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CONTINUITY OF FUZZY MULTIFUNCTIONS

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We define the upper and lower inverse of a fuzzy multifunction and prove basic identities. Then by using these ideas we introduce the concept of hemicontinuity and obtain many interesting properties of lower and upper hemicontinuous fuzzy multifunctions. Using the notion of hemicontinuity, we also characterize closed and open fuzzy mappings.

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1. Introduction

The theory of fuzzy sets provides a framework for mathematical modeling of those real world situations, which involve an element of uncertainty, imprecision, or vagueness in their description. Since its inception thirty years ago by Zadeh [6], this theory has found wide applications in engineering, economics, information sciences, medicine, etc.; for details the reader is referred to [5, 7]. A *fuzzy multifunction* is a fuzzy set valued function [1, 3, 4]. Fuzzy multifunctions arise in many applications, for instance, the budget multifunction occurs in economic theory, noncooperative games, artificial intelligence, and decision theory. The biggest difference between fuzzy functions and fuzzy multifunctions has to do with the definition of an inverse image. For a fuzzy multifunction there are two types of inverses. These two definitions of the inverse then leads to two definitions of continuity. In this paper our purpose is twofold. First, we define upper and lower inverse of a fuzzy multifunction and study their various properties. Next, we use these ideas to introduce upper hemicontinuous and lower hemicontinuous fuzzy multifunctions.

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2. Preliminaries

Let X be an arbitrary (nonempty) set. A fuzzy set (in X) is a function with domain X and values in $[0,1]$. If A is a fuzzy set and $x \in X$, the value $A(x)$ is called the grade of membership of x in A. The fuzzy set A^c defined by $A^c(x) = 1 - A(x)$ is called the complement of A. Let A and B be fuzzy sets in X. We write $A \subseteq B$ if $A(x) \leq B(x)$ for each $x \in X$. For any family $\{A_i\}_{i \in I}$ of fuzzy sets in X, we define

$$\left[\bigcap_{i \in I} A_i\right](x) = \inf_{i \in I} A_i(x)$$

and

$$\left[\bigcup_{i \in I} A_i\right](x) = \sup_{i \in I} A_i(x).$$

A family τ of fuzzy sets in X is called a *fuzzy topology for X* and the pair (X, τ)-a *fuzzy topological space* if:

(*i*) $\chi_x \in \tau$ and $\chi_\phi \in \tau$;

(*ii*) $\bigcup_{i \in I} A_i \in \tau$, whenever each $A_i \in \tau (i \in I)$; and

(*iii*) $A \cap B \in \tau$, whenever $A, B \in \tau$.

The elements of τ are called *open* and their complements-*closed*. For details, see [2, 7].

3. Upper and Lower Inverses

Definition 3.1: A *fuzzy multifunction* f from a set X into a set Y assigns to each x in X a fuzzy subset $f(x)$ of Y. We denote it by $f: X \to Y$. We can identify f with a fuzzy subsets G_f of $X \times Y$ and

$$f(x)(y) = G_f(x, y).$$

If A is a fuzzy subset of X, then the fuzzy set $f(A)$ in Y is defined by

$$f(A)(y) = \sup_{x \in X} [G_f(x, y) \wedge A(x)].$$

The *graph* G_f *of* f is the fuzzy subset of $X \times Y$ associated with f,

$$G_f = \{(x, y) \in X \times Y: [f(x)](y) \neq 0\}.$$

Definition 3.2: The *upper inverse* f^u of a fuzzy multifunction $f: X \to Y$, is defined by

$$f^u(A)(x) = \inf_{y \in Y} [(1 - G_f(x, y)) \vee A(y)].$$

Definition 3.3: The *lower inverse* f^ℓ of a fuzzy multifunction $f: X \to Y$ is defined by

$$f^\ell(A)(x) = \sup_{y \in Y} [G_f(x, y) \wedge A(y)].$$

Theorem 3.4: [Basic Identities] *Let $f: X \to Y$ be a fuzzy multifunction, then:*

(*i*) $f^u(A) = (f^\ell(A^c))^c$,

(*ii*) $f^\ell(A) = (f^u(A^c))^c$,

(*iii*) $f^u\Big(\bigcap_{i \in I} A_i\Big) = \bigcap_{i \in I} f^u(A_i)$, *and*

(*iv*) $f^\ell\Big(\bigcup_{i \in I} A_i\Big) = \bigcup_{inI} f^\ell(A_i)$.

