Publ. Mat. **61** (2017), 239–258
DOI: 10.5565/PUBLMAT_61117_09

CONTINUITY OF SOLUTIONS TO SPACE-VARYING POINTWISE LINEAR ELLIPTIC EQUATIONS

Lashi Bandara

Abstract: We consider pointwise linear elliptic equations of the form $\mathrm{L}_x u_x = \eta_x$ on a smooth compact manifold where the operators L_x are in divergence form with real, bounded, measurable coefficients that vary in the space variable x. We establish L^2-continuity of the solutions at x whenever the coefficients of L_x are L^∞-continuous at x and the initial datum is L^2-continuous at x. This is obtained by reducing the continuity of solutions to a homogeneous Kato square root problem. As an application, we consider a time evolving family of metrics g_t that is tangential to the Ricci flow almost-everywhere along geodesics when starting with a smooth initial metric. Under the assumption that our initial metric is a rough metric on $\mathcal{M}$ with a C^1 heat kernel on a "non-singular" nonempty open subset $\mathcal{N}$, we show that $x \mapsto \mathrm{g}_t(x)$ is continuous whenever $x \in \mathcal{N}$.

2010 Mathematics Subject Classification: 58J05, 58J60, 47J35, 58D25.

Key words: Continuity equation, rough metrics, homogeneous Kato square root problem.

Contents

1. Introduction

The object of this paper is to consider the continuity of solutions to certain linear elliptic partial differential equations, where the differential operators themselves vary from point to point. To fix our setting, let

$\mathcal{M}$ be a smooth compact Riemannian manifold, and g a smooth metric. Near some point $x_0 \in \mathcal{M}$, we fix an open set U_0 containing x_0. We assume that $U_0 \ni x \mapsto \mathrm{L}_x$, are space-varying, elliptic, second-order divergence form operators with real, bounded, measurable coefficients. The equation at the centre of our study is the following *pointwise* linear problem

$$\mathrm{L}_x u_x = \eta_x \tag{PE}$$

for suitable source data $\eta_x \in \mathrm{L}^2(\mathcal{M})$. Our goal is to establish the continuity of solutions $x \mapsto u_x$ (in $\mathrm{L}^2(\mathcal{M})$) under sufficiently general hypotheses on $x \mapsto \mathrm{L}_x$ and $x \mapsto \eta_x$.

There are abundant equations of the form (PE) that arise naturally. An important and large class of such equations arise as *continuity equations.* These equations are typically of the form

$$-\operatorname{div}_{\mathrm{g},y} f_x(y) \nabla u_{x,v}(y) = \mathrm{d}_x(f_x(y))(v), \tag{CE}$$

where $\operatorname{div}_{\mathrm{g},y}$ is the divergence operator in the variable y with respect to the metric g and d_x is the exterior derivative in the x variable. This equation holds in a suitable weak sense in y. These equations play an important role in geometry, and more recently, in mass transport and the geometry of measure metric spaces. See the book [**20**] by Villani, the paper [**3**] by Ambrosio and Trevisan, and references therein.

The operators L_x have the added complication that their domain may vary as the point x varies. That being said, a redeeming quality is that they facilitate a certain *disintegration.* That is, considerations in x (such as continuity and differentiability), can be obtained via weak solutions in y. This structural feature facilitates attack by techniques from operator theory and harmonic analysis as we demonstrate in this paper.

A very particular instance of the continuity equation that has been a core motivation is where, in the equation (CE), the term $f_x(y) = \rho_t^{\mathrm{g}}(x,y)$, the heat kernel associated to the Laplacian Δ_{g}. In this situation, Gigli and Mantegazza in [**12**] define a metric tensor $\mathrm{g}_t(x)(v,w) = \langle \mathrm{L}_x u_{x,v}, u_{x,w}\rangle$ for vectors $v, w \in \mathrm{T}_x\mathcal{M}$. The regularity of the metric is the regularity in x, and for an initial smooth metric, the aforementioned authors show that this evolving family of metrics are smooth. More interestingly, they demonstrate that

$$\partial_t \mathrm{g}_t(\dot\gamma(s), \dot\gamma(s))|_{t=0} = -2\operatorname{Ric}_{\mathrm{g}}(\dot\gamma(s), \dot\gamma(s))$$

for almost-every s along geodesics γ. That is, this flow g_t is *tangential* to the Ricci flow almost-everywhere along geodesics.

In [**7**], Bandara, Lakzian, and Munn study a generalisation of this flow by considering divergence form elliptic equations with bounded measurable coefficients. They obtain regularity properties for g_t when the heat kernel is Lipschitz and improves to a C^k map ($k \geq 2$) on some non-empty open set in the manifold. Their study was motivated by attempting to describe the evolution of geometric conical singularities as well as other singular spaces. As an application we return to this work and consider the case when $k = 1$.

To describe the main theorem of this paper, let us give an account of some useful terminology. We assume that L_x are defined through a space-varying symmetric form $J_x[u, v] = \langle A_x \nabla u, \nabla v\rangle$ consisting of *coefficients* A_x, where each A_x is a bounded, measurable, symmetric $(1, 1)$ tensor field elliptic at x: there exist $\kappa_x > 0$ such that $J_x[u, u] \geq \kappa_x \|\nabla u\|^2$. Next, let us be precise about the notion of L^p-continuity. We say that $x \mapsto u_x$ is L^p-continuous if, given an $\varepsilon > 0$, there exists an open set $V_{x,\varepsilon}$ containing x such that, whenever $y \in V_{x,\varepsilon}$, we have that $\|u_y - u_x\|_{L^p} < \varepsilon$. With this in mind, we showcase our main theorem.

Theorem 1.1. *Let $\mathcal{M}$ be a smooth manifold and g a smooth metric. Fix $x \in \mathcal{M}$ and suppose that near x, $y \mapsto A_y$ are real, symmetric, elliptic, bounded measurable coefficients that are L^∞-continuous at x. Furthermore, suppose that $\eta_y \in L^2(\mathcal{M})$ for y near x and $y \mapsto \eta_y$ is L^2-continuous at x. If $y \mapsto u_y$ satisfies* (PE) *near x with $\int_{\mathcal{M}} u_y \, d\mu_g = 0$, then $x \mapsto u_x$ is L^2-continuous at x.*

As aforementioned, a complication that arises in proving this theorem is that domains $\mathcal{D}(L_x)$ may vary with x. However, since the solutions $x \mapsto u_x$ live at the level of the resolvent of L_x, there is hope to reduce this problem to the difference of its square root, which incidentally has the fixed domain $W^{1,2}(\mathcal{M})$. As a means to this end, we make connections between the study of the L^2-continuity of these solutions and solving a *homogeneous Kato square root problem.*

Let B be complex and in general, non-symmetric coefficients and let $J_B[u, v] = \langle B\nabla u, \nabla v\rangle$ whenever $u, v \in W^{1,2}(\mathcal{M})$. Suppose that there exists $\kappa > 0$ such that $\operatorname{Re} J_B[u, u] \geq \kappa\|\nabla u\|$. Then, the Lax–Milgram theorem yields a closed, densely-defined operator $L_B u = -\operatorname{div}_g B\nabla u$ (see Chapter 6 in [**14**]). The homogeneous Kato square root problem is to assert that $\mathcal{D}(\sqrt{-\operatorname{div}_g B\nabla}) = W^{1,2}(\mathcal{M})$ with the estimate $\|\sqrt{-\operatorname{div}_g B\nabla} u\| \simeq \|\nabla u\|$.

The Kato square root problem on $\mathbb{R}^n$ resisted resolution for almost forty years before it was finally settled in 2002 by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian in [**4**]. Later, this problem was rephrased from a first-order point of view by Axelsson, Keith, and McIntosh in [**5**]. This seminal paper contained the first Kato square root result for compact manifolds, but the operator in consideration was inhomogeneous.

In the direction of non-compact manifolds, this approach was subsequently used by Morris in [**15**] to solve a similar inhomogeneous problem on Euclidean submanifolds. Later, in the intrinsic geometric setting, this problem was solved by McIntosh and the author in [**8**] on smooth manifolds (possibly non-compact) assuming a lower bound on injectivity radius and a bound on Ricci curvature. Again, these results were for inhomogeneous operators and are unsuitable for our setting where we deal with the homogeneous kind. In §4, we use the framework and other results in [**8**] to solve the homogeneous problem.

The solution to the homogeneous Kato square root problem is relevant to us for the following reason. Underpinning the Kato square root estimate is a *functional calculus* and due to the fact that we allow for complex coefficients, we obtain holomorphic dependency of this calculus. This, in turn, provides us with Lipschitz estimates for small perturbations of the (non-linear) operator $B \mapsto \sqrt{-\operatorname{div}_{\mathrm{g}} B\nabla}$. This is the crucial estimate that yields the continuity result in our main theorem.

