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Continuity of Wigner-type operators on Lorentz spaces and Lorentz mixed normed modulation spaces

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Abstract: We study various continuity properties for τ-Wigner transform on Lorentz spaces and τ-Weyl operators W_τ^a with symbols belonging to appropriate Lorentz spaces. We also study the action of τ-Wigner transform on Lorentz mixed normed modulation spaces.

Key words: τ-Wigner transform, τ-Weyl operators, Lorentz spaces, Lorentz mixed normed spaces, Lorentz mixed normed modulation spaces

1. Introduction

In this paper we will work on $\mathbb{R}^d$ with Lebesgue measure dx. We denote $\mathcal{S}(\mathbb{R}^d)$ as the space of complex-valued continuous functions on $\mathbb{R}^d$ rapidly decreasing at infinity. Let f be a complex valued measurable function on $\mathbb{R}^d$. The operators $T_x f(t) = f(t-x)$ and $M_w f(t) = e^{2\pi i w t} f(t)$ are called translation and modulation operators for $x, w \in \mathbb{R}^d$, respectively. The compositions

$$T_x M_w f(t) = e^{2\pi i w(t-x)} f(t-x) \quad \text{or} \quad M_w T_x f(t) = e^{2\pi i w t} f(t-x)$$

are called time-frequency shifts (see [9]). We write $\left(L^p\left(\mathbb{R}^d\right), \|.\|_p\right)$ as the Lebesgue spaces for $1 \le p \le \infty$.

For $f \in L^1\left(\mathbb{R}^d\right)$ the Fourier transform $\hat{f}$ (or $\mathcal{F}f$) is defined as

$$\hat{f}(t) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x t} dx,$$

where $xt = \sum_{i=1}^{d} x_i t_i$ is the usual scalar product on $\mathbb{R}^d$.

Fix a function $g \neq 0$ (called the window function). The short-time Fourier transform (STFT) of a function f with respect to g is given by

$$V_g f(x, w) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i t w} dt,$$

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for $x, w \in \mathbb{R}^d$. It is known that if $f, g \in L^2(\mathbb{R}^d)$ then $V_g f \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $V_g f$ is uniformly continuous (see [9]).

Let define $V_g^\tau f$ as the function

$$V_g^\tau f(x, w) = V_g f\left(\frac{1}{1-\tau}x, \frac{1}{\tau}w\right)$$

for $\tau \in (0,1)$ and $(x, w) \in \mathbb{R}^{2d}$. The generalized spectrogram depending on 2 windows ϕ, ψ is also defined as

$$Sp_{\phi\psi}(f, g)(x, w) = V_\phi f(x, w)\overline{V_\psi g(x, w)}.$$

The cross-Wigner distribution of $f, g \in L^2(\mathbb{R}^d)$ is defined to be

$$W(f, g)(x, w) = \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right)\overline{g\left(x - \frac{t}{2}\right)}e^{-2\pi i t w}dt.$$

If $f = g$, then $W(f, f) = Wf$ is called the Wigner distribution of $f \in L^2(\mathbb{R}^d)$.

For $\tau \in [0,1]$ and $f, g \in \mathcal{S}(\mathbb{R}^d)$, the τ-Wigner transform is defined as

$$W_\tau(f, g)(x, w) = \int_{\mathbb{R}^d} f(x + \tau t)\overline{g(x - (1-\tau)t)}e^{-2\pi i t w}dt.$$

If $\tau = \frac{1}{2}$, then the τ-Wigner transform is the cross-Wigner distribution. Moreover, for $\tau = 0$, W_0 is the Rihaczek transform,

$$W_0(f, g)(x, w) = R(f, g)(x, w) = e^{-2\pi i x w} f(x)\overline{\hat{g}(w)},$$

and for $\tau = 1$, W_1 is the conjugate Rihaczek transform,

$$W_1(f, g)(x, w) = \overline{R(g, f)}(x, w) = e^{2\pi i x w}\overline{g(x)}\hat{f}(w).$$

For $\tau \in (0,1)$, the τ-Wigner transform can be rewritten as

$$W_\tau(f, g)(x, w) = \frac{1}{|\tau|^d} e^{2\pi i \frac{1}{\tau} x w} V_{A_\tau g} f\left(\frac{1}{1-\tau}x, \frac{1}{\tau}w\right), \tag{1.1}$$

where the operator A_τ is defined by

$$A_\tau : h(t) \to \widetilde{h}\left(\frac{1-\tau}{\tau}t\right)$$

with $\widetilde{h}(t) = h(-t)$ (see [4, 5]).

Let $a \in \mathcal{S}(\mathbb{R}^{2d})$, and then for $\tau \in [0,1]$, the τ-Weyl pseudo-differential operators with τ-symbol a

$$W_\tau^a : f \to W_\tau^a f(x) = \int\limits_{\mathbb{R}^{2d}} e^{2\pi i (x-y) w} a((1-\tau)x + \tau y, w) f(y)\, dy dw$$

are defined as a continuous map from $\mathcal{S}(\mathbb{R}^d)$ to itself (see [5]).

Fix a nonzero window $g \in \mathcal{S}\left(\mathbb{R}^d\right)$ and $1 \leq p, q \leq \infty$. Then the modulation space $M^{p,q}\left(\mathbb{R}^d\right)$ consists of all tempered distributions $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$ such that the short-time Fourier transform $V_g f$ is in the mixed-norm space $L^{p,q}\left(\mathbb{R}^{2d}\right)$. The norm on $M^{p,q}\left(\mathbb{R}^d\right)$ is $\|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}}$. If $p = q$, then we write $M^p\left(\mathbb{R}^d\right)$ instead of $M^{p,p}\left(\mathbb{R}^d\right)$. Modulation spaces are Banach spaces whose definitions are independent of the choice of the window g (see [7, 9]).

$L(p,q)$ spaces are function spaces that are closely related to L^p spaces. We consider complex valued measurable functions f defined on a measure space (X, μ). The measure μ is assumed to be nonnegative. We assume that the functions f are finite valued a.e. and some $y > 0$, $\mu(E_y) < \infty$, where $E_y = E_y[f] = \{x \in X \mid |f(x)| > y\}$. Then, for $y > 0$,

$$\lambda_f(y) = \mu(E_y) = \mu(\{x \in X \mid |f(x)| > y\})$$

is the distribution function of f. The rearrangement of f is given by

$$f^*(t) = \inf\{y > 0 \mid \lambda_f(y) \leq t\} = \sup\{y > 0 \mid \lambda_f(y) > t\}$$

for $t > 0$. The average function of f is also defined by

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)\, dt.$$

Note that λ_f, f^*, and f^{**} are nonincreasing and right continuous functions on $(0, \infty)$. If $\lambda_f(y)$ is continuous and strictly decreasing then $f^*(t)$ is the inverse function of $\lambda_f(y)$. The most important property of f^* is that it has the same distribution function as f. It follows that

$$\left(\int_X |f(x)|^p\, d\mu(x)\right)^{\frac{1}{p}} = \left(\int_0^\infty [f^*(t)]^p\, dt\right)^{\frac{1}{p}}. \tag{1.2}$$

The Lorentz space denoted by $L(p,q)(X,\mu)$ (shortly $L(p,q)$) is defined to be vector space of all (equivalence classes) of measurable functions f such that $\|f\|_{pq}^* < \infty$, where

$$\|f\|_{pq}^* = \begin{cases} \left(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}[f^*(t)]^q\, dt\right)^{\frac{1}{q}}, & 0 < p, q < \infty \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t), & 0 < p \leq q = \infty. \end{cases}$$

By (1.2), it follows that $\|f\|_{pp}^* = \|f\|_p$ and so $L(p,p) = L^p$. Also, $L(p,q)(X,\mu)$ is a normed space with the norm

$$\|f\|_{pq} = \begin{cases} \left(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}[f^{**}(t)]^q\, dt\right)^{\frac{1}{q}}, & 0 < p, q < \infty \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^{**}(t), & 0 < p \leq q = \infty. \end{cases}$$

For any one of the cases $p = q = 1$; $p = q = \infty$ or $1 < p < \infty$ and $1 \leq q \leq \infty$, the Lorentz space $L(p,\, q)(X,\mu)$ is a Banach space with respect to the norm $\|.\|_{pq}$. It is also known that if $1 < p < \infty$, $1 \leq q \leq \infty$ we have

$$\|.\|_{pq}^{*} \leq \|.\|_{pq} \leq \frac{p}{p-1}\|.\|_{pq}^{*},$$

(see [11, 12]).

