We adopt the latter two as our official definitions of contingency: $\mathsf{C}^+\varphi$ denotes contingent truth of φ, and $\mathsf{C}^-\varphi$ denotes contingent falsity of φ. We take these two operators as primitive, together with necessity $\mathsf{L}^+\varphi$ and impossibility $\mathsf{L}^-\varphi$. In terms of $\mathcal{L}_{\mathsf{L},\mathsf{M}}$, $\mathsf{L}^+\varphi$ is $\mathsf{L}\varphi$, $\mathsf{L}^-\varphi$ is $\mathsf{L}\neg\varphi$, $\mathsf{C}^+\varphi$ is $\varphi \wedge \neg\mathsf{L}\varphi$, and $\mathsf{C}^-\varphi$ is $\neg\varphi \wedge \neg\mathsf{L}\neg\varphi$. In our language the negation operator is superfluous because $\neg\varphi$ is going to have the same interpretation as $\mathsf{L}^-\varphi \vee \mathsf{C}^-\varphi$.

In this paper we focus on the fragment of formulas whose modal depth is at most one. The paper is organized as follows. We first give syntax and semantics and study some properties. We then show that in models with at most two points, every formula is equivalent to a formula of depth at most one. We finally establish the link with the intermediate logic of here and there as studied in answer set programming.

The logic of contingency

Our language $\mathcal{L}_{pos}$ is *without negation*, and has four primitive operators of contingent truth C^+, contingent falsehood C^-, necessity L^+ and impossibility L^-. Its BNF is:

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathsf{C}^+\varphi \mid \mathsf{C}^-\varphi \mid \mathsf{L}^+\varphi \mid \mathsf{L}^-\varphi$$

where p ranges over the set of propositional variables $\mathbb{P}$. L^+ and C^+ are the positive operators and L^- and C^- are the negative operators. $\mathcal{L}^1_{pos}$ is the set of formulas of $\mathcal{L}_{pos}$ of modal depth at most 1.

The *modal depth* of a formula φ is the maximum number of nested modal operators in φ. The set of propositional variables occurring in φ is written $\mathbb{P}_\varphi$.

Given a Kripke model $M = \langle W, R, V\rangle$, the truth conditions are as follows:

$$\begin{array}{l}
M, w \Vdash \mathsf{L}^+\varphi \text{ iff } M, v \Vdash \varphi \text{ for every } v \in R(w)\\
M, w \Vdash \mathsf{L}^-\varphi \text{ iff } M, v \nVdash \varphi \text{ for every } v \in R(w)\\
M, w \Vdash \mathsf{C}^+\varphi \text{ iff } M, w \Vdash \varphi \text{ and } M, w \nVdash \mathsf{L}^+\varphi\\
M, w \Vdash \mathsf{C}^-\varphi \text{ iff } M, w \nVdash \varphi \text{ and } M, w \nVdash \mathsf{L}^-\varphi
\end{array}$$

Validity and satisfiability are defined as usual.

The modal operators C^+ and C^- are neither normal boxes nor normal diamonds in Chellas's sense [1]. However, the following distribution properties hold.

Proposition 1. *The following equivalences are valid in the class of all models.*

$$\begin{array}{lcl}
\mathsf{L}^+(\varphi \wedge \psi) & \leftrightarrow & \mathsf{L}^+\varphi \wedge \mathsf{L}^+\psi\\
\mathsf{L}^-(\varphi \vee \psi) & \leftrightarrow & \mathsf{L}^-\psi \wedge \mathsf{L}^-\psi\\
\mathsf{C}^+(\varphi \wedge \psi) & \leftrightarrow & \varphi \wedge \psi \wedge (\mathsf{C}^+\varphi \vee \mathsf{C}^+\psi)\\
\mathsf{C}^-(\varphi \vee \psi) & \leftrightarrow & \neg\varphi \wedge \neg\psi \wedge (\mathsf{C}^-\varphi \vee \mathsf{C}^-\psi)
\end{array}$$

There are no similar equivalences for the other combinations of contingency operators and Boolean connectors.