To demonstrate the usefulness of our results, we give an application of Theorem 1.1 to the aforementioned geometric flow introduced by Gigli and Mantegazza. In §3, we demonstrate under a very weak hypothesis that this flow is continuous. We remark that this is the first instance known to us where the Kato square root problem has been used in the context of geometric flows. We hope that this paper provides an impetus to further investigate the relevance of Kato square root results to geometry, particularly given the increasing prevalence of the continuity equation in geometric problems.

Throughout this paper, we will use the notation $a \lesssim b$ to mean that there exists a $C > 0$ independent of a and b such that $a \leq Cb$. On writing $a \simeq b$, we mean $a \lesssim b$ and $b \lesssim a$. The notation $\langle \cdot , \cdot \rangle$ will always be reserved to mean the L^2 inner product in question, $\|\cdot\|$ its norm, whereas $|\cdot|$ will be reserved to denote the pointwise norm on a finite dimensional vector space.

Acknowledgements

This research was conducted during the "Junior Trimester Program on Optimal Transport" at the Hausdorff Research Institute for Mathematics in Bonn, Germany. The author acknowledges the institute for funding and support.

The author thanks Sajjad Lakzian, Mike Munn, Rupert McCallum, and Alex Amenta for useful discussions that lead to this work. Moreover, the anonymous referee deserves acknowledgement for contributing to the enhancement of this paper by providing detailed suggestions and remarks. The author would also like to acknowledge and thank Alan McIntosh for his continual encouragement and support in fostering connections between harmonic analysis and geometry.

2. The structure and solutions of the equation

Throughout this paper, let us fix the manifold $\mathcal{M}$ to be a smooth, compact manifold and, unless otherwise stated, let g be a smooth Riemannian metric. By $\mathrm{T}\mathcal{M}$, $\mathrm{T}^*\mathcal{M}$, and $\mathcal{T}^{(p,q)}\mathcal{M}$, we denote the tangent, cotangent, and (p,q)-tensor bundles respectively. We regard $\nabla \colon \mathrm{W}^{1,2}(\mathcal{M}) \subset \mathrm{L}^2(\mathcal{M}) \to \mathrm{L}^2(\mathrm{T}^*\mathcal{M})$ to be the closed, densely-defined extension of the exterior derivative on functions with domain $\mathrm{W}^{1,2}(\mathcal{M})$, the first-order L^2-Sobolev space on $\mathcal{M}$. Moreover, we let $\mathrm{div}_{\mathrm{g}} = -\nabla^*$, the operator adjoint of ∇, with domain $\mathcal{D}(\mathrm{div}_{\mathrm{g}}) \subset \mathrm{L}^2(\mathrm{T}^*\mathcal{M})$. Indeed, operator theory yields that this is a densely-defined and closed operator (see, for instance, Theorem 5.29 in [**14**] by Kato). The L^2-Laplacian on $(\mathcal{M},\mathrm{g})$ is then $\Delta_{\mathrm{g}} = -\mathrm{div}_{\mathrm{g}} \nabla$ which can easily be checked to be a non-negative self-adjoint operator with energy $\mathscr{E}[u] = \|\nabla u\|^2$. We remark that our Lebesgue and Sobolev spaces are complex, but the operators we consider are have real, symmetric coefficients and hence, for real valued source data, we obtain real solutions.

In their paper [**7**], the authors prove existence and uniqueness to elliptic problems of the form

$$\mathrm{L}_A u = -\mathrm{div}_{\mathrm{g}} A \nabla u = f, \tag{E}$$

for suitable source data $f \in \mathrm{L}^2(\mathcal{M})$, where the coefficients A are real, symmetric, bounded, measurable and for which there exists a $\kappa > 0$ satisfying $\langle A\nabla u, \nabla u\rangle \geq \kappa \|\nabla u\|^2$. The key to relating this equation to (PE) is that the source data f can be chosen independent of the coefficients A. See §4, and in particular Proposition 4.5 in [**7**] for details.

The operator L_A is self-adjoint on the domain $\mathcal{D}(\mathrm{L}_A)$ supplied via the Lax–Milgram theorem by considering the symmetric form $J_A[u,v] = \langle A\nabla u, \nabla v\rangle$ whenever $u, v \in \mathrm{W}^{1,2}(\mathcal{M})$. Since the coefficients are real-symmetric, we are able to write $J_A[u,v] = \langle \sqrt{\mathrm{L}_A} u, \sqrt{\mathrm{L}_A} v\rangle$. By the operator theory of self-adjoint operators, we obtain that $\mathrm{L}^2(\mathcal{M}) = \mathcal{N}(\mathrm{L}_A) \oplus^{\perp} \overline{\mathcal{R}(\mathrm{L}_A)}$, where by $\mathcal{N}(\mathrm{L}_A)$ and $\mathcal{R}(\mathrm{L}_A)$, we denote the *null space* and *range* of L_A respectively. Moreover, L_A restricted to $\mathcal{N}(\mathrm{L}_A)$ and $\overline{\mathcal{R}(\mathrm{L}_A)}$ preserves each subspace respectively. Similarly, $\mathrm{L}^2(\mathcal{M}) = \mathcal{N}(\sqrt{\mathrm{L}_A}) \oplus^{\perp} \overline{\mathcal{R}(\sqrt{\mathrm{L}_A})}$. We refer the reader to the paper [**9**] by Cowling, Doust, McIntosh, and Yagi, in particular their Theorem 3.8, for further details.

First, we note that, due to the divergence structure of this equation, an easy operator theory argument yields $\mathcal{N}(\mathrm{L}_A) = \mathcal{N}(\nabla) = \mathcal{N}(\sqrt{\mathrm{L}_A})$ (see Proposition 4.1 in [**7**]). The characterisation of $\overline{\mathcal{R}(\mathrm{L}_A)}$ independent of L_A rests on the fact that, by the compactness of $\mathcal{M}$ and smoothness of g, there exists a Poincaré inequality of the form

$$\text{(P)} \qquad \|u - u_{\mathcal{M},\mathrm{g}}\|_{\mathrm{L}^2} \leq C\|\nabla u\|_{\mathrm{L}^2},$$

where $u_{\mathcal{M},\mathrm{g}} = \fint_{\mathcal{M}} u \, d\mu_{\mathrm{g}}$ (see Theorem 2.10 in [**13**] by Hebey). The constant C can be taken to be $\lambda_1(\mathcal{M}, \mathrm{g})$, the lowest non-zero eigenvalue of the Laplacian Δ_{g} of $(\mathcal{M}, \mathrm{g})$. The space $\overline{\mathcal{R}(\mathrm{L}_A)}$ and $\overline{\mathcal{R}(\sqrt{\mathrm{L}_A})}$ can then be characterised as the set

$$\mathcal{R} = \left\{ u \in \mathrm{L}^2(\mathcal{M}) : \int_{\mathcal{M}} u \, d\mu_{\mathrm{g}} = 0 \right\}.$$

A proof of this can be found as Proposition 4.1 in [**7**].

Recall that, again as a consequence of the fact that $(\mathcal{M}, \mathrm{g})$ is smooth and compact, the embedding $E \colon \mathrm{W}^{1,2}(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M})$ is compact (see Theorem 2.9 in [**13**]). In [**7**], the authors use this fact to show that the the spectrum of L_A is *discrete*, i.e., $\sigma(\mathrm{L}_A) = \{\lambda_0, \lambda_1, \dots\}$ with $0 = \lambda_0$ and $\lambda_j \leq \lambda_{j+1}$. Coupled with the Poincaré inequality, we can obtain that the operator exhibits a spectral gap between the zero and the first-nonzero eigenvalues. That is, $\lambda_0 < \lambda_1$. Moreover, $\kappa\lambda_1(\mathcal{M}, \mathrm{g}) \leq \lambda_1$. See Proposition 4.4 in [**7**] for details.

As aforementioned, the operator L_A restricted to $\mathcal{N}(\mathrm{L}_A)$ and $\overline{\mathcal{R}(\mathrm{L}_A)}$ preserves each subspace respectively. Consequently, the operator $\mathrm{L}_A^R = \mathrm{L}_A|_{\overline{\mathcal{R}(\mathrm{L}_A)}}$ has spectrum $\sigma(\mathrm{L}_A^R) = \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$. Thus, the operator L_A^R is invertible on $\overline{\mathcal{R}(\mathrm{L}_A)}$ and $(\mathrm{L}_A^R)^{-1} \colon \overline{\mathcal{R}(\mathrm{L}_A)} \to \mathcal{D}(\mathrm{Ł}_A) \cap \overline{\mathcal{R}(\mathrm{L}_A)}$. Upon collating and combining the aforementioned facts, we obtain the following conclusion.