It is known from [11] that $L(\infty, q) = \{0\}$ if $q \neq \infty$ and $L(\infty, q) = L^{\infty}$ if $q = \infty$. However, in [1, 2], $L(\infty, q)$ are defined as the class of all measurable functions f for which $f^{*}(t) < \infty$ for all $t > 0$ and for which $f^{**}(t) - f^{*}(t)$ is a bounded function of t such that

$$\|f\|_{\infty q} = \left(\int_{0}^{\infty} [f^{**}(t) - f^{*}(t)]^{q} \frac{dt}{t}\right)^{\frac{1}{q}} < \infty, \qquad 0 < q < \infty.$$

Moreover, if $q = 1$, $L(\infty, 1) = L^{\infty}$ and the norms coincide.

Let X and Y be 2 measure spaces with σ-finite measures μ and ν, respectively, and let f be a complex-valued measurable function on $(X \times Y, \mu \times \nu)$, $1 < P = (p_1, p_2) < \infty$ and $1 \leq Q = (q_1, q_2) \leq \infty$. The Lorentz mixed norm space $L(P,Q) = L(P,Q)(X \times Y)$ is defined by

$$L(P,Q) = L(p_2, q_2)[L(p_1, q_1)] = \left\{ f : \|f\|_{PQ} = \|f\|_{L(p_2,q_2)(L(p_1,q_1))} = \left\| \|f\|_{p_1 q_1} \right\|_{p_2 q_2} < \infty \right\}.$$

Thus, $L(P,Q)$ occurs by taking an $L(p_1, q_1)$-norm with respect to the first variable and an $L(p_2, q_2)$-norm with respect to the second variable. The $L(P,Q)$ space is a Banach space under the norm $\|.\|_{PQ}$ (see [3, 8]).

Fix a window function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, $1 \leq P = (p_1, p_2) < \infty$ and $1 \leq Q = (q_1, q_2) \leq \infty$. We let $M(P,Q)(\mathbb{R}^d)$ denote the subspace of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ consisting of $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the Gabor transform $V_g f$ of f is in the Lorentz mixed norm space $L(P,Q)(\mathbb{R}^{2d})$. We endow it with the norm $\|f\|_{M(P,Q)} = \|V_g f\|_{PQ}$, where $\|.\|_{PQ}$ is the norm of the Lorentz mixed norm space. It is known that $M(P,Q)(\mathbb{R}^d)$ is a Banach space and different windows yield equivalent norms. If $p_1 = q_1 = p$ and $p_2 = q_2 = q$, then the space $M(P,Q)(\mathbb{R}^d)$ is the standard modulation space $M^{p,q}(\mathbb{R}^d)$, and if $P = p$ and $Q = q$, in this case $M(P,Q)(\mathbb{R}^d) = M(p,q)(\mathbb{R}^d)$ (see [14]), where the space $M(p,q)(\mathbb{R}^d)$ is Lorentz type modulation space (see [10]). Furthermore, the space $M(p,q)(\mathbb{R}^d)$ was generalized to $M(p,q,w)(\mathbb{R}^d)$ by taking weighted Lorentz space rather than Lorentz space (see [15, 16]).

In this paper, we will denote the Lorentz space by $L(p,q)$, the Lorentz mixed norm space by $L(P,Q)$, the standard modulation space by $M^{p,q}$, the Lorentz type modulation space by $M(p,q)$, and the Lorentz mixed normed modulation space by $M(P,Q)$.

In Section 2, we consider continuity for generalized spectrogram, τ-Wigner transform, and τ-Weyl pseudo-differential operators acting on Lorentz spaces. We extend the results in [4, 5] to the Lorentz spaces. In Section 3, we also study continuity properties of τ-Wigner transform on Lorentz mixed normed modulation spaces. This result extends Proposition 2.5 in [6] and Proposition 15 in [14] since the τ-Wigner transform is the cross-Wigner transform for $\tau = \frac{1}{2}$ and the similar sufficient conditions provide boundedness on both classical and Lorentz mixed normed modulation spaces.

2. Continuity of some operators on Lorentz spaces

In this section, we present $L(p,q)$-boundedness of generalized spectrogram, τ-Wigner transform, and τ-Weyl pseudo-differential operators with τ-symbol a.

We begin with the following 2 Lemmas, will be used later on.

Lemma 1 *If $\tau \in (0,1)$, $1 < p < \infty$, $1 \le q \le \infty$ and $f \in L(p,q)\left(\mathbb{R}^d, \mu\right)$, then we have*

$$\|A_\tau f\|_{pq} = \left(\frac{|\tau|}{|1-\tau|}\right)^{\frac{d}{p}} \|f\|_{pq}.$$

Proof Let $\tau \in (0,1)$ and $f \in L(p,q)\left(\mathbb{R}^d, \mu\right)$. Then we have

$$\begin{aligned}
\lambda_{A_\tau f}(y) &= \mu\left\{x \in \mathbb{R}^d \mid |A_\tau f(x)| > y\right\} = \mu\left\{x \in \mathbb{R}^d \mid \left|\tilde{f}\left(\frac{1-\tau}{\tau}x\right)\right| > y\right\} \\
&= \mu\left\{x \in \mathbb{R}^d \mid \left|f\left(\frac{\tau-1}{\tau}x\right)\right| > y\right\} \\
&= \mu\left\{\frac{\tau}{\tau-1}u \in \mathbb{R}^d \mid |f(u)| > y\right\} \\
&= \left|\frac{\tau}{\tau-1}\right|^d \mu\left\{u \in \mathbb{R}^d \mid |f(u)| > y\right\} = \left|\frac{\tau}{1-\tau}\right|^d \lambda_f(y)
\end{aligned}$$

for $y > 0$. Thus, the rearrangement of $A_\tau f$ is

$$\begin{aligned}
(A_\tau f)^*(t) &= \inf\left\{y > 0 \mid \lambda_{A_\tau f}(y) \le t\right\} = \inf\left\{y > 0 \mid \left|\frac{\tau}{\tau-1}\right|^d \lambda_f(y) \le t\right\} \\
&= \inf\left\{y > 0 \mid \lambda_f(y) \le \left|\frac{1-\tau}{\tau}\right|^d t\right\} = f^*\left(\left|\frac{1-\tau}{\tau}\right|^d t\right)
\end{aligned}$$

for $t > 0$. Additionally, the average function of $A_\tau f$ is

$$\begin{aligned}
(A_\tau f)^{**}(x) &= \frac{1}{x}\int_0^x (A_\tau f)^*(t)\,dt = \frac{1}{x}\int_0^x f^*\left(\left|\frac{1-\tau}{\tau}\right|^d t\right)dt \\
&= \frac{1}{\left|\frac{1-\tau}{\tau}\right|^d x}\int_0^{\left|\frac{1-\tau}{\tau}\right|^d x} f^*(u)\,du = f^{**}\left(\left|\frac{1-\tau}{\tau}\right|^d x\right).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|A_\tau f\|_{pq} &= \left(\frac{q}{p}\int_0^\infty x^{\frac{q}{p}-1}\left[(A_\tau f)^{**}(x)\right]^q dx\right)^{\frac{1}{q}} = \left(\frac{q}{p}\int_0^\infty x^{\frac{q}{p}-1}\left[f^{**}\left(\left|\frac{1-\tau}{\tau}\right|^d x\right)\right]^q dx\right)^{\frac{1}{q}} \\ &= \left(\frac{q}{p}\int_0^\infty \left|\frac{\tau}{1-\tau}\right|^{d(\frac{q}{p}-1)} t^{\frac{q}{p}-1}\left[f^{**}(t)\right]^q \left|\frac{\tau}{1-\tau}\right|^d dt\right)^{\frac{1}{q}} \\ &= \left|\frac{\tau}{1-\tau}\right|^{\frac{d}{p}}\left(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}\left[f^{**}(t)\right]^q dt\right)^{\frac{1}{q}} \\ &= \left|\frac{\tau}{1-\tau}\right|^{\frac{d}{p}}\|f\|_{pq}.\end{aligned}$$

□

Lemma 2 *For $\tau\in(0,1)$ and $1<p<\infty,\ 1\le q\le\infty,$ and then*

$$\left\|V_g^\tau f\right\|_{pq} = (|1-\tau|\cdot|\tau|)^{\frac{d}{p}}\|V_g f\|_{pq},$$

when $V_g f\in L(p,q)\left(\mathbb{R}^{2d}\right)$.