Our results in this paper are mainly for models where the accessibility relation is *reflexive*, i.e. where $w \in R(w)$ for every world w. If M is reflexive then $M, w \nVdash \varphi$ iff $M, w \Vdash \mathsf{L}^-\varphi \vee \mathsf{C}^-\varphi$ for every w in M. We can therefore define $\neg\varphi$ to be an abbreviation of $\mathsf{L}^-\varphi \vee \mathsf{C}^-\varphi$. The operators $\bot$, $\rightarrow$, and $\leftrightarrow$ can then also be defined as abbreviations: $\bot$ is $\mathsf{L}^+p \wedge \mathsf{L}^-p$, for some $p \in \mathbb{P}$; $\varphi \rightarrow \psi$ is $\mathsf{L}^-\varphi \vee \mathsf{C}^-\varphi \vee \psi$; and $\varphi \rightarrow \psi$ is $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. In reflexive models we can also define $\mathsf{M}\varphi$ as an abbreviation of $\mathsf{L}^+\varphi \vee \mathsf{C}^+\varphi \vee \mathsf{C}^-\varphi$. Moreover, our four modal operators are exclusive and exhaustive.

Proposition 2. *The formula* $\mathsf{L}^+\varphi \vee \mathsf{L}^-\varphi \vee \mathsf{C}^+\varphi \vee \mathsf{C}^-\varphi$ *and the formulas*

$$\begin{array}{lll}
\neg(\mathsf{L}^+\varphi \wedge \mathsf{L}^-\varphi) & \neg(\mathsf{L}^+\varphi \wedge \mathsf{C}^-\varphi) & \neg(\mathsf{L}^-\varphi \wedge \mathsf{C}^-\varphi)\\
\neg(\mathsf{L}^+\varphi \wedge \mathsf{C}^+\varphi) & \neg(\mathsf{L}^-\varphi \wedge \mathsf{C}^+\varphi) & \neg(\mathsf{C}^+\varphi \wedge \mathsf{C}^-\varphi)
\end{array}$$

are valid in the class of reflexive Kripke models.

Finally, note that the equivalence $\varphi \leftrightarrow \mathsf{L}^+\varphi \vee \mathsf{C}^+\varphi$ is valid in the class of reflexive models. It follows that every $\mathcal{L}_{pos}$ formula can be rewritten to a formula such that every propositional variable is in the scope of at least one modal operator.

Models with at most two points

In the rest of the paper we consider reflexive models with at most two points. For that class we are going to establish a strong normal form.

Proposition 3. *The equivalences*

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\mathsf{L}^+\mathsf{L}^+\varphi & \leftrightarrow & \mathsf{L}^+\varphi & \mathsf{C}^+\mathsf{L}^+\varphi & \leftrightarrow & \bot \\
\mathsf{L}^+\mathsf{L}^-\varphi & \leftrightarrow & \mathsf{L}^-\varphi & \mathsf{C}^+\mathsf{L}^-\varphi & \leftrightarrow & \bot \\
\mathsf{L}^+\mathsf{C}^+\varphi & \leftrightarrow & \bot & \mathsf{C}^+\mathsf{C}^+\varphi & \leftrightarrow & \mathsf{C}^+\varphi \\
\mathsf{L}^+\mathsf{C}^-\varphi & \leftrightarrow & \bot & \mathsf{C}^+\mathsf{C}^-\varphi & \leftrightarrow & \mathsf{C}^-\varphi \\
\mathsf{L}^-\mathsf{L}^+\varphi & \leftrightarrow & \mathsf{L}^-\varphi \vee \mathsf{C}^+\varphi & \mathsf{C}^-\mathsf{L}^+\varphi & \leftrightarrow & \mathsf{C}^-\varphi \\
\mathsf{L}^-\mathsf{L}^-\varphi & \leftrightarrow & \mathsf{L}^+\varphi \vee \mathsf{C}^-\varphi & \mathsf{C}^-\mathsf{L}^-\varphi & \leftrightarrow & \mathsf{C}^+\varphi \\
\mathsf{L}^-\mathsf{C}^+\varphi & \leftrightarrow & \mathsf{L}^+\varphi \vee \mathsf{L}^-\varphi \vee \mathsf{C}^-\varphi & \mathsf{C}^-\mathsf{C}^+\varphi & \leftrightarrow & \bot \\
\mathsf{L}^-\mathsf{C}^-\varphi & \leftrightarrow & \mathsf{L}^+\varphi \vee \mathsf{L}^-\varphi \vee \mathsf{C}^+\varphi & \mathsf{C}^-\mathsf{C}^-\varphi & \leftrightarrow & \bot
\end{array}$$

are valid in the class of Kripke models having at most two points.