Theorem 2.1. *For every $f \in \mathrm{L}^2(\mathcal{M})$ satisfying $\int_{\mathcal{M}} f \, d\mu_{\mathrm{g}} = 0$, we obtain a unique solution $u \in \mathcal{D}(\mathrm{L}_A) \subset \mathrm{W}^{1,2}(\mathcal{M})$ with $\int_{\mathcal{M}} u \, d\mu_{\mathrm{g}} = 0$ to the equation $\mathrm{L}_A u = f$. This solution is given by $u = (\mathrm{L}_A^R)^{-1} f$.*

For the purposes of legibility, we write L_A^{-1} in place of $(\mathrm{L}_A^R)^{-1}$.

3. An application to a geometric flow

In this section, we describe an application of Theorem 1.1 to a geometric flow first proposed by Gigli and Mantegazza in [**12**]. In their paper, they consider solving the continuity equation

$$\text{(GMC)} \qquad -\operatorname{div}_{\mathrm{g},y} \rho^{\mathrm{g}}_t(x,y)\nabla\varphi_{t,x,v}(y) = \mathrm{d}_x(\rho^{\mathrm{g}}_t(x,y))(v),$$

for each fixed x, where ρ^{g}_t is the heat kernel of Δ_{g}, $\operatorname{div}_{\mathrm{g},y}$ denotes the divergence operator acting on the variable y, where $v \in \mathrm{T}_x\mathcal{M}$, and $\mathrm{d}_x(\rho^{\mathrm{g}}_t(x,y))(v)$ is the directional derivative of $\rho^{\mathrm{g}}_t(x,y)$ in the variable x in the direction v. They define a new family of metrics evolving in time by the expression

$$\text{(GM)} \quad \mathrm{g}_t(x)(u,v) = \int_{\mathcal{M}} \mathrm{g}(y)(\nabla\varphi_{t,x,u}(y), \nabla\varphi_{t,x,v}(y))\rho^{\mathrm{g}}_t(x,y)\, d\mu_{\mathrm{g}}(y).$$

As aforementioned, this flow is of importance since it is tangential (a.e. along geodesics) to the Ricci flow when starting with a smooth initial metric. Moreover, in [**12**], the authors demonstrate that this flow is equal to a certain heat flow in the Wasserstein space, and define a distance metric flow for the recently developed RCD-spaces. These are metric spaces that have a notion of lower bound of a generalised Ricci curvature (formulated in the language of mass transport) and for which their Sobolev spaces are Hilbert. We refer the reader to the seminal work of Ambrosio, Gigli, and Savaré in [**2**] as well as the work of Gigli in [**11**] for a detailed description of these spaces and their properties.

In [**7**], the authors were interested in the question of proving existence and regularity of this flow when the metric g was no longer assumed to be smooth or even continuous. The central geometric objects for them are *rough metrics*, which are a sufficiently large class of symmetric tensor fields which are able to capture singularities, including, but not limited to, Lipschitz transforms and certain conical singularities. The underlying differentiable structure of the manifold is always assumed to be smooth, and hence, rough metrics capture *geometric* singularities.

More precisely, let $\tilde{\mathrm{g}}$ be a measurable, symmetric $(2,0)$ tensor field and suppose at each point $x \in \mathcal{M}$, there exists a chart (ψ_x, U_x) near x and a constant $C = C(U_x) \geq 1$ satisfying

$$C^{-1}|u|_{{\psi_x}^*\delta(y)} \leq |u|_{\tilde{\mathrm{g}}(y)} \leq C|u|_{{\psi_x}^*\delta(y)},$$

for y almost-everywhere (with respect to ${\psi_x}^*\mathscr{L}$, the pullback of the Lebesgue measure) inside U_x, where $u \in \mathrm{T}_y\mathcal{M}$, and where ${\psi_x}^*\delta$ is the pullback of the Euclidean metric inside (ψ_x, U_x). A tensor field $\tilde{\mathrm{g}}$ satisfying this condition is called a *rough metric*. Such a metric may not, in general, induce a length structure, but (on a compact manifold) it will induce an n-dimensional Radon measure.

Two rough metrics $\tilde{g}_1$ and $\tilde{g}_2$ are said to be C-close (for $C \geq 1$) if

$$C^{-1}|u|_{\tilde{g}_1(x)} \leq |u|_{\tilde{g}_2(x)} \leq C|u|_{\tilde{g}_1(x)},$$

for almost-every x and where $u \in T_x\mathcal{M}$. For any two rough metrics, there exists a symmetric measurable $(1,1)$-tensor field B such that $\tilde{g}_1(Bu, v) = \tilde{g}_2(u, v)$. For C-close rough metrics, $C^{-2}|u| \leq |B(x)u| \leq C^2|u|$ in either induced norm. In particular, this means that their L^p-spaces are equal with equivalent norms. Moreover, Sobolev spaces exist and are equal with comparable norms, and for $p = 2$, these are Hilbert spaces. On writing $\theta = \sqrt{\det B}$, which denotes the density for the change of measure $d\mu_{\tilde{g}_2} = \sqrt{\det B}\, d\mu_{\tilde{g}_1}$, the divergence operators satisfy $\operatorname{div}_{\tilde{g}_2} = \theta^{-1} \operatorname{div}_{\tilde{g}_1} \theta B$, and the Laplacian $\Delta_{\tilde{g}_2} = \theta^{-1} \operatorname{div}_{\tilde{g}_1} \theta B \nabla$. Since we assume $\mathcal{M}$ is compact, for any rough metric $\tilde{g}$, there exists a $C \geq 1$ and a smooth metric g that is C-close.

As far as the author is aware, the notion of a rough metric was first introduced by the author in his investigation of the geometric invariances of the Kato square root problem in [**6**]. However, a notion close to this exists in the work of Norris in [**16**] and the notion of C-closeness between two continuous metrics can be found in [**19**] by Simon and in [**18**] by Saloff-Coste.

There is an important connection between divergence form operators and rough metrics, and this is crucial to the analysis carried out in [**7**]. The authors noticed that equation (GMC) and the flow (GM) still makes sense if the initial metric g was replaced by a rough metric $\tilde{g}$. To fix ideas, let us denote a rough metric by $\tilde{g}$ and by g, a smooth metric that is C-close. In this situation, we can write the equation (GMC) equivalently in the form

$$\text{(GMC}'\text{)} \qquad -\operatorname{div}_{g,y} \rho_t^{\tilde{g}}(x,y) B\theta\nabla\varphi_{t,x,v} = \theta\, d_x(\rho_t^{\tilde{g}}(x,y))(v).$$

Indeed, it is essential to understand the heat kernel of $\Delta_{\tilde{g}}$ and its regularity to make sense of the right hand side of this equation. In [**7**], the authors assume $\rho_t^{\tilde{g}} \in C^{0,1}(\mathcal{M})$ and further assuming $\rho_t^{g} \in C^k(\mathcal{N}^2)$, for $k \geq 2$ and where $\varnothing \neq \mathcal{N} \subset \mathcal{M}$ represents a "non-singular" open set, they show the existence of solutions to (GMC′) and provide a time evolving family of metrics g_t defined via the equation (GM) on $\mathcal{N}$ of regularity $C^{k-2,1}$. We remark that this set typically arises as $\mathcal{N} = \mathcal{M} \setminus \mathcal{S}$ where $\mathcal{S}$ is some singular part of g. For instance, for a cone attached to a sphere at the north pole, we have that $\mathcal{S} = \{p_{\text{north}}\}$, and on $\mathcal{N}$, both the metric and heat kernel are smooth.

The aforementioned assumptions are not a restriction to the applications that the authors of [**7**] consider as their primary goal was to consider geometric conical singularities, and spaces like a box in Euclidean space. All these spaces are, in fact, RCD-spaces and such spaces have been shown to always have Lipschitz heat kernels. General rough metrics may fail to be RCD, and more seriously, even fail to induce a metric. However, for such metrics, the following still holds.

Proposition 3.1. *For a rough metric* $\tilde{\mathrm{g}}$*, the heat kernel* $\rho^{\tilde{\mathrm{g}}}_t$ *for* $\Delta_{\tilde{\mathrm{g}}}$ *exists and for every* $t > 0$*, there exists some* $\alpha > 0$ *such that* $\rho^{\tilde{\mathrm{g}}}_t \in \mathrm{C}^{\alpha}(\mathcal{M})$*.*

Proof: We follow a similar argument to the proof of Theorem 5.2.1 in [**10**] for a smooth $\tilde{\mathrm{g}}$. The crux of his argument is to assert $|\mathrm{e}^{-t\Delta_{\tilde{\mathrm{g}}}} f(x)| \leq C_{t,x}\|f\|$, so that $f \mapsto \mathrm{e}^{-t\Delta_{\tilde{\mathrm{g}}}} f(x)$ is a bounded functional on $\mathrm{L}^2(\mathcal{M})$ for each $(t, x) \in (0,\infty)\times\mathcal{M}$, which allows us to invoke the Riesz representation theorem to obtain $a(t, x) \in \mathrm{L}^2(\mathcal{M})$ satisfying $\mathrm{e}^{-t\Delta_{\tilde{\mathrm{g}}}} f(x) = \langle a(t, x), f\rangle$. The heat kernel is the readily checked to be given by $\rho^{\tilde{\mathrm{g}}}_t(x, y) = \langle a(t/2, x), a(t/2, y)\rangle$. Thus, we prove that for our rough metric $\tilde{\mathrm{g}}$, the estimate $|\mathrm{e}^{-t\Delta_{\tilde{\mathrm{g}}}} f(x)| \leq C_{t,x}\|f\|$ holds.