Proof: (*i*)

$$\begin{aligned}(f^\ell(A^c))^c(x) &= 1 - f^l(A^c)(x)\\ &= 1 - \sup_{y \in Y} [G_f(x,y) \wedge A^c(Y)]\\ &= \inf_{y \in Y} [(1 - G_f(x,y)) \vee (1 - A^c(y))]\\ &= \inf_{y \in Y} [(1 - G_f(x,y)) \vee A(y)]\\ &= f^u(A)(x).\end{aligned}$$

(*ii*)

$$\begin{aligned}(f^u(A^c))^c(x) &= 1 - f^u(A^c)(x)\\ &= 1 - \inf_{y \in Y} [(1 - G_f(x,y)) \vee A^c(y)]\\ &= \sup_{y \in Y} [(1 - (1 - G_f(x,y))) \wedge (1 - A^c)(y)]\\ &= \sup_{y \in Y} [G_f(x,y) \wedge A(y)]\\ &= f^\ell(A)(x).\end{aligned}$$

(*iii*)

$$\begin{aligned}f^u(\bigcap_{i \in I} A_i)(x) &= \inf_{y \in Y}\Big[(1 - G_f(x,y)) \vee (\bigcap_{i \in I} A_i)(y)\Big]\\ &= \inf_{y \in Y} \inf_{i \in I} [(1 - G_f(x,y)) \vee A_i(y)]\\ &= \inf_{i \in I} \inf_{y \in Y} [(1 - G_f(x,y)) \vee A_i(y)]\\ &= \inf_{i \in I} f^u(A_i)(x).\end{aligned}$$

(*iv*)

$$\begin{aligned}f^\ell(\bigcup_{i \in I} A_i)(x) &= \sup_{y \in Y} [G_f(x,y) \wedge (\bigcup_{i \in I} A_i)(y)]\\ &= \sup_{y \in Y} \sup_{i \in I} (G_f(x,y) \wedge A_i(y))\\ &= \sup_{i \in I} \sup_{y \in Y} (G_f(x,y) \wedge A_i(y))\\ &= \sup_{i \in I} (f^\ell(A_i)(x)).\end{aligned}$$

Remark 3.5: Let $f: X \to Y$ be a fuzzy multifunction, which has only nonempty values then $f^u(A)$ may not be a fuzzy subset of $f^\ell(A)$.

Example 3.6: Let $f: X \to Y$ be defined as follows:

$$f(x) = \begin{cases} f(x_0), & x = x_0 \\ Y, & x \neq x_0 \end{cases}$$

and $[f(x_0)](y) = \frac{1}{3}$, for every $y \in Y$. Then for arbitrary fuzzy subset A of Y,

$$f^u(A)(x_0) = \inf_{y \in Y}[(1 - G_f(x_0, y)) \vee A(y)] \geq \frac{2}{3},$$

and

$$f^\ell(A)(x_0) = \sup_{y \in Y}[G_f(x_0, y) \wedge A(y)] \leq \frac{1}{3}.$$

Therefore $f^u(A)(x_0) > f^l(A)(x_0)$. Hence $f^u(A)$ is not a fuzzy subset of $f^l(A)$.

Remark 3.7: For the inverse of a singleton, we introduce

$$f^{-1}(y) = \{x \in X: G_f(x, y) \neq 0\} = f^\ell\{y\}.$$

4. Continuity of Fuzzy Multifunctions

Let X be a fuzzy topological space. A *neighborhood of a fuzzy set* $A \subset X$ is any fuzzy set B, for which there is an open fuzzy set V satisfying $A \subset V \subset B$. Any open fuzzy set V that satisfies $A \subset V$ is called an *open neighborhood of* A.

Definition 4.1: A fuzzy multifunction $f: X \to Y$ between two fuzzy topological spaces X and Y is:

(a) *upper hemicontinuous* at a point x, if for every open neighborhood U of $f(x)$, $f^u(U)$ is a neighborhood of x in X. The fuzzy multifunction f is *upper hemicontinuous* on X, if it is upper hemicontinuous at every point of X;

(b) *lower hemicontinuous* at x, if for every open fuzzy set U, which intersects $f(x)$, $f^\ell(U)$ is a neighborhood of x. As above, f is *lower hemicontinuous* on X if it is lower hemicontinuous at each point of X;

(c) *continuous* if it is both upper and lower hemicontinuous.

Throughout this paper, if we assert that a fuzzy multifunction is hemicontinuous, it should be understood that its domain and range space are fuzzy topological spaces.

Lemma 4.2: *Let $f: X \to Y$ be a fuzzy multifunction. Then:*

(i) *if f is upper hemicontinuous then $f^u(\phi)$ is open;*

(ii) *if f is lower hemicontinuous then $f^\ell(Y)$ is open.*

Proof: (i) Let $f: X \to Y$ be an upper hemicontinuous multifunction, then for any open neighborhood U of $f(x)$, $f^u(U)$ is a neighborhood of x. Therefore, there is an open fuzzy set V such that $\{x\} \subset V \subset \{x \in X: f(x) \subset U\}$. It further implies that if x is any point in $f^u(\phi)$, then $\{x\} \subset V \subset \{x \in X: f(X) = \phi\} = f^u(\phi)$. Hence $f^u(\phi)$ is open.

