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CONTINUITY OF EXTREMAL TRANSITIONS AND FLOPS FOR CALABI–YAU MANIFOLDS

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Abstract

In this paper, we study the behavior of Ricci-flat Kähler metrics on Calabi–Yau manifolds under algebraic geometric surgeries: extremal transitions or flops. We prove a version of Candelas and de la Ossa's conjecture: Ricci-flat Calabi–Yau manifolds related by extremal transitions and flops can be connected by a path consisting of continuous families of Ricci-flat Calabi–Yau manifolds and a compact metric space in the Gromov–Hausdorff topology. In an essential step of the proof of our main result, the convergence of Ricci-flat Kähler metrics on Calabi–Yau manifolds along a smoothing is established, which can be of independent interest.

1. Introduction

A Calabi–Yau manifold M is a simply connected projective manifold with trivial canonical bundle $\mathcal{K}_M \cong \mathcal{O}_M$. In the 1970s, S.T. Yau proved Calabi's conjecture in [**61**], which says that, for any Kähler class $\alpha \in H^{1,1}(M, \mathbb{R})$, there exists a unique Ricci-flat Kähler metric g on M with Kähler form $\omega \in \alpha$. The study of Calabi–Yau manifolds became very interesting in the last three decades (cf. [**63**]). The convergence of Ricci-flat Calabi–Yau manifolds was studied from various perspectives (cf. [**2**], [**8**], [**9**], [**13**], [**30**], [**37**], [**41**], [**56**], [**57**], [**58**], [**49**], [**60**], [**64**]). The goal of the present paper is to study the metric behavior of Calabi–Yau manifolds under some algebraic geometric surgeries.

Let M_0 be a singular projective normal variety with singular set S. Usually there are two types of desingularizations: one is a *resolution* $(\bar{M}, \bar{\pi})$, i.e., $\bar{M}$ is a projective manifold, and $\bar{\pi}$ is a morphism such that $\bar{\pi} : \bar{M}\backslash\bar{\pi}^{-1}(S) \to M_0\backslash S$ is bi-holomorphic. The other is a *smoothing* $(\mathcal{M}, \pi)$ over the unit disc $\Delta \subset \mathbb{C}$, i.e., $\mathcal{M}$ is an $(n+1)$-dimensional variety, π is a proper flat morphism, $M_0 = \pi^{-1}(0)$, and $M_t = \pi^{-1}(t)$ is a smooth projective n-dimensional manifold for any $t \in \Delta\backslash\{0\}$. If M_0 admits a resolution $(\bar{M}, \bar{\pi})$ and a smoothing $(\mathcal{M}, \pi)$, the process of going from $\bar{M}$ to M_t, $t \neq 0$, is called an *extremal transition*, denoted by $\bar{M} \to M_0 \rightsquigarrow M_t$. We call this process a *conifold transition* if M_0 is a

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conifold, which is a normal variety M_0 with only finite ordinary double points as singularities, i.e., any singular point is locally given by

$$z_0^2 + \cdots + z_n^2 = 0, \quad \text{where} \quad \dim_{\mathbb{C}} M_0 = n.$$

If M_0 admits two different resolutions $(\bar{M}_1, \bar{\pi}_1)$ and $(\bar{M}_2, \bar{\pi}_2)$ with both exceptional subvarieties of codimension at least 2, the process of going from $\bar{M}_1$ to $\bar{M}_2$ is called a *flop*, denoted by $\bar{M}_1 \to M_0 \dashrightarrow \bar{M}_2$.

Extremal transitions and flops are algebraic geometric surgeries providing ways to connect two topologically distinct projective manifolds, which are interesting in both mathematics and physics. In the minimal model program, all smooth minimal models of dimension 3 in a birational equivalence class are connected by a sequence of flops (cf. [**38**], [**39**], [**35**]). The famous Reid's fantasy conjectures that all Calabi–Yau threefolds are connected to each other by extremal transitions, possibly including non-Kähler Calabi–Yau threefolds, so as to form a huge connected web (cf. [**46**], [**48**]). There is also a projective version of this conjecture, the connectedness conjecture for moduli spaces for Calabi–Yau threefolds (cf. [**28**], [**29**], [**48**]). Furthermore, in physics, flops and extremal transitions are related to the topological change of the space-time in string theory (cf. [**7**], [**16**], [**27**], [**24**], [**48**]). Readers are referred to the survey article [**48**] for topology, algebraic geometry, and even physics properties of extremal transitions.

In [**6**], physicists P. Candelas and X.C. de la Ossa conjectured that extremal transitions and flops should be "continuous in the space of Ricci-flat Kähler metrics," even though these processes involve topologically distinct Calabi–Yau manifolds. This conjecture was verified in [**6**] for the non-compact quadric cone $M_0 = \{(z_0, \cdots, z_3) \in \mathbb{C}^4 | z_0^2 + \cdots + z_3^2 = 0\}$.

In the 1980s, Gromov introduced the notion of Gromov–Hausdorff distance d_{GH} on the space $\mathfrak{X}$ of isometric classes of all compact metric spaces (cf. [**22**]), such that $(\mathfrak{X}, d_{GH})$ is a complete metric space (cf. [**22**] and Appendix A). This notion provides a framework to study the continuity of a family of compact metric spaces with possibly different topologies. The Gromov–Hausdorff topology provides a natural mathematical formulation of Candelas and de la Ossa's conjecture as follows:

i) If $\bar{M} \to M_0 \rightsquigarrow M_t$, $t \in \Delta \backslash \{0\} \subset \mathbb{C}$, is an extremal transition among Calabi–Yau manifolds, then there exists a family of Ricci-flat Kähler metrics $\bar{g}_s$, $s \in (0,1)$, on $\bar{M}$, and a family of Ricci-flat Kähler metrics $\tilde{g}_t$ on M_t satisfying that $\{(\bar{M}, \bar{g}_s)\}$ and $\{(M_t, \tilde{g}_t)\}$ converge to a single compact metric space (X, d_X) in the Gromov–Hausdorff topology,

$$(M_t, \tilde{g}_t) \xrightarrow{d_{GH}} (X, d_X) \xleftarrow{d_{GH}} (\bar{M}, \bar{g}_s), \qquad s \to 0, \ t \to 0.$$

ii) If $\bar{M}_1 \to M_0 \dashrightarrow \bar{M}_2$ is a flop between two Calabi–Yau manifolds, then there are families of Ricci-flat Kähler metrics $\bar{g}_{i,s}$, $s \in (0,1)$

on $\bar{M}_i$ $(i = 1, 2)$ such that

$$(\bar{M}_1, \bar{g}_{1,s}) \xrightarrow{d_{GH}} (X, d_X) \xleftarrow{d_{GH}} (\bar{M}_2, \bar{g}_{2,s}), \qquad s \to 0,$$

for a single compact metric space (X, d_X).

Furthermore, in both cases X is homeomorphic to M_0 and d_X is induced by a Ricci-flat Kähler metric on $M_0 \backslash S$. In the present paper, we shall prove i) and ii) of the above version of Candelas and de la Ossa's conjecture.

Let M_0 be a projective normal Cohen-Macaulay n-dimensional variety with singular set S, and let $\mathcal{K}_{M_0}$ be the canonical sheaf of M_0 ([**33**]). In this paper, all varieties are assumed to be Cohen-Macaulay. We call M_0 Gorenstein if $\mathcal{K}_{M_0}$ is a rank one locally free sheaf. Assume that M_0 has only canonical singularities, i.e., M_0 is Gorenstein, and for any resolution $(\bar{M}, \bar{\pi})$,

$$\mathcal{K}_{\bar{M}} = \bar{\pi}^* \mathcal{K}_{M_0} + \sum a_E E, \quad a_E \geq 0,$$

where E are effective exceptional divisors. Consider a resolution $(\bar{M}, \bar{\pi})$ of M_0. If α is an ample class in the Picard group of M_0, $\bar{\pi}^*\alpha$ belongs to the boundary of the Kähler cone of $\bar{M}$. A resolution $(\bar{M}, \bar{\pi})$ of M_0 is called a *crepant resolution* if $\mathcal{K}_{\bar{M}} = \bar{\pi}^* \mathcal{K}_{M_0}$ and is called a *small resolution* if the exceptional subvariety $\bar{\pi}^{-1}(S)$ satisfies $\dim_{\mathbb{C}} \bar{\pi}^{-1}(S) \leq n-2$. It is obvious that $(\bar{M}, \bar{\pi})$ is crepant if it is a small resolution. If M_0 admits a smoothing $(\mathcal{M}, \pi)$ over a unit disc $\Delta \subset \mathbb{C}$ with an ample line bundle $\mathcal{L}$ on $\mathcal{M}$, then there is an embedding $\mathcal{M} \hookrightarrow \mathbb{CP}^N \times \Delta$ such that $\mathcal{L}^m = \mathcal{O}_\Delta(1)|_{\mathcal{M}}$ for some $m \geq 1$, π is a proper surjection given by the restriction of the projection from $\mathbb{CP}^N \times \Delta$ to Δ, and the rank of π_* is 1 on $\mathcal{M} \backslash S$. This implies that M_t, $t \in \Delta \backslash \{0\}$, have the same underlying differential manifold $\tilde{M}$. Moreover, if $\mathcal{L}$ is a line bundle on $\mathcal{M}$ such that the restriction of $\mathcal{L}$ on M_0 is ample, then by proposition 1.41 in [**39**], $\mathcal{L}$ is ample on $\pi^{-1}(\Delta')$ where $\Delta' \subseteq \Delta$ is a neighborhood of 0.

A Calabi–Yau variety is a simply connected projective normal variety M_0 with trivial canonical sheaf $\mathcal{K}_{M_0} \cong \mathcal{O}_{M_0}$ and only canonical singularities. If a Calabi–Yau variety M_0 admits a crepant resolution $(\bar{M}, \bar{\pi})$, then $\bar{M}$ is a Calabi–Yau manifold. Our first result proves i) in the above version of Candelas and de la Ossa's conjecture.

Theorem 1.1. *Let M_0 be a Calabi–Yau n-variety with singular set S. Assume that*

i) *M_0 admits a smoothing $\pi : \mathcal{M} \to \Delta$ over the unit disc $\Delta \subset \mathbb{C}$ such that the relative canonical bundle $\mathcal{K}_{\mathcal{M}/\Delta}$ is trivial, i.e., $\mathcal{K}_{\mathcal{M}/\Delta} \cong \mathcal{O}_{\mathcal{M}}$ and $\mathcal{M}$ admits an ample line bundle $\mathcal{L}$. For any $t \in \Delta \backslash \{0\}$, let $\tilde{g}_t$ be the unique Ricci-flat Kähler metric on $M_t = \pi^{-1}(t)$ with Kähler form $\tilde{\omega}_t \in c_1(\mathcal{L})|_{M_t}$.*

ii) *M_0 admits a crepant resolution $(\bar{M}, \bar{\pi})$. Let $\{\bar{g}_s\}$ $(s \in (0,1])$ be a family of Ricci-flat Kähler metrics with Kähler classes $\lim_{s\to 0}[\bar{\omega}_s] = \bar{\pi}^* c_1(\mathcal{L})|_{M_0}$ in $H^{1,1}(\bar{M}, \mathbb{R})$, where $\bar{\omega}_s$ denotes the corresponding Kähler form of $\bar{g}_s$.*

Then there exists a compact length metric space (X, d_X) such that

$$\lim_{t\to 0} d_{GH}((M_t, \tilde{g}_t), (X, d_X)) = \lim_{s\to 0} d_{GH}((\bar{M}, \bar{g}_s), (X, d_X)) = 0.$$

Furthermore, (X, d_X) is isometric to the metric completion $\overline{(M_0 \backslash S, d_g)}$ where g is a Ricci-flat Kähler metric on $M_0 \backslash S$, and d_g is the Riemannian distance function of g.

The following is a simple example from [**27**] for which Theorem 1.1 can apply. Let $\bar{M}$ be the complete intersection in $\mathbb{CP}^4 \times \mathbb{CP}^1$ given by

$$y_0 \mathfrak{g}(z_0, \ldots, z_4) + y_1 \mathfrak{h}(z_0, \ldots, z_4) = 0, \quad y_0 z_4 - y_1 z_3 = 0,$$

where $z_0, \ldots, z_4$ are homogeneous coordinates of $\mathbb{CP}^4$, y_0, y_1 are homogeneous coordinates of $\mathbb{CP}^1$, and $\mathfrak{g}$ and $\mathfrak{h}$ are generic homogeneous polynomials of degree 4. Then $\bar{M}$ is a crepant resolution of the quintic conifold M_0 given by

$$z_3 \mathfrak{g}(z_0, \ldots, z_4) + z_4 \mathfrak{h}(z_0, \ldots, z_4) = 0$$

(cf. [**48**]). Hence there is a conifold transition $\bar{M} \to M_0 \rightsquigarrow \tilde{M}$ for any smooth quintic $\tilde{M}$ in $\mathbb{CP}^4$. Theorem 1.1 implies that there is a family of Ricci-flat Kähler metrics $\bar{g}_s$ $(s \in (0,1])$ on $\bar{M}$ and a family of Ricci-flat smooth quintic $(M_t, \tilde{g}_t)$ $(t \in \Delta \backslash \{0\})$ such that $M_1 = \tilde{M}$, and

$$(M_t, \tilde{g}_t) \xrightarrow{d_{GH}} (X, d_X) \xleftarrow{d_{GH}} (\bar{M}, \bar{g}_s),$$

for a compact metric space (X, d_X).

Our second result proves ii) in the above version of Candelas and de la Ossa's conjecture.

Theorem 1.2. *Let M_0 be an n-dimensional Calabi–Yau variety with singular set S, and $\mathcal{L}$ be an ample line bundle. Assume that M_0 admits two crepant resolutions $(\bar{M}_1, \bar{\pi}_1)$ and $(\bar{M}_2, \bar{\pi}_2)$. Let $\{\bar{g}_{1,s}\}$ (resp. $\{\bar{g}_{2,s}\}$ $s \in (0,1]$) be a family of Ricci-flat Kähler metrics on $\bar{M}_1$ (resp. $\bar{M}_2$) with Kähler classes $\lim_{s\to 0}[\bar{\omega}_{\alpha,s}] = \bar{\pi}_\alpha^* c_1(\mathcal{L})$, $\alpha = 1, 2$. Then there exists a compact length metric space (X, d_X) such that*

$$\lim_{s\to 0} d_{GH}((\bar{M}_1, \bar{g}_{1,s}), (X, d_X)) = \lim_{s\to 0} d_{GH}((\bar{M}_2, \bar{g}_{2,s}), (X, d_X)) = 0.$$

Furthermore, (X, d_X) is isometric to the metric completion $\overline{(M_0 \backslash S, d_g)}$ where g is a Ricci-flat Kähler metric on $M_0 \backslash S$, and d_g is the Riemannian distance function of g.

REMARK 1.3. The present arguments are inadequate to prove that X is homeomorphic to M_0 in both Theorem 1.1 and Theorem 1.2. Additional work is required. However, if M_0 has only orbifold singularities, and $c_1(\mathcal{L})$ can be represented by an orbifold Kähler metric on M_0, then X is homeomorphic to M_0 by corollary 1.1 in [**49**].