First, we note that the semigroup $\mathrm{e}^{-t\Delta_{\tilde{\mathrm{g}}}}$ is positive: whenever $f \in \mathrm{L}^2(\mathcal{M})$ with $f \geq 0$, we have $\mathrm{e}^{-t\Delta_{\tilde{\mathrm{g}}}} f \geq 0$. The positivity of $\mathrm{e}^{-t\Delta_{\tilde{\mathrm{g}}}}$ is equivalent to the following Beurling–Deny type criterion: $f \in \mathcal{D}(\Delta_{\tilde{\mathrm{g}}}^{\frac{1}{2}})$ implies $|f| \in \mathcal{D}(\Delta_{\tilde{\mathrm{g}}}^{\frac{1}{2}})$ with $\|(\Delta_{\tilde{\mathrm{g}}} - \lambda_0)^{\frac{1}{2}}|f|\| \lesssim \|(\Delta_{\tilde{\mathrm{g}}} - \lambda_0)^{\frac{1}{2}} f\|$, where $\lambda_0 = \inf \sigma(\Delta_{\tilde{\mathrm{g}}})$ (see Theorem X11.50 in [**17**] by Reed and Simon). On writing $\Delta_{\tilde{\mathrm{g}}} = -\theta^{-1} \operatorname{div}_{\mathrm{g}} A\theta\nabla$ against a smooth background g and using the compactness of $\mathcal{M}$, we have that $\lambda_0 = 0$ and by self-adjointness of $\Delta_{\tilde{\mathrm{g}}}$, we have $\mathcal{D}(\sqrt{\Delta_{\tilde{\mathrm{g}}}}) = \mathrm{W}^{1,2}(\mathcal{M})$ with $\|\nabla f\| \simeq \|\sqrt{\Delta_{\tilde{\mathrm{g}}}} f\|$. The inequality $\|\nabla|f|\| \leq \|\nabla f\|$ follows immediately from the product rule. Thus, we conclude that $\mathrm{e}^{-t\Delta_{\tilde{\mathrm{g}}}}$ is positive.

Now, let $f \in \mathrm{L}^2(\mathcal{M})$ and note that $f = f_+ - f_-$, where $f_\pm = \max\{0, \pm f\} \geq 0$ respectively. So, letting $u(t, x) = (\mathrm{e}^{-t\Delta_{\tilde{\mathrm{g}}}} f)(x)$ and $u_\pm(t, x) = (\mathrm{e}^{-t\Delta_{\tilde{\mathrm{g}}}} f_\pm)(x)$, we have that $u(t, x) = u_+(t, x) - u_-(t, x)$ since $\mathrm{e}^{-t\Delta_{\tilde{\mathrm{g}}}}$ is positive. Note that $|\mathrm{e}^{-tL} f(x)| = |u(t, x)| \leq u_+(t, x) + u_-(t, x)$. Invoking Theorem 5.3 in [**18**] by Saloff-Coste upon viewing $\Delta_{\tilde{\mathrm{g}}} = \theta^{-1} \operatorname{div}_{\mathrm{g}} A\theta\nabla$ against a smooth background g (and by noting that $\mathcal{M}$ is compact), we have a parabolic Harnack estimate for non-negative solutions $v(t, x)$ to $(\partial_t - \Delta_{\tilde{\mathrm{g}}})v(t, x) = 0$ on a small g-ball $B_\delta(x)$ around x of radius $\delta > 0$. In particular, we obtain $u_\pm(t, x) \lesssim u_\pm(t + \varepsilon_0, y)$ for $y \in B_\delta(x)$ and by integrating with respect to y and invoking the Cauchy–Schwarz inequality,

$$u_\pm(t, x) \lesssim \mu(B_\delta(x))^{-\frac{1}{2}} \|u_\pm(t + \varepsilon_0, \cdot)\|.$$

By the properties of the semigroup, we obtain that

$$\|u_\pm(s,\cdot)\| = \|\mathrm{e}^{-s\Delta_{\tilde{\mathrm{g}}}} f_\pm\| \lesssim \|f_\pm\| \lesssim \|f\|$$

and so we obtain the existence of the heat kernel.

The C^α-regularity of $(x,y) \mapsto \rho_t(x,y)$ is again a consequence of Theorem 5.3 in [**18**] upon noting that $v(t,x) = \rho_t(x,y)$ solves the heat equation $(\partial_t - \Delta_{\tilde{\mathrm{g}}})v(t,x) = 0$ for each y, and by using the fact that $\mathcal{M}$ is compact. $\square$

In order to proceed, we note the following existence and uniqueness result to solutions of the equation (GMC′).

Proposition 3.2. *Suppose that* $\rho^{\mathrm{g}}_t \in \mathrm{C}^1(\mathcal{N}^2)$ *where* $\varnothing \neq \mathcal{N} \subset \mathcal{M}$ *is an open set. Then, for each* $x \in \mathcal{N}$*, the equation* (GMC′) *has a unique solution* $\varphi_{t,x,v} \in \mathrm{W}^{1,2}(\mathcal{M})$ *satisfying* $\int_{\mathcal{M}} \varphi_{t,x,v}\, d\mu_{\tilde{\mathrm{g}}} = 0$*. This solution is given by*

$$\varphi_{t,x,v} = \mathrm{L}_x^{-1}(\theta\eta_{t,x,v}) - \fint_{\mathcal{M}} \mathrm{L}_x^{-1}(\theta\eta_{t,x,v})\, d\mu_{\tilde{\mathrm{g}}},$$

where $\mathrm{L}_x u = -\operatorname{div}_{\mathrm{g},y} \rho^{\mathrm{g}}_t(x,y)\nabla u$ *and* $\eta_{t,x,v} = \mathrm{d}_x(\rho^{\mathrm{g}}_t(x,y))(v)$*.*

Proof: We note that the proof of this proposition runs in a very similar way to Propositions 4.6 and 4.7 in [**7**]. Note that the first proposition simply requires that $\rho^{\mathrm{g}}_t \in \mathrm{C}^0(\mathcal{M}^2)$, and that $\rho^{\mathrm{g}}_t > 0$. This latter inequality is yielded by Lemma 5.4 in [**7**], which again, only requires that $\rho^{\mathrm{g}}_t \in \mathrm{C}^0(\mathcal{M}^2)$. $\square$

Remark 3.3. When inverting this operator L_x as a divergence form operator on the nearby smooth metric g, the solutions $\psi_{t,x,v} = \mathrm{L}_x^{-1}(\theta\eta_{t,x,v})$ satisfy $\int_{\mathcal{M}} \psi_{t,x,v}\, d\mu_{\mathrm{g}} = 0$. The adjustment by subtracting $\fint_{\mathcal{M}} \psi_{t,x,v}\, d\mu_{\tilde{\mathrm{g}}}$ to this solution is to ensure that $\int_{\mathcal{M}} \varphi_{t,x,v}\, d\mu_{\tilde{\mathrm{g}}} = 0$. That is, the integral with respect to $\mu_{\tilde{\mathrm{g}}}$, rather than μ_{g}, is zero.

Collating these results together, and invoking Theorem 1.1, we obtain the following.

Theorem 3.4. *Let* $\mathcal{M}$ *be a smooth, compact manifold, and* $\varnothing \neq \mathcal{N} \subset \mathcal{M}$*, an open set. Suppose that* $\tilde{\mathrm{g}}$ *is a rough metric and that* $\rho^{\tilde{\mathrm{g}}}_t \in \mathrm{C}^1(\mathcal{N}^2)$*. Then,* g_t *as defined by* (GM) *exists on* $\mathcal{N}$ *and it is continuous.*

Proof: By Proposition 3.2, we obtain existence of $\mathrm{g}_t(x)$ for each $x \in \mathcal{N}$ as a Riemannian metric. The fact that it is a non-degenerate inner product follows from similar argument to that of the proof of Theorem 3.1 in [**7**], which only requires the continuity of ρ^{g}_t.