Proposition 3 allows to reduce every modality to a Boolean combination of modalities of length at most one (starting from outermost operators). Beyond the reduction of modalities, reflexive models with at most two points also allow for the distribution of modal operators over conjunctions and disjunctions. Proposition 1 allows us to distribute the positive operators L^+ and C^+ over conjunctions and the negative operators L^- and C^- over disjunctions. The next proposition deals with the remaining cases.

Proposition 4. *The equivalences*

$$\begin{array}{rcl}
\mathsf{L}^+(\varphi \vee \psi) & \leftrightarrow & \mathsf{L}^+\varphi \vee \mathsf{L}^+\psi \vee (\mathsf{C}^+\varphi \wedge \mathsf{C}^-\psi) \vee (\mathsf{C}^-\varphi \wedge \mathsf{C}^+\psi) \\
\mathsf{L}^-(\varphi \wedge \psi) & \leftrightarrow & \mathsf{L}^-\varphi \vee \mathsf{L}^-\psi \vee (\mathsf{C}^+\varphi \wedge \mathsf{C}^-\psi) \vee (\mathsf{C}^-\varphi \wedge \mathsf{C}^+\psi) \\
\mathsf{C}^+(\varphi \vee \psi) & \leftrightarrow & (\mathsf{C}^+\varphi \wedge \mathsf{C}^+\psi) \vee (\mathsf{C}^+\varphi \wedge \mathsf{L}^-\psi) \vee (\mathsf{L}^-\varphi \wedge \mathsf{C}^+\psi) \\
\mathsf{C}^-(\varphi \wedge \psi) & \leftrightarrow & (\mathsf{C}^-\varphi \wedge \mathsf{C}^-\psi) \vee (\mathsf{C}^-\varphi \wedge \mathsf{L}^+\psi) \vee (\mathsf{L}^+\varphi \wedge \mathsf{C}^-\psi)
\end{array}$$

are valid in the class of Kripke models having at most two points.

Distributing the modal operators over conjunctions and disjunctions results in a formula made up of modal atoms —modalities followed by a propositional variable— that are combined by conjunctions and disjunctions. These modal atoms can then be reduced by Proposition 3. If we moreover use that $\varphi \leftrightarrow \mathsf{L}^+\varphi \vee \mathsf{C}^+\varphi$ is valid in the class of reflexive models then we obtain a very simple normal form.

Definition 1. *A formula is in* strong normal form *if and only if it is built according to the following BNF:*

$$\varphi ::= \mathsf{L}^+p \mid \mathsf{L}^-p \mid \mathsf{C}^+p \mid \mathsf{C}^-p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$$

where p ranges over the set of propositional variables $\mathbb{P}$.

Theorem 1. *In the class of reflexive Kripke models with at most two points, every $\mathcal{L}_{pos}$ formula is equivalent to a formula in strong normal form.*

We now focus on models that are not only reflexive but also *antisymmetric*. For that class of models we give a validity-preserving translation from $\mathcal{L}_{pos}$ to propositional logic. We define two functions t^H and t^T by mutual recursion.

