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Continuity Properties of Solution Concepts for Cooperative Games

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Summary. A survey is given of known continuity properties of solution concepts for cooperative games. Further continuity properties are derived for the bargaining set, the kernel, the equal division core, the least core, the least tax core, the τ-value and also for the core of non sidepayment games.

Zusammenfassung. Die Arbeit gibt einen Überblick über bekannte Kontinuitätseigenschaften von Lösungsansätzen für kooperative Spiele. Weiter Kontinuitätseigenschaften werden abgeleitet für die Aushandlungsmenge, den Kernel, den Kern gleicher Aufteilungen, den kleinsten Kern, den kleinsten "tax"-Kern, den τ-Wert und den Kern von Spielen ohne Seitenzahlungen.

1. Introduction

In this paper we concentrate on continuity properties of solution concepts of sidepayment games and of the core of non sidepayment games. Let $N = \{1, ..., n\}$ be the player set and 2^N the family of all possible coalitions of players. An n-person sidepayment game is a map $v : 2^N \to \mathbb{R}$, which associates the real number $v(S)$ to every possible coalition $S \in 2^N$ such that $v(\phi) = 0$.

The $(2^n - 1)$-dimensional linear space of n-person games is denoted by G^n. We endow G^n with the distance d, where

$$d(u, v) = \max_{S \in 2^N} |u(S) - v(S)| \quad \text{for } u, v \in G^n.$$

We recall that a game $v \in G^n$ is balanced (cf. [12]) if and only if its core is a nonempty set. Denote by B^n the set of balanced games. Clearly, B^n is a full dimensional closed convex cone in G^n. An n-person non sidepayment game is a multifunction $v : 2^N \twoheadrightarrow \mathbb{R}^n$, which associates to every coalition $S \neq \phi$ a subset

$$v(S) \subset \mathbb{R}^S = \{x \in \mathbb{R}^n : x_i = 0 \text{ if } i \notin S\};$$

$v(S)$ is assumed non empty, closed and S-comprehensive (i.e. $v(S) = v(S) + \mathbb{R}^S_-$) (Note that $v(\phi) = \{0\}$). Denote by V^n the set of n-person non sidepayment games.

On the set V^n we shall consider the following convergence for sequences: $v_k \to v$ if, for every $S \subset N$

$$\limsup v_k(S) \subset v(S) \text{ (closedness property)} \tag{1}$$

$$\liminf v_k(S) \supset v(S) \text{ (lower semicontinuity property)} \tag{2}$$

We recall that for a sequence $K_1, K_2, \ldots$ of (closed) subsets of a Euclidean space E:

$\limsup_{k \to \infty} K_k$ consists of those points $x \in E$, for which there is a subsequence $t(1), t(2), \ldots$ of $1, 2, \ldots$ and a sequence $x_{t(1)}, x_{t(2)}, \ldots$ converging to x with $x_{t(m)} \in K_{t(m)}$ for each $m \in \mathbb{N}$. (3)

$\liminf_{k \to \infty} K_k$ consists of those points $x \in E$, for which there is a sequence $x_1, x_2, \ldots$ in E such that $\lim_{k \to \infty} x_k = x$ and $x_k \in K_k$ for each $k \in \mathbb{N}$. (4)

Observe that, if the closed sets $v_k(S)$ are contained in a given compact set (for all large k), then (1) is equivalent to the usual upper semicontinuity property for multifunctions, in the sequential case: this can be deduced, e.g. from [3], Theorem 1, page 24. For the motivations of the choice of this convergence, and for its remarkable properties, we refer to [9]. For what topological notions and properties of multifunctions are concerned, see for instance [3] or [4].

The organization of the paper is as follows: in Sect. 2 we collect some well-known facts about continuity properties of several solution concepts of sidepayment games. Section 3 is devoted to the study of the bargaining set, the kernel and the equal division core. In Sect. 4 we consider the least core and the least tax core. Section 5 is dedicated to the study of the τ-value. Finally Sect. 6 deals with non side payment games: we recall quickly continuity properties of the λ-transfer value and of the Harsanyi solution and we state some results about the core.

2. Well-known Facts about Sidepayment Games

For a sidepayment game $v \in G^n$, the set of preimputations is defined by:

$$I^*(v) = \left(x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = v(N) \right) \tag{5}$$

and the imputation set by

$$I(v) = \{x \in I^*(v) : x_i \geqslant v(\{i\}) \text{ for each } i \in N\}. \tag{6}$$

We shall denote by I^n the family of the n-person games v such that $I(v) \neq \phi$. It is straightforward to show that $I^* : G^n \twoheadrightarrow \mathbb{R}^n$ is a closed and lower semicontinuous multifunction, while $I : I^n \twoheadrightarrow \mathbb{R}^n$ is upper and lower semicontinuous.

The core multifunction $C : B^n \to \mathbb{R}^n$, which associates to every game $v \in B^n$ the nonempty set

$$C(v) = \{x \in I(v) : \sum_{i \in S} x_i \geqslant v(S) \text{ for each } S \in 2^N\} \tag{7}$$

is upper and lower semicontinuous: this is a consequence of the fact that it is defined by a system of linear inequalities and equalities and the related stability theorems in this field, see [4] p. 103. The Shapley value [15] $\phi : G^n \to \mathbb{R}^n$ is a Lipschitz function, with Lipschitz constant 2, so ϕ is also continuous.

Schmeidler [13] introduced the nucleolus n for sidepayment games and proved that $n : G^n \to \mathbb{R}^n$ is a continuous (one-point) solution concept. Another proof of the continuity can be found in [7]. The result is extended to the f-nucleolus in [20].