We now begin to describe our approach to Theorem 1.1 and Theorem 1.2. Let M_0 be a normal n-dimensional projective variety with singular set S. For any $p \in S$ and a small neighborhood $U_p \subset M_0$ of p, a pluri-subharmonic function v (resp. strongly pluri-subharmonic, and pluri-harmonic) on U_p is an upper semi-continuous function with value in $\mathbb{R} \cup \{-\infty\}$ (v is not locally $-\infty$) such that v extends to a pluri-subharmonic function $\tilde{v}$ (resp. strongly pluri-subharmonic, and pluri-harmonic) on a neighborhood of the image of some local embedding $U_p \hookrightarrow \mathbb{C}^m$. We call v smooth if $\tilde{v}$ is smooth. A form ω on M_0 is called a Kähler form, if ω is a smooth Kähler form in the usual sense on $M_0 \backslash S$ and, for any $p \in S$, there is a neighborhood U_p and a continuous strongly pluri-subharmonic function v on U_p such that $\omega = \sqrt{-1}\partial\overline{\partial}v$ on $U_p \bigcap (M_0 \backslash S)$. We call ω smooth if v is smooth in the above sense. Otherwise, we call ω a singular Kähler form. If $\mathcal{PH}_{M_0}$ denotes the sheaf of pluri-harmonic functions on M_0, then any Kähler form ω represents a class $[\omega]$ in $H^1(M_0, \mathcal{PH}_{M_0})$ (cf. section 5.2 in [**18**]). We also have an analogue of Chern-Weil theory for line bundles on M_0 (see [**18**] for details). If $\mathcal{L}_0$ is an ample line bundle on M_0, then there is an embedding $M_0 \hookrightarrow \mathbb{CP}^N$ such that $\mathcal{L}_0^m = \mathcal{O}(1)|_{M_0}$, and the first Chern class $c_1(\mathcal{L}_0)$ can be presented by a smooth Kähler form: $c_1(\mathcal{L}_0) = \frac{1}{m}[\omega_{FS}|_{M_0}] \in H^1(M_0, \mathcal{PH}_{M_0})$, where ω_{FS} denotes the standard Fubini-Study Kähler form on $\mathbb{CP}^N$.

In [**18**] (see also [**66**]), a generalized Calabi–Yau theorem was obtained, which says that if M_0 is a Calabi–Yau variety, then for any ample line bundle $\mathcal{L}_0$ there is a unique Ricci-flat Kähler form $\omega \in c_1(\mathcal{L}_0)$. We denote by g the corresponding Kähler metric of ω on $M_0 \backslash S$. If M_0 admits a crepant resolution $(\bar{M}, \bar{\pi})$, and $\alpha_s \in H^{1,1}(\bar{M}, \mathbb{R})$, $s \in (0,1)$, is a family of Kähler classes with $\lim\limits_{s \to 0} \alpha_s = \bar{\pi}^* c_1(\mathcal{L}_0)$, [**56**] proved that

$$\bar{g}_s \longrightarrow \bar{\pi}^* g, \quad \bar{\omega}_s \longrightarrow \bar{\pi}^* \omega, \quad \text{when} \quad s \to 0$$

in the C^∞-sense on any compact subset K of $\bar{M} \backslash \bar{\pi}^{-1}(S)$, where $\bar{g}_s$ is the unique Ricci-flat Kähler metric with Kähler form $\bar{\omega}_s \in \alpha_s$. Assume that M_0 is a Calabi–Yau conifold and M_0 admits a smoothing $(\mathcal{M}, \pi)$ satisfying that the relative canonical bundle $\mathcal{K}_{\mathcal{M}/\Delta}$ is trivial and that $\mathcal{M}$ admits an ample line bundle $\mathcal{L}$ such that $\mathcal{L}|_{M_0} = \mathcal{L}_0$. For any $t \in \Delta \backslash \{0\}$, if $\tilde{g}_t$ denotes the unique Ricci-flat Kähler metric on $M_t = \pi^{-1}(t)$ with Kähler form $\tilde{\omega}_t \in c_1(\mathcal{L})|_{M_t}$, [**49**] proved that

$$F_t^* \tilde{g}_t \longrightarrow g, \quad F_t^* \tilde{\omega}_t \longrightarrow \omega, \quad \text{when} \quad t \to 0,$$

in the C^∞-sense on any compact subset $K \subset M_0\backslash S$, where $F_t : M_0\backslash S \longrightarrow M_t$ is a family of embeddings. If M_0 is a Calabi–Yau variety (not necessarily a conifold) and $\mathcal{M}$ is smooth, then a subsequence-C^∞ convergence theorem for the Ricci-flat Kähler metric $\tilde{g}_t$ on M_t was obtained in [**49**], i.e., there is a sequence $t_k \in \Delta\backslash\{0\}$ such that $t_k \to 0$, and $F_{t_k}^*\tilde{g}_{t_k}$ converges to g ($k \to \infty$) in the C^∞-sense on any compact subset $K \subset M_0\backslash S$.

In the proof of Theorem 1.1, the following generalization of the convergence results in [**49**] plays a significant role.

Theorem 1.4. *Let M_0 be a Calabi–Yau n-variety ($n \geq 2$) with singular set S. Assume that M_0 admits a smoothing $\pi : \mathcal{M} \to \Delta$ such that $\mathcal{M}$ admits an ample line bundle $\mathcal{L}$ and the relative canonical bundle is trivial, i.e., $\mathcal{K}_{\mathcal{M}/\Delta} \cong \mathcal{O}_{\mathcal{M}}$. If $\tilde{g}_t$ denotes the unique Ricci-flat Kähler metric with Kähler form $\tilde{\omega}_t \in c_1(\mathcal{L})|_{M_t} \in H^{1,1}(M_t, \mathbb{R})$ ($t \in \Delta\backslash\{0\}$), and ω denotes the unique singular Ricci-flat Kähler form on M_0 with $\omega \in c_1(\mathcal{L})|_{M_0} \in H^1(M_0, \mathcal{PH}_{M_0})$, then*

$$F_t^*\tilde{g}_t \longrightarrow g, \quad F_t^*\tilde{\omega}_t \longrightarrow \omega, \quad \text{when} \quad t \to 0,$$

in the C^∞-sense on any compact subset $K \subset M_0\backslash S$, where $F_t : M_0\backslash S \longrightarrow M_t$ is a smooth family of embeddings and g is the corresponding Kähler metric of ω on $M_0\backslash S$. Furthermore, the diameter of $(M_t, \tilde{g}_t)$ ($t \in \Delta\backslash\{0\}$) satisfies

$$\mathrm{diam}_{\tilde{g}_t}(M_t) \leq D,$$

where $D > 0$ is a constant independent of t.

Our proof of Theorem 1.1 is to show that $(M_0\backslash S, g)$ has a metric completion (X, d_X) satisfying the property that both $\{(M, \bar{g}_s)\}$ and $\{(M_t, \tilde{g}_t)\}$ converge to (X, d_X) in the Gromov–Hausdorff topology when $s \to 0$ and $t \to 0$. The same method also proves Theorem 1.2.

As an application of Theorem 1.1 and Theorem 1.2, we shall explore the path connectedness properties of a certain class of Ricci-flat Calabi–Yau threefolds. Inspired by string theory in physics, some physicists made a projective version of Reid's fantasy (cf. [**7**], [**24**], and [**48**]), the so-called connectedness conjecture, which is formulated more precisely in [**28**] (see also [**29**]). This conjecture says that there is a huge connected web Γ such that nodes of Γ consist of all deformation classes of Calabi–Yau threefolds, and two nodes are connected $\mathfrak{D}_1 - \mathfrak{D}_2$ if $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are related by an extremal transition, i.e., there is a Calabi–Yau 3-variety M_0 that admits a crepant resolution $\bar{M} \in \mathfrak{D}_1$ and a smoothing $(\mathcal{M}, \pi)$ satisfying $\pi^{-1}(t) = M_t \in \mathfrak{D}_2$ for any $t \in \Delta\backslash\{0\}$. It was shown in [**24**], [**16**], [**5**], and [**28**] that many Calabi–Yau threefolds are connected to each other in the above sense. By combining the connectedness conjecture and Theorem 1.1 and Theorem 1.2, we reach a metric version of connectedness conjecture as follows: if $\mathfrak{CY}_3$ denotes the set of Ricci-flat Calabi–Yau threefolds (M, g) with volume 1, then the closure $\overline{\mathfrak{CY}}_3$ of $\mathfrak{CY}_3$ in $(\mathfrak{X}, d_{GH})$ is path connected, i.e., for any two points p_1 and $p_2 \in \overline{\mathfrak{CY}}_3$, there is a path

$$\gamma : [0,1] \longrightarrow \overline{\mathfrak{CY}}_3 \subset (\mathfrak{X}, d_{GH})$$

such that $p_1 = \gamma(0)$ and $p_2 = \gamma(1)$.

Given a class of Calabi–Yau 3-manifolds known to be connected by extremal transitions and flops in algebraic geometry, Theorem 1.1 and Theorem 1.2 can be used to show that the closure of the class of Calabi–Yau 3-manifolds is path connected in $(\mathfrak{X}, d_{GH})$. In the minimal model program, it was proved that for any two Calabi–Yau 3-manifolds M and M' birational to each other, there is a sequence of flops connecting M and M' (cf. [**38**], [**39**], [**35**]). In [**24**], it was shown that all complete intersection Calabi–Yau manifolds (CICY) of dimension 3 in products of projective spaces are connected by conifold transitions. Furthermore, in [**5**] and [**16**] a large number of complete intersection Calabi–Yau 3-manifolds in toric varieties were verified to be connected by extremal transitions, which include Calabi–Yau hypersurfaces in all toric manifolds obtained by resolving weighted projective 4-spaces. As a corollary of Theorem 1.1 and Theorem 1.2, we obtain the following result.

Corollary 1.5. *For any Calabi–Yau manifold M, let*

$$\mathfrak{M}_M = \{(M,g) \in \mathfrak{X} |\ g \textit{ is a Ricci flat Kähler metric on } M \textit{ with } \mathrm{Vol}_g(M) = 1\}.$$

i) *If M is a three-dimensional Calabi–Yau manifold, and*

$$\mathfrak{BM}_M = \bigcup_{\textit{all Calabi–Yau manifolds } M' \textit{ birational to } M} \mathfrak{M}_{M'},$$

then the closure $\overline{\mathfrak{BM}}_M$ of $\mathfrak{BM}_M$ in $(\mathfrak{X}, d_{GH})$ is path connected.

ii) *Let*

$$\mathfrak{CP} = \bigcup_{\textit{all CICY 3-manifolds } M' \textit{ in products of projective spaces}} \mathfrak{M}_{M'}.$$

Then the closure $\overline{\mathfrak{CP}}$ of $\mathfrak{CP}$ in $(\mathfrak{X}, d_{GH})$ is path connected.

iii) *There is a path connected component $\overline{\mathfrak{CT}}$ of $\overline{\mathfrak{CY}}_3 \subset (\mathfrak{X}, d_{GH})$ such that $\mathfrak{CP} \subset \overline{\mathfrak{CT}}$, and $\overline{\mathfrak{CT}}$ contains all (M, g), where M is a Calabi–Yau hypersurface in a toric 4-manifold obtained by resolving a weighted projective 4-space, and g is a Ricci-flat Kähler metric of volume 1 on M.*

The study of metric behaviors under some algebraic geometric surgeries also arises from other perspectives, such as Kähler-Ricci flow (cf.

[**51**] [**52**] [**53**] and [**54**]) and balanced metrics on non-Kähler Calabi–Yau threefolds (cf. [**20**]).

The rest of the paper is organized as follows: In Section 2, we bound from above the diameters of Ricci-flat Calabi–Yau manifolds along a smoothing. In Section 3, we prove Theorem 1.4. In Section 4, we establish a link between point-wise C^∞-convergence of Riemannian metrics on a 'big' open subset and global Gromov–Hausdorff convergence. In Section 5, we prove Theorem 1.1, Theorem 1.2, and Corollary 1.5. In Appendix A, we supply basic properties on Gromov–Hausdorff convergence used in Section 4. In Appendix B (written by Mark Gross), some bounds for volumes of Calabi–Yau manifolds along a smoothing are provided which are used in the proof of Theorem 1.4.

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2. A Priori Estimate

In this section, we obtain an estimate for diameters of Ricci-flat Calabi–Yau manifolds along a smoothing, which plays a key role in our C^0-estimate in the proof of Theorem 1.4.

Theorem 2.1. *Let M_0 be a projective n-dimensional variety with singular set S. Assume that M_0 admits a smoothing $\pi : \mathcal{M} \to \Delta$ over the unit disc $\Delta \subset \mathbb{C}$ such that $\mathcal{M}$ admits an ample line bundle $\mathcal{L}$, and the relative canonical bundle is trivial, i.e., $\mathcal{K}_{\mathcal{M}/\Delta} \cong \mathcal{O}_{\mathcal{M}}$. Let Ω_t be a relative holomorphic volume form, i.e., a nowhere vanishing section of $\mathcal{K}_{\mathcal{M}/\Delta}$, and let $\tilde{g}_t$ be the unique Ricci-flat Kähler metric with Kähler form $\tilde{\omega}_t \in c_1(\mathcal{L})|_{M_t} \in H^{1,1}(M_t, \mathbb{R})$, for $t \in \Delta\backslash\{0\}$. Then the diameter of $(M_t, \tilde{g}_t)$ satisfies that*

$$\mathrm{diam}_{\tilde{g}_t}(M_t) \leq 2 + D(-1)^{\frac{n^2}{2}} \int_{M_t} \Omega_t \wedge \bar{\Omega}_t,$$

where D is a constant independent of t.

Proof. Recall that $\mathcal{M}$ is an $(n+1)$-dimensional variety with an embedding $\mathcal{M} \hookrightarrow \mathbb{CP}^N \times \Delta$ such that $\mathcal{L}^m = \mathcal{O}_\Delta(1)|_\mathcal{M}$ for an $m \geq 1$, π is the restriction to $\mathcal{M}$ of the projection from $\mathbb{CP}^N \times \Delta$ to Δ, which is a proper surjection such that the rank of π_* is 1 on $\mathcal{M}\backslash S$. Then $M_t = \pi^{-1}(t)$ is a smooth Calabi–Yau manifold for any $t \in \Delta\backslash\{0\}$. Denote

$$\omega_t = \frac{1}{m}\omega_{FS}|_{M_t},$$

where ω_{FS} is the standard Fubini-Study metric on $\mathbb{CP}^N$, and g_t is the corresponding Kähler metric of ω_t. Note that $\tilde{\omega}_t$ satisfies the Monge-Ampère equation
(2.1)

$$\tilde{\omega}_t^n = (-1)^{\frac{n^2}{2}} e^{\sigma_t}\Omega_t \wedge \overline{\Omega}_t, \quad \text{where} \quad e^{\sigma_t} = V\left((-1)^{\frac{n^2}{2}} \int_{M_t} \Omega_t \wedge \overline{\Omega}_t\right)^{-1},$$

where $V = n!\mathrm{Vol}_{g_t}(M_t)$ is a constant independent of t.

For $p \in M_0\backslash S$, there are coordinates $z_0, \ldots, z_n$ on a neighborhood U of p in $\mathcal{M}$ such that $t = \pi(z_0, \ldots, z_n) = z_0$, $p = (0, \ldots, 0)$, and the closure $\overline{U}$ of U is a compact subset of $\mathcal{M}\backslash S$. There is an $r_0 > 0$ such that $\Delta^1 \times \Delta^n \subset U$, where $\Delta^1 = \{|t| < r_0\} \subset \Delta$, $\Delta^n = \{|z_j| < r_0, j = 1, \ldots, n\} \subset \mathbb{C}^n$, and $\{t\} \times \Delta^n \subset M_t$. Note that locally ω_t and $\tilde{\omega}_t$ are families of Kähler forms on $\Delta^n \subset \mathbb{C}^n$, and there is a constant C_1 independent of t such that

$$C_1^{-1}\omega_E \leq \omega_t \leq C_1\omega_E, \tag{2.2}$$

where $\omega_E = \sqrt{-1}\partial\bar{\partial}\sum\limits_{i=1}^{n}|z_i|^2$ is the standard Euclidean Kähler form on Δ^n, and g_E denotes the corresponding Euclidean Kähler metric. We need the following fact, which is a simplified version of lemma 1.3 in [**17**]. For completeness, we shall sketch a proof.

Lemma 2.2 (lemma 1.3 in [**17**])**.** *For any $\delta > 0$, and any $t \in \Delta^1\backslash\{0\}$, there is an open subset $U_{t,\delta}$ of Δ^n such that*

$$\mathrm{Vol}_{g_t}(U_{t,\delta}) \geq \mathrm{Vol}_{g_t}(\Delta^n) - \delta, \quad \mathrm{diam}_{\tilde{g}_t}(U_{t,\delta}) \leq \hat{C}\delta^{-\frac{1}{2}},$$

where $\hat{C}$ is a constant independent of t.