Proof Let υ is a measure on $\mathbb{R}^d$. Then $\mu=\upsilon\times\upsilon$ is a measure on $\mathbb{R}^{2d}$. Thus, the distribution function of $V_g^\tau f$ is

$$\begin{aligned}\lambda_{V_g^\tau f}(y) &= \mu\left\{(x,w)\in\mathbb{R}^{2d}\mid\left|V_g^\tau f(x,w)\right|>y\right\} \\ &= \mu\left\{(x,w)\in\mathbb{R}^{2d}\mid\left|V_g f\left(\frac{1}{1-\tau}x,\frac{1}{\tau}w\right)\right|>y\right\} \\ &= \mu\left[\left\{x\in\mathbb{R}^d\mid\left|V_g f\left(\frac{1}{1-\tau}x,.\right)\right|>y\right\}\times\left\{w\in\mathbb{R}^d\mid\left|V_g f\left(.,\frac{1}{\tau}w\right)\right|>y\right\}\right] \\ &= \upsilon\left\{x\in\mathbb{R}^d\mid\left|V_g f\left(\frac{1}{1-\tau}x,.\right)\right|>y\right\}\upsilon\left\{w\in\mathbb{R}^d\mid\left|V_g f\left(.,\frac{1}{\tau}w\right)\right|>y\right\} \\ &= (|1-\tau|\cdot|\tau|)^d\upsilon\left\{u\in\mathbb{R}^d\mid|V_g f(u,.)|>y\right\}\upsilon\left\{v\in\mathbb{R}^d\mid|V_g f(.,v)|>y\right\} \\ &= (|1-\tau|\cdot|\tau|)^d\mu\left\{(u,v)\in\mathbb{R}^{2d}\mid|V_g f(u,v)|>y\right\} \\ &= (|1-\tau|\cdot|\tau|)^d\lambda_{V_g f}(y)\end{aligned}$$

for $y>0$. Then the rearrangement function of $V_g^\tau f$ is

$$\begin{aligned}\left(V_g^\tau f\right)^*(t) &= \inf\left\{y>0\mid\lambda_{V_g^\tau f}(y)\le t\right\} \\ &= \inf\left\{y>0\mid(|1-\tau|\cdot|\tau|)^d\lambda_{V_g f}(y)\le t\right\} \\ &= \inf\left\{y>0\mid\lambda_{V_g f}(y)\le\frac{t}{(|1-\tau|\cdot|\tau|)^d}\right\} = (V_g f)^*\left(\frac{t}{(|1-\tau|\cdot|\tau|)^d}\right)\end{aligned}$$

for $t>0$. Also, the average function of $V_g^{\tau}f$ is

$$\begin{aligned}\left(V_g^{\tau}f\right)^{**}(x) &= \frac{1}{x}\int_0^x \left(V_g^{\tau}f\right)^{*}(t)\,dt = \frac{1}{x}\int_0^x \left(V_gf\right)^{*}\left(\frac{t}{(|1-\tau|\cdot|\tau|)^d}\right)dt \\ &= \frac{(|1-\tau|\cdot|\tau|)^d}{x}\int_0^{\frac{x}{(|1-\tau|\cdot|\tau|)^d}} \left(V_gf\right)^{*}(u)\,du = \left(V_gf\right)^{**}\left(\frac{x}{(|1-\tau|\cdot|\tau|)^d}\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}\left\|V_g^{\tau}f\right\|_{pq} &= \left(\frac{q}{p}\int_0^{\infty} x^{\frac{q}{p}-1}\left[\left(V_g^{\tau}f\right)^{**}(x)\right]^q dx\right)^{\frac{1}{q}} \\ &= \left(\frac{q}{p}\int_0^{\infty} x^{\frac{q}{p}-1}\left[\left(V_gf\right)^{**}\left(\frac{x}{(|1-\tau|\cdot|\tau|)^d}\right)\right]^q dx\right)^{\frac{1}{q}} \\ &= (|1-\tau|\cdot|\tau|)^{\frac{d}{p}}\left(\frac{q}{p}\int_0^{\infty} t^{\frac{q}{p}-1}\left[\left(V_gf\right)^{**}(t)\right]^q dt\right)^{\frac{1}{q}} \\ &= (|1-\tau|\cdot|\tau|)^{\frac{d}{p}}\left\|V_gf\right\|_{pq}.\end{aligned}$$

□

We shall need the following continuity property of the short-time Fourier transform on Lorentz spaces in order to prove the continuity properties concerning the generalized spectrogram and τ-Wigner transform.

Proposition 3 *Let* $1<p<2$, $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{p_1}+\frac{1}{p_2}<1$, $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p'}$, *and* $q\geq 1$ *be any number such that* $\frac{1}{q_1}+\frac{1}{q_2}\geq\frac{1}{q}$. *Then the Gabor transform*

$$V:(f,g)\in L(p_1,q_1)\left(\mathbb{R}^d\right)\times L(p_2,q_2)\left(\mathbb{R}^d\right)\to V_gf\in L(p,q)\left(\mathbb{R}^{2d}\right)$$

is bounded. In particular,

$$\left\|V_gf\right\|_{pq}\leq C\left\|f\right\|_{p_1q_1}\left\|g\right\|_{p_2q_2}.$$

Proof Let $f\in L(p_1,q_1)\left(\mathbb{R}^d\right)$ and $g\in L(p_2,q_2)\left(\mathbb{R}^d\right)$. Using the equality $V_gf(x,w)=(f\cdot T_xg)^{\wedge}(w)$, Theorem 4.3. in [11], and a generalization of Hölder's inequality for Lorentz spaces (see [12]), we obtain

$$\begin{aligned}\left\|V_gf\right\|_{pq} &= \left\|(f\cdot T_xg)^{\wedge}\right\|_{pq}\leq\left\|f\cdot T_xg\right\|_{p'q} \\ &\leq C\left\|f\right\|_{p_1q_1}\left\|T_xg\right\|_{p_2q_2}=C\left\|f\right\|_{p_1q_1}\left\|g\right\|_{p_2q_2}.\end{aligned}$$

This is the desired result.

□

Now we will state the continuity of $Sp_{\phi\psi}$ on the Lorentz spaces.

Theorem 4 *Let* $1<p,p_3,p_4<2$, $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{p_3}+\frac{1}{p_3'}=1$, $\frac{1}{p_4}+\frac{1}{p_4'}=1$, $\frac{1}{p_i}+\frac{1}{p_i'}=1$ $(i=1,2)$, $\frac{1}{p_1}+\frac{1}{p_1'}=\frac{1}{p_3'}$, $\frac{1}{p_2}+\frac{1}{p_2'}=\frac{1}{p_4'}$, *and* $q,q_3,q_4\geq 1$ *be numbers such that* $\frac{1}{q_3}+\frac{1}{q_4}\geq\frac{1}{q}$, $\frac{1}{q_1}+\frac{1}{q_1'}\geq\frac{1}{q_3}$, *and* $\frac{1}{q_2}+\frac{1}{q_2'}\geq\frac{1}{q_4}$. *Then* $(f,\phi,g,\psi)\rightarrow Sp_{\phi\psi}(f,g)=V_\phi f\overline{V_\psi g}$ *is continuous from* $L(p_1,q_1)(\mathbb{R}^d)\times L(p_1',q_1')(\mathbb{R}^d)\times L(p_2,q_2)(\mathbb{R}^d)\times L(p_2',q_2')(\mathbb{R}^d)$ *into* $L(p,q)(\mathbb{R}^{2d})$. *In particular,*

$$\|Sp_{\phi\psi}(f,g)\|_{pq}=\left\|V_\phi f\overline{V_\psi g}\right\|_{pq}\leq C\|f\|_{p_1q_1}\|\phi\|_{p_1'q_1'}\|g\|_{p_2q_2}\|\psi\|_{p_2'q_2'}.$$

Proof By using Proposition 3, we write that

$$V_\phi f: L(p_1,q_1)(\mathbb{R}^d)\times L(p_1',q_1')(\mathbb{R}^d)\rightarrow L(p_3,q_3)(\mathbb{R}^{2d}),$$

with

$$\|V_\phi f\|_{p_3q_3}\leq C\|f\|_{p_1q_1}\|\phi\|_{p_1'q_1'}$$

and

$$V_\psi g: L(p_2,q_2)(\mathbb{R}^d)\times L(p_2',q_2')(\mathbb{R}^d)\rightarrow L(p_4,q_4)(\mathbb{R}^{2d})$$

with

$$\left\|\overline{V_\psi g}\right\|_{p_4q_4}\leq C\|g\|_{p_2q_2}\|\psi\|_{p_2'q_2'}$$

being continuous. Hence, we get that $Sp_{\phi\psi}(f,g)=V_\phi f\overline{V_\psi g}$ is continuous from $L(p_1,q_1)(\mathbb{R}^d)\times L(p_1',q_1')(\mathbb{R}^d)\times L(p_2,q_2)(\mathbb{R}^d)\times L(p_2',q_2')(\mathbb{R}^d)$ into $L(p_3,q_3)(\mathbb{R}^{2d})\cdot L(p_4,q_4)(\mathbb{R}^{2d})$ with

$$\|V_\phi f\|_{p_3q_3}\left\|\overline{V_\psi g}\right\|_{p_4q_4}\leq C\|f\|_{p_1q_1}\|\phi\|_{p_1'q_1'}\|g\|_{p_2q_2}\|\psi\|_{p_2'q_2'}. \tag{2.3}$$