Now, to prove that $x \mapsto \mathrm{g}_t(x)$ is continuous, it suffices to prove that $x \mapsto |u|^2_{\mathrm{g}_t(x)}$ as a consequence of polarisation. Here, we fix a coordinate chart (ψ_x, U_x) near x and consider $u = {\psi_x^{-1}}_* \tilde{u}$, where $\tilde{u} \in \mathbb{R}^n$ is a constant vector inside (ψ_x, U_x). In this situation, we note that (GM) can be written in the following way:

$$|u|^2_{\mathrm{g}_t(x)} = \langle \mathrm{L}_x \varphi_{t,x,u}, \varphi_{t,x,u} \rangle = \langle \eta_{t,x,u}, \varphi_{t,x,u} \rangle.$$

Now, to prove continuity, we need to prove that $||u|_{\mathrm{g}_t(x)} - |u|_{\mathrm{g}_t(y)}|$ can be made small when y is sufficiently close to x. This is obtained if, each of $|\langle \eta_{t,x,u} - \eta_{t,y,u}, \varphi_{t,x,u} \rangle|$ and $|\langle \eta_{t,y,u}, \varphi_{t,x,u} - \varphi_{t,y,u} \rangle|$ can be made small.

The first quantity is easy:

$$|\langle \eta_{t,x,u} - \eta_{t,y,u}, \varphi_{t,x,u} \rangle| \leq \|\eta_{t,x,u} - \eta_{t,y,u}\| \|\varphi_{t,x,u}\|,$$

and by our assumption on $\rho^{\mathrm{g}}_t(x,z)$ that it is continuously differentiable for $x \in \mathcal{N}$ and C^α in z, we have that $(x,y) \mapsto \eta_{x,t,u}(y)$ is uniformly continuous on $K \times \mathcal{M}$ for every $K \Subset \mathcal{N}$ (open subset, compactly contained in $\mathcal{N}$) by the compactness of $\mathcal{M}$. Thus, on fixing $K \Subset \mathcal{N}$, we have that for $x, y \in K$,

$$\|\eta_{t,x,u} - \eta_{t,y,u}\| \leq \mu_{\tilde{\mathrm{g}}}(\mathcal{M}) \sup_{z \in \mathcal{M}} |\eta_{t,x,u}(z) - \eta_{t,y,u}(z)|$$

and the right hand side can be made small for y sufficiently close to x.

Now, the remaining term can be estimated in a similar way:

$$|\langle \eta_{t,y,u}, \varphi_{t,x,u} - \varphi_{t,y,u} \rangle| \leq \|\eta_{t,y,u}\| \|\varphi_{t,x,u} - \varphi_{t,y,u}\|.$$

First, observe that $\|\eta_{t,y,u}\| = \|\eta_{t,y,u} - \eta_{t,x,u}\| + \|\eta_{t,x,u}\|$ and hence, by our previous argument, the first term can be made small and the second term only depends on x. Thus, it suffices to prove that $\|\varphi_{t,x,u} - \varphi_{t,y,u}\|$ can be made small. Note then that

$$\begin{aligned}
\|\varphi_{t,x,u} - \varphi_{t,y,u}\| &\leq \|\mathrm{L}_x^{-1} \theta \eta_{t,x,u} - \mathrm{L}_x^{-1} \theta \eta_{t,y,u}\| \\
&\qquad + \mu_{\tilde{\mathrm{g}}}(\mathcal{M}) \left(\fint_{\mathcal{M}} \right) \mathrm{L}_x^{-1} \theta \eta_{t,x,u} - \mathrm{L}_x^{-1} \theta \eta_{t,y,u} \, d\mu_{\tilde{\mathrm{g}}} \\
&\leq (1 + \mu_{\tilde{\mathrm{g}}}(\mathcal{M})) \|\mathrm{L}_x^{-1} \theta \eta_{t,x,u} - \mathrm{L}_x^{-1} \theta \eta_{t,y,u}\|,
\end{aligned}$$

where the last inequality follows from the Cauchy–Schwarz inequality applied to the average.

Again, by the assumptions on $\rho^{\tilde{\mathrm{g}}}_t$,

$$\|\mathrm{B} \theta \rho^{\tilde{\mathrm{g}}}_t(x, \cdot) - B \theta \rho^{\tilde{\mathrm{g}}}_t(y, \cdot)\|_\infty \lesssim \|B\theta\|_\infty \sup_{z \in \mathcal{M}} |\rho^{\tilde{\mathrm{g}}}_t(x,z) - \rho^{\tilde{\mathrm{g}}}_t(y,z)|$$

which shows that $x \mapsto B(\cdot)\theta(\cdot)\rho^{\tilde{\mathrm{g}}}_t(x, \cdot)$ is L^∞-continuous. Moreover, we have already shown that $(w,z) \mapsto \eta_{t,x,u}(z)$ is uniformly continuous on $K \times \mathcal{M}$ for $K \Subset \mathcal{N}$ and hence, since θ is essentially bounded from above and below, $x \mapsto \theta\eta_{t,x,u}$ is L^2-continuous. Thus, we apply Theorem 1.1 to obtain the conclusion. □

Remark 3.5. If we assume that $\tilde{\mathrm{g}}$ is a rough metric on $\mathcal{M}$, but away from some singular piece $\mathcal{S}$, we assume that the metric is C^1, then, by the results in §6 of [**7**], we are able to obtain that the heat kernel $\rho_t^{\tilde{\mathrm{g}}} \in \mathrm{C}^2(\mathcal{M} \setminus \mathcal{S})$. Hence, we can apply this theorem to obtain that the flow is continuous on $\mathcal{M} \setminus \mathcal{S}$. In [**7**] a similar theorem is obtained (Theorem 3.2) but requires the additional assumption that $\rho_t^{\tilde{\mathrm{g}}} \in \mathrm{C}^1(\mathcal{M}^2)$.

4. Proof of the theorem

In this section, we prove the main theorem by first proving a homogeneous Kato square root result. We begin with a description of functional calculus tools required phrase and resolve the problem.

4.1. Functional calculi for sectorial operators. Let $\mathscr{H}$ be a *complex* Hilbert space and $T\colon \mathcal{D}(T) \subset \mathscr{H} \to \mathscr{H}$ a linear operator. Recall that the *resolvent set* of T denoted by $\rho(T)$ consists of $\zeta \in \mathbb{C}$ such that $(\zeta\mathrm{I} - T)$ has dense range and a bounded inverse on its range. It is easy to see that $(\zeta\mathrm{I} - T)^{-1}$ extends uniquely to bounded operator on the whole space. The *spectrum* is then $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

Fix $\omega \in [0, \pi/2)$ and define the ω-bisector and open ω-bisector respectively as

$$\mathrm{S}_\omega = \{\zeta \in \mathbb{C} : |\arg \zeta| \le \omega \text{ or } |\arg(-\zeta)| \le \omega \text{ or } \zeta = 0\} \qquad \text{and}$$
$$\mathrm{S}_\omega^{\mathrm{o}} = \{\zeta \in \mathbb{C} : |\arg \zeta| < \omega \text{ or } |\arg(-\zeta)| < \omega \text{ and } \zeta \neq 0\}\,.$$

An operator T is said to be *ω-bi-sectorial* if it is closed, $\sigma(T) \subset \mathrm{S}_\omega$, and whenever $\mu \in (\omega, \pi/2)$, there exists a $C_\mu > 0$ satisfying the *resolvent bounds*: $|\zeta| \|(\zeta\mathrm{I} - T)^{-1}\| \le C_\mu$ for all $\zeta \in \mathbb{C} \setminus \mathrm{S}_\mu$. Bi-sectorial operators naturally generalise self-adjoint operators: a self-adjoint operator is 0-bi-sectorial. Moreover, bi-sectorial operators admit a spectral decomposition of the space $\mathscr{H} = \mathcal{N}(T) \oplus \overline{\mathcal{R}(T)}$. This sum is not, in general, orthogonal, but it is always topological. By $\mathbf{P}_{\mathcal{N}(T)}\colon \mathscr{H} \to \mathcal{N}(T)$ we denote the continuous projection from $\mathscr{H}$ to $\mathcal{N}(T)$ that is zero on $\overline{\mathcal{R}(T)}$.

Fix some $\mu \in (\omega, \pi/2)$ and by $\Psi(\mathrm{S}_\mu^{\mathrm{o}})$ denote the class of holomorphic functions $\psi\colon \mathrm{S}_\mu^{\mathrm{o}} \to \mathbb{C}$ for which there exists an $\alpha > 0$ satisfying

$$|\psi(\zeta)| \lesssim \frac{|\zeta|^\alpha}{1 + |\zeta|^{2\alpha}}.$$

For an ω-bi-sectorial operator T, we define a bounded operator $\psi(T)$ via

$$\psi(T)u = \frac{1}{2\pi\imath}\oint_{\gamma} \psi(\zeta)(\zeta \mathrm{I} - T)^{-1}u\, d\zeta,$$

where γ is an unbounded contour enveloping S_{ω} counter-clockwise inside $\mathrm{S}_{\mu}^{\mathrm{o}}$ and where the integral is defined via Riemann sums. The resolvent bounds for the operator T coupled with the decay of the function ψ yields the absolute convergence of this integral.

