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CONTINUITY OF GAUSSIAN PROCESSES

BY
M. B. MARCUS AND L. A. SHEPP

Abstract. We give a proof of Fernique's theorem that if X is a stationary Gaussian process and $\sigma^2(h)=E(X(h)-X(0))^2$ then X has continuous sample paths provided that, for some $\varepsilon>0$, $\sigma(h)\leqq\psi(h)$, $0\leqq h\leqq\varepsilon$, where ψ is any increasing function satisfying

$$(*)\qquad \int_0^\varepsilon \frac{\psi(h)}{h(\log(1/h))^{1/2}}\,dh<\infty.$$

We prove the partial converse that if $\sigma(h)\geqq\psi(h)$, $0\leqq h\leqq\varepsilon$ and ψ is any increasing function not satisfying (*) then the paths are not continuous. In particular, if σ is monotonic we may take $\psi=\sigma$ and (*) is then necessary and sufficient for sample path continuity. Our proof is based on an important lemma of Slepian.

Finally we show that if σ is monotonic and convex in $[0,\varepsilon]$ then $\sigma(h)(\log 1/h)^{1/2}\to 0$ as $h\to 0$ iff the paths are *incrementally continuous*, meaning that for each monotonic bounded sequence $t=t_1,t_2,\ldots,X(t_{n+1})-X(t_n)\to 0$, w.p.1.

1. **Introduction.** Let X be a zero-mean separable stationary Gaussian process with continuous covariance $\rho(h)=EX(s)X(s+h)$. Xavier Fernique (1964) made considerable progress toward a solution of the well-known problem [6], [3], [1] of finding necessary and sufficient conditions on ρ in order that X have continuous sample paths. Define $\sigma^2(h)=E(X(h)-X(0))^2=2(\rho(0)-p(h))$. Fernique obtained that if $\sigma(h)\leqq\psi(h)$, $0\leqq h\leqq 1$, where ψ is monotonic $\uparrow$ (nondecreasing) and satisfies

$$(1.1)\qquad \int_0^1 \frac{\psi(h)}{h(\log 1/h)^{1/2}}\,dh<\infty$$

then X has continuous paths. He also obtained (under some additional conditions) that if ψ is monotonic $\uparrow$ and (1.1) fails to hold then there exists a discontinuous process X (which he gave as a random lacunary Fourier series) with $\sigma(h)\leqq\psi(h)$, $0\leqq h\leqq\varepsilon$. Modifying his method of random lacunary Fourier series, we obtain (§3) using a basic inequality of D. Slepian (1962) that if for some $\varepsilon>0$, $\sigma(h)\geqq\psi(h)$, $0\leqq h\leqq\varepsilon$, where ψ is monotonic $\uparrow$ and (1.1) fails to hold then the paths of X are discontinuous. As an immediate consequence we find that if σ is monotonic $\uparrow$ then

$$(1.2)\qquad \int_0^\varepsilon \frac{\sigma(h)}{h(\log 1/h)^{1/2}}\,dh<\infty$$

is the necessary and sufficient condition for X to have continuous paths. In particular if the convariance of X is of Polya's type, the question of whether the paths are continuous is settled completely.

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M. Nisio [9] has obtained conditions for continuity of X based on the spectral distribution function F. As a corollary to the results of [9], it has been proved [10] that if $F(2^{n+1})-F(2^n)=s_n$ is eventually decreasing then $\sum(s_n)^{1/2}<\infty$ is a necessary and sufficient condition for continuity of X. We point out by examples (§5) that neither (1.2) nor $\sum(s_n)^{1/2}<\infty$ is a nasc for continuity among all covariances.

Finally, we observe that a discontinuous process X may or may not possess the following property: call X *incrementally continuous* if for *each* monotonic (either increasing or decreasing) bounded sequence $\mathbf{t}=t_1, t_2, \ldots$

$$(1.3) \qquad X(t_{n+1})-X(t_n)\to 0 \quad \text{as } n\to\infty$$

except for a null set (which may depend on $\mathbf{t}$) of paths X. We show (§4) that a Gaussian process X *with covariance of Polya's type* is incrementally continuous if and only if

$$(1.4) \qquad \sigma(h)(\log(1/h))^{1/2}\to 0 \quad \text{as } h\to 0.$$

As an immediate consequence we find that there exists a process X which is incrementally continuous yet not sequentially continuous. (A process X is sequentially continuous if $t_n\to t$ implies $X(t_n)\to$ limit a.s.; for Gaussian processes sequential continuity is equivalent to sample path continuity.)

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2. **Fernique's sufficient condition for continuity of paths.** The purpose of this section is to prove the following theorem of Fernique [5]. Alternate proofs can be found in [13]–[16].

THEOREM. *Let X be a separable stationary Gaussian process with zero mean and let $\sigma^2(h)=E(X(h)-X(0))^2$. Let ψ be any nondecreasing majorant of σ, so that*

$$(2.1) \qquad \sigma(h)\leqq\psi(h), \qquad 0\leqq h\leqq 1,$$

$$(2.2) \qquad \psi(h)\uparrow, \qquad 0\leqq h\leqq 1.$$

If (1.1) *holds, then the paths of X are continuous. Note*: *it is clear that without loss of generality we could take ψ to be the least nondecreasing majorant of σ,*

$$\psi(h) = \max\,[\sigma(u) : 0\leqq u\leqq h].$$

Proof. By Belyaev's theorem [1] (see [4] for an alternate proof) discontinuous Gaussian processes are unbounded on every set S dense(1) in some interval with probability one, so we need only prove that for some M and S

$$(2.3) \qquad P\{|X(t)|\leqq M,\ t\in S\}>0$$

(1) See Appendix 3.