We thus obtain that

$$\left\|V_\phi f\overline{V_\psi g}\right\|_{pq}\leq\|V_\phi f\|_{p_3q_3}\left\|\overline{V_\psi g}\right\|_{p_4q_4} \tag{2.4}$$

by the generalized Hölder inequality for Lorentz spaces. Moreover, (2.4) means that $L(p_3,q_3)(\mathbb{R}^{2d})\cdot L(p_4,q_4)(\mathbb{R}^{2d})$ is continuously embedded into $L(p,q)(\mathbb{R}^{2d})$. Then by (2.3) and (2.4), we have the desired result. □

Theorem 5 *i. Assume that* $1<p<2$, $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{p_1}+\frac{1}{p_2}<1$, $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p'}$, *and* $q\geq 1$ *is any number such that* $\frac{1}{q_1}+\frac{1}{q_2}\geq\frac{1}{q}$. *Then for* $\tau\in(0,1)$,

$$W_\tau: L(p_1,q_1)(\mathbb{R}^d)\times L(p_2,q_2)(\mathbb{R}^d)\rightarrow L(p,q)(\mathbb{R}^{2d})$$

is continuous.
ii. Let $1<p'<2$, $\frac{1}{p}+\frac{1}{p'}=1$, $1<r\leq\infty$, $0<q,s\leq\infty$ *and let the following 2 inequalities be satisfied:*

$$1)\quad \max(q,r)\leq s$$

$$2)\quad \frac{1}{p}+\frac{1}{s}\leq\frac{1}{q}+\frac{1}{r}.$$

Then for $\tau=0$,

$$W_0: L(p,q)(\mathbb{R}^d)\times L(p',r)(\mathbb{R}^d)\rightarrow L(p,s)(\mathbb{R}^{2d})$$

is continuous and holds

$$\left\|W_0(f,g)\right\|_{ps} \le C\left\|f\right\|_{pq}\left\|g\right\|_{p'r}.$$

iii. Let $1<p'<2,\ \frac{1}{p}+\frac{1}{p'}=1,\ 1<q\le\infty,\ 0<r,s\le\infty$ *and let the following 2 inequalities be satisfied:*

$$1)\quad \max(q,r)\le s$$

$$2)\quad \frac{1}{p}+\frac{1}{s}\le\frac{1}{q}+\frac{1}{r}.$$

Then for $\tau=1$,

$$W_1 : L(p',q)\left(\mathbb{R}^d\right)\times L(p,r)\left(\mathbb{R}^d\right)\to L(p,s)\left(\mathbb{R}^{2d}\right)$$

is continuous. In particular,

$$\left\|W_1(f,g)\right\|_{ps} \le C\left\|f\right\|_{p'q}\left\|g\right\|_{pr}.$$

Proof i. Using Lemma 1, Lemma 2, and Proposition 3, we have

$$\begin{aligned}
\left\|W_\tau(f,g)\right\|_{pq}^q &= \left\|\frac{1}{|\tau|^d}e^{2\pi i\frac{1}{\tau}xw}V_{A_\tau g}^{\tau}f\right\|_{pq}^q = \frac{1}{|\tau|^{dq}}\left\|V_{A_\tau g}^{\tau}f\right\|_{pq}^q \\
&= \frac{1}{|\tau|^{dq}}\left(|1-\tau|\cdot|\tau|\right)^{\frac{dq}{p}}\left\|V_{A_\tau g}f\right\|_{pq}^q \\
&\le \frac{1}{|\tau|^{dq}}\left(|1-\tau|\cdot|\tau|\right)^{\frac{dq}{p}}C\left\|f\right\|_{p_1q_1}^q\left\|A_\tau g\right\|_{p_2q_2}^q \\
&= \frac{1}{|\tau|^{dq}}\left(|1-\tau|\cdot|\tau|\right)^{\frac{dq}{p}}C\left\|f\right\|_{p_1q_1}^q\left|\frac{\tau}{1-\tau}\right|^{\frac{dq}{p_2}}\left\|g\right\|_{p_2q_2}^q \\
&= |\tau|^{dq\left(\frac{1}{p}+\frac{1}{p_2}-1\right)}|1-\tau|^{dq\left(\frac{1}{p}-\frac{1}{p_2}\right)}C\left\|f\right\|_{p_1q_1}^q\left\|g\right\|_{p_2q_2}^q.
\end{aligned}$$

This completes the proof.

ii. Let $f\in L(p,q)\left(\mathbb{R}^d\right)$ and $g\in L(p',r)\left(\mathbb{R}^d\right)$. Then $\hat{g}\in L(p,r)\left(\mathbb{R}^d\right)$ and

$$\left\|\hat{g}\right\|_{pr}\le B\left\|g\right\|_{p'r} \tag{2.5}$$

by Theorem 4.3 in [11]. By using the equality $W_0(f,g)(x,w)=e^{-2\pi ixw}f(x)\overline{\hat{g}}(w)=R(f,g)(x,w)$, inequality (2.5), and Theorem 7.7 in [13], we get

$$\begin{aligned}
\left\|W_0(f,g)\right\|_{ps} &= \left\|R(f,g)\right\|_{ps}\le K\left\|f\right\|_{pq}\left\|\hat{g}\right\|_{pr} \\
&\le C\left\|f\right\|_{pq}\left\|g\right\|_{p'r}.
\end{aligned}$$

This is the desired result.

iii. Let $f\in L(p',q)\left(\mathbb{R}^d\right)$ and $g\in L(p,r)\left(\mathbb{R}^d\right)$. Then $\hat{f}\in L(p,q)\left(\mathbb{R}^d\right)$ and

$$\left\|\hat{f}\right\|_{pq}\le B\left\|f\right\|_{p'q} \tag{2.6}$$

by Theorem 4.3 in [11]. By using the equality $W_1(f,g)(x,w) = e^{2\pi ixw}\overline{g(x)}\hat{f}(w) = \overline{R(g,f)}(x,w)$, inequality (2.6), and Theorem 7.7 in [13], we have

$$\begin{aligned}\|W_1(f,g)\|_{ps} &= \left\|\overline{R(g,f)}\right\|_{ps} \leq K \left\|\hat{f}\right\|_{pq} \|g\|_{pr} \\ &\leq C\|f\|_{p'q}\|g\|_{pr}.\end{aligned}$$

□

If $(0,\infty)$ is taken instead of $\mathbb{R}^d$ in Theorem 5 (ii) and (iii), then the boundedness of $W_0(f,g)$ and $W_1(f,g)$ is equivalent to conditions (1) and (2) by Theorem 7.7 in [13].
In the next theorem the Lorentz mixed normed space $L(P,Q)(\mathbb{R}^{2d})$ is taken, where $P=(p_1,p_2)$ and $Q=(q_1,q_2)$, instead of the Lorentz space $L(p,s)(\mathbb{R}^{2d})$ as the range of W_0 and W_1.

Proposition 6 *Let* $1<p_1<\infty$, $1<p_2'<2$, $\frac{1}{p_2}+\frac{1}{p_2'}=1$, $P=(p_1,p_2)$, $1\leq Q=(q_1,q_2)\leq\infty$. *For* $\tau=0,1$,

$$W_0 : L(p_1,q_1)(\mathbb{R}^d)\times L(p_2',q_2)(\mathbb{R}^d)\to L(P,Q)(\mathbb{R}^{2d})$$

and

$$W_1 : L(p_2',q_2)(\mathbb{R}^d)\times L(p_1,q_1)(\mathbb{R}^d)\to L(P,Q)(\mathbb{R}^{2d})$$

are continuous. In particular,

$$\|W_0(f,g)\|_{PQ}\leq B\|f\|_{p_1q_1}\|g\|_{p_2'q_2}$$

and

$$\|W_1(f,g)\|_{PQ}\leq B\|g\|_{p_1q_1}\|f\|_{p_2'q_2}.$$

Proof If $g\in L(p_2',q_2)(\mathbb{R}^d)$, then $\hat{g}\in L(p_2,q_2)(\mathbb{R}^d)$ and $\left\|\hat{g}\right\|_{p_2q_2}\leq\|g\|_{p_2'q_2}$. By using the equality $W_0(f,g)(x,w)=e^{-2\pi ixw}f(x)\overline{\hat{g}}(w)=R(f,g)(x,w)$, we have

$$\begin{aligned}\|W_0(f,g)\|_{PQ} &= \|R(f,g)\|_{PQ} = \left\|\|f\|_{p_1q_1(\mathbb{R}^d_x)}\hat{g}\right\|_{p_2q_2(\mathbb{R}^d_w)} \\ &= \|f\|_{p_1q_1}\left\|\hat{g}\right\|_{p_2q_2} \\ &\leq B\|f\|_{p_1q_1}\|g\|_{p_2'q_2},\end{aligned}$$

which proves the continuity of W_0. The continuity of W_1 is proved in a similar way to the continuity of W_0. □

The following Theorem is proven from Proposition 5.1 in [5] and Theorem 5.