Now, suppose there exists a $C > 0$ so that $\|\psi(T)\| \leq C\|\psi\|_{\infty}$. In this situation we say that T has a *bounded functional calculus.* Let $\mathrm{Hol}^{\infty}(\mathrm{S}_{\mu}^{\mathrm{o}})$ be the class of bounded functions $f\colon \mathrm{S}_{\mu}^{\mathrm{o}} \cup \{0\} \to \mathbb{C}$ for which $f|_{\mathrm{S}_{\mu}^{\mathrm{o}}} \colon \mathrm{S}_{\mu}^{\mathrm{o}} \to \mathbb{C}$ is holomorphic. For such a function, there is always a sequence of functions $\psi_n \in \Psi(\mathrm{S}_{\mu}^{\mathrm{o}})$ which converges to $f|_{\mathrm{S}_{\mu}^{\mathrm{o}}}$ uniformly on compact subsets of $\mathrm{S}_{\mu}^{\mathrm{o}}$. Moreover, if T has a bounded functional calculus, the limit $\lim_{n\to\infty} \psi_n(T)$ exists in the strong operator topology, and hence, we define

$$f(T)u = f(0)\mathbf{P}_{\mathcal{N}(T)}u + \lim_{n\to\infty} \psi_n(T)u.$$

The operator $f(T)$ is independent of the sequence ψ_n, it is bounded, and moreover, satisfies $\|f(T)\| \leq C\|f\|_{\infty}$. By considering the function χ^{+}, which takes the value 1 for $\operatorname{Re}\zeta > 0$ and 0 otherwise, and χ^{-} taking 1 for $\operatorname{Re}\zeta < 0$ and 0 otherwise, we define $\mathrm{sgn} = \chi^{+} - \chi^{-}$. It is readily checked that $\mathrm{sgn} \in \mathrm{Hol}^{\infty}(\mathrm{S}_{\mu}^{\mathrm{o}})$ for any μ and hence, for T with a bounded functional calculus, the $\chi^{\pm}(T)$ define projections. In addition to the spectral decomposition, we obtain $\mathscr{H} = \mathcal{N}(T) \oplus \mathcal{R}(\chi^{+}(T)) \oplus \mathcal{R}(\chi^{-}(T))$.

Lastly, we remark that a quantitative criterion for demonstrating that T has a bounded functional calculus is to find $\psi \in \Psi(\mathrm{S}_{\mu}^{\mathrm{o}})$ satisfying the *quadratic estimate*

$$\int_0^{\infty} \|\psi(tT)u\|^2 \frac{dt}{t} \simeq \|u\|^2, \quad u \in \overline{\mathcal{R}(T)}.$$

In particular, this criterion facilitates the use of harmonic analysis techniques to prove the boundedness of the functional calculus. For a more complete treatment of these ideas, we refer the reader to [**1**] by Albrecht, Duong, and McIntosh for the treatment of one-one operators, and the paper [**9**] (in particular the text following Theorem 3.8) where the authors demonstrate how to extend these ideas to general operators on reflexive spaces.

4.2. Homogeneous Kato square root problem. We have already given a brief historical overview of the Kato square root problem in the introduction. An important advancement, from the point of view of proving such results on manifolds, was the development of the first-order Dirac-type operator approach by Axelsson, Keith, and McIntosh in [**5**]. Their set of hypotheses (H1)–(H8) is easily accessed in the literature, and therefore, we shall omit repeating them here. For the benefit of the reader, we remark that the particular form that we use here is listed in [**8**].

Let $\mathscr{H} = \mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M})$,

$$\Gamma = \begin{pmatrix} 0 & 0 \\ \nabla & 0 \end{pmatrix}, \quad \text{and} \quad \Gamma^* = \begin{pmatrix} 0 & -\operatorname{div}_{\mathrm{g}} \\ 0 & 0 \end{pmatrix}.$$

Then, for elliptic (possibly complex and non-symmetric) coefficients $B \in \mathrm{L}^\infty(\mathcal{T}^{(1,1)}\mathcal{M})$ (where we recall that $\mathcal{T}^{(1,1)}\mathcal{M}$ are the bundle of $(1,1)$-tensors) satisfying $\operatorname{Re}\langle Bu, u\rangle \geq \kappa_1 \|u\|^2$, and $b \in \mathrm{L}^\infty(\mathcal{M})$ with $\operatorname{Re} b(x) \geq \kappa_2$, define $B_1, B_2 \colon \mathscr{H} \to \mathscr{H}$ by

$$B_1 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$

Define the Dirac-type operators $\Pi_B = \Gamma + B_1 \Gamma^* B_2$ and $\Pi = \Gamma + \Gamma^*$. The first operator is bi-sectorial and the second is self-adjoint (but with spectrum possibly on the whole real line).

First, we note that by bi-sectoriality and by invoking Theorem 3.8 of [**9**],

$$\mathscr{H} = \mathcal{N}(\Pi) \oplus^\perp \overline{\mathcal{R}(\Pi)} = \mathcal{N}(\Pi_B) \oplus \overline{\mathcal{R}(\Pi_B)},$$

where the second direct sum is topological but not necessarily orthogonal. In particular, the first direct sum yields that $\mathrm{L}^2(\mathcal{M}) = \mathcal{N}(\nabla) \oplus^\perp \overline{\mathcal{R}(\operatorname{div})}$ and $\mathrm{L}^2(\mathrm{T}^*\mathcal{M}) = \mathcal{N}(\operatorname{div}) \oplus^\perp \overline{\mathcal{R}(\nabla)}$. We observe the following.

Lemma 4.1. *The space* $\overline{\mathcal{R}(\operatorname{div})} = \left\{u \in \mathrm{L}^2(\mathcal{M}) : \int_{\mathcal{M}} u = 0\right\}$.

Proof: Let $u \in \overline{\mathcal{R}(\operatorname{div})}$. Then, there is a sequence $u_n \in \mathcal{R}(\operatorname{div})$ such that $u_n \to u$ in $\mathrm{L}^2(\mathcal{M})$. Indeed, $u_n = \operatorname{div} v_n$, for some vectorfield $v_n \in \mathcal{D}(\operatorname{div})$. Thus,

$$\int_{\mathcal{M}} u \, d\mu_{\tilde{\mathrm{g}}} = \langle u, 1\rangle = \lim_{n\to\infty} \langle u_n, 1\rangle = \lim_{n\to\infty} \langle \operatorname{div} v_n, 1\rangle = 0,$$

where the second equality follows from the fact that strong convergence implies weak convergence.

Now, suppose that $\int_{\mathcal{M}} u \, d\mu_{\tilde{\mathrm{g}}} = 0$. Then, since $(\mathcal{M}, \tilde{\mathrm{g}})$ admits a Poincaré inequality, we have that $\langle u, v\rangle = 0$ for all $v \in \mathcal{N}(\nabla)$. But since we have that $\mathrm{L}^2(\mathcal{M}) = \mathcal{N}(\nabla) \oplus^\perp \overline{\mathcal{R}(\operatorname{div})}$ via spectral theory, we obtain that $u \in \overline{\mathcal{R}(\operatorname{div})}$. □

With this lemma, we obtain the following coercivity estimate.

Lemma 4.2. *Let $u \in \mathcal{R}(\Pi) \cap \mathcal{D}(\Pi)$. Then, there exists a constant $C > 0$ such that $\|u\| \leq C\|\Pi u\|$.*

Proof: Fix $u = (u_1, u_2) = \mathcal{R}(\Pi) = \mathcal{R}(\mathrm{div}) \oplus^{\perp} \mathcal{R}(\nabla)$. Then, $\|\Pi u\| = \|\nabla u_1\| + \|\mathrm{div}\, u_2\|$. By the Poincaré inequality along with the previous lemma, we obtain that $\|\nabla u_1\| \geq C_1\|u_1\|$. For the other term, note that $\mathrm{div}\, u_2 = \mathrm{div}\, \nabla v = \Delta v$ for some $v \in \mathcal{D}(\nabla)$. Thus,

$$\|\Delta v\| = \|\sqrt{\Delta}\sqrt{\Delta} v\| = \|\nabla(\sqrt{\Delta} v)\| \geq C_1\|\sqrt{\Delta} v\| = C_1\|\nabla v\| = C_1\|u_2\|.$$

On setting $C = C_1$, we obtain the conclusion. □

Indeed, this is the key ingredient to obtain a bounded functional calculus for the operator Π_B.