where S is dense in some interval. Oddly enough, the choice of the subset S is the crucial part of the proof. We shall choose $S=\{k/2_p : 0 \leqq k/2_p \leqq 1\}$ where k and p are nonnegative integers and $2_p=2^{2^p}$, following Fernique [5] (see also [8]). S is sparse enough so that (2.6) (below) holds and yet dense enough so that (2.8) (below) also holds. If $t \in S$, let $k(p, t)$ be the largest integer k for which $k/2_p \leqq t$, so that $k(p, t)=[2_p t]$. Writing $\tau(p, t)=k(p, t)/2_p$ we have

$$0 \leqq \tau(p+1, t)-\tau(p, t) < 1/2_p \tag{2.4}$$

and

$$X(t) = X(\tau(0, t))+ \sum_{p=1}^{\infty} (X(\tau(p+1, t))-X(\tau(p, t)) \tag{2.5}$$

where the series converges because only finitely many terms are nonzero. To prove the right side of (2.5) bounded in t for X in a set of positive measure we proceed as follows. For $p=0, 1, 2, \ldots$; $k=0, 1, 2, \ldots, 2_p-1$; $q=0, 1, 2, \ldots, 2_p-1$, define

$$\zeta(p, k, q) = X(k/2_p+q/2_{p+1})-X(k/2_p).$$

Let $D^2(\zeta)=E\zeta^2$ and set $c(p)=b2^{p/2}$, $p=0, 1, 2, \ldots$, where b will be chosen later. Let A be the event that for some (p, k, q) in the range above, $|\zeta(p, k, q)| > c(p)D(\zeta(p, k, q))$. Thus we have

$$P(A) \leqq \sum_{p=0}^{\infty} \sum_{k=0}^{2_p-1} \sum_{q=0}^{2_p-1} P\{|\eta| > c(p)\},$$

where $\eta=\zeta/D(\zeta)$ is standard normal. Since $P\{|\eta|>c\} \leqq \exp(-c^2/2)$ for $c>1$ we find choosing b large enough that

$$P(A) \leqq \sum_{p=0}^{\infty} (2_p)^2 \exp(-b^2 2^{p-1}) < 1. \tag{2.6}$$

Thus comp (A) has positive measure and if $X \in$ comp (A) we have

$$|X(\tau(p+1, t))-X(\tau(p, t))| \leqq c(p)D(\zeta), \tag{2.7}$$

where $\zeta=\zeta(p, k, q)$ and $k=k(p, t)$, $q=(\tau(p+1, t)-\tau(p, t))2_{p+1}$. We have

$$\begin{aligned} D^2(\zeta) &= D^2(\zeta(p, k, q)) = \sigma^2(q/2_{p+1}) = \sigma^2(\tau(p+1, t)-\tau(p, t)) \\ &\leqq \psi^2(\tau(p+1, t)-\tau(p, t)) \leqq \psi^2(1/2_p) \end{aligned}$$

by (2.1), (2.2), and (2.4). Thus from (2.7) and (2.5)

$$|X(t)| \leqq |X(\tau(0, t))|+ \sum_{p=1}^{\infty} c(p)\psi(1/2_p). \tag{2.8}$$

Again by monotonicity of ψ we have

$$2^{p/2}\psi(1/2_p) \leqq \int_{2^{p-1}}^{2^p} \psi(2^{-u}) \frac{du}{\sqrt{u}}$$

and so by (1.1) the series in (2.8) converges:

$$\sum_{p=1}^{\infty} c(p)\psi(1/2_p) \leqq b \int_1^{\infty} \psi(2^{-u}) \frac{du}{\sqrt{u}} < \infty. \tag{2.9}$$

Finally, since $\tau(0, t)$ has only three possible values, it follows directly from (2.8) and (2.9) that (2.3) holds for some value of M.

3. A sufficient condition for the paths to be discontinuous.

THEOREM. *Let X be a stationary Gaussian process with zero mean and let $\sigma^2(h) = E(X(t+h)-X(t))^2$. Let ψ be any nondecreasing local minorant of σ, that is for some $\varepsilon>0$*

$$\sigma(h) \geqq \psi(h) \geqq 0, \qquad 0 \leqq h \leqq \varepsilon, \tag{3.1}$$

$$\psi(h)\uparrow, \qquad 0 \leqq h \leqq \varepsilon. \tag{3.2}$$

If (1.1) *does not hold for ψ then the paths of X are not continuous.*

Proof. We will show that if the hypothesis holds for σ there is a separable, zero mean, stationary Gaussian process Y for which, in the range $s \in I$, $t \in I$, $I=[0, \varepsilon]$

$$E(Y(t)-Y(s))^2 \leqq K\sigma^2(|t-s|), \tag{3.3}$$

$$P(\sup [Y(t) : t \in I] = \infty) = 1 \tag{3.4}$$

where K is some constant. In other words we will obtain an unbounded process Y whose incremental variance is bounded by a constant times the incremental variance of X. Given such a Y the following inequality of Slepian shows that the paths of X are also unbounded and hence discontinuous.

LEMMA (SLEPIAN). *Let X and Z be separable zero mean Gaussian processes such that $EX^2(t)=EZ^2(t)$, $EX(s)X(t) \leqq EZ(s)Z(t)$ for $s, t \in I$. Then*

$$P(\sup [X(t) : t \in I] \geqq M) \geqq P(\sup [Z(t) : t \in I] \geqq M). \tag{3.5}$$

We include Slepian's proof for completeness, but defer it to the appendix.

To see that X is discontinuous if Y exists satisfying (3.3) and (3.4), take $Z(t) =(a\eta+Y(t))/b$, where η is a standard normal variable independent of Y, and a and b are constants to be chosen. If b is large enough so that $EY^2(t)<b^2EX(t)^2$ and $K \leqq b^2$ we may choose a so that $EX^2(t)=EZ^2(t)$. We then have $E(Z(t)-Z(s))^2 =E(Y(t)-Y(s))^2/b^2 \leqq \sigma^2(|t-s|)$ and the hypothesis of Slepian's lemma is satisfied. Since the right side of (3.5) is unity for every M, X is unbounded also.