Theorem 7 *i. Let* $\tau\in(0,1)$. *A necessary and sufficient condition that*

$$a\in L(p',q')(\mathbb{R}^{2d})\to W_\tau^a\in B\left(L(p_2,q_2)(\mathbb{R}^d), L(p_1',q_1')(\mathbb{R}^d)\right)$$

is continuous is that $1<p<2$, $\frac{1}{p_1}+\frac{1}{p_2}<1$, $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p'}$, *and* $q\geq 1$ *be any number such that* $\frac{1}{q_1}+\frac{1}{q_2}\geq\frac{1}{q}$, *where* p',q',p_1', *and* q_1' *are the conjugates of* p,q,p_1, *and* q_1.

ii. Let $\tau = 0$. *A necessary and sufficient condition that*

$$a \in L(p', s')(\mathbb{R}^{2d}) \to W_0^a \in B\left(L(p', r)(\mathbb{R}^d), L(p', q')(\mathbb{R}^d)\right)$$

is continuous is that $1 < p' < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{s} + \frac{1}{s'} = 1$, $1 < r \leq \infty$, $0 < q, s \leq \infty$, *and also that the following 2 inequalities be satisfied:*

$$1) \quad \max(q, r) \leq s$$

$$2) \quad \frac{1}{p} + \frac{1}{s} \leq \frac{1}{q} + \frac{1}{r}.$$

iii. Let $\tau = 1$. *A necessary and sufficient condition that*

$$a \in L(p', s')(\mathbb{R}^{2d}) \to W_0^a \in B\left(L(p, r)(\mathbb{R}^d), L(p, q')(\mathbb{R}^d)\right)$$

is continuous is that $1 < p' < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{s} + \frac{1}{s'} = 1$, $1 < q \leq \infty$, $0 < r, s \leq \infty$ *and also that the following 2 inequalities be satisfied:*

$$1) \quad \max(q, r) \leq s$$

$$2) \quad \frac{1}{p} + \frac{1}{s} \leq \frac{1}{q} + \frac{1}{r}.$$

3. Boundedness of τ-Wigner transform on Lorentz mixed normed modulation spaces

The aim of this section is to study continuity properties of the τ-Wigner transform when acting on the Lorentz mixed normed modulation spaces.

In Proposition 8-10 below we have listed some properties for $\tau-$Wigner transform. From these results we then prove the continuity of the τ-Wigner transform.

Proposition 8 *For* $f, g \in \mathcal{S}(\mathbb{R}^d)$, $\tau \in [0,1]$ *and* $u, v, \eta, \gamma \in \mathbb{R}^d$, *we have*

$$\begin{aligned} W_\tau(T_u M_\eta f, T_v M_\gamma g)(x, w) &= e^{2\pi i x(\eta-\gamma)} e^{2\pi i w(v-u)} e^{2\pi i(\gamma-\eta)(\tau v+(1-\tau)u)} \\ &\quad W_\tau(f, g)(x - (\tau v + (1-\tau)u), w - (\tau\eta + (1-\tau)\gamma)). \end{aligned}$$

In particular,

$$W_\tau(T_u M_\eta f)(x, w) = W_\tau f(x - u, w - \eta). \tag{3.7}$$

Proof For $\tau \in (0,1)$ and $u, v, \eta, \gamma \in \mathbb{R}^d$, we have

$$\begin{aligned} & W_\tau(T_u M_\eta f, T_v M_\gamma g)(x, w) \\ = & \int_{\mathbb{R}^d} T_u M_\eta f(x + \tau t)\overline{T_v M_\gamma g(x - (1-\tau)t)} e^{-2\pi i t w} dt \\ = & \int_{\mathbb{R}^d} f((x-u) + \tau t)\overline{g((x-v) - (1-\tau)t)} e^{2\pi i \eta((x-u)+\tau t)} e^{-2\pi i \gamma((x-v)-(1-\tau)t)} e^{-2\pi i t w} dt. \end{aligned}$$

We make the substitution $z = x - u + \tau t$ and obtain

$$\begin{aligned}
& W_\tau\left(T_u M_\eta f, T_v M_\gamma g\right)(x,w) \\
= \; & \frac{1}{|\tau|^d} \int\limits_{\mathbb{R}^d} f(z) \overline{g\left(-\left(\frac{1-\tau}{\tau} z - \left(\frac{x}{\tau} - v - \frac{1-\tau}{\tau} u\right)\right)\right)} e^{2\pi i \eta z} e^{-2\pi i \gamma\left(-\frac{1-\tau}{\tau} z + \frac{x}{\tau} - v - \frac{1-\tau}{\tau} u\right)} \\
& e^{-2\pi i \frac{w}{\tau}(z-x+u)} dz \\
= \; & \frac{1}{|\tau|^d} e^{-2\pi i \gamma\left(\frac{x}{\tau} - v - \frac{1-\tau}{\tau} u\right) + 2\pi i \frac{w}{\tau}(x-u)} \int\limits_{\mathbb{R}^d} f(z) \overline{g^{\sim}\left(\frac{1-\tau}{\tau}\left(z - \left(\frac{x}{1-\tau} - \frac{\tau}{1-\tau} v - u\right)\right)\right)} \\
& e^{-2\pi i z\left(\frac{1}{\tau} w - \eta - \frac{1-\tau}{\tau}\gamma\right)} dz \\
= \; & \frac{1}{|\tau|^d} e^{-2\pi i \gamma\left(\frac{x}{\tau} - v - \frac{1-\tau}{\tau} u\right) + 2\pi i \frac{w}{\tau}(x-u)} \int\limits_{\mathbb{R}^d} f(z) \overline{A_\tau g\left(z - \left(\frac{x}{1-\tau} - \frac{\tau}{1-\tau} v - u\right)\right)} \\
& e^{-2\pi i z\left(\frac{1}{\tau} w - \eta - \frac{1-\tau}{\tau}\gamma\right)} dz \\
= \; & \frac{1}{|\tau|^d} e^{-2\pi i \gamma\left(\frac{x}{\tau} - v - \frac{1-\tau}{\tau} u\right) + 2\pi i \frac{w}{\tau}(x-u)} V_{A_\tau g} f\left(\frac{1}{1-\tau}\left(x - \tau v - (1-\tau) u\right), \frac{1}{\tau}\left(w - \tau\eta - (1-\tau)\gamma\right)\right).
\end{aligned}$$

Now, equality (1.1) is applied, we have the desired equality for $\tau \in (0,1)$.

