CONTINUITY PROPERTIES OF GAUSSIAN PROCESSES WITH MULTIDIMENSIONAL TIME PARAMETER

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In this paper we shall present a strengthening and generalization to higher dimensions of the real variable lemma presented in [4].

As a consequence we shall obtain a criterion for the continuity of sample functions of Gaussian processes with a multidimensional time parameter.

Remarkably enough, the difficulty of the arguments here is almost independent of dimensions, indeed the proofs in this paper are considerably simpler and yield stronger results than those in [4].

As in [4] our point of departure is a real variable lemma giving an *a priori* modulus of continuity for functions satisfying certain integral inequalities.

As in [4], the basic ingredients are two functions $p(u)$, defined in $[-1, 1]$ and $\Psi(u)$, defined in $(-\infty, +\infty)$. However here, in addition to the conditions

$$p(u) = p(-u) \downarrow 0 \quad \text{as} \quad |u| \downarrow 0, \tag{1}$$

$$\Psi(u) = \Psi(-u) \uparrow \infty \quad \text{as} \quad |u| \uparrow \infty, \tag{2}$$

we shall assume that $\Psi(u)$ is *convex*.

Let then I_0 denote the unit hypercube in d dimensional cartesian space. In this paper, by "hypercube" we mean a hypercube with edges parallel to the coordinate axes.

For every hypercube I we denote by $|I|$ its volume and by $e(I)$ the common length of its edges.

This given, we can state our basic lemma in the following form.

LEMMA 1. *Let $f(x)$ be measurable in I_0 and such that*

$$\int_I \int_I \Psi\left(\frac{f(x) - f(y)}{p(e(I))}\right) dx dy \leqq B, \qquad \text{for all } I \subset I_0. \tag{3}$$

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Then, if f is continuous in I_0, for all $x, y \in I_0$ we have

$$|f(x) - f(y)| \leqq 8 \int_0^{|x-y|} \Psi^{-1}\left(\frac{B}{u^{2d}}\right) dp(u), \tag{4}$$

otherwise, (4) *holds at least for almost all $x, y \in I_0$.*

PROOF. Given $x, y \in I_0$ we let Q_0 be the smallest hypercube containing both of them. Further, let $\{Q_n\}$ be a sequence of hypercubes such that

$$\text{(5)} \quad \begin{aligned} &\text{(a)} \quad Q_n \subset Q_{n-1}, \qquad n = 1, 2, \cdots, \\ &\text{(b)} \quad p(e(Q_n)) = \tfrac{1}{2} p(e(Q_{n-1})). \end{aligned}$$

From (3) and the convexity of Ψ, we deduce

$$\begin{aligned} \Psi\left(\frac{f_{Q_n} - f_{Q_{n-1}}}{p(e(Q_{n-1}))}\right) &\leqq \frac{1}{|Q_n||Q_{n-1}|} \int_{Q_n} \int_{Q_{n-1}} \Psi\left(\frac{f(x) - f(y)}{p(e(Q_{n-1}))}\right) dx dy \\ &\leqq \frac{B}{|Q_n||Q_{n-1}|}, \end{aligned} \tag{6}$$

where, for any hypercube Q, we set $f_Q = (1/|Q|) \int_Q f(x)\, dx$. Inverting Ψ in the inequality (6) we get

$$|f_{Q_n} - f_{Q_{n-1}}| \leqq \Psi^{-1}\left(\frac{B}{|Q_n||Q_{n-1}|}\right) p(e(Q_{n-1})). \tag{7}$$

For convenience set $x_n = e(Q_n)$.

This given, from (5b) it is easily seen that

$$p(e(Q_{n-1})) = p(x_{n-1}) = 4[p(x_n) - p(x_{n+1})]. \tag{8}$$

So therefore, from (7) we easily deduce

$$|f_{Q_n} - f_{Q_{n-1}}| \leqq 4 \int_{x_{n+1}}^{x_n} \Psi^{-1}\left(\frac{B}{u^{2d}}\right) dp(u). \tag{9}$$

On summing this inequality for $n = 1, 2, \cdots$, we derive

$$\limsup_{n \to \infty} |f_{Q_n} - f_{Q_0}| \leqq \int_0^{e(Q_1)} \Psi^{-1}\left(\frac{B}{u^{2d}}\right) dp(u). \tag{10}$$

If the sequence $\{Q_n\}$ is in addition chosen to decrease to x and f is continuous at x, from (10) we obtain

$$|f(x) - f_{Q_0}| \leqq 4 \int_0^{|x-y|} \Psi^{-1}\left(\frac{B}{u^{2d}}\right) dp(u). \tag{11}$$

The same inequality of course must hold with x replaced by y if f is continuous at y.