To produce a process Y satisfying (3.3) and (3.4) we take Y to be a random lacunary Fourier series, the stationary Gaussian process defined by

$$Y(t) = \sum_{n=0}^{\infty} a_n(\eta_n \cos 2^n t+\eta_n' \sin 2^n t), \tag{3.6}$$

where $\{\eta_n\}$ and $\{\eta_n'\}$ are independent standard normal sequences and $\sum a_n^2 < \infty$. Note that

$$EY(s)Y(t) = \sum_{n=0}^{\infty} a_n^2 \cos 2^n(t-s), \tag{3.7}$$

$$E(Y(t)-Y(s))^2 = 2\sum_{n=0}^{\infty} a_n^2(1-\cos 2^n(t-s)). \tag{3.8}$$

We now use the following theorem of Szidon [12, §6.4] to show that $Y(t)$ satisfies (3.4) if

$$\sum |a_n| = \infty. \tag{3.9}$$

LEMMA (SZIDON). *If* $\sum (b_n^2+c_n^2)<\infty$, $\sum (|b_n|+|c_n|)=\infty$, *and* $\theta_{n+1}/\theta_n>\lambda>1$ *then the* L^2 *function with Fourier series* $\sum (b_n \cos \theta_n t + c_n \sin \theta_n t)$ *is unbounded.*

To show that (3.4) holds if (3.9) holds let $b_n = a_n\eta_n$, $c_n = a_n\eta_n'$ and $\theta_n = 2^n$ so that the series of the lemma is Y. Note that $E\sum (b_n^2+c_n^2) = 2\sum a_n^2$ and so $\sum (b_n^2+c_n^2)<\infty$ a.s. Similarly, $E[\exp(-\sum(|b_n|+|c_n|)] = \prod (E\exp - |b_n|)E(\exp(-|c_n|)) = 0$ by (3.9) and so $\sum (|b_n|+|c_n|)=\infty$ a.s. It follows from the lemma that $Y(t)$ is a.s. unbounded on $[0, 2\pi]$ and hence [1] on every interval $[0, \varepsilon]$. Thus the problem reduces to finding an l^2 sequence a_n satisfying (3.9) for which the function Y defined by (3.6) satisfies (3.3).

We shall assume first that σ^2 is monotonically increasing and that $\sigma^2(2^{-n}) \leqq 2\sigma^2(2^{-n-1})$, $n=1, 2, \ldots$, (call this assumption A) because if A holds (which it does if the covariance is of Polya-type, σ^2 then being monotonic and convex) the construction of the a_n sequence is simpler. Later we give a construction which works in general. Define $f_n^2 = \sigma^2(2^{-n})$ and observe that f_n^2 eventually decreases. Define g_n^2 as the largest convex minorant of f_n^2, so that $g_n^2 = f_n^2$ for certain values of n and between these values, g_n^2 is linear and lies strictly below f_n^2. We claim first that $g_{n+1}^2 \leqq g_n^2 \leqq 2g_{n+1}^2$. The first inequality holds because g_n^2 is convex and tends to zero. To prove the second inequality observe that either $g_{n+1}^2 = f_{n+1}^2$ or $g_{n+1}^2 < f_{n+1}^2$. In the first case, $2g_{n+1}^2 = 2f_{n+1}^2 \geqq f_n^2 \geqq g_n^2$, while in the second case $g_{n+1}^2 - g_{n+2}^2 = g_n^2 - g_{n+1}^2$ by the definition of g^2 and so $2g_{n+1}^2 - g_n^2 = g_{n+2}^2 \geqq 0$. Thus we have that $g_n^2 \leqq 2g_{n+1}^2$ for all values of n.

The hypothesis of the theorem requires that (1.1) not hold for ψ. By A, $\psi = \sigma$ and by monotonicity of σ it is easy to see that (1.1) is the same as

$$\infty = \sum_{n=1}^{\infty} \frac{\sigma(2^{-n})}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{f_n}{\sqrt{n}}. \tag{3.10}$$

We claim next that $\sum g_n/\sqrt{n} = \infty$ also. To see this let r and s, defined as above, be any two consecutive points where f and g are equal. Then if $r \leqq n \leqq (r+s)/2$ we have by linearity

$$\begin{aligned} g_n^2 = g_r^2+(g_s^2-g_r^2)(n-r)/(s-r) &\geqq g_r^2-g_r^2(n-r)/(s-r) = g_r^2(s-n)/(s-r) \\ &= f_r^2(s-n)/(s-r) \geqq f_n^2(s-n)/(s-r) \geqq f_n^2/2 \end{aligned}$$

and since $f_n/\sqrt{n}$ decreases,

$$\sum_{r\leqq n<s} \frac{f_n}{\sqrt{n}} \leqq 2 \sum_{r\leqq n\leqq (r+s)/2} \frac{f_n}{\sqrt{n}} \leqq 2\sqrt{2} \sum_{r\leqq n\leqq (r+s)/2} \frac{g_n}{\sqrt{n}} \leqq 2\sqrt{2} \sum_{r\leqq n<s} \frac{g_n}{\sqrt{n}}. \tag{3.11}$$

Now we define $a_n^2 = g_n^2 - g_{n+1}^2$. Since g_n^2 is convex, a_n decreases monotonically. We now use the following inequality of Boas, proved in the appendix.

LEMMA. *Let $g_n^2 = \sum_{j=n}^{\infty} a_j^2$ where a_j decreases. Then*

$$\sum_{n=1}^{\infty} \frac{g_n}{\sqrt{n}} \leqq 2 \sum_{n=1}^{\infty} a_n.$$

It follows from the lemma that (3.9) holds and we need only verify (3.3). Fix $t<1$ and let N be the largest integer for which $2^N t \leqq 1$. By (3.8) we see that

$$E(Y(s+t)-Y(s))^2 \leqq \sum_{n=1}^{N} a_n^2 2^{2n} t^2 + 4 \sum_{N+1}^{\infty} a_n^2. \tag{3.12}$$

The second term on the right in (3.12) is $4g_{N+1}^2 \leqq 4f_{N+1}^2 = 4\sigma^2(2^{-N-1}) \leqq 4\sigma^2(t)$ since σ is monotonic by A. The first term on the right of (3.12) is less than

$$\begin{aligned} t^2 \sum_{n=1}^{N} g_n^2 2^{2n} &= t^2 g_N^2 \sum_{j=1}^{N} 2^{2j} + t^2 \sum_{n=1}^{N-1} (g_n^2 - g_{n+1}^2) \sum_{j=1}^{n} 2^{2j} \\ &\leqq \tfrac{1}{2} t^2 g_N^2 2^{2N} \tfrac{4}{3} + \tfrac{1}{2} t^2 \sum_{n=1}^{N} g_n^2 2^{2n} \tfrac{4}{3} \end{aligned} \tag{3.13}$$

since $g_n^2 - g_{n+1}^2 \leqq g_n^2/2$ as we proved above, and since $\sum_{j=1}^{n} 2^{2j} \leqq 2^{2n}(4/3)$. From (3.13) we get upon subtracting the last term that

$$t^2 \sum_{n=1}^{N} g_n^2 2^{2n} \leqq 2t^2 g_N^2 2^{2N} \leqq 2g_N^2. \tag{3.14}$$