Let $\tau = 0$. For $u, v, \eta, \gamma \in \mathbb{R}^d$, by using the equality $\left(T_v M_\gamma g\right)^{\wedge} = M_{-v} T_\gamma \hat{g}$, we get

$$\begin{aligned}
W_0\left(T_u M_\eta f, T_v M_\gamma g\right)(x,w) &= R\left(T_u M_\eta f, T_v M_\gamma g\right)(x,w) = e^{-2\pi i x w}\left(T_u M_\eta f\right)(x) \overline{\left(T_v M_\gamma g\right)^{\wedge}(w)} \\
&= e^{-2\pi i x w} e^{2\pi i \eta(x-u)} f(x-u) \overline{\left(M_{-v} T_\gamma \hat{g}\right)(w)} \\
&= e^{2\pi i x(\eta-\gamma)} e^{2\pi i w(v-u)} e^{2\pi i u(\gamma-\eta)} e^{-2\pi i (x-u)(w-\gamma)} f(x-u) \overline{\hat{g}(w-\gamma)} \\
&= e^{2\pi i x(\eta-\gamma)} e^{2\pi i w(v-u)} e^{2\pi i u(\gamma-\eta)} W_0(f,g)(x-u, w-\gamma).
\end{aligned}$$

Similarly, if $\tau = 1$, for $u, v, \eta, \gamma \in \mathbb{R}^d$, we obtain

$$\begin{aligned}
W_1\left(T_u M_\eta f, T_v M_\gamma g\right)(x,w) &= R^*\left(T_u M_\eta f, T_v M_\gamma g\right)(x,w) = e^{2\pi i x w} \overline{\left(T_v M_\gamma g\right)(x)} \left(T_u M_\eta f\right)^{\wedge}(w) \\
&= e^{2\pi i x w} e^{2\pi i \gamma(x-v)} \overline{g(x-v)} \left(M_{-u} T_\eta \hat{f}\right)(w) \\
&= e^{2\pi i x(\eta-\gamma)} e^{2\pi i w(v-u)} e^{2\pi i v(\gamma-\eta)} e^{2\pi i (x-v)(w-\eta)} \overline{g(x-v)} \hat{f}(w-\eta) \\
&= e^{2\pi i x(\eta-\gamma)} e^{2\pi i w(v-u)} e^{2\pi i v(\gamma-\eta)} W_1(f,g)(x-v, w-\eta).
\end{aligned}$$

□

Proposition 9 *Let $f, g \in \mathcal{S}\left(\mathbb{R}^d\right)$ and $\tau \in [0,1]$. Then we have*

$$V_{T_{\tau\xi_2} M_{-(1-\tau)\xi_1} A_\tau g} T_{\tau\xi_2} M_{-(1-\tau)\xi_1} f(x,w) = e^{-2\pi i \left(x(1-\tau)\xi_1 + w\tau\xi_2\right)} V_{A_\tau g} f(x,w), \tag{3.8}$$

for $x, w, \xi_1, \xi_2 \in \mathbb{R}^d$.

Proof Assume that $f,g\in\mathcal{S}\left(\mathbb{R}^d\right)$ and $\tau\in[0,1]$. Then we write

$$\begin{aligned} V_{T_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g}T_{\tau\xi_2}M_{-(1-\tau)\xi_1}f(x,w) &= \left\langle T_{\tau\xi_2}M_{-(1-\tau)\xi_1}f, M_wT_xT_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g\right\rangle \\ &= \left\langle f, M_{(1-\tau)\xi_1}T_{-\tau\xi_2}M_wT_xT_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g\right\rangle. \end{aligned}$$

Since

$$\begin{aligned} M_{(1-\tau)\xi_1}T_{-\tau\xi_2}M_wT_xT_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g(t) &= e^{2\pi it(1-\tau)\xi_1}M_wT_xT_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g(t+\tau\xi_2) \\ &= e^{2\pi it(1-\tau)\xi_1}e^{2\pi iw(t+\tau\xi_2)}M_{-(1-\tau)\xi_1}A_\tau g(t-x) \\ &= e^{2\pi it(1-\tau)\xi_1}e^{2\pi iw(t+\tau\xi_2)}e^{-2\pi i(1-\tau)\xi_1(t-x)}A_\tau g(t-x) \\ &= e^{2\pi i(x(1-\tau)\xi_1+w\tau\xi_2)}e^{2\pi iwt}A_\tau g(t-x) \\ &= e^{2\pi i(x(1-\tau)\xi_1+w\tau\xi_2)}M_wT_xA_\tau g(t), \end{aligned}$$

we have

$$\begin{aligned} V_{T_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g}T_{\tau\xi_2}M_{-(1-\tau)\xi_1}f(x,w) &= \left\langle f, e^{2\pi i(x(1-\tau)\xi_1+w\tau\xi_2)}M_wT_xA_\tau g\right\rangle \\ &= e^{-2\pi i(x(1-\tau)\xi_1+w\tau\xi_2)}\left\langle f, M_wT_xA_\tau g\right\rangle \\ &= e^{-2\pi i(x(1-\tau)\xi_1+w\tau\xi_2)}V_{A_\tau g}f(x,w). \end{aligned}$$

□

Proposition 10 *i) If $f,g\in\mathcal{S}\left(\mathbb{R}^d\right)$ and $\tau\in[0,1]$, then $W_\tau(f,g)\in\mathcal{S}\left(\mathbb{R}^{2d}\right)$.*
ii) Let $\varphi\in\mathcal{S}\left(\mathbb{R}^d\right)$ and set $\Phi=W_\tau(\varphi,\varphi)=W_\tau(\varphi)\in\mathcal{S}\left(\mathbb{R}^{2d}\right)$. For $\tau\in(0,1)$, we have

$$V_\Phi(W_\tau(f,g))(z,\xi)=e^{-4\pi iz_2\tau\xi_2}V_\varphi f(z_1-\tau\xi_2,z_2+(1-\tau)\xi_1)\overline{V_\varphi g(z_1+\tau\xi_2,z_2-(1-\tau)\xi_1)},$$

where $z=(z_1,z_2)$ and $\xi=(\xi_1,\xi_2)$.
iii) For $\tau=0$, $\Phi=W_0(\varphi,\varphi)=W_0(\varphi)=R(\varphi)\in\mathcal{S}\left(\mathbb{R}^{2d}\right)$, and $W_0(f,g)=R(f,g)$, we have

$$\begin{aligned} V_\Phi(W_0(f,g))(z,\xi) &= V_{W_0(\varphi)}(W_0(f,g))(z,\xi)=V_{R(\varphi)}(R(f,g))(z,\xi) \\ &= e^{-2\pi iz_2\xi_2}V_\varphi f(z_1,z_2+\xi_1)\overline{V_\varphi g(z_1+\xi_2,z_2)}. \end{aligned}$$

iv) For $\tau=1$, $\Phi=W_1(\varphi,\varphi)=W_1(\varphi)=\overline{R(\varphi)}\in\mathcal{S}\left(\mathbb{R}^{2d}\right)$, and $W_1(f,g)=\overline{R(f,g)}$, we have

$$\begin{aligned} V_\Phi(W_1(f,g))(z,\xi) &= V_{W_1(\varphi)}(W_1(f,g))(z,\xi)=V_{\overline{R(\varphi)}}\left(\overline{R(f,g)}\right)(z,\xi) \\ &= e^{-2\pi iz_2\xi_2}V_\varphi f(z_1-\xi_2,z_2)\overline{V_\varphi g(z_1,z_2-\xi_1)}. \end{aligned}$$

Proof i) Since

$$W_\tau(f,g)(x,w)=\frac{1}{|\tau|^d}e^{2\pi i\frac{1}{\tau}xw}V_{A_\tau g}f\left(\frac{1}{1-\tau}x,\frac{1}{\tau}w\right),$$

for $\tau\in(0,1)$, we obtain $W_\tau(f,g)\in\mathcal{S}\left(\mathbb{R}^{2d}\right)$ by using Theorem 11.2.5 in [9].

If $\tau=0,$ let $f\otimes g$ be the tensor product $\left(f\otimes g\right)\left(x,t\right)=f\left(x\right)g\left(t\right),$ and set $\mathcal{T}_{a}F\left(x,t\right)=F\left(x,x-t\right).$ Then we write

$$\begin{aligned} W_0\left(f,g\right)\left(x,w\right) &= R\left(f,g\right)\left(x,w\right)=\int\limits_{\mathbb{R}^d} f\left(x\right)\overline{g\left(x-t\right)}e^{-2\pi itw}dt \\ &= \int\limits_{\mathbb{R}^d}\left(f\otimes\overline{g}\right)\left(x,x-t\right)e^{-2\pi itw}dt \\ &= \int\limits_{\mathbb{R}^d}\mathcal{T}_a\left(f\otimes\overline{g}\right)\left(x,t\right)e^{-2\pi itw}dt \\ &= \mathcal{F}_2\mathcal{T}_a\left(f\otimes\overline{g}\right)\left(x,w\right), \end{aligned}$$

where $\mathcal{F}_2$ is the Fourier transform with respect to the second variable. So, since $f,g\in\mathcal{S}\left(\mathbb{R}^d\right),$ then $f\otimes\overline{g}\in\mathcal{S}\left(\mathbb{R}^{2d}\right).$ Also, since $\mathcal{S}\left(\mathbb{R}^{2d}\right)$ is invariant under the transformation and the Fourier transform, then $W_0\left(f,g\right)\in\mathcal{S}\left(\mathbb{R}^{2d}\right).$

For $\tau=1,$ if we set $\mathcal{T}_bF\left(x,t\right)=F\left(x+t,x\right),$ we get

$$W_1\left(f,g\right)\left(x,w\right)=R^*\left(f,g\right)\left(x,w\right)=\mathcal{F}_2\mathcal{T}_b\left(f\otimes\overline{g}\right)\left(x,w\right).$$