We can thus derive (4) for all $x, y \in I_0$ when f is continuous in I.

However, in the general case, we know from classical real variable theory, that for almost all $x \in I_0$ we can still choose the hypercubes Q_n so that in addition we have

$$\lim_{n\to\infty} f_{Q_n} = f(x). \tag{12}$$

Thus the lemma is established in full.

We are now in a position to derive our application to Gaussian processes. Our arguments here do not differ in a substantial way from those presented in [4] or [5], however, since they are quite short, for the sake of completeness we shall present them in full detail.

Let again I_0 denote the unit hypercube in d dimensions. Let $R(s, t)$ be continuous and positive definite in $S = I_0 \times I_0$ and let

$$R(s, t) = \sum_{\nu=1}^{\infty} \lambda_\nu \varphi_\nu(s)\varphi_\nu(t) \tag{13}$$

be its Mercer's expansion in S.

Set

$$\Delta R(s, t) = \sum_{\nu=1}^{\infty} \lambda_\nu [\varphi_\nu(s) - \varphi_\nu(t)]^2 \tag{14}$$

and

$$p(u) = \max_{|s-t| \leqq |u|\sqrt{d}} [\Delta R(s, t)]^{1/2}. \tag{15}$$

Finally, let $\{\theta_n(\omega)\}$ be a sequence of independent standard Gaussian variables and set

$$X_t^{(n)}(\omega) = \sum_{\nu=1}^{n} \sqrt{\lambda_\nu}\, \varphi_\nu(t)\theta_\nu(\omega). \tag{16}$$

The following result holds.

THEOREM 1. *Suppose*

$$\int_0^1 \left(\log \frac{1}{u}\right)^{1/2} dp(u) < \infty. \tag{17}$$

Then, with probability one the partial sums $X_t^{(n)}(\omega)$ converge uniformly in I_0, indeed they are almost surely equicontinuous in I_0. More precisely for all n we have

$$|X_s^{(n)}(\omega) - X_t^{(n)}(\omega)| \leqq 16[\log B(\omega)]^{1/2} p(s-t) + 16\,(2d)^{1/2} \int_0^{|s-t|} \left(\log \frac{1}{u}\right)^{1/2} dp(u), \tag{18}$$

where

$$B(\omega) = \sup_n \int_{I_0}\int_{I_0} \exp \frac{1}{4}\left\{\frac{X_s^{(n)}(\omega) - X_t^{(n)}(\omega)}{p((s-t)/\sqrt{d})}\right\}^2 ds\, dt \tag{19}$$

and

$$E[B(\omega)] \leqq 4\sqrt{2}. \tag{20}$$

PROOF. For fixed s and t in I_0 we introduce the random variables

$$P_n(\omega) = \exp \frac{1}{8}\left\{\frac{X_s^{(n)}(\omega) - X_t^{(n)}(\omega)}{p\big((s-t)/\sqrt{d}\big)}\right\}^2. \tag{21}$$

We note that

(a) $\{P_n\}$ is a submartingale,
(b) $E(P_n^2) \leqq \sqrt{2}$.

Indeed, (a) follows from the convexity of $\exp\{u^2\}$ and the definition of $\{X_t^{(n)}(\omega)\}$, while (b) holds because for each n the random variable

$$\frac{X_s^{(n)}(\omega) - X_t^{(n)}(\omega)}{p\big((s-t)/\sqrt{d}\big)} \tag{22}$$

is Gaussian, has mean zero and variance less than or equal to one.