As before, $2g_N^2 \leqq 4g_{N+1}^2 < 4f_{N+1}^2 = 4\sigma^2(2^{-N-1}) \leqq 4\sigma^2(t)$ and so each term in (3.12) $\leqq 4\sigma^2(t)$ and (3.3) holds with $K=8$. This proves the theorem in the case when σ is monotonic and $\sigma^2(2^{-n}) \leqq 2\sigma^2(2^{-n-1})$. We point out that the proof would work as well even if $\sigma^2(2^{-n}) \leqq \alpha\sigma^2(2^{-n-1})$ as long as $\alpha<4$. The case $\alpha=4$ is the general case because for any σ and h we have

$$\sigma^2(2h) = 2\int (1-\cos 2h\lambda)\, dF(\lambda) = 2\int (2-2\cos^2 h\lambda)\, dF(\lambda) \leqq 4\sigma^2(h). \tag{3.15}$$

We now give another proof of the theorem, slightly more involved, but which works in general. We start from scratch at the expense of some repetition. Since the incremental variance $\sigma^2(t)$ has a least positive zero there is an $\varepsilon > 0$ for which

$$0 < \sigma(\varepsilon/2) = \min [\sigma(u) : \varepsilon/2 \leqq u \leqq \varepsilon], \tag{3.16}$$

that is, the minimum of σ on $[\varepsilon/2, \varepsilon]$ is taken at the left endpoint. To see that such an ε exists, let $2z$ be the least positive zero of σ, and let $m(u) = \min [\sigma(x) : u \leqq x \leqq z]$. We now choose ε as the largest number $\leqq z$ for which $\sigma(\varepsilon/2) = m(z/2)$, which exists because $0 < m(z/2) \leqq \sigma(z/2)$ and σ tends to zero at zero. Then $0 < \sigma(\varepsilon/2) \leqq \sigma(x)$ for $\varepsilon/2 < x \leqq z/2$ since ε was largest and $\sigma(\varepsilon/2) \leqq \sigma(x)$ for $z/2 \leqq x \leqq \varepsilon$ because $\sigma(\varepsilon/2) = m(z/2)$ and $\varepsilon \leqq z$. Thus (3.16) holds for ε.

Having chosen ε, we next define

$$\psi(h) = \min [\sigma(u) : h \leqq u \leqq \varepsilon]. \tag{3.17}$$

ψ dominates any nondecreasing minorant ψ' of σ on $[0, \varepsilon]$ and so if the hypothesis of the theorem holds then (1.1) fails to hold for ψ, since

$$\psi'(h) = \min [\psi'(u) : h \leqq u \leqq \varepsilon] \leqq \min [\sigma(u) : h \leqq u \leqq \varepsilon] = \psi(h)$$

for $0 \leqq h \leqq \varepsilon$. Fixing ψ as in (3.17) we have that if $2t \leqq \varepsilon$,

$$\begin{aligned} \psi^2(2t) &= \min [\sigma^2(2u) : t \leqq u \leqq \varepsilon/2] \\ &= 4 \min [\sigma^2(u) : t \leqq u \leqq \varepsilon] \leqq 4 \min [\sigma^2(u) : t \leqq u \leqq \varepsilon/2] \end{aligned}$$

by (3.15) and so by (3.16) we have

$$\psi^2(2t) \leqq 4\psi^2(t). \tag{3.18}$$

Let now $f_n^2 = \psi^2(2^{-n}\varepsilon)$, $n = 1, 2, \ldots$, so that $f_n^2 \downarrow$ and let g_n^2 be the largest convex minorant of f_n^2 defined as before. By hypothesis, $\int^\infty \psi(2^{-u}\varepsilon)u^{-1/2}\,du = \infty$ and so $\sum f_n/\sqrt{n} = \infty$ since ψ is monotonic. By (3.11) we see that $\sum g_n/\sqrt{n} = \infty$ as well.

We next define a *convex* subminorant h^2 of g^2 by extending certain edges of the graph of g^2, viewed as a convex polygon. Let $s_0 = 0$, and define $0 < r_1 < s_1 < r_2 < s_2 < \cdots$ as follows. Suppose that s_{j-1} has been defined. Let r_j be the first $n > s_{j-1}$ for which $g_n^2 > 2g_{n+1}^2$ if there are such n. (Let $r_j = \infty$ if $g_n^2 \leqq 2g_{n+1}^2$ for all $n > s_{j-1}$.) If $r_j < \infty$, let s_j be the first $n > r_j$ for which $g_n^2 \leqq 2g_{n+1}^2$, noting that $s_j < \infty$ since otherwise g_n decreases exponentially contradicting $\sum g_n/\sqrt{n} = \infty$. Now define $h_n = g_n$ for $s_{j-1} \leqq n \leqq r_j + 1$ and for $r_j + 1 < n < s_j$ define h so that $h_n^2 - h_{n+1}^2 = h^2(s_j) - h^2(s_j + 1) = g^2(s_j) - g^2(s_j + 1)$ (where $h(n)$ is written for h_n to avoid double subscripts). If $r_j = \infty$, set $h_n = g_n$ for all $n > s_{j-1}$. Since h^2 has been defined in $r_j + 1 < n < s_j$ by extending the edge between s_j and $s_j + 1$ of the graph of g^2 backwards throughout $r_j + 1 < n < s_j$, and since g^2 is convex, we must have $h_n^2 \leqq g_n^2$ for all n and we see that h^2 is also convex. We next show that $\sum h_n/\sqrt{n} = \infty$. For $r_j \leqq n < s_j$, $g_{n+1} < g_n/\sqrt{2}$ and so $g_n \leqq g(r_j)\sqrt{2}^{r_j - n}$ for $r_j \leqq n < s_j$. Hence with $r = r_j$, $s = s_j$, we have

$$\sum_{n=r}^{s-1} g_n/\sqrt{n} \leqq (g(r)/\sqrt{r}) \sum_{n=r}^{\infty} \sqrt{2}^{r-n} \leqq 4g(r)/\sqrt{r} = 4h(r)/\sqrt{r} \leqq 4 \sum_{n=r}^{s-1} h(n)/\sqrt{n}.$$

Since $h_n = g_n$ for $s \leqq n < r_{j+1}$, it follows that $\sum g_n/\sqrt{n} \leqq 4 \sum h_n/\sqrt{n}$ and so $\sum h_n/\sqrt{n} = \infty$.