Then, similarly to the case $\tau=0,$ we have $W_1\left(f,g\right)\in\mathcal{S}\left(\mathbb{R}^{2d}\right).$

ii) If we use the equalities (1.1) and $(3.7),$ then we write

$$\begin{aligned} & V_\Phi\left(W_\tau\left(f,g\right)\right)\left(z,\xi\right) \\ = & \iint\limits_{\mathbb{R}^{2d}} W_\tau\left(f,g\right)\left(x,w\right)\overline{W_\tau\left(\varphi\right)\left(x-z_1,w-z_2\right)}e^{-2\pi i\left(x\xi_1+w\xi_2\right)}dxdw \\ = & \frac{1}{\left|\tau\right|^d}\iint\limits_{\mathbb{R}^{2d}} e^{2\pi i\frac{1}{\tau}xw}V_{A_\tau g}f\left(\frac{1}{1-\tau}x,\frac{1}{\tau}w\right)\overline{W_\tau\left(T_{z_1}M_{z_2}\varphi\right)\left(x,w\right)}e^{-2\pi i\left(x\xi_1+w\xi_2\right)}dxdw \\ = & \frac{1}{\left|\tau\right|^{2d}}\iint\limits_{\mathbb{R}^{2d}} V_{A_\tau g}f\left(\frac{1}{1-\tau}x,\frac{1}{\tau}w\right)e^{2\pi i\left(-x\xi_1-w\xi_2\right)}\overline{V_{A_\tau\left(T_{z_1}M_{z_2}\varphi\right)}\left(T_{z_1}M_{z_2}\varphi\right)\left(\frac{1}{1-\tau}x,\frac{1}{\tau}w\right)}dxdw \\ = & \frac{\left|1-\tau\right|^d}{\left|\tau\right|^d}\iint\limits_{\mathbb{R}^{2d}} V_{A_\tau g}f\left(x,w\right)e^{2\pi i\left(-x\left(1-\tau\right)\xi_1-w\tau\xi_2\right)}\overline{V_{A_\tau\left(T_{z_1}M_{z_2}\varphi\right)}\left(T_{z_1}M_{z_2}\varphi\right)\left(x,w\right)}dxdw. \end{aligned}$$

Additionally, if equality (3.8) and orthogonality relations (see Theorem 3.2.1 in [9]) are applied, then we get

$$\begin{aligned} & V_\Phi\left(W_\tau\left(f,g\right)\right)\left(z,\xi\right) \\ = & \frac{\left|1-\tau\right|^d}{\left|\tau\right|^d}\iint\limits_{\mathbb{R}^{2d}} V_{T_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g}T_{\tau\xi_2}M_{-(1-\tau)\xi_1}f\left(x,w\right)\overline{V_{A_\tau\left(T_{z_1}M_{z_2}\varphi\right)}\left(T_{z_1}M_{z_2}\varphi\right)\left(x,w\right)}dxdw \\ = & \frac{\left|1-\tau\right|^d}{\left|\tau\right|^d}\left\langle T_{\tau\xi_2}M_{-(1-\tau)\xi_1}f,T_{z_1}M_{z_2}\varphi\right\rangle\overline{\left\langle T_{\tau\xi_2}M_{-(1-\tau)\xi_1}A_\tau g,A_\tau\left(T_{z_1}M_{z_2}\varphi\right)\right\rangle}. \end{aligned}$$

The first factor on the right side of the equality is

$$\begin{aligned}
& \left\langle T_{\tau\xi_2} M_{-(1-\tau)\xi_1} f, T_{z_1} M_{z_2} \varphi \right\rangle \\
= & \left\langle f, M_{(1-\tau)\xi_1} T_{z_1-\tau\xi_2} M_{z_2} \varphi \right\rangle \\
= & \int\limits_{\mathbb{R}^d} f(x)\, e^{-2\pi i(1-\tau)\xi_1 x} e^{-2\pi i z_2(x-z_1+\tau\xi_2)} \overline{\varphi(x-z_1+\tau\xi_2)} dx \\
= & e^{2\pi i z_2(z_1-\tau\xi_2)} \int\limits_{\mathbb{R}^d} f(x)\, \overline{\varphi(x-(z_1-\tau\xi_2))} e^{-2\pi i x((1-\tau)\xi_1+z_2)} dx \\
= & e^{2\pi i z_2(z_1-\tau\xi_2)} V_\varphi f\left(z_1-\tau\xi_2, z_2+(1-\tau)\xi_1\right).
\end{aligned}$$

Also, since

$$\begin{aligned}
A_\tau (T_u M_\eta g)(x) &= (T_u M_\eta g)^\sim \left(\frac{1-\tau}{\tau} x\right) = (T_u M_\eta g)\left(-\frac{1-\tau}{\tau} x\right) \\
&= e^{-2\pi i \eta\left(\frac{1-\tau}{\tau} x + u\right)} g^\sim \left(\frac{1-\tau}{\tau} x + u\right) \\
&= T_{-u} M_{-\eta} g^\sim \left(\frac{1-\tau}{\tau} x\right) = T_{-u} M_{-\eta} A_\tau g(x),
\end{aligned}$$

the second factor is

$$\begin{aligned}
& \left\langle T_{\tau\xi_2} M_{-(1-\tau)\xi_1} A_\tau g, A_\tau (T_{z_1} M_{z_2} \varphi) \right\rangle \\
= & \left\langle A_\tau \left(T_{-\tau\xi_2} M_{(1-\tau)\xi_1} g\right), A_\tau (T_{z_1} M_{z_2} \varphi) \right\rangle \\
= & \int\limits_{\mathbb{R}^d} \left(T_{-\tau\xi_2} M_{(1-\tau)\xi_1} g\right)^\sim \left(\frac{1-\tau}{\tau} x\right) \overline{(T_{z_1} M_{z_2} \varphi)^\sim \left(\frac{1-\tau}{\tau} x\right)} dx \\
= & \frac{|\tau|^d}{|1-\tau|^d} \int\limits_{\mathbb{R}^d} M_{(1-\tau)\xi_1} g(-u+\tau\xi_2)\, \overline{M_{z_2} \varphi(-u-z_1)} du \\
= & \frac{|\tau|^d}{|1-\tau|^d} \int\limits_{\mathbb{R}^d} e^{2\pi i(1-\tau)\xi_1(-u+\tau\xi_2)} g(-u+\tau\xi_2)\, \overline{\varphi(-u-z_1)} e^{-2\pi i z_2(-u-z_1)} du \\
= & \frac{|\tau|^d}{|1-\tau|^d} e^{2\pi i z_2(z_1+\tau\xi_2)} \int\limits_{\mathbb{R}^d} g(x)\, \overline{\varphi(x-(z_1+\tau\xi_2))} e^{-2\pi i x(z_2-(1-\tau)\xi_1)} dx \\
= & \frac{|\tau|^d}{|1-\tau|^d} e^{2\pi i z_2(z_1+\tau\xi_2)} V_\varphi g\left(z_1+\tau\xi_2, z_2-(1-\tau)\xi_1\right).
\end{aligned}$$

Hence, we have

$$
\begin{aligned}
& V_{\Phi}\left(W_{\tau}\left(f,g\right)\right)\left(z,\xi\right) \\
= & \frac{\left|1-\tau\right|^{d}}{\left|\tau\right|^{d}} e^{2\pi i z_2\left(z_1-\tau\xi_2\right)} V_{\varphi} f\left(z_1-\tau\xi_2, z_2+\left(1-\tau\right)\xi_1\right) \cdot \\
& \overline{\frac{\left|\tau\right|^{d}}{\left|1-\tau\right|^{d}} e^{2\pi i z_2\left(z_1+\tau\xi_2\right)} V_{\varphi} g\left(z_1+\tau\xi_2, z_2-\left(1-\tau\right)\xi_1\right)} \\
= & e^{-4\pi i z_2\tau\xi_2} V_{\varphi} f\left(z_1-\tau\xi_2, z_2+\left(1-\tau\right)\xi_1\right) \overline{V_{\varphi} g\left(z_1+\tau\xi_2, z_2-\left(1-\tau\right)\xi_1\right)}.
\end{aligned}
$$

iii) For $\tau=0,$ by using the equality $V_g f\left(x,w\right)=e^{-2\pi i x w} V_{\hat{g}} \hat{f}\left(w,-x\right),$ we get

