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CONTINUITY OF NONLINEAR MONOTONE OPERATORS

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ABSTRACT. If a Banach space E has an equivalent norm such that weak* sequential convergence and norm convergence coincide on the dual unit sphere, then every monotone operator on E is single-valued and norm-norm continuous on a dense G_δ subset of E. In particular, this holds for reflexive spaces.

Let E be a real Banach space with dual E^*. A multivalued mapping $T: E \to E^*$ is called a *monotone operator* if $\langle x^* - y^*, x - y \geqslant 0\rangle$ whenever $x^* \in Tx$ and $y^* \in Ty$. It is called *maximal monotone* if, in addition, its graph, $\{(x, x^*): x \in E, x^* \in Tx\}$, is not properly contained in the graph of any monotone operator on E.

We say that a monotone operator $T: E \to E^*$ is *locally bounded* at $x \in E$ if there is a neighborhood U of x such that $T(U) = \bigcup\{Ty: y \in U\}$ is a bounded subset of E^*. This does not demand that $x \in D(T) = \{y \in E: Ty \neq \varnothing\}$. We say that T is *continuous* at a point $x \in D(T)$ if, whenever $x_n \to x$, $x_n^* \in Tx_n$ and $x^* \in Tx$, we have $\|x_n^* - x^*\| \to 0$. If T is continuous at x, then it is necessarily *single-valued* at x, that is, Tx has exactly one element.

We will assume from now on that T is a maximal monotone operator on E, with $D(T) = E$. This latter hypothesis, while not strictly necessary, simplifies both the statements and the proofs of our results. All the proofs can actually be extended to the case where $D(T) \neq E$, provided $\operatorname{int}\operatorname{conv} D(T) \neq \varnothing$. The reason for this stems from the first part of the following result of Rockafellar [9], which will be of further use to us.

PROPOSITION A. *Let $T: E \to E^*$ be a maximal monotone operator with* $\operatorname{int}\operatorname{conv} D(T) \neq \varnothing$. *Then* $\operatorname{int} D(T)$ *is convex,* $\operatorname{cl} D(T) = \operatorname{cl}\operatorname{int} D(T)$, *$T$ is locally bounded at each point of* $\operatorname{int} D(T)$ *and T is not locally bounded at any point of the boundary of $D(T)$.*

The motivation for studying monotone operators comes from the study of

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[2] The author notes that the same result has recently been announced by P. Kenderov and R. Robert, in C. R. Acad. Sci. Paris **282** (1976), No. 16.

integral and partial differential equations, and from the theory of convex functions. The subdifferential ∂f of a lower semicontinuous convex function $f\colon E \to \mathbf{R}$, where

$$\partial f(x) = \{x^* \in E^*\colon \langle x^*, x - y\rangle \geqslant f(x) - f(y), \text{ all } y \in E\},$$

is a maximal monotone operator (Rockafellar [10]; later, Taylor [11] gave an easier proof), and maximal monotone operators which are *not* the subdifferentials of lower semicontinuous convex functions (that is, are not *cyclically monotone* [10]) have some of the desirable properties of those which are. For instance, if E is separable (Zarantonello [12]), or if E^* has an equivalent strictly convex dual norm (Kenderov [5]), then the set $S(T)$ of points where T is single-valued is a dense G_δ subset of E, hence topologically "almost all" of E. In the finite dimensional case, Zarantonello showed that $S(T)$ is almost all of E with respect to Lebesgue measure, too. Using Aronszajn's generalization of the notion of sets of measure zero [1], this result remains valid for separable spaces. These results generalize known results about lower semicontinuous convex functions f on E, for f is Gâteaux (resp. Fréchet) differentiable at x if and only if ∂f is single-valued (resp. continuous) at x.

As in Asplund [2] and Namioka and Phelps [6], we define E to be a *weak differentiability space* (resp. *Asplund space*) if every continuous convex function $f\colon E \to \mathbf{R}$ is Gâteaux (resp. Fréchet) differentiable on a dense G_δ subset of E.

We now ask

Question (A). If E is an Asplund space, is T necessarily continuous on a dense G_δ subset of E?

Question (B). If E is a weak differentiability space, is T necessarily single-valued on a dense G_δ subset of E?

The results of Kenderov and Zarantonello answer Question (B) affirmatively for most of the known weak differentiability spaces, but the general case for both questions seems difficult.

Our main result is the following partial answer to Question (A).