Set $a_n^2 = h_n^2 - h_{n+1}^2$ in (3.6). Since a_n decreases (3.9) holds by Boas' lemma. By Szidon's lemma the process Y of (3.6) satisfies (3.4). By Slepian's lemma the theorem follows from (3.3) and (3.4) and the only thing left to show is that the right side of (3.12) is bounded by $K\sigma^2(t)$ for $t < \varepsilon$, where N is the largest integer for which $2^N t \leqq \varepsilon$, ε as above. The second sum in (3.12) is $4h_{N+1}^2 \leqq 4g_{N+1}^2 \leqq 4f_{N+1}^2 = 4\psi^2(2^{-N-1}\varepsilon) \leqq 4\psi^2(t) \leqq 4\sigma^2(t)$. The first sum in (3.12) will be bounded by $K\sigma^2(t)$ if we can show that

$$\sum_{n=1}^{N} (h_n^2 - h_{n+1}^2) 2^{2n} t^2 \leqq K g_N^2, \tag{3.19}$$

because $g_N^2 \leqq f_N^2 = \psi^2(2^{-N}\varepsilon) \leqq 4\psi^2(2^{-N-1}\varepsilon) \leqq 4\psi^2(t) \leqq 4\sigma^2(t)$, where we have used (3.18). It is the construction of h that plays the crucial role in proving (3.19) and (3.19) is not necessarily true with the right side replaced by Kh_N^2.

We begin with the observation that

$$4g_{n+1}^2 \geqq g_n^2 \quad \text{for all } n = 1, 2, \ldots. \tag{3.20}$$

To prove (3.20) observe that either $g_{n+1}^2 = f_{n+1}^2$ or $g_{n+1}^2 < f_{n+1}^2$. In the first case, $4g_{n+1}^2 = 4f_{n+1}^2 \geqq f_n^2 \geqq g_n^2$ while in the second case, $g_{n+1}^2 - g_{n+2}^2 = g_n^2 - g_{n+1}^2$ by the definition of g^2 and so $4g_{n+1}^2 \geqq 2g_{n+1}^2 = g_n^2 + g_{n+2}^2 \geqq g_n^2$. Thus (3.20) holds.

Now suppose that N in (3.19) satisfies $s_k \leqq N < s_{k+1}$. For $j < k$ we have with $s = s_{j-1}$, $r = r_j$, $t = s_j$,

$$\sum_{n=s+1}^{r} (h_n^2 - h_{n+1}^2) 2^{2n} = \sum_{n=s+1}^{r} (g_n^2 - g_{n+1}^2) 2^{2n} \leqq \sum_{n=s+1}^{r} g_n^2 2^{2n}. \tag{3.21}$$

But summing by parts we get

$$\begin{aligned} \sum_{n=s+1}^{r} g_n^2 2^{2n} &= g_r^2 \sum_{n=s+1}^{r} 2^{2k} + \sum_{n=s+1}^{r-1} (g_n^2 - g_{n+1}^2) \sum_{k=s+1}^{n} 2^{2k} \\ &\leqq \tfrac{4}{3} g_r^2 2^{2r} + \tfrac{4}{3} \sum_{n=s+1}^{r-1} (g_n^2 - g_{n+1}^2) 2^{2n}. \end{aligned} \tag{3.22}$$

Since $g_n^2 \leqq 2g_{n+1}^2$ for $s < n \leqq r-1$, $g_n^2 - g_{n+1}^2 \leqq g_n^2/2$ for $s+1 \leqq n \leqq r-1$ and substituting in (3.22) we get

$$\sum_{n=s+1}^{r} g_n^2 2^{2n} \leqq \tfrac{4}{3} g_r^2 2^{2r} + \tfrac{2}{3} \sum_{n=s+1}^{r} g_n^2 2^{2n}. \tag{3.23}$$

Subtracting the last term in (3.23) from both sides and using (3.21) gives

$$\sum_{n=s+1}^{r} (h_n^2 - h_{n+1}^2) 2^{2n} \leqq 2g_r^2 2^{2r}. \tag{3.24}$$

Since h is linear in the range $r+1<n\leqq t+1$ we have

$$\sum_{n=r+2}^{t} (h_n^2-h_{n+1}^2)2^{2n} = (h_t^2-h_{t+1}^2) \sum_{n=r+2}^{t} 2^{2n} \tag{3.25}$$
$$\leqq (g_t^2-g_{t+1}^2)2^{2t}\tfrac{4}{3} \leqq 2g_t^2 2^{2t}.$$

Since $(h_{r+1}^2-h_{r+2}^2)2^{2(r+1)}\leqq 4h_{r+1}^2 2^{2r}\leqq 4h_r^2 2^{2r}=4g_r^2 2^{2r}$ we have from (3.24) and (3.25) that

$$\sum_{n=s_{j-1}+1}^{s_j} (h_n^2-h_{n+1}^2)2^{2n} \leqq 6g(r_j)^2 2^{2r_j}+2g^2(s_j)2^{2s_j} \tag{3.26}$$
$$\leqq 8g^2(s_j)2^{2s_j},$$

the last inequality being valid because by (3.20) $g_n^2\leqq 4g_{n+1}^2$ for all n and so $g(n)^2 2^{2n}$ is monotonically increasing in n. Now again by (3.20), $g_n^2\leqq 4g_{n+1}^2$ for $s_{j-1}\leqq n<s_j$ but there is at least one value of n in the range $s_{j-1}\leqq n<s_j$ for which $g_n^2\leqq 2g_{n+1}^2$, namely $n=s_{j-1}$. Hence

$$g(s_{j-1})^2 2^{2s_{j-1}} \leqq \tfrac{1}{2}g(s_j)^2 2^{2s_j}. \tag{3.27}$$

Summing the inequalities (3.26) and using (3.27) we get

$$\sum_{n=1}^{s_k} (h_n^2-h_{n+1}^2)2^{2n} \leqq 8g(s_k^2)2^{2s_k}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^k}\right). \tag{3.28}$$

The same proof as in (3.24) shows that if $s_k\leqq N\leqq r_k$ that

$$\sum_{n=s_k+1}^{N} (h_n^2-h_{n+1}^2)2^{2n} \leqq 2g_N^2 2^{2N} \tag{3.29}$$

and the same proof as in (3.25) shows that (3.29) is valid also for $r_k<N<s_{k+1}$. Thus we have from (3.28) and (3.29)

$$\sum_{n=1}^{N} (h_n^2-h_{n+1}^2)2^{2n} \leqq 18g_N^2 2^{2N} \tag{3.30}$$

again using the monotonicity of $g_n^2 2^{2n}$. Since $t^2 2^{2N}\leqq \varepsilon^2$ we see that (3.19) is valid with $K=18\varepsilon^2$. The theorem is proved.