Proof. Let $dv_E = (-1)^{\frac{n}{2}} dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n$ be the standard Euclidean volume form on Δ^n and for any $x_1, x_2 \in \Delta^n$, let $[x_1, x_2] \subset \Delta^n$

be the segment connecting x_1 and x_2. By Fubini's Theorem, the Cauchy-Schwarz inequality, and (2.2), we have

$$\begin{aligned}
&\int_{\Delta^n\times\Delta^n} \mathrm{length}_{\tilde g_t}([x_1,x_2])^2 dv_E(x_1)dv_E(x_2)\\
\leq\ & \|x_2-x_1\|_E^2\int_0^1 ds\int_{\Delta^n\times\Delta^n}\mathrm{tr}_{\omega_E}\tilde\omega_t((1-s)x_1+sx_2)dv_E(x_1)dv_E(x_2)\\
\leq\ & 2^{2n}\mathrm{diam}^2_{g_E}(\Delta^n)\mathrm{Vol}_{g_E}(\Delta^n)\int_{\Delta^n}\tilde\omega_t\wedge\omega_E^{n-1}\\
\leq\ & C_2\int_{\Delta^n}\tilde\omega_t\wedge\omega_t^{n-1}\\
\leq\ & C_2\int_{M_t}\tilde\omega_t\wedge\omega_t^{n-1}=\bar C,
\end{aligned}$$

where $\bar C$ is a constant independent of t. The second inequality is obtained by integrating first with respect to $y=(1-s)x_1$ when $s\leq\frac12$, then with respect to $y=sx_2$ when $s\geq\frac12$, since $dv_E(x_i)\leq 2^{2n}dv_E(y)$. If

$$S_t=\{(x_1,x_2)\in\Delta^n\times\Delta^n|\mathrm{length}^2_{\tilde g_t}([x_1,x_2])>\bar C\delta^{-1}\},$$

then $\mathrm{Vol}_{g_E\times g_E}(S_t)<\delta$. Let $S_t(x_1)=\{x_2\in\Delta^n|\ (x_1,x_2)\in S_t\}$, and let $Q_t=\{x_1\in\Delta^n|\ \mathrm{Vol}_{g_E}(S_t(x_1))\geq\frac12\mathrm{Vol}_{g_E}(\Delta^n)\}$. By Fubini's Theorem,

$$\mathrm{Vol}_{g_E}(Q_t)<2\delta\mathrm{Vol}^{-1}_{g_E}(\Delta^n),\qquad \mathrm{Vol}_{g_E}(S_t(x_j))<\frac12\mathrm{Vol}_{g_E}(\Delta^n),$$

for any $x_1,x_2\in\Delta^n\backslash Q_t$. Thus $(\Delta^n\backslash S_t(x_1))\bigcap(\Delta^n\backslash S_t(x_2))$ is not empty. If $y\in(\Delta^n\backslash S_t(x_1))\cap(\Delta^n\backslash S_t(x_2))$, then $(x_1,y),(x_2,y)\in(\Delta^n\times\Delta^n)\backslash S_t$, and

$$\mathrm{length}^2_{\tilde g_t}([x_1,y]\cup[y,x_2])\leq 2\bar C\delta^{-1},$$

and therefore

$$\mathrm{diam}^2_{\tilde g_t}(\Delta^n\backslash Q_t)\leq 2\bar C\delta^{-1}.$$

If we denote $U_{t,\delta}=\Delta^n\backslash Q_t$, then by (2.2) we derive

$$\mathrm{Vol}_{g_t}(\Delta^n\backslash U_{t,\delta})=\mathrm{Vol}_{g_t}(Q_t)\leq C_3\mathrm{Vol}_{g_E}(Q_t)<2C_3\mathrm{Vol}^{-1}_{g_E}(\Delta^n)\delta,$$

where $C_3>0$ is a constant independent of t. By replacing δ with $(2C_3)^{-1}\mathrm{Vol}_{g_E}(\Delta^n)\delta$, we obtain the desired conclusion. q.e.d.

We return to the proof of Theorem 2.1. Let $\delta_t=\frac12\mathrm{Vol}_{g_t}(\Delta^n)$, and let $p_t\in U_{t,\delta_t}$. By (2.2), we get

$$\delta_t\geq\frac{C_4}{2}\mathrm{Vol}_{g_E}(\Delta^n)=\bar{\bar\delta}$$

and thus $U_{t,\delta_t}\subset B_{\tilde g_t}(p_t,r)$, where $r=\max\{1,2\hat C\bar{\bar\delta}^{-\frac12}\}$ and $\hat C$ is the constant in Lemma 2.2. Since $U\subset\mathcal M\backslash S$, there is a constant $\kappa_U>0$ such that

$$(-1)^{\frac{n^2}{2}}\Omega_t\wedge\overline\Omega_t\geq\kappa_U\omega_t^n$$

on $U \cap M_t$. Note that $\Delta^1 \times \Delta^n \subset U$, and $\{t\} \times \Delta^n \subset U \cap M_t$. Since $U_{t,\delta_t} \subset \{t\} \times \Delta^n$, we have $U_{t,\delta_t} \subset U \cap M_t$. By (2.1), we derive

$$\begin{aligned}
\mathrm{Vol}_{\tilde{g}_t}(B_{\tilde{g}_t}(p_t, r)) \geq \mathrm{Vol}_{\tilde{g}_t}(U_{t,\delta_t}) &= \frac{(-1)^{\frac{n^2}{2}}}{n!} e^{\sigma_t} \int_{U_{t,\delta_t}} \Omega_t \wedge \overline{\Omega}_t \\
&\geq \frac{\kappa_U e^{\sigma_t}}{n!} \int_{U_{t,\delta_t}} \omega_t^n \\
&= \kappa_U e^{\sigma_t} \mathrm{Vol}_{g_t}(U_{t,\delta_t}) \\
&\geq \frac{\kappa_U e^{\sigma_t}}{2} \mathrm{Vol}_{g_t}(\Delta^n) \\
&\geq C_5 e^{\sigma_t} \mathrm{Vol}_{g_E}(\Delta^n) = C_6 e^{\sigma_t},
\end{aligned}$$

where C_6 is a constant independent of t. By Bishop-Gromov relative volume comparison, we obtain

$$\mathrm{Vol}_{\tilde{g}_t}(B_{\tilde{g}_t}(p_t, 1)) \geq \frac{1}{r^{2n}} \mathrm{Vol}_{\tilde{g}_t}(B_{\tilde{g}_t}(p_t, r)) \geq \frac{C_6}{r^{2n}} e^{\sigma_t}.$$

In the rest of the proof, we need the following lemma.

Lemma 2.3 (theorem 4.1 of chapter 1 in [**50**] and lemma 2.3 in [**44**]). *Let (M, g) be a $2n$-dimensional compact Riemannian manifold with nonnegative Ricci curvature. Then for any $p \in M$ and any $1 < R < \mathrm{diam}_g(M)$, we have*

$$\frac{\mathrm{Vol}_g(B_g(p, 2R+2))}{\mathrm{Vol}_g(B_g(p, 1))} \geq \frac{R-1}{2n}.$$

By letting $R = \frac{1}{2}\mathrm{diam}_{\tilde{g}_t}(M_t)$, we obtain

$$\mathrm{diam}_{\tilde{g}_t}(M_t) \leq 2 + 8n \frac{\mathrm{Vol}_{\tilde{g}_t}(M_t)}{\mathrm{Vol}_{\tilde{g}_t}(B_{\tilde{g}_t}(p_t, 1))} \leq 2 + De^{-\sigma_t},$$

where D is a constant independent of t. We conclude the proof by (2.1). q.e.d.

The following is a consequence of Theorem 2.1 and Theorem B.1.

Corollary 2.4. *Let M_0, $\mathcal{M}$, $\mathcal{L}$, Ω_t, and $\tilde{g}_t$ be as in Theorem 2.1. If in addition we assume that M_0 is a Calabi–Yau n-variety, then the diameter of $(M_t, \tilde{g}_t)$ has a uniform bound*

$$\mathrm{diam}_{\tilde{g}_t}(M_t) \leq D,$$

where D is a constant independent of t.

3. Proof of Theorem 1.4

Let M_0 be an n-dimensional Calabi–Yau variety with singular set S. Assume that M_0 admits a smoothing $\pi : \mathcal{M} \to \Delta$ over the unit disc $\Delta \subset \mathbb{C}$ such that $\mathcal{M}$ admits an ample line bundle $\mathcal{L}$, and the relative canonical bundle is trivial, i.e., $\mathcal{K}_{\mathcal{M}/\Delta} \cong \mathcal{O}_{\mathcal{M}}$. Following the discussion at the beginning of the proof of Theorem 2.1, let

$$\omega_t = \omega_h|_{M_t} = \frac{1}{m}\omega_{FS}|_{M_t}, \quad \text{and} \quad \omega_h = \sqrt{-1}\partial\bar{\partial}|t|^2 + \frac{1}{m}\omega_{FS},$$

for any $t \in \Delta$, where ω_{FS} is the standard Fubini-Study metric on $\mathbb{CP}^N$, and g_t is the corresponding Kähler metric of ω_t. Let Ω_t be a relative holomorphic volume form, i.e., a nowhere vanishing section of $\mathcal{K}_{\mathcal{M}/\Delta}$. Yau's proof of Calabi's conjecture ([**61**]) asserts that there is a unique Ricci-flat Kähler metric $\tilde{g}_t$ with Kähler form $\tilde{\omega}_t \in [\omega_t] = c_1(\mathcal{L})|_{M_t} \in H^{1,1}(M_t, \mathbb{R})$ for $t \in \Delta\backslash\{0\}$, i.e., there is a unique function φ_t on M_t satisfying that $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi_t$, and

$$(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = (-1)^{\frac{n^2}{2}} e^{\sigma_t}\Omega_t \wedge \overline{\Omega}_t, \quad \text{with} \quad \sup_{M_t}\varphi_t = 0 \tag{3.1}$$

where

$$\sigma_t = \log\left(n!V((-1)^{\frac{n^2}{2}}\int_{M_t}\Omega_t \wedge \overline{\Omega}_t)^{-1}\right)$$

and $V = \mathrm{Vol}_{\tilde{g}_t}(M_t)$.

By Theorem B.1, on M_t we have

$$(-1)^{\frac{n^2}{2}}\Omega_t \wedge \overline{\Omega}_t \geq \kappa\omega_t^n, \tag{3.2}$$

and

$$\int_{M_t}(-1)^{\frac{n^2}{2}}\Omega_t \wedge \overline{\Omega}_t \leq \Lambda, \tag{3.3}$$

where $\kappa > 0$ and $\Lambda > 0$ are constants independent of $t \in \Delta\backslash\{0\}$. Thus there is a constant $C_1 > 0$ independent of t such that

$$-C_1 \leq \sigma_t \leq C_1. \tag{3.4}$$

Note that $(M_t, \tilde{g}_t)$ satisfies that

$$\mathrm{Ric}_{\tilde{g}_t} \equiv 0, \quad \mathrm{Vol}_{\tilde{g}_t}(M_t) \equiv V, \quad \text{and} \quad \mathrm{diam}_{\tilde{g}_t}(M_t) \leq D,$$

where the upper bound of diameters is from Corollary 2.4. By [**15**], [**21**], and [**40**], $(M_t, \tilde{g}_t)$ has uniform Sobolev constants, i.e., constants $\bar{C}_{S,1} > 0$ and $\bar{C}_{S,2} > 0$ independent of t such that for any $t \neq 0$ and any smooth function χ on M_t,

$$\|\chi\|^2_{L^{\frac{4n}{2n-2}}(\tilde{g}_t)} \leq \bar{C}_{S,1}(\|d\chi\|^2_{L^2(\tilde{g}_t)} + \|\chi\|^2_{L^2(\tilde{g}_t)}), \tag{3.5}$$

and if $\int_{M_t} \chi dv_{\tilde{g}_t} = 0$,

$$\|\chi\|^2_{L^{\frac{4n}{2n-2}}(\tilde{g}_t)} \leq \bar{C}_{S,2} \|d\chi\|^2_{L^2(\tilde{g}_t)}. \tag{3.6}$$

Now following the standard Moser iteration argument in [**61**] with a trick inspired by [**56**], we are able to get a uniform C^0-estimate of the potential function φ_t.

Lemma 3.1. *There is a constant $C > 0$ independent of $t \in \Delta\backslash\{0\}$ such that*

$$\|\varphi_t\|_{C^0(M_t)} \leq C.$$

Proof. Let

$$f_t = \log\left(\frac{(-1)^{\frac{n^2}{2}} e^{\sigma_t} \Omega_t \wedge \overline{\Omega}_t}{\omega_t^n}\right), \qquad \tilde{\varphi}_t = \int_{M_t} \varphi_t dv_{\tilde{g}_t} - \varphi_t.$$

Then (3.1) shows that

$$\omega_t^n = e^{-f_t} \tilde{\omega}_t^n = (\tilde{\omega}_t + \sqrt{-1}\partial\overline{\partial}\tilde{\varphi}_t)^n, \quad \text{with} \quad \int_{M_t} \tilde{\varphi}_t dv_{\tilde{g}_t} = 0.$$

By (3.2) and (3.4), there is a constant $C_2 > 0$ independent of t such that

$$e^{-f_t} = \left(\frac{(-1)^{\frac{n^2}{2}} e^{\sigma_t} \Omega_t \wedge \overline{\Omega}_t}{\omega_t^n}\right)^{-1} \leq C_2.$$

Now we follow the standard Moser iteration argument in [**61**] (cf. [**4**]).

A direct calculation shows that

$$\int_{M_t} |d|\tilde{\varphi}_t|^{\frac{p}{2}}|^2 dv_{\tilde{g}_t} \leq \frac{np^2}{4(p-1)} \int_{M_t} |1-e^{-f_t}||\tilde{\varphi}_t|^{p-1} dv_{\tilde{g}_t} \leq Ap \int_{M_t} |\tilde{\varphi}_t|^{p-1} dv_{\tilde{g}_t}, \tag{3.7}$$

for any $p \geq 2$ (cf. (15) in chapter 7 of [**4**]), where $A > 0$ is a constant independent of t. For $p = 2$, by (3.6), (3.7), and Hölder's inequality we see that

$$\begin{aligned} \|\tilde{\varphi}_t\|^2_{L^{\frac{4n}{2n-2}}(\tilde{g}_t)} &\leq \bar{C}_{S,2} \|d\tilde{\varphi}_t\|^2_{L^2(\tilde{g}_t)} \\ &\leq 2A\bar{C}_{S,2} \int_{M_t} |\tilde{\varphi}_t| dv_{\tilde{g}_t} \\ &\leq 2A\bar{C}_{S,2} V^{\frac{2n+2}{4n}} \|\tilde{\varphi}_t\|_{L^{\frac{4n}{2n-2}}(\tilde{g}_t)}, \end{aligned}$$

and thus

$$\|\tilde{\varphi}_t\|_{L^{\frac{4n}{2n-2}}(\tilde{g}_t)} \leq \hat{C},$$

where $\hat{C}$ is a constant independent of t. For $p > 2$, by (3.5), (3.7), and Hölder's inequality we see that

$$\begin{aligned}\|\tilde{\varphi}_t\|^p_{L^{\frac{2np}{2n-2}}(\tilde{g}_t)} &= \||\tilde{\varphi}_t|^{\frac{p}{2}}\|^2_{L^{\frac{4n}{2n-2}}(\tilde{g}_t)} \\ &\leq \bar{C}_{S,1}(\|d|\tilde{\varphi}_t|^{\frac{p}{2}}\|^2_{L^2(\tilde{g}_t)} + \||\tilde{\varphi}_t|^{\frac{p}{2}}\|^2_{L^2(\tilde{g}_t)}) \\ &\leq \bar{C}_{S,1}(pAV^{\frac{1}{p}} + \|\tilde{\varphi}_t\|_{L^p(\tilde{g}_t)})\|\tilde{\varphi}_t\|^{p-1}_{L^p(\tilde{g}_t)}.\end{aligned}$$

Let $p_0 = \frac{4n}{2n-2}$, $p_{k+1} = \frac{2n}{2n-2}p_k$ $(k \geq 0)$, let $\hat{C}_0 = \hat{C}$ and let $\hat{C}_{k+1} = \bar{C}_{S,1}^{\frac{1}{p_k}}(p_kAV^{\frac{1}{p_k}}+1)^{\frac{1}{p_k}}\hat{C}_k$ if $\hat{C}_k > 1$. Otherwise, let $\hat{C}_{k+1} = \bar{C}_{S,1}^{\frac{1}{p_k}}(p_kAV^{\frac{1}{p_k}} + 1)^{\frac{1}{p_k}}$. Then $\|\tilde{\varphi}_t\|_{L^{p_k}(\tilde{g}_t)} \leq \hat{C}_k < C_3$, a constant $C_3 > 0$ independent of k and t. By letting $k \to \infty$, we have

$$\|\tilde{\varphi}_t\|_{C^0(M_t)} \leq C_3.$$

Since there is a $p_t \in M_t$ such that $\varphi_t(p_t) = 0$, we have

$$\left|\int_{M_t} \varphi_t dv_{\tilde{g}_t}\right| \leq C_3, \quad \text{and} \quad \|\varphi_t\|_{C^0(M_t)} \leq C,$$

where $C > 0$ is a constant independent of t. q.e.d.