$$
\begin{array}{rlrl}
t^H(p) &= p^H, \quad \text{for } p \in \mathbb{P} & t^T(p) &= p^T, \quad \text{for } p \in \mathbb{P} \\
t^H(\varphi \wedge \psi) &= t^H(\varphi) \wedge t^H(\psi) & t^T(\varphi \wedge \psi) &= t^T(\varphi) \wedge t^T(\psi) \\
t^H(\varphi \vee \psi) &= t^H(\varphi) \vee t^H(\psi) & t^T(\varphi \vee \psi) &= t^T(\varphi) \vee t^T(\psi) \\
t^H(\mathsf{L}^+\varphi) &= t^H(\varphi) \wedge t^T(\varphi) & t^T(\mathsf{L}^+\varphi) &= t^T(\varphi) \\
t^H(\mathsf{L}^-\varphi) &= \neg t^H(\varphi) \wedge \neg t^T(\varphi) & t^T(\mathsf{L}^-\varphi) &= \neg t^T(\varphi) \\
t^H(\mathsf{C}^+\varphi) &= t^H(\varphi) \wedge \neg t^T(\varphi) & t^T(\mathsf{C}^+\varphi) &= \bot \\
t^H(\mathsf{C}^-\varphi) &= \neg t^H(\varphi) \wedge t^T(\varphi) & t^T(\mathsf{C}^-\varphi) &= \bot
\end{array}
$$

Theorem 2. *Let φ be a $\mathcal{L}_{pos}$ formula. φ is valid in the class of reflexive and antisymmetric models with at most two points if and only if $t^H(\varphi)$ is propositionally valid (in a vocabulary made up of the set of p^H, t^H such that p is in the vocabulary of $\mathcal{L}_{pos}$).*

If φ is in strong normal form then the translation is linear. The problem of checking validity of such formulas in reflexive and antisymmetric models is therefore in coNP.

Contingency and the logic of here-and-there

In the rest of the paper we consider models with at most two points where the accessibility relation is reflexive and *persistent*, aka *hereditary*. Just as in intuitionistic logic, we say that R is persistent if $\langle u, v\rangle \in R$ implies $V(u) \subseteq V(v)$. Such models were studied since Gödel in order to give semantics to an implication $\Rightarrow$ with strength between intuitionistic and material implication [5]. More recently these models were baptized *here-and-there models*: triples $M = \langle W, R, V\rangle$ with $W = \{H, T\}$, $V(H) \subseteq V(T)$, and $R = \{\langle H, H\rangle, \langle H, T\rangle, \langle T, T\rangle\}$.[1] Particular such models —*equilibrium models*— were investigated as a basis for answer set programming by Pearce, Cabalar, Lifschitz, Ferraris and others as a semantical framework for answer set programming [7].[2]

In persistent models C^+p is unsatisfiable for every propositional variable $p \in \mathbb{P}$. In the normal form of Theorem 1 we may therefore replace every subformula C^+p by $\mathsf{L}^+p \wedge \mathsf{L}^-p$ (which is unsatisfiable in reflexive models), resulting in a formula built from modal atoms of depth one where C^+ does not occur. This observation leads also to a polynomial transformation allowing to check in reflexive and antisymmetric two-points models whether a given $\mathcal{L}_{pos}$ formula is valid in here-and-there models.

Theorem 3. *A $\mathcal{L}_{pos}$ formula φ is valid in the class of here-and-there models if and only if $\left(\bigwedge_{p\,:\,p\in\mathbb{P}_\varphi} \neg\mathsf{C}^+p\right) \to \varphi$ is valid in reflexive and antisymmetric two-points models.*

[1] The other reflexive and persistent models with at most two points are bisimilar to here-and-there models: (1) (pointed) two points models with a symmetric relation are bisimilar to one point models due to persistence; (2) (pointed) models where the two points are not related are bisimilar to one point models; (3) one point models are bisimilar to here-and-there models.

[2] `http://www.equilibriumlogic.net`

In the rest of the section we relate our language $\mathcal{L}_{pos}$ to the customary language of equilibrium logic. Let us call $\mathcal{L}_{pos}^{\Rightarrow}$ the language resulting from the addition of a binary modal connector $\Rightarrow$ to our language $\mathcal{L}_{pos}$. The language of equilibrium logic is the fragment $\mathcal{L}^{\Rightarrow}$ of $\mathcal{L}_{pos}^{\Rightarrow}$ without our four unary modal connectors.

The truth condition for the intermediate implication $\Rightarrow$ can be written as:

$$M, w \Vdash \varphi \Rightarrow \psi \text{ iff } \forall v \in R(w),\ M, v \nVdash \varphi \text{ or } M, v \Vdash \psi$$

Given that $\mathsf{C}^+ p$ is unsatisfiable for every $p \in \mathbb{P}$, Theorem 1 shows that the above language made up of Boolean combinations of modal atoms of the form $\mathsf{L}^+ p$, $\mathsf{L}^- p$, $\mathsf{C}^- p$ is an alternative to the traditional Horn clause language.