4. **Discontinuous processes which are incrementally continuous but not sequentially continuous.** A process X will be called incrementally continuous if for every monotonic (either increasing or decreasing) bounded sequence $\mathbf{t}=t_1, t_2, \ldots,$

$$X(t_{n+1})-X(t_n)\to 0 \quad \text{as } n\to\infty \tag{4.1}$$

except for a set $A(\mathbf{t})$ of paths X with $P(A(\mathbf{t}))=0$. Of course if X consists of a single function then that function must be continuous, but the Poisson process is an example of a discontinuous process which is incrementally continuous. On the other hand, it seems to be difficult to find a process which is incrementally continuous without also being sequentially continuous, that is whenever $t_n \to t$, $X(t_n) \to$ limit, except for a null set $A(t)$. Since it can be shown(2) that a sequentially continuous Gaussian process always has continuous sample paths, examples of processes with the desired properties can be constructed by using the theorem of §3 and the following theorem.

THEOREM. *If X has covariance ρ of Polya's type then X is incrementally continuous if and only if* (1.4) *holds.*

Proof. The property of Polya-type processes that we make use of is that if $s_1 < s_2 < t_1 < t_2$, then

$$E(X(s_2)-X(s_1))(X(t_2)-X(t_1)) \leqq 0. \tag{4.2}$$

Indeed, if $t_2-s_2 < t_1-s_1$ we write the left side of (4.2) as

$$\begin{aligned}(\rho(t_2-s_2)-\rho(t_1-s_2))-(\rho(t_2-s_1)-\rho(t_1-s_1)) &= (\rho'(\theta_2)-\rho'(\theta_1))(t_2-t_1)\\ &= \rho''(\theta_3)(\theta_2-\theta_1)(t_2-t_1) \leqq 0\end{aligned}$$

since $0 < \theta_2 < t_2-s_2 < t_1-s_1 < \theta_1$. On the other hand if $t_1-s_1 < t_2-s_2$ we write the left side of (4.2) as $(\rho(t_1-s_1)-\rho(t_1-s_2))-(\rho(t_2-s_1)-\rho(t_2-s_2))$ and proceed analogously. Thus (4.2) holds.

Now we use two lemmas from [7], omitting their proofs.

LEMMA 1. *Let X and Y be Gaussian variables with zero mean and suppose* $EXY \leqq 0$. *Then if* $a \geqq 0$, $b \geqq 0$,

$$P(X \geqq a,\ Y \geqq b) \leqq P(X \geqq a)P(Y \geqq b). \tag{4.3}$$

LEMMA 2. *Let* $B_1, B_2, \ldots$ *be events with* $P(B_jB_k) \leqq P(B_j)P(B_k)$ *for all* $j \neq k$. *If* $\sum P(B_n) = \infty$ *then* $P(B_n$ *infinitely often*$) = 1$.

Let $\boldsymbol{t}$ be a monotone sequence and $\varepsilon > 0$. Define

$$B_n = \{X(t_{n+1}) - X(t_n) \geqq \varepsilon\}, \qquad n = 1, 2, \ldots.$$

By (4.2), Lemmas 1 and 2, and the usual Borel-Cantelli lemma, we see that X is incrementally continuous if and only if $\sum P(B_n) < \infty$ for every monotonic sequence $\boldsymbol{t}$. Supposing that $t_n \uparrow t$ which involves no loss of generality, we see that $\delta_n = t_{n+1} - t_n \geqq 0$ must satisfy $\sum \delta_n < \infty$. We have

$$P(B_n) = P\{\eta \geqq \varepsilon/\sigma(\delta_n)\} \tag{4.4}$$

where η is a standard normal variable. If (1.4) fails to hold then there exist $u_n \downarrow 0$ and $\theta > 0$ for which $\sigma^2(u_n) > \theta/\log(1/u_n)$. If we let $(\delta_1, \delta_2, \ldots) = (u_1, u_1, \ldots, u_1, u_2, u_2, \ldots, u_2, u_3, \ldots)$ where the block of u_1's is of length $A_1 \geqq 0$, the block of u_2's

(2) See Appendix 4.

is of length $A_2 \geqq 0$, and so on, we can make $\sum \delta_n < \infty$ and $\sum P(B_n) = \infty$ provided that we can choose the A's so that

$$\sum A_n u_n < \infty \tag{4.5}$$

and

$$\sum A_n \exp(-\varepsilon^2/\sigma^2(u_n)) = \infty \tag{4.6}$$

hold. The sum in (4.6) is at least $\sum A_n u_n^{\varepsilon^2/\theta}$. If we choose $\varepsilon^2 = \theta/2$, it is clear that there can be found nonnegative integers $A_1, A_2, \ldots$, for which $\sum A_n u_n < \infty$ and $\sum A_n u_n^{1/2} = \infty$. Thus if (1.4) fails for X of Polya type then X is not incrementally continuous.

On the other hand, if (1.4) holds, then X is incrementally continuous even if X is not of Polya's type because $(\varepsilon^2/2\sigma^2(\delta_n))^{-1} = o(1/\log(1/\delta_n))$ and it follows that $\sum P(B_n) < \infty$ since $P(B_n) \leqq \delta_n$ for sufficiently large n.

It is easy to find covariances which satisfy (1.4) but not (1.1) and thereby construct incrementally continuous processes which are not sequentially continuous.

