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CONTINUITY OF LINEAR FRACTIONAL TRANSFORMATIONS ON AN OPERATOR ALGEBRA[1]

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ABSTRACT. An operator coefficient linear fractional automorphism $\mathscr{F}$ on the unit ball of operators is continuous in the weak operator topology if and *only if* $\mathscr{F}(0)$ is compact.

Let $\mathscr{B}$ denote the set of bounded operators on the Hilbert space H which have norm not greater than one. A map $\mathscr{F}:\mathscr{B}\to\mathscr{B}$ of the form

$$\mathscr{F}(J) = (C + DJ)(A + BJ)^{-1} \quad \text{for each } J \in \mathscr{B} \tag{1}$$

is called *general symplectic* when A, B, C, and D are operators on H which satisfy

$$AA^* - BB^* = I = DD^* - CC^*, \tag{2}$$

$$AC^* = BD^*. \tag{3}$$

Fixed point theorems for general symplectic maps are of particular interest. The best proof of the main fixed point theorem (Pontryagin-H. Langer) for these maps is an application of the Schauder-Tychonoff theorem and is due to M. G. Krein [**1**]. His proof consists of showing that if $\mathscr{F}$ is a general symplectic map of form (1) where the operators B and C are compact operators, then $\mathscr{F}$ is continuous in the weak operator topology (abbreviated w.o.t.). In this note, we prove that the converse is true, namely

PROPOSITION. *If $\mathscr{F}$ is continuous in the weak operator topology then B and C are compact operators.*

PROOF. The following facts will be useful: Equation (2) implies that A and D are invertible. Set $S=A^{-1}B$. Equation (3) implies that $S^*=D^{-1}C$. Equations (2) and (3) combined imply $\|S\|<1$; consequently $1-S^*S$ is invertible.

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Suppose that $\mathscr{F}(J)$ is continuous in the w.o.t. Then

$$\begin{aligned} D^{-1}[\mathscr{F}(J) - \mathscr{F}(0)]A &= (D^{-1}C + J)(I + A^{-1}BJ)^{-1} - D^{-1}C \\ &= (S^* + J)(1 + SJ)^{-1} - S^* \\ &= [I - S^*S]J(I + SJ)^{-1} \end{aligned} \tag{4}$$

is w.o.t. continuous. Thus $G(J)=J(I+SJ)^{-1}$ is continuous in the weak operator topology. Let $S=RU$ be the polar decomposition of S; R is nonnegative and U is a partial isometry. The fact that $G(J)$ is w.o.t. continuous implies that $T(M)=M(I+RM)^{-1}$ is w.o.t. continuous, since $UG(J)=T(UJ)$.

If we assume that S is not compact then R has an infinite dimensional and separable invariant subspace τ on which R is an invertible operator. Now we restrict our attention to τ. Let $W=R|_\tau$. The map $T^1(N)=N(I+WN)^{-1}$ defined for contraction operators N on τ must be w.o.t. continuous. Since W is invertible, the map B defined by $B(K)=K(1+K)^{-1}=WT(W^{-1}K)$ is w.o.t. continuous on

$$\{K: \|S^{-1}K\| \leqq 1\} \subset \{K: \|K\| \leqq 1/\|W^{-1}\|\} \triangleq \mathscr{U}.$$

We now produce a contradiction to our assumption that S is not compact and $\mathscr{F}$ is w.o.t. continuous by proving that $B(K)$ is not w.o.t. continuous at the origin. Since τ is separable we may, with no loss of generality, assume that τ is $L^2(-\pi, \pi)$. Let $\alpha=1/\|W^{-1}\|<1$ and let K_n denote the operator on τ given by

$$[K_n f](x) = k_n(x)f(x),$$

where $k_n(x)=\alpha \sin nx$. By the Riemann-Lebesgue lemma K_n converges to 0 in the weak operator topology. However,

$$(1, K_n(1 + K_n)^{-1}1) = -\alpha\int_{-\pi}^{\pi} \frac{\sin nx}{1 - \alpha \sin nx}\, dx = \sum_{k=1}^{\infty} \alpha^{2k-1}\binom{2k}{k}\left(\frac{1}{4}\right)^k$$

which is independent of n and which is not equal to 0.

Bibliography

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