The C^2-estimate for φ_t is obtained by the same arguments as in the proof of lemma 5.2 in [**49**]. For completeness, we present it here.

Lemma 3.2. *For any compact subset $K \subset \mathcal{M}\backslash S$, there exists a constant $C_K > 0$ independent of t such that on $K \cap M_t$,*

$$C\omega_t \leq \tilde{\omega}_t \leq C_K\omega_t,$$

where $C > 0$ is a constant independent of t and K.

Proof. Let $\psi_t : (M_t, \tilde{\omega}_t) \longrightarrow (\mathbb{CP}^N, \frac{1}{m}\omega_{FS})$ be the inclusion map induced by $\mathcal{M} \subset \mathbb{CP}^N \times \Delta$. The Chern-Lu inequality says

$$\Delta_{\tilde{\omega}_t} \log|\partial\psi_t|^2 \geq \frac{\mathrm{Ric}_{\tilde{\omega}_t}(\partial\psi_t, \overline{\partial\psi_t})}{|\partial\psi_t|^2} - \frac{\mathrm{Sec}(\partial\psi_t, \overline{\partial\psi_t}, \partial\psi_t, \overline{\partial\psi_t})}{|\partial\psi_t|^2},$$

where Sec denotes the holomorphic bi-sectional curvature of $\frac{1}{m}\omega_{FS}$ (cf. [**62**]). Note that $\frac{1}{m}\psi_t^*\omega_{FS} = \omega_t$, $|\partial\psi_t|^2 = \frac{1}{m}\mathrm{tr}_{\tilde{\omega}_t}\psi_t^*\omega_{FS} = \mathrm{tr}_{\tilde{\omega}_t}\omega_t = n - \Delta_{\tilde{\omega}_t}\varphi_t$ and $\mathrm{Ric}_{\tilde{\omega}_t} = 0$. Thus we have that

$$\Delta_{\tilde{\omega}_t}(\log \mathrm{tr}_{\tilde{\omega}_t}\omega_t - 2\overline{R}\varphi_t) \geq -2\overline{R}n + \overline{R}\mathrm{tr}_{\tilde{\omega}_t}\omega_t.$$

where $\overline{R}$ is a constant depending only on the upper bound of Sec. By the maximum principle and Lemma 3.1, there is an $x \in M_t$ such that $\mathrm{tr}_{\tilde{\omega}_t}\omega_t(x) \leq 2n$,

$$\mathrm{tr}_{\tilde{\omega}_t}\omega_t \leq 2ne^{2\overline{R}(\varphi_t - \varphi_t(x))} \leq C \quad \text{and} \quad \omega_t \leq C\tilde{\omega}_t,$$

where $C > 0$ is a constant independent of t. Note that for any compact subset $K \subset \mathcal{M}\backslash S$, by (3.4) and the compactness of K there exists a constant $C'_K > 0$ independent of t such that on $K \cap M_t$

$$\tilde{\omega}_t^n = e^{\sigma_t}(-1)^{\frac{n^2}{2}}\Omega_t \wedge \overline{\Omega}_t \leq C'_K \omega_t^n.$$

Then we obtain that

$$C\omega_t \leq \tilde{\omega}_t \leq C_K \omega_t.$$

q.e.d.

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. In [**18**], it is proved that there is a unique bounded function $\hat{\varphi}_0$ on M_0 such that $\hat{\varphi}_0$ is smooth on $M_0\backslash S$ and satisfies

$$(3.8) \qquad (\omega_0 + \sqrt{-1}\partial\overline{\partial}\hat{\varphi}_0)^n = (-1)^{\frac{n^2}{2}} e^{\hat{\sigma}_0}\Omega_0 \wedge \overline{\Omega}_0, \quad \sup_{M_0} \hat{\varphi}_0 = 0,$$

in the distribution sense, where $\hat{\sigma}_0$ is a constant. Note that $\omega = \omega_0 + \sqrt{-1}\partial\overline{\partial}\hat{\varphi}_0$ is the unique singular Ricci-flat Kähler form with $\|\hat{\varphi}_0\|_{L^\infty} \leq C$. Let $F : (M_0\backslash S) \times \Delta \longrightarrow \mathcal{M}$ be a smooth embedding such that $F((M_0\backslash S) \times \{t\}) \subset M_t$ and $F|_{(M_0\backslash S)\times\{0\}} : M_0\backslash S \longrightarrow M_0\backslash S$ is the identity map. Let $K_1 \subset \cdots \subset K_i \subset \cdots \subset M_0\backslash S$ be a sequence of compact subsets such that $M_0\backslash S = \bigcup_i K_i$. On a fixed K_i, the embedding map

$$F_{K_i,t} = F|_{K_i\times\{t\}} : K_i \longrightarrow M_t$$

satisfies that $F^*_{K_i,t}\omega_t$ C^∞-converges to ω_0, and $dF^{-1}_{K_i,t} J_t dF_{K_i,t}$ C^∞-converges to J_0, where J_t (resp. J_0) is the complex structure on M_t (resp. M_0).

For a fixed K_i, let K be a compact subset of $\mathcal{M}\backslash S$ such that $F_{K_i,t_k}(K_i) \subset K$ for $|t_k| \ll 1$. By (3.4), Lemma 3.1, and Lemma 3.2, there exist constants $C > 0$ and $C_K > 0$ independent of t such that $C^{-1} \leq \sigma_t \leq C$, $\|\varphi_t\|_{C^0(M_t)} \leq C$, and $C^{-1}\omega_t \leq \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi_t \leq C_K\omega_t$ on K. By theorem 17.14 in [**31**], we have that $\|\varphi_t\|_{C^{2,\alpha}(M_t\cap K)} \leq C''_K$ for a constant $C''_K > 0$. Furthermore, by the standard bootstrapping argument we have that for any $l > 0$, $\|\varphi_t\|_{C^{l,\alpha}(M_t\cap K)} \leq C_{K,l}$ for constants $C_{K,l} > 0$ independent of t. By the standard diagonal arguments and passing to a subsequence, we see that $F^*_{K_{i_k},t_k}\varphi_{t_k}$ C^∞-converges to a smooth function φ_0 on $M_0\backslash S$ with $\|\varphi_0\|_{L^\infty} < C$ and that σ_{t_k} converges to a σ_0, which satisfies

$$(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_0)^n = (-1)^{\frac{n^2}{2}} e^{\sigma_0}\Omega_0 \wedge \overline{\Omega}_0.$$

Hence $\tilde{\omega}_0 = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_0$ is a Ricci-flat Kähler form on $M_0\backslash S$ with $\|\varphi_0\|_{L^\infty} < C$. By the uniqueness of the solution of (3.8), $\varphi_0 = \hat{\varphi}_0$ and $\sigma_0 = \hat{\sigma}_0$. The uniqueness of ω and the standard compactness argument imply that $F|_{M_0\backslash S\times\{t\}}^*\tilde{\omega}_t$ (resp. $F|_{M_0\backslash S\times\{t\}}^*\tilde{g}_t$) C^∞-converges to ω (resp. g) when $t\to 0$.

The diameter estimate is obtained by Corollary 2.4. q.e.d.

4. An Almost Gauge Fixing Theorem

Let M be a compact n-manifold, and let g_k be a sequence of Riemannian metrics on M. Assume that the Ricci curvature, volume, and diameter of g_k satisfy

i) $|\mathrm{Ric}(g_k)|\le 1, \mathrm{Vol}_{g_k}(M)\ge V>0$ and $\mathrm{diam}_{g_k}(M)\le D$.

By the Gromov's pre-compactness theorem, we may assume

ii) $(M,g_k)\xrightarrow{d_{GH}}(X,d_X)$, where (X,d_X) is a compact metric space.

Suppose, in addition,

iii) E is a closed subset of Hausdorff dimension $\le n-2$, and there is a (non-complete) Riemannian metric g_∞ on $M\backslash E$ such that g_k converges to g_∞ in the C^∞-sense on any compact subset $K\subset M\backslash E$.

Because $M\backslash E$ is path connected, g_∞ induces the Riemannian distance structure defined by

$$\begin{aligned}d_{g_\infty}(x,y)=\inf_{\gamma\text{ continuous}}&\{\mathrm{length}_{g_\infty}(\gamma),\ \ \gamma:[0,1]\to M\backslash E,\ \gamma(0)\\ &=x,\gamma(1)=y\}.\end{aligned}$$

Let $\overline{(M\backslash E,g_\infty)}$ denote the metric completion of $(M\backslash E,d_{g_\infty})$. Let $S_X\subset X$ denote the subset consisting of points $x\in X$ such that there is a sequence $x_k\in E\subset(M,g_k)$ and $x_k\to x$ (see comments at the end of Appendix A). It is clear that $S_X\subset X$ is a closed subset and thus S_X is compact.

The main effort of this section is to prove the following result.

Theorem 4.1. *Let M, g_k, g_∞, d_{g_∞}, E, (X,d_X), and S_X be as above. Then there is a continuous surjection $f:\overline{(M\backslash E,d_{g_\infty})}\to(X,d_X)$ such that $f:(M\backslash E,d_{g_\infty})\to(X\backslash S_X,d_X)$ is a homeomorphism and a local isometry, i.e., for any $x\in M\backslash E$, there is an open neighborhood of x, $U\subset M\backslash E$, such that $f:(U,d_{g_\infty}|_U)\to(f(U),d_X|_{f(U)})$ is an isometry.*

Proof. We first construct a dense subset $A\subseteq X\backslash S_X$ and define a local isometric embedding $h:(A,d_X)\to(M\backslash E,d_{g_\infty})$ such that $f(A)$ is dense. Then we will show that $f=h^{-1}:h(A)\to X\backslash S_X$ extends uniquely to a continuous surjection $f:\overline{(M\backslash E,g_\infty)}\to(X,d_X)$ such that f is a homeomorphism and a local isometric embedding on $(M\backslash E,d_{g_\infty})$.

Without loss of generality, we may assume that for all $k\ge j$, d_{GH} $((M,g_k),(M,g_j))<2^{-j}$. Let $\phi_j:(M,g_{j+1})\to(M,g_j)$ denote an 2^{-j}-Gromov–Hausdorff approximation. Then $\phi_j^{j+s}=\phi_j\circ\cdots\circ\phi_{j+s-1}:(M,g_{j+s})\to(M,g_{j+s-1})\to\cdots\to(M,g_j)$ is an 2^{-j+1}-Gromov–

Hausdorff approximation. Recall that there is an admissible metric d_Z on the disjoint union $Z = (\coprod_{k=1}^{\infty}(M, g_k)) \coprod (X, d_X)$ such that $(M, g_k) \xrightarrow{d_{Z,H}} (X, d_X)$ (see Appendix A, Proposition A.1).

Let $\epsilon_j = j^{-1}$, $j = 1, 2, \ldots$. For ϵ_1 and each g_k, take a finite ϵ_1-net $\{x^k_{i_1}\} \subset (M\backslash E, g_k)$ such that

$$|\{x^k_{i_1}\}| \leq c'_1, \quad \text{and} \tag{4.1}$$

$$\begin{aligned} d_{g_k}(\{x^k_{i_1}\}, E) \geq \frac{\epsilon_1}{2}, \quad \text{where } \ & d_{g_k}(\{x^k_{i_1}\}, E) = \min\{d_{g_k}(x^k_j, y), \\ & x^k_j \in \{x^k_{i_1}\},\ y \in E\}. \end{aligned} \tag{4.2}$$

We may assume, passing to a subsequence if necessary, that $\{x^k_{i_1}\} \xrightarrow{d_{H,Z}} \{x_{i_1}\}_{i_1=1}^{c_1} \subset (X, d_X)$, where $c_1 = |\{x_{i_1}\}|$. Then by (4.2), $\{x_{i_1}\}_{i_1=1}^{c_1} \subset X\backslash S_X$. We claim that there is $\bar{k}_1 > 0$ such that for all $k \geq \bar{k}_1$, $\{\phi^k_{\bar{k}_1}(x^k_{i_1})\} \subset K_1 = M\backslash B_{g_{\bar{k}_1}}(E, \frac{\epsilon_1}{4})$, a compact subset. Here $B_{g_{\bar{k}_1}}(E, \frac{\epsilon_1}{4}) = \{y \in M | d_{g_{\bar{k}_1}}(y, E) < \frac{\epsilon_1}{4}\}$. Assuming the claim, by iii) we may assume that passing to a subsequence $\{\phi^k_{\bar{k}_1}(x^k_{i_1})\} \to \{y_{i_1}\}_{i_1=1}^{c_1} \subset (M\backslash E, g_\infty)$ point-wise, and we denote the corresponding subsequence by $\{g_{k_1}\} \subset \{g_k\}$.