5. **Examples.** We give first a continuous process with $\limsup (\log 1/h)\sigma^2(h) \neq 0$ thereby showing that the condition that X be of Polya type cannot be dropped entirely in proving that X is not incrementally continuous in §4. At the same time, σ of the example also fails to satisfy (1.2) and thereby shows that (1.2) cannot be necessary for sample continuity in general. The example is

$$X(t) = \sum_1^\infty \frac{1}{n^2}(\eta_n \cos 2^{2^n} t + \eta_n' \sin 2^{2^n} t) \tag{5.1}$$

where η and η' are independent standard normal sequences. X is continuous because $\sum (|\eta_n| + |\eta_n'|)/n^2 < \infty$, a.s. On the other hand, we have

$$(\log 2^{2^k})\sigma^2(1/2^{2^k}) \geqq (1-\cos 1)(\log 2)2^k/k^4 \to \infty \tag{5.2}$$

and so $\limsup (\log 1/h)\sigma^2(h) \neq 0$. A short calculation shows that (1.2) fails to hold. Of course, σ^2 is not monotonic.

We remark that an extension of the above example shows that there is no simple analogue of Slepian's lemma in terms of spectral distribution functions (s.d.f.). In fact if F is the s.d.f. of any discontinuous process then there is a s.d.f. G of a continuous process Y with $G(x) \leqq F(x)$ for all $x \geqq 0$, $G(\infty) = F(\infty)$. Indeed this is trivially seen by taking $Y(t) = \sum a_n(\eta_n \cos \theta_n t + \eta_n' \sin \theta_n t)$ where η and η' are as in (3.6). If $\sum |a_n| < \infty$, Y is continuous no matter what values the θ's take and if $\theta_n \to \infty$ sufficiently fast, it is clear that G will have uniformly fatter tails than F. Similarly, it is shown that if F is *any* s.d.f. with $F \not\equiv 0$ then G_1 and G_2 may be found with $G_1 \leqq F \leqq G_2$ and $G_1(\infty) = G_2(\infty)$ where the corresponding processes are either continuous or discontinuous.

The next example shows that the condition

$$\sum s_n^{1/2} < \infty, \qquad s_n = F(2^{n+1}) - F(2^n), \tag{5.3}$$

where F is the spectral cumulative distribution function of X, fails to be sufficient for continuity of X in general. Our example, which is of Polya type, is defined as follows:

Let μ be the measure having a point mass

$$\lambda_m = (1/2^m)^{1/2} \text{ at } p_m = 1/2^{2^m}, \tag{5.4}$$

$m=1, 2, \ldots$. Define $F(x)$ by setting $F'(x)=f(x)$ for all x, where

$$f(x) = \int_0^1 \frac{1-\cos xt}{x^2t^2}\, t\, d\mu(t). \tag{5.5}$$

If $p_m < x^{-1} < p_{m+1}$,

$$f(x) \leqq \int_0^{1/x} t\, d\mu(t) + \int_{1/x}^1 \frac{2}{x^2 t}\, d\mu(t) \leqq 2\lambda_{m+1}p_{m+1} + 4x^{-2}\lambda_m p_m^{-1} \tag{5.6}$$

and so, for $0 \leqq k < 2^m$,

$$F(2^{2^m+k+1}) - F(2^{2^m+k}) = S_{2^{2^m+k}} \leqq 2\lambda_{m+1}p_{m+1}2^{2^m+k} + 4\lambda_m p_m^{-1}2^{-2^m-k}. \tag{5.7}$$

It follows easily that (5.3) holds. To prove that X is not continuous it suffices to check that (1.2) fails to hold because X is of Polya type, as is seen by noting that

$$\begin{aligned} f(x) &= \int_0^1 \rho(t)\cos xt\, dt = -\frac{1}{x}\int_0^1 \rho'(t)\sin xt\, dt \\ &= \int_0^1 \frac{1-\cos xt}{x^2t^2}\, t^2\, d\rho'(t) \end{aligned} \tag{5.8}$$

where we have set $td\rho'(t)=d\mu(t)$, $\rho(1)=\rho'(1)=0$, μ as in (5.4). We have

$$\begin{aligned} \rho(0)-\rho(t) &= \int_0^t -\rho'(u)\, du = -\rho'(t)t + \int_0^t d\mu(u) \\ &= t\int_t^1 \frac{1}{u}\, d\mu(u) + \int_0^t d\mu(u) \geqq \int_0^t d\mu(u). \end{aligned} \tag{5.9}$$

Thus if $p_{m+1} < t < p_m$, $\rho(0)-\rho(t) \geqq \lambda_{m+1}$ and it is a simple matter to check that (1.2) fails to hold. Thus X is discontinuous in spite of (5.3). Of course, s_n in this case is not a monotonic sequence.

Appendix.

1. *Proof of Slepian's lemma.* The intuitive explanation of Slepian's inequality is that although X and Z are instanteously of equal variance, X oscillates more than Z because it is less correlated. It is enough to prove (3.5) for finite sets $I=\{t_1, \ldots, t_n\}$. The left side of (3.5) is then $1-Q$ where $Q=Q(\boldsymbol{r})$ is given by

$$Q(\boldsymbol{r}) = \int_{-\infty}^M dx_1 \cdots \int_{-\infty}^M dx_n g(x_1, \ldots, x_n; \boldsymbol{r}) \tag{A.1}$$

and the Gaussian density $g(\boldsymbol{x}, \boldsymbol{r})$ is given in terms of its characteristic function by

$$\text{(A.2)} \quad g(\boldsymbol{x}, \boldsymbol{r}) = \int_{-\infty}^{\infty} d\xi_1 \cdots \int_{-\infty}^{\infty} d\xi_n (2\pi)^{-n} \exp\left(i \sum x_j \xi_j - (1/2) \sum \sum r_{jk} \xi_j \xi_k\right)$$

and $r_{jk} = EX(t_j)X(t_k)$, $\boldsymbol{r} = \{r_{jk}\}$. From (A.2) we obtain immediately that $\partial g/\partial r_{jk} = \partial^2 g/\partial x_j \partial x_k$ for $j \neq k$ and so differentiating (A.1) with respect to r_{12} say we get

$$\frac{\partial}{\partial r_{12}} Q(\boldsymbol{r}) = \int_{-\infty}^{M} dx_1 \cdots \int_{-\infty}^{M} dx_n \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1, x_2, \ldots, x_n, \boldsymbol{r}).$$

Integrating we get

$$\frac{\partial}{\partial r_{12}} Q(\boldsymbol{r}) = \int_{-\infty}^{M} dx_3 \cdots \int_{-\infty}^{M} dx_n g(M, M, x_3, \ldots, x_n, \boldsymbol{r}).$$