This given, from the classical martingale inequalities we deduce that for each s and t in I_0

$$E(\max_{m \leqq n} P_m^2) \leqq 4E(P_n^2) \leqq 4\sqrt{2}. \tag{23}$$

Integrating over $S = I_0 \times I_0$ and using Fubini's theorem, we then obtain

$$E\left(\int_{I_0}\int_{I_0} \max_{m \leqq n} \exp \frac{1}{4}\left\{\frac{X_s^{(n)}(\omega) - X_t^{(n)}(\omega)}{p\big((s-t)/\sqrt{d}\big)}\right\}^2 ds\, dt\right) \leqq 4\sqrt{2}. \tag{24}$$

Upon letting $n \uparrow \infty$ in this inequality and using the monotone convergence theorem, we can easily deduce the assertion in (20). Thus we almost surely have for $n = 1, 2, \cdots$,

$$\int_{I_0}\int_{I_0} \exp \frac{1}{4}\left\{\frac{X_s^{(n)}(\omega) - X_t^{(n)}(\omega)}{p\big((s-t)/\sqrt{d}\big)}\right\}^2 ds\, dt \leqq B(\omega) < \infty. \tag{25}$$

Lemma 1 then gives

$$|X_s^{(n)}(\omega) - X_t^{(n)}(\omega)| \leqq 16 \int_0^{|s-t|} (\log B(\omega)/u^{2d})^{1/2}\, dp(u). \tag{26}$$

This inequality implies (18).

Finally, note that since

$$E\left(\sum_{\nu=1}^{\infty} \lambda_\nu \theta_\nu^2(\omega)\right) = \sum_{\nu=1}^{\infty} \lambda_\nu = \int_{I_0} R(s, s)\, ds < \infty, \tag{27}$$

with probability one we have

$$\sum_{\nu=1}^{\infty} \lambda_\nu \theta_\nu^2(\omega) < \infty. \tag{28}$$

This implies the almost sure uniform convergence of $\{X_t^{(n)}(\omega)\}$ in I_0. For, at each ω where (25) and (28) hold simultaneously, the functions $X_t^{(n)}(\omega)$ are both equicontinuous and convergent in square mean in I_0.

Thus the proof of the theorem is complete.

REMARK 1. It is worthwhile observing that in the proof of Theorem 1 we have only made use of the following weaker form of Lemma 1.

LEMMA 2. *Let $f(x)$ be continuous in I_0 and such that*

$$\int_{I_0}\int_{I_0} \Psi\left(\frac{f(x) - f(y)}{p((x - y)/\sqrt{d})}\right) dx\, dy \leqq B < \infty. \tag{29}$$

Then for all $x, y \in I_0$ we have

$$|f(x) - f(y)| \leqq 8 \int_0^{|x-y|} \Psi^{-1}\left(\frac{B}{u^{2d}}\right) dp(u). \tag{30}$$

Now, for $d = 1$ this is precisely the real variable lemma established in [4]. We have thus obtained here a new (and somewhat simpler) proof of that result.

REMARK 2. From Theorem 1 we can immediately deduce the following.

THEOREM 2. *A sufficient condition for a Gaussian process $X_t(\omega)$ on I_0 to admit a separable and measurable model whose sample functions are continuous with probability one is that if we set*

$$p(u) = \max_{|s-t| \leqq |u|\sqrt{d}} [E(|X_s - X_t|^2)]^{1/2} \tag{31}$$

we have

$$\int_0^1 \left(\log \frac{1}{u}\right)^{1/2} dp(u) < \infty. \tag{32}$$

Furthermore, if this holds, then with probability one the modulus of continuity $\Delta(\delta)$ of the sample functions can be majorized by an expression of the form

$$Cp(\delta) + D \int_0^\delta \left(\log \frac{1}{u}\right)^{1/2} dp(u), \tag{33}$$

where in general C is random and D is a universal constant.

In the case $d = 1$ this theorem reduces to Theorem 2.1 of [4]. It can also be shown that in this case our condition (32) is equivalent to the conditions considered by Delporte [1] and Fernique [2]. It thus follows, from some of the considerations in [3], that our condition (32) cannot in general be replaced by any weaker condition concerning $p(u)$ itself.

Finally, we observe that Lemma 1 implies also the Hölder continuity lemma of N. G. Meyers [6]. We thus obtain here also a new proof and extension of that result.

It is to be noted that Meyers' result has interesting applications in the theory of partial differential equations.

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