To verify the claim, we may assume $\bar{k}_1$ large so that for all $k \geq \bar{k}_1$,

$$d_{Z,H}(\{x^k_{i_1}\}, \{x_{i_1}\}) < \frac{\epsilon_1}{9}, \qquad 2^{-\bar{k}_1} \ll \epsilon_1.$$

For the sake of distinction, let $E_0 = E \subset (M, g_{\bar{k}_1})$. Then

$$\begin{aligned} d_{g_{\bar{k}_1}}(\{\phi^k_{\bar{k}_1}(x^k_{i_1})\}, E_0) &= d_{Z,H}(\{\phi^k_{\bar{k}_1}(x^k_{i_1})\}, E_0) \\ &\geq d_{Z,H}(\{x^{\bar{k}_1}_{i_1}\}, E_0) - d_{Z,H}(\{x^{\bar{k}_1}_{i_1}\}, \{\phi^k_{\bar{k}_1}(x^k_{i_1})\}) \\ &\geq \frac{\epsilon_1}{2} - [d_{Z,H}(\{x^{\bar{k}_1}_{i_1}\}, \{x_{i_1}\}) + d_{Z,H}(\{\phi^k_{\bar{k}_1}(x^k_{i_1})\}, \{x_{i_1}\})] \\ &\geq \frac{\epsilon_1}{2} - \left[\frac{\epsilon_1}{9} + d_{Z,H}(\{\phi^k_{\bar{k}_1}(x^k_{i_1})\}, \{x^k_{i_1}\}) + d_{Z,H}(\{x^k_{i_1}\}, \{x_{i_1}\})\right] \\ &\geq \frac{\epsilon_1}{2} - \left(\frac{\epsilon_1}{9} + \frac{\epsilon_1}{9} + 2^{-\bar{k}_1}\right) \geq \frac{\epsilon_1}{4}. \end{aligned}$$

For ϵ_2 and each g_{k_1}, extend $\{x^{k_1}_{i_1}\}$ to an ϵ_2-dense subset of $(M\backslash E, g_{k_1})$, $\{x^{k_1}_{i_1}\} \subset \{x^{k_1}_{i_2}\}$, such that for all g_{k_1},

$$d_{g_{k_1}}(x^{k_1}_{i_2}, x^{k_1}_{i'_2}) \geq \frac{\epsilon_2}{4}, \quad |\{x^{k_1}_{i_2}\}| \leq c'_2, \quad \text{and} \tag{4.3}$$

$$d_{g_{k_1}}(\{x^{k_1}_{i_2}\}, E) \geq \frac{\epsilon_2}{2}. \tag{4.4}$$

Similarly, by (4.3) and (4.4), passing to a subsequence we may assume that $\{x^{k_1}_{i_2}\} \xrightarrow{d_{H,Z}} \{x_{i_2}\} \subset (X\backslash S_X, d_X)$. Clearly, $\{x_{i_1}\} \subset \{x_{i_2}\}_{i_2=1}^{c_2}$, where $c_2 = |\{x_{i_2}\}|$. By the argument as in the above, we may assume large $\bar{k}_2 > \bar{k}_1$ such that for all $k \geq \bar{k}_2$, $\{\phi^k_{\bar{k}_2}(x^k_{i_2})\} \subset K_2 = (M\backslash B_{g_{k_1}}(E, \frac{\epsilon_2}{4}))$. By the compactness of K_2 and iii), we may assume that $\{\phi^k_{\bar{k}_2}(x^k_{i_2})\} \to \{y_{i_2}\}^{c_1}_{i_2=1} \subset (M\backslash E, d_{g_\infty})$ point-wise. The natural identification $\phi^k_{\bar{k}_1}(x^k_{i_1}) \leftrightarrow \phi^k_{\bar{k}_2}(x^k_{i_1})$ induces an injective map, $\{y_{i_1}\} \hookrightarrow \{y_{i_2}\}$.

Repeating this process and together with a standard diagonal argument, we obtain a sequence of finite subsets of $(X\backslash S_X, d_X)$:

$$\{x_{i_1}\}^{c_1}_{i_1=1} \subset \cdots \subset \{x_{i_s}\}^{c_s}_{i_s=1} \subset \cdots,$$

and a sequence of finite subsets of $(M\backslash E, g_\infty)$:

$$\{y_{i_1}\}^{c_1}_{i_1=1} \hookrightarrow \cdots \hookrightarrow \{y_{i_s}\}^{c_s}_{i_s=1} \hookrightarrow \cdots. \tag{4.5}$$

Let $A = \bigcup\limits_{s=1}^{\infty}\{x_{i_s}\}^{c_s}_{i_s=1}$, and $A_{k_l} = \bigcup\limits_{s=1}^{\infty}\{x^{k_l}_{i_s}\}^{c_s}_{i_s=1}$. Since A_{k_l} is dense in $(M\backslash E, g_{k_l})$ for all k_l, $A \subset (X\backslash S_X, d_X)$ is a dense subset. Let Y denote the direct limit of (4.5). Then $Y \subseteq M - E$. We now define a map, $f : A \to (M\backslash E, g_\infty)$, by

$$f(x_{i_s}) = [y_{i_s}] = \{y_{i_s} \to \cdots \to \cdots\}.$$

It is clear that f is injective since f is injective on each $\{x_{i_s}\}^{c_s}_{i_s=1}$, and $f(A)$ is dense in $(M\backslash E, g_\infty)$. From the construction of f, we see that f is a local isometric embedding: for $x \in A$ we may assume that $x = x_{i_s}$. Since $x_{i_s} \notin S_X$ which is a compact subset of X, there is an $r > 0$ such that $\bar{B}_{d_X}(x_{i_s}, r) \cap S_X = \emptyset$. Recall that we may assume $\bar{k}_v$ large and $\phi^{k_l}_{\bar{k}_v}(x^{k_l}_{i_s}) \subset K$ and $\phi^{k_l}_{\bar{k}_v}(x^{k_l}_{i_s}) \to y_{i_s}$ point-wise with respect to d_{g_∞}, where $K \subset M$ is compact such that $K \cap E = \emptyset$. Clearly, we may assume that r small and a compact subset $K' \supseteq K$ such that $B_{g_\infty}([y_{i_s}], r) \subset K'$ and $K' \cap E = \emptyset$. By iii), $(K', g_{k_l}) \to (K', g_\infty)$ in the C^∞-sense. Observe the following two facts:

(4.5) For $z, z' \in B_{g_\infty}([y_{i_s}], \frac{r}{2})$, any g_∞-minimal geodesic from z to z' is contained in $B_{g_\infty}([y_{i_s}], r)$.

(4.6) $d_{g_\infty}|_{B_{g_\infty}([y_{i_s}], \frac{r}{2})}$ (resp. $d_X|_{B(x, \frac{r}{2})}$) is determined by the lengths of curves in $B_{g_\infty}([y_{i_s}], r)$ (resp. $B_{d_X}(x_{i_s}, r)$). The two length structures coincide, because $(K', g_k) \to (K', g_\infty)$ in the C^∞ sense. As a consequence of (4.5) and (4.6), we conclude that

$$f : \left(B_{d_X}\left(x_{i_s}, \frac{r}{2}\right), d_X|_{B_{d_X}\left(x_{i_s}, \frac{r}{2}\right)}\right) \to \left(B_{g_\infty}\left([y_{i_s}], \frac{r}{2}\right), d_{g_\infty}|_{B_{d_{g_\infty}}\left([y_{i_s}], \frac{r}{2}\right)}\right)$$

is an isometry.

To uniquely extend $f : A \to (M\backslash E, g_\infty)$ to a continuous surjection $f : (X, d_X) \to \overline{(M\backslash E, d_{g_\infty})}$, one needs to show that $\{x_j\}, \{y_j\} \subset A$ such that $d_X(x_j, y_j) \to 0$ implies that $d_{g_\infty}(f(x_j), f(y_j)) \to 0$, which may require that $S_X \subset X$ has codimension at least 2. Because we do not know whether $\dim_{\mathcal{H}}(S_X) \leq \dim_{\mathcal{H}}(X) - 2$, we will instead extend $f^{-1} : f(A) \to A$ to a continuous map, $f^{-1} : \overline{(M\backslash E, d_{g_\infty})} \to (X, d_X)$. So, we may assume that $\{x_j\}, \{y_j\} \subset f(A)$ such that $d_{g_\infty}(x_j, y_j) \to 0$. Since d_{g_∞} is a length metric, there is a path $\gamma_i \subset M\backslash E$ from x_j to y_j such that $\mathrm{length}_{d_{g_\infty}}(\gamma_i) = d_{g_\infty}(x_j, y_j) + \delta_j$ and $\delta_j \to 0$. Since $f^{-1} : (M\backslash E, d_{g_\infty}) \to (X\backslash S_X, d_X)$ is a local isometric embedding,

$$\begin{aligned} d_X(f^{-1}(x_j), f^{-1}(y_j)) &\leq \mathrm{length}_{d_X}(f(\gamma_i)) = \mathrm{length}_{d_{g_\infty}}(\gamma_i) \\ &= d_{g_\infty}(x_j, y_j) + \delta_j \to 0. \end{aligned}$$

q.e.d.

5. Proofs of Theorem 1.1, Theorem 1.2, and Corollary 1.5

Let M_0 be a Calabi–Yau n-variety with singular set S that admits a crepant resolution $(\bar{M}, \bar{\pi})$, and let $\mathcal{L}_0$ be an ample line bundle on M_0. Note that there is an embedding $M_0 \hookrightarrow \mathbb{CP}^N$ such that $\mathcal{L}_0^m = \mathcal{O}(1)|_{M_0}$ for an $m \geq 1$, and that the restriction of the Fubini-Study metric $\omega_{FS}|_{M_0}$ represents $mc_1(\mathcal{L}_0)$ in $H^1(M_0, \mathcal{PH}_{M_0})$. By theorem 7.5 of [**18**], there is a unique Ricci-flat Kähler metric g on M_0 with Kähler form $\omega \in c_1(\mathcal{L}_0)$. Let $\{\bar{g}_s\}$ $(s \in (0,1])$ be a family of Ricci-flat Kähler metrics with Kähler classes $\lim\limits_{s\to 0}[\bar{\omega}_s] = \bar{\pi}^* c_1(\mathcal{L}_0)$ in $H^{1,1}(\bar{M}, \mathbb{R})$, where $\bar{\omega}_s$ denotes the corresponding Kähler form of $\bar{g}_s$. Then

$$\lim_{s\to 0} \mathrm{Vol}_{\bar{g}_s}(\bar{M}) = \frac{1}{n!} c_1^n(\mathcal{L}_0) = \frac{1}{m^n n!} \int_{M_0} \omega_{FS}^n > 0. \tag{5.1}$$

Furthermore, it is proved in [**56**] that

$$\bar{g}_s \longrightarrow \bar{\pi}^* g, \quad \text{and} \quad \bar{\omega}_s \longrightarrow \bar{\pi}^* \omega, \quad \text{when} \quad s \to 0,$$

in the C^∞-sense on any compact subset $K \subset\subset \bar{M}\backslash\bar{\pi}^{-1}(S)$. By [**49**] and [**56**], the diameter of $(\bar{M}, \bar{g}_s)$ has a uniform bound, i.e.,

$$\mathrm{diam}_{\bar{g}_s}(\bar{M}) \leq C \tag{5.2}$$

where C is a constant independent of s. By the Bishop-Gromov relative volume comparison and (5.1), $(\bar{M}, \bar{g}_s)$ is non-collapsed, i.e., there is a constant $\kappa > 0$ independent of s such that

$$\mathrm{Vol}_{\bar{g}_s}(B_{\bar{g}_s}(p, r)) \geq \kappa r^{2n}, \tag{5.3}$$

for any metric ball $B_{\bar{g}_s}(p, r) \subset (M, \bar{g}_s)$. Gromov's pre-compactness theorem (cf. [**22**]) implies that, for any sequence $s_k \to 0$, a subsequence of $(\bar{M}, \bar{g}_{s_k})$ converges to a compact length metric space (X, d_X) in the Gromov–Hausdorff topology. First, we explore some metric properties of (X, d_X).

Lemma 5.1. *Let (X, d_X) be as in the above. Then the following properties hold:*

i) *There is a closed subset* $S_X \subset X$ *of Hausdorff dimension* $\dim_{\mathcal{H}} S_X \leq 2n-4$, *and* $(X \backslash S_X, d_X|_{X \backslash S_X})$ *is a path metric space, i.e., for any* $\delta > 0$ *and any two points* $x_1, x_2 \in X \backslash S_X$, *there is a cure* $\gamma_\delta \subset X \backslash S_X$ *connecting* x_1 *and* x_2 *satisfying*

$$\mathrm{length}_{d_X}(\gamma_\delta) \leq d_X(x_1, x_2) + \delta.$$

ii) *There is a homeomorphic local isometry* $f : (X \backslash S_X, d_X) \to (M_0 \backslash S, d_g)$, *i.e., for* $x \in X \backslash S_X$, *there is an open subset* $U_x \subset\subset X \backslash S_X$ *such that for any* x_1, $x_2 \in U_x$, $d_X(x_1, x_2) = d_g(f(x_1), f(x_2))$.

iii) (X, d_X) *is isometric to the metric completion* $\overline{(M_0 \backslash S, d_g)}$.

Proof. Applying general theorems in [**10**], [**13**], and [**12**] to our situation, i.e., $(\bar{M}, \bar{g}_{s_k}) \overset{d_{GH}}{\longrightarrow} (X, d_X)$, we see the following properties:

a) There is a closed subset $S' \subset X$ of Hausdorff dimension $\dim_{\mathcal{H}} S' \leq 2n-4$ such that for any $x \in S'$, there is a tangent cone $T_x X$ that is not isometric to $\mathbb{R}^{2n}$.

b) $X \backslash S'$ is a smooth open complex manifold, and $d_X|_{X \backslash S'}$ is induced by a Ricci-flat Kähler metric g_∞ on $X \backslash S'$.

From section 3 of [**11**], we see that for any $x_1, x_2 \in X \backslash S'$, and any $\delta > 0$, there is a curve γ_δ connecting x_1 and x_2 in $X \backslash S'$ such that

$$\mathrm{length}_{d_X}(\gamma_\delta) \leq \delta + d_X(x_1, x_2).$$

Note that $\bar{\pi}^{-1}(S)$ is a finite disjoint union of complex subvarieties E_i, i.e., $\bar{\pi}^{-1}(S) = \coprod E_i$. If $S_X \subset X$ denotes the subset consisting of points $x \in X$ such that for each k there is an $\bar{x}_k$ in the smooth part of $\bar{\pi}^{-1}(S) \subset (\bar{M}, \bar{g}_{s_k})$ and $\bar{x}_k \to x$ under the Gromov–Hausdorff convergence of $\{(\bar{M}, \bar{g}_{s_k})\}$ to (X, d_X), then by Theorem 4.1 there is a homeomorphic local isometry $f : (X \backslash S_X, d_X) \to (M_0 \backslash S, g)$. Thus, for any $x \in X \backslash S_X$, the tangent cone $T_x X$ is unique and isometric to $\mathbb{R}^{2n}$, which implies that $X \backslash S_X \subseteq X \backslash S'$, i.e., $S' \subseteq S_X$.

We claim that $S_X = S'$. If false, there is an $x \in S_X \backslash S'$ and there is a $\sigma > 0$ such that the metric ball $B_{g_\infty}(x, \sigma) \subset X \backslash S'$. By the volume convergence theorem due to Cheeger and Colding (cf. [**9**], [**10**]) and from $\bar{x}_k \to x$, we derive that for any $0 < \rho \leq \sigma$,

$$\lim_{k\to\infty} \mathrm{Vol}_{\bar{g}_{s_k}}(B_{\bar{g}_{s_k}}(\bar{x}_k, \rho)) = \mathrm{Vol}_{g_\infty}(B_{g_\infty}(x, \rho)).$$

Since g_∞ is a smooth metric, $\lim_{\rho\to 0} |\frac{\mathrm{Vol}_{g_\infty}(B_{g_\infty}(x,\rho))}{\varpi_{2n}\rho^{2n}} - 1| = 0$, where ϖ_{2n} denotes the volume of the metric 1-ball in the Euclidean space $\mathbb{R}^{2n}$. Thus for any $\varepsilon > 0$ we can find a $\rho \ll 1$ and a $k(\rho) \gg 1$ such that for any $k \geq k(\rho)$ we have

$$\left| \frac{\mathrm{Vol}_{\bar{g}_{s_k}}(B_{\bar{g}_{s_k}}(\bar{x}_k, \rho))}{\varpi_{2n}\rho^{2n}} - 1 \right| \leq \varepsilon.$$

By the proof of theorem 3.2 in [**3**], we see that there is a uniform lower bound $0 < \rho_h < \rho$ (independent of s_k) for the harmonic radius of $\bar{g}_{s_k}$ at $\bar{x}_k$, i.e., there are harmonic coordinates $h^1, \dots, h^{2n}$ on $B_{\bar{g}_{s_k}}(\bar{x}_k, \rho_h)$ such that $\bar{g}_{s_k} = \sum_{ij} \bar{g}_{s_k,ij} dh^i dh^j$,

$$2^{-1}(\delta_{ij}) \leq (\bar{g}_{s_k,ij}) \leq 2(\delta_{ij}), \qquad \rho^{1+\alpha}\|\bar{g}_{s_k,ij}\|_{C^{1,\alpha}} \leq 2,$$

where $\alpha \in (0,1)$. Furthermore, by Ricci flatness there are constants $C_l > 0$ independent of k such that

$$\|\bar{g}_{s_k,ij}\|_{C^l} \leq C_l$$

on $B_{\bar{g}_{s_k}}(\bar{x}_k, \frac{\rho_h}{2})$ (cf. section 4 in [**1**]). Hence the sectional curvature $\mathrm{Sec}_{\bar{g}_{s_k}}$ of $\bar{g}_{s_k}$ on $B_{\bar{g}_{s_k}}(\bar{x}_k, \frac{\rho_h}{2})$ and the injectivity radius $i_{\bar{g}_{s_k}}(\bar{x}_k)$ have uniform bounds,

$$\sup_{B_{\bar{g}_{s_k}}(\bar{x}_k, \frac{\rho_h}{2})} |\mathrm{Sec}_{\bar{g}_{s_k}}| \leq \Lambda, \qquad i_{\bar{g}_{s_k}}(\bar{x}_k) > \iota,$$

where Λ and ι are two constants independent of k.