Thus $\partial Q(\boldsymbol{r})/\partial r_{jk} \geqq 0$ for all $j \neq k$. Now suppose that $s_{jk} = EZ(t_j)Z(t_k)$ and let

$$r_{jk}(\lambda) = \lambda r_{jk} + (1-\lambda) s_{jk}.$$

Then $\boldsymbol{r}(\lambda) = \{r_{jk}(\lambda)\}$ is positive definite and so $q(\lambda) = 1 - Q(\boldsymbol{r}(\lambda))$ is defined. We see that

$$\begin{aligned} q'(\lambda) &= -\sum_j \sum_k (\partial/\partial r_{jk}) Q(\boldsymbol{r}(\lambda)) \cdot dr_{jk}(\lambda)/d\lambda \\ &= -\sum_j \sum_k (\partial/\partial r_{jk}) Q(\boldsymbol{r}(\lambda)) \cdot (r_{jk} - s_{jk}) \geqq 0 \end{aligned}$$

since $r_{jj} = s_{jj}$ and for $j \neq k$, $\partial Q/\partial r_{jk} \geqq 0$ and $s_{jk} \geqq r_{jk}$. Thus

$$q(1) = 1 - Q(\boldsymbol{r}) = P\{\max X(t_j) \geqq M\} \geqq q(0) = 1 - Q(\boldsymbol{s}) = P\{\max Z(t_j) \geqq M\},$$

proving (3.5).

2. *Proof of Boas' inequality.* The following proof was shown to us by H. O. Pollak. We have since $a_1 \geqq a_2 \geqq \cdots$, for each $m \geqq n$

$$\text{(B.1)} \quad \begin{aligned} \left(\sum_{k=n}^{m} a_k^2\right)^{1/2} - \left(\sum_{k=n}^{m-1} a_k^2\right)^{1/2} &= a_m^2 \left(\left(\sum_{k=n}^{m} a_k^2\right)^{1/2} + \left(\sum_{k=n}^{m-1} a_k^2\right)^{1/2} \right)^{-1} \\ &\leqq a_m^2/(m+1-n)^{1/2} a_m = a_m/(m+1-n)^{1/2}. \end{aligned}$$

Adding on m, we get

$$\text{(B.2)} \quad \begin{aligned} \sum_{m=n}^{\infty} \left(\left(\sum_{k=n}^{m} a_k^2\right)^{1/2} - \left(\sum_{k=n}^{m-1} a_k^2\right)^{1/2} \right) &= \left(\sum_{k=n}^{\infty} a_k^2\right)^{1/2} = g_n \\ &\leqq \sum_{m=n}^{\infty} a_m/(m+1-n)^{1/2}. \end{aligned}$$

Dividing by $\sqrt{n}$ and adding, we get

$$(\text{B.3}) \qquad \begin{aligned} \sum_{n=1}^{\infty} g_n/\sqrt{n} &\leqq \sum_{n=1}^{\infty} (1/\sqrt{n}) \sum_{m=n}^{\infty} a_m/(m+1-n)^{1/2} \\ &= \sum_{m=1}^{\infty} a_m \sum_{n=1}^{m} 1/(n(m+1-n))^{1/2} \\ &\leqq 2 \sum_{m=1}^{\infty} a_m, \end{aligned}$$

proving the lemma.

We remark finally that the reverse inequality, $\sum a_n \leqq 2 \sum g_n/\sqrt{n}$ is valid even without assuming that the a's are monotonic. Indeed, we may write

$$(\text{B.4}) \qquad \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{1}{m} a_m \leqq \sum_{n=1}^{\infty} g_n \left(\sum_{m=n}^{\infty} 1/m^2 \right)^{1/2}$$

and the remark follows.

3. *Footnote* 1, §2. Actually Belyaev's theorem shows only that a discontinuous Gaussian process is a.s. unbounded on any *interval I*. To see that X is in fact a.s. unbounded on any set S dense in I, we proceed as follows. If $S' \subset I$ is a countable set which acts as a set of separability for X, then, since X is unbounded on I, $\sup [|X(t)| : t \in S \cup S'] = \infty$, a.s. But $\sup [|X(t)| : t \in S] = \sup [|X(t)| : t \in S \cup S']$ w.p.1. because every point $t \in S'$ is a limit of points $t_n \in S$ and so $X(t_{n'}) \to X(t)$ a.s. along some subsequence $\{t_{n'}\}$ of $\{t_n\}$ since $X(t_n) \to X(t)$ in the mean. Since S' is countable we see that $\sup [|X(t)| : t \in S] = \sup [|X(t)| : t \in S \cup S'] = \infty$ w.p.1. Thus the claim that (2.3) is sufficient for the theorem is proved.

4. *Footnote* 2, §4. To see that a discontinuous Gaussian process is not sequentially continuous, note first that as a result of Appendix 3 above if X is discontinuous and $a < b$,

$$(\text{D.1}) \qquad P(|X(a+(b-a)j/n)| \leqq M, j = 0, 1, \ldots, n) \to 0, \qquad \text{as } n \to \infty.$$

Define $t_2 = t_4 = \cdots = 0$ and choose $0 < a_j < b_j$, $j = 1, 2, \ldots$, with $b_j \downarrow 0$. Define $t_1, t_3, \ldots$, and $n(0) = 0 < n(1) < \cdots$ inductively as follows: Choose for $j = 1, 2, \ldots$, $t_{2n(j-1)+1}, t_{2n(j-1)+3}, \ldots, t_{2n(j)-1}$ so each belong to (a_j, b_j) and together satisfy for each $j = 1, 2, \ldots$

$$(\text{D.2}) \qquad P(|A(t_{2i+1})| \leqq j, n(j-1) \leqq i < n(j)) < 1/j,$$

which can be done on account of (D.1). We see that $t_n \to 0$ but $X(t_n)$ do not have a limit because $P(X(t_{2n+1}) - X(t_{2n}) \to 0) = P(X(t_{2n+1}) - X(0) \to 0) \leqq P(|X(t_{2n+1})|$, $n = 1, 2, \ldots$ is bounded$) = 0$ by (D.2). Thus X is not sequentially continuous.

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Northwestern University,
Evanston, Illinois 60201
Bell Telephone Laboratories, Inc.,
Murray Hill, New Jersey 07974