Theorem 4.3 (Homogenous Kato square root problem for compact manifolds)**.** *On a compact manifold $\mathcal{M}$ with a smooth metric* g*, the operator Π_B admits a bounded functional calculus. In particular, $\mathcal{D}(\sqrt{b\, \mathrm{div}\, B\nabla}) = \mathrm{W}^{1,2}(\mathcal{M})$ and $\|\sqrt{b\, \mathrm{div}\, B\nabla u}\| \simeq \|\nabla u\|$. Moreover, whenever $\|\tilde{b}\|_\infty < \eta_1$ and $\|\tilde{B}\|_\infty < \eta_2$, where $\eta_i < \kappa_i$, we have the following Lipschitz estimate*

$$\|\sqrt{b\, \mathrm{div}\, B\nabla} u - \sqrt{(b + \tilde{b})\, \mathrm{div}(B + \tilde{B})\nabla} u\| \lesssim (\|\tilde{b}\|_\infty + \|\tilde{B}\|_\infty)\|\nabla u\|$$

whenever $u \in \mathrm{W}^{1,2}(\mathcal{M})$. The implicit constant depends on b, B, and η_i.

Proof: Our goal is to check the Axelsson–Keith–McIntosh hypotheses (H1)–(H8) as listed in [**8**] to invoke Theorem 4.2 in that paper and obtain a bounded functional calculus for Π_B.

To avoid unnecessary repetition by listing this framework, we leave it to the reader to consult [**8**]. However, for completeness of the proof, we will remark on why the bulk of these hypothesis are automatically true.

First, by virtue of the fact that we are on a smooth manifold with a smooth metric, we have that $|\mathrm{Ric}| \lesssim 1$, and $\mathrm{inj}(\mathcal{M}, \mathrm{g}) > \kappa > 0$. Coupling this with the fact that Γ is a first-order differential operator makes their hypotheses (H1)–(H7) valid immediately. The hypotheses (H1)–(H6) are valid as a consequence of their Theorem 6.4 and Corollary 6.5 in [**8**], and the proof of (H7) is contained in their Theorem 6.2. The hypothesis (H8) splits into two parts, (H8)-1 and (H8)-2. The first part is a direct consequence of their Theorem 6.2, along with bootstrapping the Poincaré inequality (P) and coupling this with their Proposition 5.3. It only remains to prove their (H8)-2: that there exists a $C > 0$ such that $\|\nabla u\| + \|u\| \leq C\|\Pi u\|$, whenever $u \in \mathcal{R}(\Pi) \cap \mathcal{D}(\Pi)$.

Fix such a $u = (u_1, u_2)$ and note that $u_1 = \operatorname{div} v_2$ for some $v_2 \in \mathcal{D}(\operatorname{div})$ and $u_2 = \nabla v_1$ for some $v_1 \in \mathcal{D}(\nabla)$. Then,

$$\|\nabla u\|^2 = \|\nabla u_1\|^2 + \|\nabla u_2\|^2 = \|\nabla \operatorname{div} v_2\|^2 + \|\nabla^2 v_1\|^2.$$

Also,

$$\|\Pi u\|^2 = \|\operatorname{div} \nabla v_1\|^2 + \|\nabla \operatorname{div} v_2\|^2.$$

Thus, it suffices to estimate the term $\|\nabla^2 v_1\|$ above from $\|\Delta v_1\| + \|v_1\|$. By exploiting the fact that $\mathrm{C}^\infty_\mathrm{c}$ functions are dense in both $\mathcal{D}(\Delta)$ and $\mathrm{W}^{2,2}(\mathcal{M})$ on a compact manifold, the Bochner–Weitzenböck identity yields $\|\nabla^2 v_1\|^2 \lesssim \|\Delta v_1\|^2 + \|v_1\|^2$. Now, $u_2 = \nabla v_1 \in \mathcal{R}(\nabla)$ and we can assume that $u_2 \neq 0$. Thus, $v_1 \not\in \mathcal{N}(\nabla)$ and hence, $\int_\mathcal{M} v_1 \, d\mu_{\tilde{g}} = 0$. Thus, by invoking the Poincaré inequality, we obtain that $\|v_1\| \leq C\|\nabla v_1\| = \|u_2\|$. On combining these estimates, we obtain that $\|\nabla u\| \lesssim \|\Pi u\|$. In Lemma 4.2, we have already proven that $\|u\| \lesssim \|\Pi u\|$.

This allows us to invoke Theorem 4.2 in [**8**], which says that the operator Π_B has a bounded functional calculus. The first estimate in the conclusion is then immediate.

For the Lipschitz estimate, by the fact that that Π_B has a bounded functional calculus, we can apply Corollary 4.6 in [**8**]. This result states that for multiplication operators A_i satisfying

(i) $\|A_i\|_\infty \leq \eta_i$,
(ii) $A_1 A_2 \mathcal{R}(\Gamma), B_1 A_2 \mathcal{R}(\Gamma), A_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$, and
(iii) $A_2 A_1 \mathcal{R}(\Gamma^*), B_2 A_1 \mathcal{R}(\Gamma^*), A_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$,

we obtain that for an appropriately chosen $\mu < \pi/2$, and for all $f \in \mathrm{Hol}^\infty(S^o_\mu)$,

$$\|f(\Pi_B) - f(\Pi_{B+A})\| \lesssim (\|A_1\|_\infty + \|A_2\|_\infty)\|f\|_\infty.$$

Setting

$$A_1 = \begin{pmatrix} \tilde{b} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{B} \end{pmatrix},$$

it is easy to see that these conditions are satisfied, and by repeating the argument in Theorem 7.2 in [**8**] for our operator Π_B, we obtain the Lipschitz estimate in the conclusion. □

4.3. The main theorem. Let us now return to the proof of Theorem 1.1. Recall the operator $\mathrm{L}_x u = -\operatorname{div} A_x \nabla u$, and that $\langle A_x \nabla u, \nabla u\rangle \geq \kappa_x \|\nabla u\|^2$, for $u \in \mathrm{W}^{1,2}(\mathrm{T}^*\mathcal{M})$. From here on, we identify κ_x by the largest such constant at x, which is given by the expression

$$\kappa_x = \inf_{u \in \mathrm{W}^{1,2}(\mathcal{M})} \frac{\langle A_x \nabla u, \nabla u\rangle}{\|\nabla u\|^2}. \tag{EL}$$

A direct consequence of the Kato square root result from our previous subsection is then the following.

Corollary 4.4. *Fix $x \in \mathcal{M}$ and that the operator $y \mapsto \mathrm{L}_y$ is defined near x. If $\|A_x - A_y\|_\infty \leq \zeta < \kappa_x$, then for $u \in \mathrm{W}^{1,2}(\mathcal{M})$,*

$$\|\sqrt{\mathrm{L}_x}u - \sqrt{\mathrm{L}_y}u\| \lesssim \|A_x - A_y\|_\infty \|\nabla u\|.$$

The implicit constant depends on ζ and A_x.

In turn, this implies the following.

Corollary 4.5. *Fix $x \in \mathcal{M}$, assume that $y \mapsto \mathrm{L}_y$ is defined near x and that $\|A_x - A_y\|_\infty \leq \zeta < \kappa_x$. Then,*

$$\|\mathrm{L}_x^{-1}\eta_x - \mathrm{L}_y^{-1}\eta_y\| \lesssim \|A_x - A_y\|_\infty \|\eta_x\| + \|\eta_x - \eta_y\|,$$

whenever $\eta_x, \eta_y \in \mathrm{L}^2(\mathcal{M})$ satisfies $\int_{\mathcal{M}} \eta_x \, d\mu_{\mathrm{g}} = \int_{\mathcal{M}} \eta_y \, d\mu_{\mathrm{g}} = 0$. The implicit constant depends on ζ, κ_x, and A_x.

Proof: First consider the operator $T_x = \sqrt{\mathrm{L}_x}$, and fix $u \in \mathrm{L}^2(\mathcal{M})$ such that $\int_{\mathcal{M}} u \, d\mu_{\mathrm{g}} = 0$. We prove that $\|T_x^{-1}u - T_y^{-1}u\| \lesssim \|A_x - A_y\|_\infty \|u\|$.

Observe that $\mathcal{D}(T_x) = \mathrm{W}^{1,2}(\mathcal{M})$ and so $T_x^{-1}u = T_x^{-1}(T_y T_y^{-1})u = (T_x^{-1}T_y)T_y^{-1}u$ since $T_y^{-1}u \in \mathrm{W}^{1,2}(\mathcal{M})$. Also, since $T_x^{-1}T_x = T_x T_x^{-1}$ on $\mathrm{W}^{1,2}(\mathcal{M})$, we have that $T_y^{-1}u = T_x^{-1}T_x T_y^{-1}u$. Thus,

$$\begin{aligned} \|T_x^{-1}u - T_y^{-1}u\| = \|T_x^{-1}T_y T_y^{-1}u - T_x^{-1}T_x T_y^{-1}u\| &= \|T_x^{-1}(T_y - T_x)T_y^{-1}u\| \\ \lesssim \|(T_y - T_x)T_y^{-1}u\| &\lesssim \|A_x - A_y\|_\infty \|\nabla T_y^{-1}u\|, \end{aligned}$$

where the final inequality follows from Corollary 4.4.