In the rest of the proof of Lemma 5.1, we need the following theorem.

Theorem 5.2. *Let (M, g, ω) be a complete Kähler n-manifold, and $p \in M$. Assume that the sectional curvature Sec_g satisfies*

$$\sup_{B_g(p, \frac{2\pi}{\sqrt{\Lambda}})} \mathrm{Sec}_g \leq \Lambda, \qquad \Lambda > 0,$$

and there is a complex subvariety E of dimension $m \leq n$ such that p belongs to the regular part of E. Then

$$\mathrm{Vol}_g(B_g(p,r) \cap E) \geq \varpi r^{2m},$$

for any $r \leq \min\{i_g(p), \frac{\pi}{2\sqrt{\Lambda}}\}$, where $i_g(p)$ denotes the injectivity radius of g at p, and $\varpi = \varpi(m, \Lambda)$ is a constant depending only on m and Λ.

Note that similar volume comparison results were obtained for smooth minimal submanifolds in [**42**] and [**23**], for complex subvarieties of $\mathbb{C}^n$ in [**25**], and for minimal currents in $\mathbb{R}^n$ (cf. [**43**]). Since the authors could not find a proof of Theorem 5.2 in the literature, we shall present a proof at the end of this section.

By Theorem 5.2 and taking $r = \min\{\iota, \frac{\pi}{2\sqrt{\Lambda}}, \frac{\rho_h}{2}\}$, we obtain

$$\mathrm{Vol}_{\bar{g}_{s_k}}(\bar{\pi}^{-1}(S)) \geq \mathrm{Vol}_{\bar{g}_{s_k}}(\bar{\pi}^{-1}(S) \cap B_{\bar{g}_{s_k}}(\bar{x}_k, r)) > C,$$

where $C > 0$ is a constant independent of k. On the other hand, since $\lim_{s \to 0} [\bar{\omega}_s] = \bar{\pi}^* c_1(\mathcal{L})|_{M_0}$ in $H^{1,1}(M, \mathbb{R})$, we have

$$\lim_{k \to \infty} \mathrm{Vol}_{\bar{g}_{s_k}}(\bar{\pi}^{-1}(S)) = \sum_i \lim_{k \to \infty} \frac{1}{\dim_{\mathbb{C}} E_i!} \int_{E_i} \bar{\omega}_{s_k}^{\dim_{\mathbb{C}} E_i} = 0,$$

which is a contradiction.

Note that by i) $(X\backslash S_X, d_X)$ coincides with the length metric structure $(X\backslash S_X, d_{d_X})$, i.e., for any two points $x_1, x_2 \in X\backslash S_X$,

$$d_X(x_1,x_2) = \inf\{\text{length}_{d_X}(\gamma)|\ \text{all curves } \gamma \subset X\backslash S_X \text{ connecting } x_1 \text{ and } x_2\}.$$

Consequently, $f : (X\backslash S_X, d_X) \to (M_0\backslash S, d_g)$ is an isometry, and (X, d_X) is the unique metric completion of $(X\backslash S_X, d_X)$ since $X\backslash S_X$ is dense in (X, d_X). We obtain iii). q.e.d.

Lemma 5.3. *Let* (X, d_X) *be as in Lemma* 5.1*, and let* (M_k, g_k, ω_k) *be any family of Ricci-flat Kähler* n*-dimensional manifolds satisfying*

i)
$$\lim_{k\to\infty} \text{Vol}_{g_k}(M_k) = \frac{1}{n!}c_1^n(\mathcal{L}_0).$$

ii) *There is a family of embeddings* $F_k : M_0\backslash S \to M_k$ *such that*
$$F_k^* g_k \to g, \quad \text{and} \quad F_k^*\omega_k \to \omega, \quad \text{when} \quad k \to \infty,$$
in the C^∞*-sense on any compact subset* $K \subset\subset M_0\backslash S$*.*

Then
$$(M_k, g_k) \xrightarrow{d_{GH}} (X, d_X) \xleftarrow{d_{GH}} (\bar{M}, \bar{g}_{s_k}).$$

Proof. By Lemma 5.1, there is a homeomorphic local isometry $f : (X\backslash S_X, d_X) \to (M_0\backslash S, g)$. For an $x \in X\backslash S_X$, ii) of Lemma 5.3 implies that $\text{Vol}_{g_k}(B_{g_k}(F_k(f(x)), 1)) \geq \upsilon$ for a constant $\upsilon > 0$ independent of k. For any $1 < R < \text{diam}_{g_k}(M_k)$, Lemma 2.3 (lemma 2.3 in [**44**]) shows that
$$R \leq 1 + 2n\frac{\text{Vol}_{g_k}(B_{g_k}(F_k(f(x)), 2R+2))}{\text{Vol}_{g_k}(B_{g_k}(F_k(f(x)), 1))}.$$
By taking $R = \frac{1}{2}\text{diam}_{g_k}(M_k)$, we obtain that
$$\text{diam}_{g_k}(M_k) < 2 + 4n\upsilon^{-1}\frac{1}{n!}c_1^n(\mathcal{L}_0).$$
By the Bishop-Gromov relative volume comparison, (M_k, g_k) is non-collapsed, i.e., there is a constant $\kappa > 0$ independent of k such that for any metric ball $B_{g_k}(p, r) \subset M_k$,

$$\text{Vol}_{g_k}(B_{g_k}(p,r)) \geq \kappa r^{2n}. \tag{5.4}$$

Gromov's pre-compactness theorem implies that a subsequence of $\{(M_k, g_k)\}$ d_{GH}-converges to a compact length metric space (Y, d_Y). Following the proof of theorem 4.1 in [**49**] with a minor modification will prove that (Y, d_Y) is isometric to (X, d_X). Because of this, we only present a sketch of the proof. (There is an analog proof for the case of Ricci-solitons in [**67**].)

First, the same arguments as in the proof of lemma 4.1 in [**49**] imply that there exists an embedding $\psi' : (M_0\backslash S, g) \to (Y, d_Y)$ which is a local isometry. Hence $\psi = \psi' \circ f : (X \backslash S_X, d_X) \to (Y, d_Y)$ is a local isometric embedding. Thus, if γ is a geodesic in $(X \backslash S_X, d_X)$, then $\psi(\gamma)$ is a geodesic in $(\psi(X \backslash S_X), d_Y)$. For any $x_1, x_2 \in X$, and two sequences $\{x_{1,j}\}, \{x_{2,j}\} \subset X \backslash S_X$ converging to x_1 and x_2 respectively, there are curves γ_j connecting $x_{1,j}$ and $x_{2,j}$ in $X \backslash S_X$ with $\mathrm{length}_g(\gamma_j) \leq d_X(x_{1,j}, x_{2,j}) + \frac{1}{j}$ by Lemma 5.1, which implies

$$\begin{aligned} d_Y(\psi(x_{1,j}), \psi(x_{2,j})) &\leq \mathrm{length}_{d_Y}(\psi(\gamma_j)) = \mathrm{length}_g(\gamma_j) \\ &\leq d_X(x_{1,j}, x_{2,j}) + \frac{1}{j}. \end{aligned} \tag{5.5}$$

If $x_1 = x_2 = x$, both $\{\psi(x_j)\}$ and $\{\psi(x'_j)\}$ are Cauchy sequences, and converge to the same limit y in Y. By defining $\tilde{\psi}(x) = y$, ψ extends to a continuous map $\tilde{\psi} : X \longrightarrow Y$ such that $\tilde{\psi}(X) \subseteq Y$ is closed.

If $\tilde{\psi}(X) \subsetneq Y$, then there is a metric ball $B_{d_Y}(y, \delta) \subset\subset Y \backslash \tilde{\psi}(X)$ for a $\delta > 0$. By (5.1), (5.4), and the volume convergence theorem due to Cheeger and Colding (cf. [**9**], [**10**]), we derive

$$\mathcal{H}^{2n}(Y) = \mathcal{H}^{2n}(X) = \mathrm{Vol}_g(X \backslash S_X) \quad \text{and} \quad \mathcal{H}^{2n}(B_{d_Y}(y, \delta)) \geq \kappa \delta^{2n},$$

where $\mathcal{H}^{2n}$ denotes the $2n$-dimensional Hausdorff measure. Thus

$$\begin{aligned} \mathcal{H}^{2n}(Y) \geq \mathcal{H}^{2n}(\psi(X \backslash S_X)) + \mathcal{H}^{2n}(B_{d_Y}(y, \delta)) &\geq \mathrm{Vol}_g(X \backslash S_X) \\ &\quad + \kappa \delta^{2n} > \mathcal{H}^{2n}(Y), \end{aligned}$$

a contradiction.

To show that $\tilde{\psi}$ is an isometry, we first check that $\tilde{\psi}$ is 1-Lipschitz. For any $x_1 \neq x_2 \in X$, there are sequences of points $\{x_{i,j}\} \subset X \backslash S_X$, $i = 1, 2$, such that $d_X(x_{i,j}, x_i) \to 0$ when $j \to \infty$. Thus $d_Y(\psi(x_{i,j}), \tilde{\psi}(x_j)) \to 0$, $i = 1, 2$, when $j \to \infty$. By (5.5) and letting $j \to \infty$, we obtain that

$$d_Y(\tilde{\psi}(x_1), \tilde{\psi}(x_2)) \leq d_X(x_1, x_2) \tag{5.6}$$

i.e., $\tilde{\psi}$ is a 1-Lipschitz map.

Because $\tilde{\psi}(S_X) \bigcup \psi(X \backslash S_X) = Y$, $\tilde{\psi}(S_X) \supseteq Y \backslash \psi(X \backslash S_X)$. Since $\dim_{\mathcal{H}} S_X \leq 2n - 2$,

$$\mathcal{H}^{2n-1}(Y \backslash \psi(X \backslash S_X)) \leq \mathcal{H}^{2n-1}(\tilde{\psi}(S_X)) \leq \mathcal{H}^{2n-1}(S_X) = 0.$$

If there is a j such that $\varrho = d_X(x_{1,j}, x_{2,j}) - d_Y(\psi(x_{1,j}), \psi(x_{2,j})) > 0$, then by section 3 of [**11**] we see that there is a curve $\bar{\gamma}$ connecting $\psi(x_{1,j}), \psi(x_{2,j})$ in $\psi(X \backslash S_X)$ and

$$\begin{aligned} d_X(x_{1,j}, x_{2,j}) \leq \mathrm{length}_{d_X}(\psi^{-1}(\bar{\gamma})) &= \mathrm{length}_{d_Y}(\bar{\gamma}) \\ &\leq d_Y(\psi(x_{1,j}), \psi(x_{2,j})) + \frac{1}{2}\varrho, \end{aligned}$$

a contradiction. Then $d_X(x_{1,j}, x_{2,j}) = d_Y(\psi(x_{1,j}), \psi(x_{2,j}))$ and thus by letting $j \to \infty$, we obtain that

$$d_Y(\tilde{\psi}(x_1), \tilde{\psi}(x_2)) = d_X(x_1, x_2).$$

By now, we have proved that $\tilde{\psi} : (X, d_X) \longrightarrow (Y, d_Y)$ is an isometry.
q.e.d.

After the above preparation, we are ready to prove Theorem 1.1, Theorem 1.2 and Corollary 1.5.

Proof of Theorem 1.1. Let M_0, $\pi : \mathcal{M} \to \Delta$, $\mathcal{K}_{\mathcal{M}/\Delta}$, $\mathcal{L}$, M_t, $\tilde{g}_t$, $\tilde{\omega}_t$, $(\bar{M}, \bar{\pi})$, $\bar{g}_s$, $\bar{\omega}_s$ be as in Theorem 1.1.

Note that

$$\lim_{s\to 0} \mathrm{Vol}_{\bar{g}_s}(\bar{M}) = \frac{1}{n!} c_1^n(\mathcal{L})|_{M_0} \equiv \frac{1}{n!} c_1^n(\mathcal{L})|_{M_t} = \mathrm{Vol}_{\tilde{g}_t}(M_t).$$

By [**56**],

$$\bar{g}_s \longrightarrow \bar{\pi}^* g, \quad \text{and} \quad \bar{\omega}_s \longrightarrow \bar{\pi}^* \omega, \quad \text{when} \quad s \to 0,$$

in the C^∞-sense on any compact subset $K \subset\subset \bar{M} \backslash \bar{\pi}^{-1}(S)$. By (5.2) and Ricci flatness, we apply Gromov's compactness theorem to conclude that for any sequence $s_k \to 0$, a subsequence of $(\bar{M}, \bar{g}_{s_k})$ d_{GH}-converges to a compact path metric space (X, d_X), which satisfies the conclusion of Lemma 5.1. By Lemma 5.3, for any other sequence $s'_k \to 0$, $(\bar{M}, \bar{g}_{s'_k})$ d_{GH}-converges to (X, d_X) too. Thus

$$\lim_{s\to 0} d_{GH}((\bar{M}, \bar{g}_s), (X, d_X)) = 0.$$

By Theorem 1.4,

$$F_t^* \tilde{g}_t \to g, \quad \text{and} \quad F_t^* \tilde{\omega}_t \to \omega, \quad \text{when} \quad t \to 0,$$

in the C^∞-sense on any compact subset $K \subset\subset M_0 \backslash S$, where $F_t : M_0 \backslash S \to M_t$ is a family of embeddings. By Lemma 5.3 and the fact that the limit is independent of convergent subsequences, we obtain the conclusion,

$$\lim_{t\to 0} d_{GH}((M_t, \tilde{g}_t), (X, d_X)) = 0.$$

q.e.d.

The same argument in the proof of Theorem 1.1 also gives a proof of Theorem 1.2.

Proof of Corollary 1.5. Let M be a Calabi–Yau manifold, and let ω_s ($s \in [0, 1]$) be a family of Ricci-flat Kähler forms. It is clear that ω_s converges to ω_0 when $s \to 0$ in the C^∞-sense, which implies that $\mathfrak{M}_M$ is path connected in $(\mathfrak{X}, d_{GH})$. Let $\pi' : \bar{\mathcal{M}} \to \Delta$ be a smooth family of Calabi–Yau manifolds over the unit disc $\Delta \subset \mathbb{C}$ with an ample line bundle $\mathcal{L}$ on $\bar{\mathcal{M}}$, and let $\tilde{\omega}_t$ be the unique Ricci-flat Kähler form on $\pi'^{-1}(t) = M_t$ with $\tilde{\omega}_t \in c_1(\mathcal{L})|_{M_t}$. It is standard that $F_t^* \tilde{\omega}_t$ converges to $\tilde{\omega}_0$ in the C^∞-sense, when $t \to 0$, where $F_t : M_0 \to M_t$ is a smooth family of diffeomorphisms. Thus, if $\pi' : \bar{\mathcal{M}} \to \mathcal{D}$ is a smooth family of Calabi–Yau manifolds over connected complex manifold $\mathcal{D}$, then

$$\bigcup_{M_t=\pi'^{-1}(t), t\in\mathcal{D}} \mathfrak{M}_{M_t} \subset (\mathfrak{X}, d_{GH})$$

is path connected.