THEOREM 1. *Let E be a Banach space which admits an equivalent norm whose dual norm satisfies* (H*):

$$(\mathrm{H}^*) \qquad \text{If } x_n^* \in E^*,\ x_n^* \to x_0^* \text{ weak}^* \text{ and } \|x_n^*\| \to \|x_0^*\|, \text{ then } \|x_n^* - x_0^*\| \to 0.$$

Then for any maximal monotone operator $T\colon E \to E^*$, the set $C(T) = \{x \in E\colon T$ is continuous at $x\}$ is a dense G_δ subset of E.

We know of no Asplund spaces whose duals are not locally uniformly convex for some equivalent dual norm, and it is easy to see that if a dual norm on E^* is locally uniformly convex, then it satisfies (H^*). We also note that (H^*) is apparently weaker then (**) of Corollary 8 of Namioka and Phelps [6], so we thus have a very different proof of that corollary.

If E is finite dimensional, then continuity follows from single-valuedness, and Zarantonello [12] shows that the points of continuity are almost all of E with respect to Lebesgue measure. Robert [7], [8], has proved Theorem 1 for separable spaces with separable duals. Theorem 1 covers all reflexive spaces, of course.

Now we turn to the proof of the theorem. We do not assume (H*) until Lemma 7; before that E can be any Banach space.

The following proposition was announced by Kenderov; we prove it for the sake of completeness.

PROPOSITION B. *If T is a maximal monotone operator and $x \in E$, then given a weak* neighborhood W of Tx, there is a neighborhood U of x such that $T(U) \subset W$.*

PROOF. Suppose $x_n \to x$ and $x_n^* \in Tx_n$. Then by local boundedness (Proposition A) we may assume that $x_n^* \to x^* \in E^*$, weak*. We show that $x^* \in Tx$: to do this, let $y \in E$ and $y^* \in Ty$. Then

$$0 \leqslant \langle x_n^* - y^*, x_n - y\rangle = \langle x_n^* - y^*, x - y\rangle + \langle x_n^* - y^*, x_n - x\rangle$$

and in the limit we get $\langle x^* - y^*, x - y\rangle \geqslant 0$, so $x^* \in Tx$ by maximality. This is sufficient to prove the proposition.

The next result is due to Browder [3].

PROPOSITION C. *If T is maximal monotone and $x \in E$, then Tx is a weak* compact convex subset of E^*.*

(We are assuming that $D(T) = E$. Otherwise these would hold only for $x \in \operatorname{int} D(T)$.) The next result was announced by Kenderov [5].

LEMMA 2. *Let T be maximal monotone and define $f\colon E \to \mathbf{R}$ by $f(x) = \inf\{\|x^*\|\colon x^* \in Tx\}$. Then f is lower semicontinuous.*

PROOF. Let $x_n \to x$, and suppose $\liminf f(x_n) < f(x)$. Then without loss of generality, there are $x_n^* \in Tx_n$ and $\alpha > 0$ such that $\|x_n^*\| < f(x) - \alpha$. Now (x_n^*) has a weak* convergent subsequence, by local boundedness (Proposition A), so we can assume that $x_n^* \to x^*$ weak*. By maximality $x^* \in Tx$. Since the norm in E^* is weak* lower semicontinuous, $\|x^*\| \leqslant \liminf\|x_n^*\| \leqslant f(x) - \alpha$, which is a contradiction.

By a Baire category result (cf. e.g. [4, Corollary 7.6]) we immediately obtain

COROLLARY 3. *The set $A(T)$ where f is continuous is a dense G_δ subset of E.*

The following lemma contains the critical point in our reconstruction of the argument in Kenderov [5].

LEMMA 4. *If T is maximal monotone, $x \in A(T)$ and $x^* \in Tx$, then $\|x^*\| = f(x)$.*

PROOF. If $\|x^*\| > \dot{f}(x)$, then choose $z \in E$, $\|z\| = 1$, such that $\langle x^*, z\rangle > f(x)$. Now by continuity of f at x, for $\varepsilon > 0$ small enough there exists $x_0^* \in T(x + \varepsilon z)$ such that $\|x_0^*\| < \langle x^*, z\rangle$. But $\langle x_0^* - x^*, (x + \varepsilon z) - x\rangle \geqslant 0$ by monotonicity, so $\langle x_0^*, z\rangle \geqslant \langle x^*, z\rangle > \|x_0^*\|$, contradicting the fact that $\|z\| = 1$.