On letting $J_x[u] = \langle A_x \nabla u, \nabla u\rangle \geq \kappa_x \|\nabla u\|^2$, and fixing $\varepsilon > 0$, we note that since κ_y is the largest constant given by (EL), there exists $u_\varepsilon \in \mathrm{W}^{1,2}(\mathcal{M})$ such that $J_y[u_\varepsilon] - \varepsilon \leq \kappa_y$. It is easy to also see that $\kappa_x \leq J_x[u_\varepsilon]$. Thus,

$$\kappa_x - \kappa_y - \varepsilon \leq \frac{J_x[u_\varepsilon] - J_y[u_\varepsilon]}{\|\nabla u_\varepsilon\|^2} \leq \|A_x - A_y\|_\infty \leq \zeta < \kappa_x.$$

Since ε is arbitrary, we get that $\kappa_x - \zeta \leq \kappa_y$, by our hypothesis $\kappa_x - \zeta > 0$ and therefore,

$$(\kappa_x - \zeta)\|\nabla u\|^2 \leq \kappa_y \|\nabla u\|^2 \leq J_y[u] = \|T_y u\|^2.$$

Thus, $\|\nabla T_y^{-1}u\| \leq (\kappa_x - \zeta)^{-1}\|u\|$, and hence,

$$\|T_x^{-1}u - T_y^{-1}u\| \lesssim \|A_x - A_y\|_\infty \|u\|,$$

where the implicit constant depends on ζ, κ_x, and A_x.

Next, let $v_x, v_y \in \mathrm{L}^2(\mathcal{M})$ satisfy $\int_{\mathcal{M}} v_x \, d\mu_{\mathrm{g}} = \int_{\mathcal{M}} v_y \, d\mu_{\mathrm{g}} = 0$ and note that

$$\begin{aligned}\|T_x^{-1}v_x - T_y^{-1}v_y\| &\leq \|T_x^{-1}v_x - T_y^{-1}v_x\| + \|T_y^{-1}(v_x - v_y)\| \\ &\lesssim \|A_x - A_y\|_\infty \|v_x\| + \|(T_x^{-1} - T_y^{-1})(v_x - v_y)\| \\ &\qquad + \|T_x^{-1}(v_x - v_y)\| \\ &\lesssim \|A_x - A_y\|_\infty \|v_x\| + \|A_x - A_y\|_\infty \|v_x - v_y\| + \|v_x - v_y\| \\ &\lesssim \|A_x - A_y\|_\infty \|v_x\| + \|v_x - v_y\|,\end{aligned}$$

where the constant depends on ζ, κ_x, and A_x. Now, putting $v_x = \mathrm{L}_x^{-\frac{1}{2}}\eta_x = T_x^{-1}\eta_x$, and similarly choosing v_y, since we assume $\int_{\mathcal{M}} \eta_x \, d\mu_{\mathrm{g}} = \int_{\mathcal{M}} \eta_y \, d\mu_{\mathrm{g}} = 0$, the same is satisfied for v_x and v_y. Hence, we apply what we have just proved to obtain

$$\begin{aligned}\|\mathrm{L}_x^{-1}\eta_x - \mathrm{L}_y^{-1}\eta_y\| &\lesssim \|A_x - A_y\|_\infty \|\mathrm{L}_x^{-\frac{1}{2}}\eta_x\| + \|T_x^{-1}\eta_x - T_y^{-1}\eta_y\| \\ &\lesssim \|A_x - A_y\|_\infty \|\eta_x\| + \|A_x - A_y\|_\infty \|\eta_x\| + \|\eta_x - \eta_y\| \\ &\lesssim \|A_x - A_y\|_\infty \|\eta_x\| + \|\eta_x - \eta_y\|.\end{aligned}$$

This proves the claim. □

With the aid of this, the proof of Theorem 1.1 is immediate.

Proof of Theorem 1.1: Fix $\varepsilon \in (0, \kappa_x/2)$ and by the assumption that $x \mapsto \eta_x$ is L^2-continuous at x and that $x \mapsto A_x$ is L^∞-continuous at x, we have a $\delta = \delta(x, \varepsilon)$ such that

$$\|\eta_x - \eta_y\| < \varepsilon \quad \text{and} \quad \|A_x - A_y\|_\infty < \varepsilon$$

uniformly for $y \in B_\delta(x)$, the ball of radius δ at x. Thus, in invoking Corollary 4.5, we obtain $\|u_x - u_y\| \lesssim \varepsilon$ where the implicit constant only depends on η_x, κ_x, and A_x. □

References

[1] D. Albrecht, X. Duong, and A. McIntosh, Operator theory and harmonic analysis, in: *"Instructional Workshop on Analysis and Geometry, Part III"* (Canberra, 1995), Proc. Centre Math. Appl. Austral. Nat. Univ. **34**, Austral. Nat. Univ., Canberra, 1996, pp. 77–136.

[2] L. Ambrosio, N. Gigli, and G. Savaré, Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below, *Invent. Math.* **195(2)** (2014), 289–391. DOI: 10.1007/s00222-013-0456-1.

[3] L. AMBROSIO AND D. TREVISAN, Well-posedness of Lagrangian flows and continuity equations in metric measure spaces, *Anal. PDE* **7(5)** (2014), 1179–1234. DOI: 10.2140/apde.2014.7.1179.

[4] P. AUSCHER, S. HOFMANN, M. LACEY, A. MCINTOSH, AND PH. TCHAMITCHIAN, The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$, *Ann. of Math. (2)* **156(2)** (2002), 633–654. DOI: 10.2307/3597201.

[5] A. AXELSSON, S. KEITH, AND A. MCINTOSH, Quadratic estimates and functional calculi of perturbed Dirac operators, *Invent. Math.* **163(3)** (2006), 455–497. DOI: 10.1007/s00222-005-0464-x.

[6] L. BANDARA, Rough metrics on manifolds and quadratic estimates, *Math. Z.* **283(3–4)** (2016), 1245–1281. DOI: 10.1007/s00209-016-1641-x.

[7] L. BANDARA, S. LAKZIAN, AND M. MUNN, Geometric singularities and a flow tangent to the Ricci flow, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* (to appear). DOI: 10.2422/2036-2145.201506_010.

[8] L. BANDARA AND A. MCINTOSH, The Kato square root problem on vector bundles with generalised bounded geometry, *J. Geom. Anal.* **26(1)** (2016), 428–462. DOI: 10.1007/s12220-015-9557-y.

[9] M. COWLING, I. DOUST, A. MCINTOSH, AND A. YAGI, Banach space operators with a bounded H^∞ functional calculus, *J. Austral. Math. Soc. Ser. A* **60(1)** (1996), 51–89. DOI: 10.1017/S1446788700037393.

[10] E. B. DAVIES, *"Heat Kernels and Spectral Theory"*, Cambridge Tracts in Mathematics **92**, Cambridge University Press, Cambridge, 1990.

[11] N. GIGLI, Nonsmooth differential geometry - An approach tailored for spaces with Ricci curvature bounded from below, Preprint (2014). arXiv:1407.0809.

[12] N. GIGLI AND C. MANTEGAZZA, A flow tangent to the Ricci flow via heat kernels and mass transport, *Adv. Math.* **250** (2014), 74–104. DOI: 10.1016/j.aim.2013.09.007.

[13] E. HEBEY, *"Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities"*, Courant Lecture Notes in Mathematics **5**, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.

[14] T. KATO, *"Perturbation Theory for Linear Operators"*, Second edition, Grundlehren der Mathematischen Wissenschaften **132**, Springer-Verlag, Berlin-New York, 1976.

[15] A. J. Morris, The Kato square root problem on submanifolds, *J. Lond. Math. Soc. (2)* **86(3)** (2012), 879–910. DOI: 10.1112/jlms/jds039.

[16] J. R Norris, Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds, *Acta Math.* **179(1)** (1997), 79–103. DOI: 10.1007/BF02392720.

[17] M. Reed and B. Simon, *"Methods of Modern Mathematical Physics. IV. Analysis of Operators"*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[18] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, *J. Differential Geom.* **36(2)** (1992), 417–450.

[19] M. Simon, Deformation of C^0 Riemannian metrics in the direction of their Ricci curvature, *Comm. Anal. Geom.* **10(5)** (2002), 1033–1074. DOI: 10.4310/CAG.2002.v10.n5.a7.

[20] C. Villani, *"Optimal Transport. Old and New"*, Grundlehren der Mathematischen Wissenschaften **338**, Springer-Verlag, Berlin, 2009. DOI: 10.1007/978-3-540-71050-9.

Mathematical Sciences
Chalmers University of Technology and
University of Gothenburg
SE-412 96, Gothenburg
Sweden
E-mail address: lashi.bandara@chalmers.se
URL: http://www.math.chalmers.se/~lashitha

Primera versió rebuda el 23 de juny de 2015,
darrera versió rebuda el 10 de febrer de 2016.