Note that Calabi–Yau manifolds are minimal models. If M and M' are two birationally equivalent three-dimensional Calabi–Yau manifolds, then M and M' are related by a sequence of flops (cf. [**38**] [**39**] [**35**]), i.e., there is a sequence of varieties $M_1, \dots, M_k$ such that $M = M_1$, $M' = M_k$, and M_{j+1} is obtained by a flop from M_j. Consequently, there are normal projective varieties $M_{0,1}, \dots, M_{0,k-1}$, and small resolutions $\bar{\pi}_j : M_j \to M_{0,j}$ and $\bar{\pi}_j^+ : M_j \to M_{0,j-1}$. By [**38**], M_j has the same singularities as M, and thus M_j is smooth. Since the exceptional loci of $\bar{\pi}_j$ and $\bar{\pi}_j^+$ are of co-dimension at least 2, $M_{0,j}$ has only canonical singularities, and the canonical bundle of $M_{0,j}$ is trivial (cf. corollary 1.5 in [**34**]). Therefore $M_{0,j}$ is a three-dimensional Calabi–Yau variety, and M_j is a three-dimensional Calabi–Yau manifold. By Theorem 1.2, for any $j > 0$,

$$\overline{\mathfrak{M}}_{M_j} \bigcup \overline{\mathfrak{M}}_{M_{j+1}}$$

is path connected, where $\overline{\mathfrak{M}}_{M_j}$ denotes the closure of $\mathfrak{M}_{M_j} \subset (\mathfrak{X}, d_{GH})$. By now we have proved i) of Corollary 1.5.

Let M_0 be a three-dimensional complete intersection Calabi–Yau conifold in $\mathbb{C}P^{m_1} \times \cdots \times \mathbb{C}P^{m_l}$, and $\bar{M}$ be a small resolution of M_0, which is a three-dimensional complete intersection Calabi–Yau (CICY) manifold in products of projective spaces. By Theorem 1.1, we see that

$$\bigcup_{\tilde{M}\in\mathfrak{D}(M_0)} \overline{\mathfrak{M}}_{\tilde{M}} \bigcup \overline{\mathfrak{M}}_{\bar{M}} \subset (\mathfrak{X}, d_{GH})$$

is path connected, where $\mathfrak{D}(M_0)$ denotes the set of three-dimensional CICY manifolds in $\mathbb{C}P^{m_1} \times \cdots \times \mathbb{C}P^{m_l}$ obtained by a smoothing of M_0. If M and M' are two three-dimensional CICY manifolds in products of projective spaces, then by [**24**] M and M' are related by a sequence of conifold transitions, or inverse conifold transitions. Precisely, there is a sequence of three-dimensional CICY manifolds $M_1, \dots, M_k$ with $M = M_1$ and $M' = M_k$ such that for any $1 \le j \le k$, there is a three-dimensional CICY conifold $M_{0,j}$ in some $\mathbb{C}P^{m_1} \times \cdots \times \mathbb{C}P^{m_l}$, M_j is a small resolution of $M_{0,j}$ and $M_{j+1} \in \mathfrak{D}(M_{0,j})$, or vice versa M_{j+1} is a small resolution of $M_{0,j}$ and $M_j \in \mathfrak{D}(M_{0,j})$. Thus ii) of Corollary 1.5 is followed, i.e., $\overline{\mathfrak{CP}}$ is path connected.

In [**5**] and [**16**], many complete intersection Calabi–Yau 3-manifolds in toric varieties were verified to be connected by extremal transitions, which include Calabi–Yau hypersurfaces in all toric 4-manifolds obtained by resolving weighted projective 4-spaces. Let $\mathfrak{CT}_0$ be the set of the above Calabi–Yau 3-manifolds with Ricci-flat Kähler metrics of volume 1. By Theorem 1.1, the closure $\overline{\mathfrak{CT}}_0$ is path connected. Note that a quintic in $\mathbb{CP}^4$ with Ricci-flat Kähler metric of volume 1 is in $\mathfrak{CT}_0 \bigcap \mathfrak{CP}$. Thus iii) of Corollary 1.5 is obtained. q.e.d.

Now we give a proof of Theorem 5.2, which can be viewed as a combination of the proof of theorem 2.0.1 in [**23**] and the proof of theorem 9.3 in [**43**].

Proof of Theorem 5.2. For any $r \leq \min\{i_g(p), \frac{\pi}{2\sqrt{\Lambda}}\}$, there are normal coordinates such that $g = d\rho^2 + g_\rho$ on $B_g(p,r)$ where g_ρ is a Riemannian metric on $\partial B_g(p,\rho)$, $0 < \rho \leq r$. Let $f(q,\rho) = \exp_p \frac{\rho}{r}(\exp_p^{-1} q)$ for any $q \in \partial B_g(p,r)$. Then $f : \partial B_g(p,r) \times (0,r] \to B_g(p,r)\backslash\{p\}$ is a diffeomorphism. For any $w \in T_q(\partial B_g(p,r))$, it is clear that $J(\rho) = df|_{(\rho,q)} w$ is a normal Jacobi field along the geodesic $\gamma(\rho) = f(q,\rho)$ with $J(0) = 0$ and $J(r) = w$. A standard Rauch comparison argument shows that

$$|J(\rho)|_{g_\rho} \leq \frac{\sin\sqrt{\Lambda}\rho}{\sin\sqrt{\Lambda}r}|J(r)|_{g_r} = \frac{\sin\sqrt{\Lambda}\rho}{\sin\sqrt{\Lambda}r}|w|_{g_r}$$

(cf. lemma 2.0.1 in [**23**]). Thus the norm of the differential $df|_{\partial B_g(p,r)\times\{\rho\}}$ corresponding to the metric g_r on $\partial B_g(p,r) \times \{\rho\}$ and g_ρ on $\partial B_g(p,\rho)$ satisfies

$$|df|_{\partial B_g(p,r)\times\{\rho\}}| \leq \frac{\sin\sqrt{\Lambda}\rho}{\sin\sqrt{\Lambda}r},$$

which implies

$$g_\rho \leq \frac{\sin^2\sqrt{\Lambda}\rho}{\sin^2\sqrt{\Lambda}r} g_r. \tag{5.7}$$

Denote

$$\Theta(r) = \mathrm{Vol}_g(B_g(p,r) \cap E).$$

Since $\Theta(r)$ is monotonically increasing, $\Theta'(r)$ exists for almost all r. By 4.11(3) in [**43**], we have

$$\mathcal{H}^{2m-1}_{g_r}(\partial B_g(p,r) \cap E) \leq \Theta'(r).$$

Let $\mathcal{C} = (\partial B_g(p,r) \cap E) \times (0,r] \subset \partial B_g(p,r) \times (0,r] = B_g(p,r)\backslash\{p\}$. By Fubini's Theorem and (5.7), we see that

$$\begin{aligned}\mathcal{H}^{2m}_g(\mathcal{C}) &= \int_0^r \mathcal{H}^{2m-1}_{g_\rho}(\partial B_g(p,r) \cap E)d\rho \\ &\leq \mathcal{H}^{2m-1}_{g_r}(\partial B_g(p,r) \cap E)\int_0^r \frac{\sin^{2m-1}\sqrt{\Lambda}\rho}{\sin^{2m-1}\sqrt{\Lambda}r}d\rho.\end{aligned}$$

Since E is a complex subvariety, E is a volume minimizer and thus

$$\Theta(r) \leq \mathcal{H}^{2m}_g(\mathcal{C}) \leq \frac{\int_0^r \sin^{2m-1}\sqrt{\Lambda}\rho d\rho}{\sin^{2m-1}\sqrt{\Lambda}r}\Theta'(r).$$

Therefore

$$\frac{d}{dr}\left(\frac{\Theta(r)}{\int_0^r \sin^{2m-1}\sqrt{\Lambda}\rho d\rho}\right) \geq 0.$$

Since p is a smooth point of E,

$$\lim_{\bar{r}\to 0}\frac{\Theta(\bar{r})}{\int_0^{\bar{r}} \sin^{2m-1}\sqrt{\Lambda}\rho d\rho} = C,$$

where C is a constant depending only on Λ and m. Thus

$$\Theta(r) \geq C\int_0^r \sin^{2m-1}\sqrt{\Lambda}\rho d\rho \geq \varpi r^{2m}.$$

q.e.d.

Appendix A. Gromov–Hausdorff Convergence of Compact Metric Spaces

In the proof of Theorem 4.1, we freely used some basic properties of the Gromov–Hausdorff convergence of compact metric spaces. For the convenience of readers, we will briefly recall related notions and proofs of these properties (cf. [**47**]).

Let (Z, d) be a metric space, and let C^Z denote the set of all compact subsets of Z. For $A, B \in C^Z$, the Hausdorff distance of A and B is

$$d_H(A, B) = \inf\{\epsilon, U_\epsilon(A) \supseteq B, U_\epsilon(B) \supseteq A\},$$

where $U_\epsilon(S)$ denotes the ϵ-neighborhood of S. Then (C^Z, d_H) is a complete metric space. The Gromov–Hausdorff distance can be viewed as an abstract extension of d_H on $\mathfrak{X}$: the space of isometry classes of all compact metric spaces. For $X, Y \in \mathfrak{X}$, the Gromov–Hausdorff distance of X and Y is

$$d_{GH}(X,Y) = \inf_Z\{d_H^Z(X,Y),$$
$$\exists \text{ isometric embeddings}, X, Y \hookrightarrow Z, \text{ a metric space}\}.$$

In the above definition, one can consider the disjoint union that $Z = X \coprod Y$ with an admissible metric d, i.e., a metric on Z such that the restriction on X (resp. Y) is the metric on X (resp. Y).

It is not hard to check that $d_{GH}(X, Y) = 0$ if and only if X is isometric to Y and d_{GH} satisfies the triangle inequality. Hence, $(\mathfrak{X}, d_{GH})$ is a metric space.

In the proof of Theorem 4.1, the following proposition is used.

Proposition A.1. *Given $\{X_i\}$ in $\mathfrak{X}$ such that $d_{GH}(X_i, X_{i+k}) < 2^{-i}$ for all i and k, let $Y = \coprod_i X_i$.*

i) There is a metric d_Y on Y such that the restriction of d_Y on each X_i is the metric on X_i and $\{X_i\}$ is a Cauchy sequence with respect to $d_{Y,H}$.

ii) Let X be the collection of equivalent Cauchy sequences, $\{\{x_i\},\ x_i \in X_i\}$, equipped with the metric $\hat{d}(\{x_i\},\{y_i\}) = \lim_{i\to\infty} d_Y(x_i,y_i)$. Then $Y \coprod X$ has an admissible metric defined by $d(x,\{x_i\}) = \lim_{i\to\infty} d(x,x_i)$.
iii) For all $\epsilon > 0$, X has a finite ϵ-dense subset (thus the completion of X is compact).
iv) $d_H(X_i, X) \to 0$ as $i \to \infty$.

Proof. i) We first take, for each i, an admissible metric $d_{i,i+1}$ on $X_i \coprod X_{i+1}$ such that $d_{i,i+1}(X_i, X_{i+1}) < d_{GH}(X_i, X_{i+1}) + 2^{-i} < 2^{-i+1}$. We then extend $\{d_{i,i+1}\}$ to an admissible metric on Y by defining, for each pair (i,j), an admissible metric $d_{i,i+j}$ on $X_i \coprod X_{i+j}$, as follows:

$$d_Y(x_i, x_{i+j}) = \inf_{x_{i+k}\in X_{i+k}} \{\sum_{k=0}^{j-1} d_{i+k,i+k+1}(x_{i+k}, x_{i+k+1})\}.$$

It is straightforward to check that d_Y satisfies the triangle inequality. Then $\{X_i\}$ is a Cauchy sequence with respect to $d_{Y,H}$, because for all j,

$$\begin{aligned} d_{Y,H}(X_i, X_{i+j}) &\le d_{Y,H}(X_i, X_{i+1}) + \cdots + d_{Y,H}(X_{i+j-1}, X_{i+j}) \\ &\le 2^{-i+1} + 2^{-i} + \cdots + 2^{-i-j+2} \\ &\le 2^{-i+2}. \end{aligned}$$

Note that (Y, d_Y) may not be complete, and if not, the unique limit point is the desired limit space X.

ii) Consider a subset of Cauchy sequences in Y,

$$\hat{X} = \{\{x_i\} : x_i \in X_i \text{ is a Cauchy sequence in } Y\},$$

and define a pseudo-metric on $\hat{X}$,

$$\hat{d}(\{x_i\},\{y_i\}) = \lim_{i\to\infty} d_Y(x_i, y_i),$$

where the existence of the limit is from

$$|d_Y(x_i,y_i) - d_Y(x_j,y_j)| \le d_Y(x_i,x_j) + d_Y(y_i,y_j) \to 0 \qquad \text{as } i,j \to \infty.$$

Then $\hat{d}$ yields a metric on the quotient space $X = \hat{X}/\sim$, where

$$\{x_i\} \sim \{y_i\} \qquad \text{iff} \qquad \hat{d}(\{x_i\},\{y_i\}) = 0.$$

We now define an admissible metric on $X \coprod Y$ by declaring

$$d(\{x_i\}, y) = \lim_{i\to\infty} d_Y(x_i, y).$$

(Because $|d_Y(x_i,y) - d_Y(x_j,y)| \le d_Y(x_i,x_j)$, $d_Y(x_i,y)$ is a Cauchy sequence.) Since $d(\{x_i\},y) \ge d_Y(x_k,y)$ for some $x_k \in \{x_i\}$, d is indeed a metric (because $d_Y(x_{k+j},y) > d_Y(x_k,y) > 0$).

iii) Given $\epsilon > 0$, we will construct a finite ϵ-dense subset of X as follows: choose i so that $2^{-i} < \frac{\epsilon}{5}$. Because X_i is compact, we may assume a finite $\frac{\epsilon}{5}$-net, $\{x_i^1, \ldots, x_i^\ell\}$, of X_i. Let $x_{i+1}^1, \ldots, x_{i+1}^\ell \in$

X_{i+1} such that $d(x_i^j, x_{i+1}^j) < 2^{-i}$. Let $x_{i+2}^1, \dots, x_{i+2}^\ell \in X_{i+2}$ such that $d(x_{i+1}^j, x_{i+2}^j) < 2^{-i-1}$. Repeating this, we obtain, for each k, $x_{i+k}^1, \dots, x_{i+k}^\ell \in X_{i+k}$ such that $d(x_{i+k-1}^j, x_{i+k}^j) < 2^{-i-k+1}$. For each $1 \le j \le \ell$, it is clear that $\{x_{i+k}^j\}_{k=1}^\infty$ is a Cauchy sequence, that is, $\{x_{i+k}^j\}_{k=1}^\infty \in X$. Moreover, for each $1 \le k < \infty$, $x_{i+k}^1, \dots, x_{i+k}^\ell$ is $\frac{3\epsilon}{5}$-dense in X_{i+k}. This is because for any $x \in X_{i+k}$, we can choose $x' \in X_i$ such that $d(x, x') < 2^{-i}$, and let $x_i^j \in \{x_i^j\}$ such that $d(x', x_i^j) < \frac{\epsilon}{5}$. Then $d(x, x_{i+k}^j) \le d(x, x') + d(x', x_i^j) + d(x_i^j, x_{i+k}^j) < \frac{\epsilon}{5} + 2 \cdot 2^{-i} < \frac{3\epsilon}{5}$.

Finally, we check that $\{\{x_{i+k}^1\}_{k=1}^\infty, \dots, \{x_{i+k}^\ell\}_{k=1}^\infty\}$ is an ϵ-dense subset in X. Given any $\{y_k\} \in X$, we may assume that for a large k, $d(y_k, y_{k+j}) < \frac{\epsilon}{5}$. Since $\{x_k^1, \dots, x_k^\ell\}$ is a $\frac{3\epsilon}{5}$-net for X_k, we may assume that $d(y_k, x_k^s) < \frac{3\epsilon}{5}$. Then

$$d(y_{k+j}, x_{k+j}^s) \le d(y_k, y_{k+j}) + d(y_k, x_k^s) + d(x_k^s, x_{k+j}^s) < \frac{\epsilon}{5} + \frac{3\epsilon}{5} + \frac{\epsilon}{5} = \epsilon,$$

and thus $d(\{y_k\}, \{x_k^s\}) < \epsilon$.

iv) We shall show that for any $\epsilon > 0$, $B_\epsilon(X) \supseteq X_i$ and $B_\epsilon(X_i) \supseteq X$ for all large i.