LEMMA 5. *If T is maximal monotone, $x_0 \in A(T)$ and $x_n \to x_0$, then $\|x_n^*\| \to \|x_0^*\|$, whenever $x_n^* \in Tx_n$ and $x_0^* \in Tx_0$.*

PROOF. Suppose $\limsup\|x_n^*\| > \|x_0^*\| + \alpha$, where $\alpha > 0$, and choose a subsequence, again called (x_n^*), such that $\|x_n^*\| > \|x_0^*\| + \alpha$. Let $z_n \in E$, $\|z_n\| = 1$, be such that $\langle x_n^*, z_n\rangle > \|x_0^*\| + \alpha$. By local boundedness, we may assume that $\|x_n^*\| < N$, for all $n \geqslant 1$. Pick $y_n \in B(z_n, (nN)^{-1}) = \{x\colon \|x - z_n\| < (nN)^{-1}\}$ such that $x_n + n^{-1}y_n \in A(T)$; this is possible since the open set $x_n + n^{-1}B(z_n, (nN)^{-1})$ intersects the dense set $A(T)$. Then we compute

$$\begin{aligned}\langle x_n^*, y_n\rangle &= \langle x_n^*, z_n\rangle - \langle x_n^*, y_n - z_n\rangle \\ &\geqslant \langle x_n^*, z_n\rangle - \|x_n^*\|(nN)^{-1} \geqslant \langle x_n^*, z_n\rangle - n^{-1}.\end{aligned}$$

Let $y_n^* \in T(x_n + n^{-1}y_n)$; from Lemma 4 it follows that

$$\|y_n^*\| = f(x_n + n^{-1}y_n).$$

By monotonicity, we have $\langle y_n^* - x_n^*, (x_n + n^{-1}y_n) - x_n\rangle \geqslant 0$, so

$$\langle y_n^*, y_n\rangle \geqslant \langle x_n^*, y_n\rangle \geqslant \langle x_n^*, z_n\rangle - n^{-1}.$$

Thus,

$$\|y_n^*\|\,\|y_n\| \geqslant \langle x_n^*, z_n\rangle - n^{-1} > \|x_0^*\| - \alpha - n^{-1},$$

while $\|y_n\| < 1 + (nN)^{-1}$, so $\limsup\|y_n^*\| \geqslant \|x_0^*\| + \alpha$. Now $x_n + n^{-1}y_n \to x_0$ and $x_n + n^{-1}y_n \in A(T)$, so $\|y_n^*\| = f(x_n + n^{-1}y_n) \to f(x_0) = \|x_0^*\|$, since $x_0 \in A(T)$. This contradiction shows that $\limsup\|x_n^*\| \leqslant \|x_0^*\|$.

Now we show that $\liminf\|x_n^*\| \geqslant \|x_0^*\|$. Suppose otherwise; then we may assume that $\|x_n^*\| < \|x_0^*\| - \alpha$, for some $\alpha > 0$. There is a weak* convergent subsequence, again called (x_n^*), such that $x_n^* \to x^* \in Tx_0$ weak*, using local boundedness and Proposition B. Since the norm in E^* is weak* lower semicontinuous, we have $\liminf\|x_n^*\| \geqslant \|x^*\|$ and $\|x^*\| = \|x_0^*\|$ by Lemma 4. This is again a contradiction, and we conclude that $\|x_n^*\| \to \|x_0^*\|$.

Next, we generalize Asplund's Lemma 6 of [2] to the present situation.

LEMMA 6. *Let T be a maximal monotone operator. Then the set of points where T is continuous, $C(T)$, is a G_δ subset of E.*

PROOF. For each $n \geqslant 1$, let $U_n = \{x \in E$: there exists $\delta > 0$ such that for

$y, z \in B(x,\delta)$, $\|y^* - z^*\| < n^{-1}$, for any $y^* \in Ty$ and $z^* \in Tz\}$. It is immediate that $C(T) \subseteq \bigcap U_n$ and it is obvious that U_n is always open, so we only have to prove that $\bigcap U_n \subseteq C(T)$.

Suppose $x \in \bigcap U_n$. Then there are $\delta_n > 0$, which we may take decreasing to 0, such that if $y, z \in B(x,\delta_n)$, $y^* \in Ty$ and $z^* \in Tz$, then $\|y^* - z^*\| < n^{-1}$. Thus $K_n = \operatorname{cl} T(B(x,\delta_n))$ has diameter at most n^{-1}, and $K_{n+1} \subseteq K_n$. By completeness, $\bigcap K_n = \{x^*\}$ for a unique $x^* \in E^*$. We claim that $x^* \in Tx_n$; to this end, let $y \in E$ and $y^* \in Ty$. Then letting $x_n \in B(x,\delta_n)$ and $x_n^* \in Tx_n$, we have

$$\begin{aligned}\langle x^* - y^*, x - y\rangle &\\ &= \langle x^* - x_n^*, x - y\rangle + \langle x_n^* - y^*, x_n - y\rangle + \langle x_n^* - y^*, x - x_n\rangle \\ &\geqslant -n^{-1}\|x - y\| + 0 - \delta_n\|x_n^* - y^*\|.\end{aligned}$$

Since n is arbitrary, we get $\langle x^* - y^*, x - y\rangle \geqslant 0$, and hence by maximality, $x^* \in Tx$. Now if $y \in B(x,\delta_n)$ and $y^* \in Ty$, then $\|y^* - x^*\| \leqslant n^{-1}$. That is, $x \in C(T)$.