$$
\begin{aligned}
& V_{W_0\varphi}\left(W_0\left(f,g\right)\right)\left(z,\xi\right) \\
= & \left\langle W_0\left(f,g\right), M_{\xi} T_z W_0\varphi\right\rangle \\
= & \iint_{\mathbb{R}^{2d}} W_0\left(f,g\right)\left(x,w\right) \overline{M_{\xi} T_z W_0\varphi\left(x,w\right)} dx dw \\
= & \iint_{\mathbb{R}^{2d}} f\left(x\right) \overline{\hat{g}\left(w\right)} e^{-2\pi i x w} \overline{W_0\varphi\left(x-z_1, w-z_2\right)} e^{-2\pi i\left(x\xi_1+w\xi_2\right)} dx dw \\
= & \iint_{\mathbb{R}^{2d}} f\left(x\right) \overline{\hat{g}\left(w\right)} \overline{\varphi\left(x-z_1\right) \overline{\hat{\varphi}\left(w-z_2\right)}} e^{-2\pi i\left(xw+x\xi_1+w\xi_2-\left(x-z_1\right)\left(w-z_2\right)\right)} dx dw \\
= & e^{2\pi i z_1 z_2} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f\left(x\right) \overline{\varphi\left(x-z_1\right)} e^{-2\pi i x\left(\xi_1+z_2\right)} dx \right) \overline{\hat{g}\left(w\right)} \hat{\varphi}\left(w-z_2\right) e^{-2\pi i w\left(\xi_2+z_1\right)} dw \\
= & e^{2\pi i z_1 z_2} V_{\varphi} f\left(z_1,\xi_1+z_2\right) \overline{\int_{\mathbb{R}^d} \hat{g}\left(w\right) \overline{\hat{\varphi}\left(w-z_2\right)} e^{2\pi i w\left(\xi_2+z_1\right)} dw} \\
= & e^{2\pi i z_1 z_2} V_{\varphi} f\left(z_1,\xi_1+z_2\right) \overline{V_{\hat{\varphi}} \hat{g}\left(z_2,-z_1-\xi_2\right)} \\
= & e^{-2\pi i z_2\xi_2} V_{\varphi} f\left(z_1,\xi_1+z_2\right) \overline{V_{\varphi} g\left(z_1+\xi_2, z_2\right)}.
\end{aligned}
$$

iv) It is proven by using the same proof technique as in iii. □

We can now prove the continuity of the τ-Wigner transform for Lorentz mixed normed modulation spaces.

Proposition 11 *Let* $P=\left(1,p_2\right),\ Q=\left(q_1,q_2\right),\ 1\leq Q<\infty$ *and* $1<p_2<\infty.$ *If* $\varphi_1\in M^1\left(\mathbb{R}^d\right),$ *and* $\varphi_2\in M\left(p_2,q_2\right)\left(\mathbb{R}^d\right);$ *then* $W_{\tau}\left(\varphi_2,\varphi_1\right)\in M\left(P,Q\right)\left(\mathbb{R}^{2d}\right)$ *and satisfies*

$$
\left\|W_{\tau}\left(\varphi_2,\varphi_1\right)\right\|_{M(P,Q)}\leq\left\|\varphi_1\right\|_{M^1}\left\|\varphi_2\right\|_{M(p_2,q_2)}
$$

for $\tau\in\left[0,1\right].$

Proof Let $\varphi_1, \varphi_2, g \in \mathcal{S}\left(\mathbb{R}^d\right)$, $\tau \in [0,1]$, and $\Phi = W_\tau g \in \mathcal{S}\left(\mathbb{R}^{2d}\right)$. Then $W_\tau(\varphi_2, \varphi_1) \in \mathcal{S}\left(\mathbb{R}^{2d}\right)$ and so $V_\Phi(W_\tau(\varphi_2, \varphi_1)) \in \mathcal{S}\left(\mathbb{R}^{4d}\right)$ by Proposition 10 (i) and Theorem 11.2.5 in [9], respectively. On the other hand, if $\varphi_1 \in \mathcal{S}\left(\mathbb{R}^d\right)$, then it is known that φ_1 is in the standard modulation space $M^1\left(\mathbb{R}^d\right)$, and if $\varphi_2 \in \mathcal{S}\left(\mathbb{R}^d\right)$, then $\varphi_2 \in M(p_2, q_2)\left(\mathbb{R}^d\right)$ by Proposition 2.1 in [10].

For $\tau \in (0,1)$, Proposition 10 (ii) says that

$$|V_\Phi W_\tau(\varphi_2, \varphi_1)(z,\xi)| = |V_g\varphi_1(z_1 + \tau\xi_2, z_2 - (1-\tau)\xi_1)|\,|V_g\varphi_2(z_1 - \tau\xi_2, z_2 + (1-\tau)\xi_1)|.$$

Write $\widetilde{\xi} = (\tau\xi_2, -(1-\tau)\xi_1)$ and

$$|V_\Phi W_\tau(\varphi_2, \varphi_1)(z,\xi)| = \left|V_g\varphi_1\left(z + \widetilde{\xi}\right)\right|\left|V_g\varphi_2\left(z - \widetilde{\xi}\right)\right|.$$

Thus, by using the inequality $\|.\|_{1q_1} \le \|.\|_{11} = \|.\|_1$ when $1 \le q_1$ and changing variables $z \to z - \widetilde{\xi}$, we have

$$\begin{aligned}
\|V_\Phi W_\tau(\varphi_2, \varphi_1)\|_{1q_1}(\xi) &\le \|V_\Phi W_\tau(\varphi_2, \varphi_1)\|_1(\xi) \\
&= \int\limits_{\mathbb{R}^{2d}} |V_g\varphi_1(z)|\left|V_g\varphi_2\left(z - 2\widetilde{\xi}\right)\right| dz \\
&= \left(|V_g\varphi_1| * |V_g\widetilde{\varphi_2}|\right)\left(2\widetilde{\xi}\right).
\end{aligned}$$

Again using the fact that the Lorentz space $L(p_2, q_2)\left(\mathbb{R}^{2d}\right)$ is an essential Banach convolution module over $L^1\left(\mathbb{R}^{2d}\right)$, we obtain

$$\begin{aligned}
\|W_\tau(\varphi_2, \varphi_1)\|_{M(P,Q)} &= \left\|\|V_\Phi W_\tau(\varphi_2, \varphi_1)\|_{1q_1}\right\|_{p_2q_2} \\
&\le \||V_g\varphi_1| * |V_g\varphi_2|\|_{p_2q_2} \le \|V_g\varphi_1\|_1 \|V_g\varphi_2\|_{p_2q_2} \\
&= \|\varphi_1\|_{M^1} \|\varphi_2\|_{p_2q_2}
\end{aligned}$$

for $\tau \in (0,1)$.

If $\tau = 0$, then we write

$$|V_\Phi W_0(\varphi_2, \varphi_1)(z,\xi)| = |V_\Phi R(\varphi_2, \varphi_1)(z,\xi)| = |V_g\varphi_1(z_1 + \xi_2, z_2)|\,|V_g\varphi_2(z_1, z_2 + \xi_1)|$$

from Proposition 10 (iii), where $\Phi = W_0 g = R(g) \in \mathcal{S}\left(\mathbb{R}^{2d}\right)$. Changing variable $z_1 \to z_1 - \xi_2$ and writing $\widetilde{\xi} = (\xi_2, -\xi_1)$, we get

$$\begin{aligned}
\|V_\Phi W_0(\varphi_2, \varphi_1)\|_{1q_1}(\xi) &\le \|V_\Phi W_0(\varphi_2, \varphi_1)\|_1(\xi) \\
&= \int\limits_{\mathbb{R}^{2d}} |V_g\varphi_1(z_1 + \xi_2, z_2)|\,|V_g\varphi_2(z_1, z_2 + \xi_1)|\, dz_1 dz_2 \\
&= \int\limits_{\mathbb{R}^{2d}} |V_g\varphi_1(z)|\left|V_g\varphi_2\left(z - \widetilde{\xi}\right)\right| dz \\
&= \left(|V_g\varphi_1| * |V_g\widetilde{\varphi_2}|\right)\left(\widetilde{\xi}\right)
\end{aligned}$$

and

$$\|W_0(\varphi_2,\varphi_1)\|_{M(P,Q)} = \|R(\varphi_2,\varphi_1)\|_{M(P,Q)} \le \|\varphi_1\|_{M^1}\|\varphi_2\|_{M(p_2,q_2)}.$$

If we apply the same proof technique above for $\tau = 1$, by using Proposition 10 (iv), we have

$$\|W_1(\varphi_2,\varphi_1)\|_{M(P,Q)} = \|R^*(\varphi_2,\varphi_1)\|_{M(P,Q)} \le \|\varphi_1\|_{M^1}\|\varphi_2\|_{M(p_2,q_2)}.$$

□

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