For any $\epsilon > 0$, let $2^{-i+1} < \epsilon$. For $x_i \in X_i$, from the condition that $d_{i,i+j,H}(X_i, X_{i+j}) < 2^{-i+1}$, we define a sequence $y_k \in X_k$ such that $d(y_k, y_{k+j}) < 2^{-k+1}$ and $y_i = x_i$ (we can choose $y_1, \dots, y_{i-1}$ arbitrarily). Clearly, $\{y_k\}$ is a Cauchy sequence and $d(x_i, \{y_k\}) < 2^{-i+1} < \epsilon$. This shows that $X_i \subseteq B_\epsilon(X)$ for $i \ge \frac{-\ln \epsilon}{\ln 2} + 1$.

For any $\{x_i\} \in X$, for i large, we can assume that $d(x_i, \{x_j\}) < \epsilon$. Note that this does not give $B_\epsilon(X_i) \supseteq X$, because how large i is may depend on $\{x_i\}$ in X. To overcome this trouble, by iii), we can assume a finite $\frac{\epsilon}{4}$-dense subset, $\{y_i^1\}_{i=1}^\infty, \dots, \{y_i^\ell\}_{i=1}^\infty$, for X. For each $1 \le j \le \ell$, we may assume some N_j such that for $i \ge N_j$, $d(y_i^j, \{y_i^j\}) < \frac{\epsilon}{4}$ and $2^{-i+1} < \frac{\epsilon}{4}$. Let $N = \max\{N_1, \dots, N_\ell\}$. For any $\{x_i\} \in X$, we may assume some $1 \le j \le \ell$ such that $d(\{x_i\}, \{y_i^j\}) < \frac{\epsilon}{4}$. From the above, for each $i \ge N$,

$$d(y_i^j, \{x_i\}) \le d(y_i^j, \{y_i^j\}) + d(\{y_i^j\}, \{x_i\}) < \epsilon,$$

and thus $X \subseteq B_\epsilon(X_i)$. q.e.d.

A direct consequence of Proposition A.1 is that $(\mathfrak{X}, d_{GH})$ is a complete metric space.

A by-product of the above proof is that an abstract convergent sequence, $X_i \xrightarrow{d_{GH}} X$, can be realized as a concrete Hausdorff convergence, $d_H(X_i, X) \to 0$, in $\coprod X_i \coprod X$ with an admissible metric d. In particular, it makes sense to say that $x_i \in X_i, x_i \to x \in X$ because $d(x, x_i) \to 0$.

Appendix B. Estimates for Volume Forms

by MARK GROSS[1]

Theorem B.1. *Let $\pi : \mathcal{M} \to \Delta$ be a flat projective family of n-dimensional Calabi–Yau varieties, with $M_t = \pi^{-1}(t)$ non-singular for $t \neq 0$ and $M_0 = \pi^{-1}(0)$ a variety with canonical singularities. After embedding the family $\mathcal{M}$ in $\mathbb{CP}^N \times \Delta$, let ω_t denote the restriction of the Fubini-Study metric on $\mathbb{CP}^N$ to M_t. Furthermore, let Ω be a nowhere vanishing holomorphic section of the relative canonical bundle $\mathcal{K}_{\mathcal{M}/\Delta}$, and set $\Omega_t = \Omega|_{M_t}$. Then*

i) *There is a κ independent of t such that*

$$(-1)^{\frac{n^2}{2}}\Omega_t \wedge \bar{\Omega}_t > \kappa\omega_t^n.$$

ii) *There is a constant Λ independent of t such that*

$$(-1)^{\frac{n^2}{2}}\int_{M_t}\Omega_t \wedge \bar{\Omega}_t < \Lambda.$$

Proof. For i), we use an argument similar to that in [**18**], lemma 6.4. Let $\mathcal{M}^{sm}$ denote the set of points of $\mathcal{M}$ where π is smooth, i.e., the set of points where $\mathcal{M}$ is non-singular and π_* is surjective. Let $p \in \mathcal{M}$ be a point, and consider an open neighborhood U_p of p which embeds into $\mathbb{C}^{N+1}$ via $\iota : U_p \to \mathbb{C}^{N+1}$, with coordinates $t, z_1, \ldots, z_N$. The Fubini-Study form is comparable to $\omega = \sqrt{-1}\sum_{i=1}^{N} dz_i \wedge d\bar{z}_i$, so we can assume that locally ω_t is the restriction of ω to $U_p \cap M_t$. Now $\omega^n = n!(-1)^{\frac{n}{2}}\sum_I dz_I \wedge d\bar{z}_I$, where the sum is over all index sets $I \subseteq \{1, \ldots, N\}$ with $\#I = n$. Now as $\iota^*(dz_I)$ is a relative holomorphic n-form on $U_p \cap \mathcal{M}^{sm}$, there is a holomorphic function f_I on $U_p \cap \mathcal{M}^{sm}$ such that $\iota^*(dz_I) = f_I\Omega$. Note that since $\mathcal{M}$ is necessarily normal and $\mathcal{M}\backslash\mathcal{M}^{sm}$ is codimension ≥ 2, we can apply Hartogs' theorem for normal analytic spaces to extend f_I to a holomorphic function on U_p. Thus

$$\iota^*\omega^n = C(-1)^{\frac{n^2}{2}}\left(\sum_I |f_I|^2\right)\Omega \wedge \bar{\Omega}.$$

On an open neighborhood $V_p \subset\subset U_p$ of p, $|f_I|$ is bounded. This gives the desired result.

For ii), we need to apply some standard results from Hodge theory. After making a base-change $\Delta \to \Delta$ given by $t \mapsto t^k$ for some k, we can assume that the monodromy operator T about the origin acting on

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$H^n(M_{t_0}, \mathbb{C})$ is unipotent, i.e., $(T-I)^m = 0$ for some m. Here $t_0 \in \Delta^* = \Delta \setminus \{0\}$ is a fixed basepoint. Let

$$N = \log(T - I);$$

this makes sense via the power series expansion. By the stable reduction theorem [**36**], one has a diagram

$$\begin{tikzcd} \widetilde{\mathcal{M}} \arrow[r, "\eta"] \arrow[dr, "\tilde{\pi}"'] & \mathcal{M} \arrow[d, "\pi"] \\ & \Delta \end{tikzcd}$$

in which η is an isomorphism outside the central fibre and $\tilde{\pi}$ is normal crossings, i.e., locally around points of $\widetilde{M}_0 = \tilde{\pi}^{-1}(0)$ there are coordinates $z_1, \ldots, z_{n+1}$ on $\widetilde{\mathcal{M}}$ such that $t = z_1 \cdots z_p$ for some $p \leq n+1$. One has the sheaf $\Omega^1_{\widetilde{\mathcal{M}}}(\log \widetilde{M}_0)$ of logarithmic differentials on $\widetilde{\mathcal{M}}$ locally generated by $\frac{dz_1}{z_1}, \ldots, \frac{dz_p}{z_p}, dz_{p+1}, \ldots, dz_{n+1}$, and the sheaf of relative logarithmic differentials $\Omega^1_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0)$ is obtained by dividing out by the relation $\frac{dt}{t} = 0$. It is standard (see for example the book [**45**] for the full background used here) that $\Omega^1_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0)$ is a rank n vector bundle, and if X is an irreducible component of $\widetilde{M}_0$, then $\Omega^1_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0)|_X = \Omega^1_X(\log \partial X)$, where $\partial X = X \cap \widetilde{S}$, and $\widetilde{S}$ is the singular set of $\widetilde{M}_0$. One then obtains the logarithmic de Rham complex $\Omega^\bullet_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0)$, with $\Omega^p_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0)$ the p-th exterior power of the sheaf of relative log differentials, and the differential d is the ordinary exterior derivative. In particular, we have the line bundle $\Omega^n_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0)$.

By [**55**], theorem 2.11, $\tilde{\pi}_*\Omega^n_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0)$ is a vector bundle whose fibre over $t \neq 0$ is $H^0(M_t, \mathcal{K}_{M_t})$. Hence this is a line bundle. On the other hand, by assumption on $\mathcal{M}$, $\mathcal{K}_{\mathcal{M}/\Delta} \cong \mathcal{O}_{\mathcal{M}}$ and so $\pi_*\mathcal{K}_{\mathcal{M}/\Delta}$ is also a line bundle. Let $\mathcal{M}^o$ be the largest open set so that $\eta^{-1}(\mathcal{M}^o) \to \mathcal{M}^o$ is an isomorphism, and let $i : \mathcal{M}^o \to \mathcal{M}$ be the inclusion. Then the codimension of $\mathcal{M} \setminus \mathcal{M}^o$ in $\mathcal{M}$ is at least two. Since $\mathcal{M} \setminus \mathcal{M}^{sm}$ has codimension at least two, $\mathcal{M} \setminus (\mathcal{M}^{sm} \cap \mathcal{M}^o)$ has codimension at least two. We have a composition of canonical sheaf homomorphisms

$$\eta_*\Omega^n_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0) \to i_*i^*\eta_*\Omega^n_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0) \cong \mathcal{K}_{\mathcal{M}/\Delta},$$

the latter isomorphism by Hartogs' theorem and the fact that the isomorphism holds over $\mathcal{M}^{sm} \cap \mathcal{M}^o$. Applying π_* then gives a map

$$\tilde{\pi}_*\Omega^n_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0) \to \pi_*\mathcal{K}_{\mathcal{M}/\Delta}. \tag{B.1}$$

This map is an isomorphism over Δ^*, and hence is necessarily an inclusion of sheaves. To show it is in fact an isomorphism, we need to show that any section of $\mathcal{K}_{\mathcal{M}/\Delta}|_{M_0} = \mathcal{K}_{M_0}$ comes from a section of $\Omega^n_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0)|_{\widetilde{M}_0}$. To see this, let X_0 be the proper transform of M_0 in $\widetilde{M}_0$. Then $\eta_0 : X_0 \to M_0$ is a resolution of singularities, and since M_0 has canonical singularities, we have

$$\mathcal{K}_{X_0} = \eta_0^* \mathcal{K}_{M_0} + \sum_E a_E E,$$

where the sum is over all exceptional divisors E of η_0 and $a_E \geq 0$. (Note a_E is an integer since M_0 is Gorenstein.) On the other hand, $\Omega^n_{X_0}(\log \partial X_0)$ is $\mathcal{K}_{X_0} + \sum_E E$, where the sum is again over all exceptional divisors of η_0. So $\eta_0^*\Omega_0$, viewed as a section of $\Omega^n_{X_0}(\log \partial X_0)$, has a zero of order at least 1 along each exceptional divisor E. Thus $\eta_0^*\Omega_0$ extends by zero to a section of $\Omega^n_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0)|_{\widetilde{M}_0}$. Thus (B.1) is surjective, hence an isomorphism.

We now recall some standard material concerning the limiting mixed Hodge structure and the nilpotent orbit theorem. Denote by $\mathcal{H}^n$ the vector bundle $\mathbb{R}^n \tilde{\pi}_* \Omega^\bullet_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0)$. The fibre of this bundle at t is isomorphic to $H^n(M_t, \mathbb{C})$. This bundle comes along with the Gauss-Manin connection, which is flat with a regular singular point at $0 \in \Delta$.

Let $j : H \to \Delta^*$ be the universal cover, with H the upper half-plane, with coordinate $w = \frac{1}{2\pi\sqrt{-1}} \log t$. Then $j^*\mathcal{H}^n$ is now canonically identified with the trivial bundle $H \times H^n(M_{t_0}, \mathbb{C})$ via parallel transport by the Gauss-Manin connection. If $e \in H^n(M_{t_0}, \mathbb{C})$, one obtains a constant section σ_e of $j^*\mathcal{H}^n$ by parallel transport, and then $e^{-wN}\sigma_e$ descends to a single-valued section of $\mathcal{H}^n$ over Δ^*. The bundle $\mathcal{H}^n$ is then the canonical extension of $\mathcal{H}^n|_{\Delta^*}$, i.e., the extension in which, for a basis $e_1, \ldots, e_s$ of $H^n(M_{t_0}, \mathbb{C})$, $e^{-wN}\sigma_{e_1}, \ldots e^{-wN}\sigma_{e_n}$ form a holomorphic frame. In particular, there is an isomorphism of the fibre $\mathcal{H}^n_0 = \mathbb{H}^n(\widetilde{M}_0, \Omega^\bullet_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0)|_{\widetilde{M}_0})$ with $H^n(M_{t_0}, \mathbb{C})$, isomorphic to the space of flat sections of $j^*\mathcal{H}^n$.

We also have an inclusion

$$\mathcal{F}^n := \tilde{\pi}_* \Omega^n_{\widetilde{\mathcal{M}}/\Delta}(\log \widetilde{M}_0) \hookrightarrow \mathcal{H}^n.$$

The fibre of $\mathcal{F}^n$ over $0 \in \Delta$ is $\mathcal{F}^n_{\lim} \subseteq H^n(M_{t_0}, \mathbb{C})$ under the above isomorphism, a piece of the limiting mixed Hodge structure. In particular, the value of the holomorphic section Ω of $\pi_*\mathcal{K}_{\mathcal{M}/\Delta}$ at 0 under the isomorphism (B.1) defines a class $\Omega_{\lim} \in \mathcal{F}^n_{\lim}$.

We now apply the nilpotent orbit theorem (see e.g., [**26**], chapter IV, for an exposition of this material). Let $\phi : H \to \mathbb{P}(H^n(M_{t_0}, \mathbb{C}))$ be the period map, with, for $w \in H$, $\phi(w)$ being the one-dimensional subspace $(j^*\mathcal{F}^n)_w \subseteq (j^*\mathcal{H}^n)_w \cong H^n(M_{t_0},\mathbb{C})$, the latter identification via the Gauss-Manin connection. Then $e^{-wN}\phi : H \to \mathbb{P}(H^n(M_{t_0},\mathbb{C}))$ descends to a map $\psi : \Delta^* \to \mathbb{P}(H^n(M_{t_0},\mathbb{C}))$ which in turn extends across the origin, with $\psi(0) = [\Omega_{\mathrm{lim}}]$. The nilpotent orbit is the map $\phi^{nil} : H \to \mathbb{P}(H^n(M_{t_0},\mathbb{C}))$ given by

$$\phi^{nil}(w) = e^{wN}\psi(0) = e^{wN}[\Omega_{\mathrm{lim}}].$$

The nilpotent orbit theorem states that ϕ^{nil} is a good approximation to ϕ, i.e., with a suitable metric on $\mathbb{P}(H^n(M_{t_0},\mathbb{C}))$ inducing a distance function ρ, we have constants A and B such that for $\mathrm{Im}w \geq A > 0$,

$$\rho(\phi(w),\phi^{nil}(w)) \leq (\mathrm{Im}w)^B e^{-2\pi \mathrm{Im}w}.$$

This implies that $\int_{M_t} \Omega_t \wedge \bar{\Omega}_t$ is bounded independently of t near 0 provided that $\int_{M_{t_0}} e^{wN}\Omega_{\mathrm{lim}} \wedge \overline{e^{wN}\Omega_{\mathrm{lim}}}$ is bounded for $\mathrm{Im}w \geq A$.

Now we apply the argument of proposition 2.3 and theorem 2.1 of [**59**]. The argument of proposition 2.3 tells us that $\widetilde{M_0}$ has an irreducible component (in fact X_0) with $H^{n,0}(X_0,\mathbb{C}) \neq 0$. Thus, by the first line of the proof of theorem 2.1, $N\mathcal{F}^n_\infty = 0$. So in particular, $e^{wN}\Omega_{\mathrm{lim}} = \Omega_{\mathrm{lim}}$, giving the desired boundedness. q.e.d.

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