We now use property (H*) defined earlier.

LEMMA 7. *Let T be a maximal monotone operator on a Banach space whose dual has an equivalent dual norm satisfying* (H*). *Suppose $x_0 \in A(T)$ and* $\operatorname{diam}(Tx_0) < \varepsilon$. *Then there is a neighborhood V of x such that* $\operatorname{diam}(Tx) < \varepsilon$ *for all $x \in V$.*

PROOF. Suppose that $x_n \to x_0$ and $\operatorname{diam}(Tx_n) \geqslant \varepsilon$. Take $v_n^*, w_n^* \in Tx_n$ such that $\|v_n^* - w_n^*\| \to \varepsilon$ as $n \to \infty$. By local boundedness, we can take a subsequence $(v_{n_j}^*)$ of (v_n^*) such that $v_{n_j}^* \to v^*$, weak*. As in the proof of Lemma 5, $v^* \in Tx_0$. By Lemma 5 and (H*), $\|v_{n_j}^* - v^*\| \to 0$. Take a subsequence $(w_{m_j}^*)$ of $(w_{n_j}^*)$ such that $w_{m_j}^* \to w^*$ weak*. Then $w^* \in Tx_0$ and $\|w_{m_j}^* - w^*\| \to 0$. Now $\|v_{m_j}^* - v^*\| \to 0$, $\|w_{m_j}^* - w^*\| \to 0$ and $\|v_{m_j}^* - w_{m_j}^*\| \to \varepsilon$. Hence $\|v^* - w^*\| = \varepsilon$ and $\varepsilon > \operatorname{diam}(Tx_0)$, a contradiction.

We now give the proof of Theorem 1.

PROOF. If $y^* \in E^*$, then $T - y^*$ is a maximal monotone operator, where

$$(T - y^*)(x) = \{x^* - y^* : x^* \in Tx\}, \qquad x \in E.$$

Let $x_0 \in A(T)$. Then Tx_0 is weak* compact, and $\|x^*\| = \|y^*\|$ for all $x^*, y^* \in Tx_0$, so (H*) implies that Tx_0 is compact. Thus there is a norm-dense sequence $(y_m^*) \subset Tx_0$.

Let $A = \bigcap_m (A(T) \cap A(T - y_m^*))$; this is again a dense G_δ. Pick $x_n \to x_0$, $x_n \in A$, and $v_n^*, w_n^* \in Tx_n$ such that $\|v_n^* - w_n^*\| > \frac{1}{2}\operatorname{diam}(Tx_n)$. The sequence (v_n^*) has a subsequence $(v_{n_j}^*)$ converging weak* to $v^* \in Tx_0$, and by (H*) and Lemma 5, $\|v_{n_j}^* - v^*\| \to 0$. By taking y_m^* close to v^*, and applying Lemma 4 to $T - y_m^*$ we see that $\|v_{n_j}^* - v^*\| = \|w_{n_j}^* - v^*\|$, since $x_{n_j} \in A$. Thus $\|v_{n_j}^* - w_{n_j}^*\| \to 0$, which means that $\operatorname{diam}(Tx_{n_j}) \to 0$.

So for every $x_0 \in A(T)$ and $\varepsilon > 0$, there is $z_0 \in A(T)$ with $\operatorname{diam}(Tz_0) < \varepsilon$ and $\|x_0 - z_0\| < \varepsilon$. Further, by Lemma 7, there is an open neighborhood V_0 of z_0 such that if $x \in V_0$ then $\operatorname{diam}(Tx) < \varepsilon$. By the density of $A(T)$, we see that $\{x \in E: \operatorname{diam}(Tx) < n^{-1}\}$ contains an open dense set for each $n \geqslant 1$. Thus $S(T) = \{x \in E: \operatorname{diam}(Tx) = 0\} = \bigcap \{x \in E: \operatorname{diam}(Tx) < n^{-1}\}$ contains a dense G_δ. Consequently, $A(T) \cap S(T)$ contains a dense G_δ, and T is continuous at each point of the latter set, by Proposition B, Lemma 5 and the property (H*). Lemma 6 then gives the theorem.

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