(ii) Let $f: X \to Y$ be a lower hemicontinuous multifunction. Then using Theorem 3.4 (i), we obtain $[f^\ell(Y)]^c = [f^\ell(\phi^c)]^c = f^u(\phi)$. Part ($i$) further implies that $f^\ell(Y)$ is open.

Remark 4.3: If f is a continuous multifunction then $f^u(\phi) = \{x \in X: f(x) = \phi\}$ is a closed and open fuzzy set.

Theorem 4.4: *For $f: X \to Y$, the following statements are equivalent:*

(1) *$f^u(V)$ is open for each open fuzzy subset V of Y.*

(2) *$f^{\ell}(W)$ is closed for each closed fuzzy subset W of Y.*

Proof: (1)⇒(2): Let W be a closed fuzzy subset of Y. Then Theorem 3.4 (*ii*) implies, $f^{\ell}(W) = (f^{u}(W^{c}))^{c}$. Now (1) further implies that $f^{l}(W)$ is closed.

(2)⇒(1): Let V be an open fuzzy subset of Y. Then Theorem 3.4 (*i*) implies $f^{u}(V) = (f^{\ell}(V^{c}))^{c}$. Now (2) further implies that $f^{u}(V)$ is open.

Theorem 4.5: *Let $f: X \rightarrow Y$ and $f^{u}(V)$ be open for each open fuzzy subset V of Y then f is upper hemicontinuous.*

Proof: Let $f^{u}(V)$ be open for each open fuzzy subset V of Y. Then $f^{u}(V)$ is a neighborhood of each of its point. Hence f is upper hemicontinuous.

Theorem 4.6: *For $f: X \rightarrow Y$, the following statements are equivalent:*

(1) *$f^{\ell}(V)$ is open for each open fuzzy subset V of Y.*

(2) *$f^{u}(W)$ is closed for each closed fuzzy subset W of Y.*

Proof: (1)⇒(2): Let W be a closed fuzzy subset of Y. Then by Theorem 3.4 (*i*), we have, $f^{u}(W) = (f^{\ell}(W^{c}))^{c}$. Now (1) further implies that $f^{u}(W)$ is closed.

(2)⇒(1): Let V be an open fuzzy subset of Y. Then by Theorem 3.4 (*ii*), we have $f^{\ell}(V) = (f^{u}(V^{c}))^{c}$. Now (2) further implies that $f^{\ell}(V)$ is open.

Theorem 4.7: *Let $f: X \rightarrow Y$ and $f^{\ell}(V)$ be open for each open fuzzy subset V of Y then f is lower hemicontinuous.*

Proof: Let $f^{\ell}(V)$ be open for each open fuzzy subset V of Y. Then $f^{\ell}(V)$ is a neighborhood of each of its point. Hence f is lower hemicontinuous.

Recall that a fuzzy function $f: X \rightarrow Y$ between two fuzzy topological spaces is:

(*a*) an open fuzzy mapping, if $f(V)$ is open in Y for each open fuzzy subset V in X;

(*b*) a closed fuzzy mapping, if $f(W)$ is closed in Y for each closed fuzzy subset W in X.

Next we characterize closed and open fuzzy mappings in terms of hemicontinuity of the inverse fuzzy multifunction.

Theorem 4.8: *Let $f: X \rightarrow Y$ be a fuzzy function between fuzzy topological spaces and the inverse fuzzy multifunction $f^{-1}: Y \rightarrow X$ defined by the formula*

$$[f^{-1}(y)](x) = G_{f}(x, y).$$

Then f is a closed fuzzy mapping if and only if f^{-1} is upper hemicontinuous fuzzy multifunction.

Proof: Assume that f is a closed fuzzy mapping. Fix $y \in Y$ and choose an open fuzzy subset V of X such that $f^{-1}(y) \subset V$. Put $E = [f(V^{c})]^{c}$. Then E is an open neighborhood of y, satisfying $f^{-1}(z) \subset V$ for each $z \in E$. Thus $E \subset (f^{-1})^{u}(V)$ and so $(f^{-1})^{u}(V)$ is a neighborhood of y.

Conversely, suppose that f^{-1} is upper hemicontinuous. Let W be a closed fuzzy subset of X and pick $y \in [f(W)]^{c}$. Then $f^{-1}(y) \subset W^{c}$. So by the upper hemicontinuity of f^{-1}, there exists an open neighborhood V of y such that $f^{-1}(z) \subset W^{c}$ for all $z \in V$. This implies $V \cap f(W) = \phi$, that is, $V \subset [f(W)]^{c}$. Hence $f(W)$ is closed.

Theorem 4.9: *Let f and f^{-1} be as in Theorem 4.8. Then f is an open mapping if and only if f^{-1} is lower hemicontinuous.*

The proof is similar to Theorem 4.8.

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