Theorem 4. *The $\mathcal{L}_{pos}^{\Rightarrow}$ formula*

$$\varphi \Rightarrow \psi \ \leftrightarrow\ \mathsf{L}^-\varphi \vee \mathsf{L}^+\psi \vee (\mathsf{C}^-\varphi \wedge \mathsf{C}^-\psi) \vee (\mathsf{C}^+\varphi \wedge \mathsf{C}^+\psi)$$

is valid in here-and-there models.

Proof. By the truth condition of $\Rightarrow$, the formula $\varphi \Rightarrow \psi$ is equivalent to $\mathsf{L}^+(\neg\varphi \vee \psi)$. By Proposition 4 the latter is equivalent to

$$\mathsf{L}^+\neg\varphi \vee \mathsf{L}^+\psi \vee (\mathsf{C}^-\neg\varphi \wedge \mathsf{C}^+\psi) \vee (\mathsf{C}^+\neg\varphi \wedge \mathsf{C}^-\psi),$$

which is equivalent to the right-hand side. ■

Together, theorems 4 and 3 say that instead of reasoning with an intermediate implication $\Rightarrow$ one might as well use our fairly simple modal logic of contingency having reflexive and antisymmetric two-points models. These models are not necessarily persistent. Once one has moved to that logic, one can put formulas in strong normal form (Theorem 1), apply Theorem 2, and work in classical propositional logic.

Contingency and equilibrium logic

When we talked about satisfiability in here-and-there models we took it for granted that this meant truth in some possible world w of some model M, and likewise for validity. The definitions of satisfiability and validity are more sophisticated in *equilibrium logic*.

Definition 2. *A here-and-there model $M = \langle\{H, T\}, R, V\rangle$ is an equilibrium model of φ if and only if*

- *$M, H \Vdash \varphi$,*
- *$V(H) = V(T)$, and*
- *there is no here-and-there model $M' = \langle\{H, T\}, R, V'\rangle$ such that $V'(T) = V(T)$, $V'(H) \subset V(H)$, and $M', H \Vdash \varphi$.*

An equilibrium model of φ is therefore isomorphic to a model of classical propositional logic, and moreover its here-valuation is minimal. This can be captured in our language.

Theorem 5. *The formula φ of $\mathcal{L}_{pos}^{\Rightarrow}$ has an equilibrium model iff there is $P \subseteq \mathbb{P}_\varphi$ s.th.*

$$\Big(\mathsf{L}^+(\bigwedge P) \land \mathsf{L}^-(\bigvee(\mathbb{P}_\varphi \setminus P))\Big) \to \varphi$$
$$\Big(\mathsf{C}^-(\bigwedge P) \land \mathsf{L}^-(\bigvee(\mathbb{P}_\varphi \setminus P))\Big) \to \neg\varphi$$

are both valid in reflexive and antisymmetric two-points models.

The premise of the first formula describes a here-and-there model bisimilar to the classical model P. The premise of the second formula describes all here-and-there models whose there-world matches the classical model and whose here-world is 'less true'.

Conclusion

We have presented a modal logic of positive and negative contingency and have studied its properties in different classes of models. We have in particular investigated models with at most two points. We have established a link with equilibrium logics as studied in answer set programming. Our negation-free language in terms of contingency provides an alternative to the usual implication-based language. Our logic can also be seen as a combination of intuitionistic and classical implication, in the spirit of [8, 3, 2, 4].

One of the perspectives is to study the first-order version of our logic. We can extend our translation from $\mathcal{L}_{pos}$ into propositional logic, to a translation from the first-order extension of $\mathcal{L}_{pos}$ into predicate logic as follows:

$$t^H(\forall x\varphi) = \forall x t^H(\varphi) \qquad\qquad t^T(\forall x\varphi) = \forall x t^T(\varphi)$$

This works for the first-order version of equilibrium logic with uniform domains. By means of such a translation the link with answer sets for programs with variables can be studied.[3]

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