Some other known results concerning the bargaining set and the kernel [16], and the equal division core, [8], are recalled in the next section.

3. The Bargaining Set, the Kernel and the Equal Division Core

In this section we deal with the bargaining set, introduced by Aumann and Maschler [1], the kernel, introduced by Davis and Maschler [2], and the equal division core, defined by Selten in [14]. As all the three are in general sets, here we shall speak about the properties of closedness (see (1)), lower semicontinuity (see (2)) and upper semicontinuity. The natural domain on which to study the corresponding multisolutions is

$$T^n = \{v \in G^n : v(S) \geqslant \sum_{i \in S} v(\{i\}) \text{ for all } S \in 2^N\}.$$

We denote by $\mathcal{I} = (I_1, \ldots, I_m)$ a coalition structure, namely a partition of the set N, and by $(x, \mathcal{I})$ a payoff configuration, namely a coalition structure $\mathcal{I}$ and a payoff vector $x \in \mathbb{R}^n$ such that $\sum_{i \in I_j} x_i = v(I_j)$ for all $I_j \in \mathcal{I}$.

Denote by $PC(v)$ the set of all the payoff configurations $(x, \mathcal{I})$ of the game v which are individually rational, which means that $x_i \geqslant v(\{i\})$ for each $i \in N$. Observe that $v \in T^n$ if and only if for every $\mathcal{I}$ there is at least one element belonging to $PC(v)$ supported by the coalition structure $\mathcal{I}$.

For a coalition $K \subset N$ we denote by $P(K, \mathcal{I})$ the set of the partners of K in $\mathcal{I}$, namely

$$P(K, \mathcal{I}) = \{i \in N : i \in I_j \text{ for some } j \text{ with } I_j \cap K \neq \phi\} \tag{8}$$

Definition 3.1. Let $(x, \mathcal{I}) \in PC(v)$ and let K and L be non-empty disjoint subsets of the same $I_j \in \mathcal{I}$. An objection of K against L is an $(y, \mathcal{U}) \in PC(v)$ such that

(a) $P(K, \mathcal{U}) \cap L = \phi$

(b) $y_i > x_i$ for all $i \in K$

(c) $y_i \geqslant x_i$ for all $i \in P(K, \mathcal{U})$

A counterobjection of L against K is an element $(z, \mathcal{V}) \in PC(v)$ such that

(d) $K \not\subset P(L, \mathcal{V})$

(e) $z_i \geqslant x_i$ for all $i \in P(L, \mathcal{V})$

(f) $z_i \geqslant y_i$ for all $i \in P(L, \mathcal{V}) \cap P(K, \mathcal{U})$

Definition 3.2. The bargaining set of the game v is: $\{(x, \mathcal{I}) \in PC(v)$: for every objection of K against L, there is a counterobjection of L against $K\}$.

Denote by M: $T^n \twoheadrightarrow \mathbb{R}^n$ the bargaining multifunction, which assigns to each $v \in T^n$ the set $M(v)$

$= \{x \in \mathbb{R}^n$: there is an $\mathcal{I}$ such that $(x, \mathcal{I})$ is in the bargaining set of $v\}$.

Theorem 3.1. *The bargaining multifunction M is upper semicontinuous.*

Proof. Let $v_k \to v$ if $k \to \infty$. It is simple to show that all the payoff vectors of $PC(v_n)$ and $PC(v)$ lie in a suitable fixed compact set. Hence to show the claimed upper semicontinuity we have only to check that $\limsup M(v_k) \subset M(v)$.

Suppose that $x_j \in M(v_{m_j})$, $\lim_{j \to \infty} x_j = x$, where $m_1, m_2, \ldots$ is a subsequence of $\mathbb{N}$.

We have to prove that $x \in M(v)$. There is a sequence $\langle \mathcal{I}_j \rangle$ of coalition structures related to $\langle x_j \rangle$: by considering, if necessary, a subsequence (here labelled with the same index), we can suppose $\mathcal{I}_j = \mathcal{I}$ for every j. As $(x, \mathcal{I})$ belongs obviously to $PC(v)$, to conclude we have to show that for every $(y, \mathcal{U})$ objection of K against L there is a counterobjection $(z, \mathcal{W})$ of L against K. At first we construct a sequence $\langle y_j \rangle$ converging to y of objections of K against L with the additional property that $(y_j)_i > x_i$ for all $i \in P(K, \mathcal{U})$ (for all large j). Let $U_1 \cup \ldots \cup U_r$ be the set of the partners $P(K, \mathcal{U})$ $(U_j \in \mathcal{U}$ for all $j)$. If $i \notin U_1 \cup \ldots \cup U_r$ let $(y_j)_i = y_i$.

In the other cases, for every $s \in \{1, \ldots, r\}$ select $i_s \in U_s \cap K$. Then define

$$(y_j)_{i_s} = y_{i_s} - \frac{|U_s| - 1}{j|U_s|}(y_{i_s} - x_{i_s}),$$

and for $i \in U_s$ unequal to i_s:

$$(y_j)_i = y_i + \frac{y_{i_s} - x_{i_s}}{j|U_s|}$$

(Here, for a finite set S, $|S|$ means the number of elements of S). It is easy to verify that y_j has the claimed properties.

Now, for every fixed j, there are a natural member t_j and an $\epsilon > 0$ such that

$$(y_j)_i > (x_t)_i + \epsilon \quad \text{for all } t \geq t_j \text{ and for all } i \in P(K, \mathcal{U}).$$

Let $\epsilon_j = \inf \{(y_j)_i - (x_t)_i : t \geq t_j, i \in P(K, \mathcal{U})\} > 0$.

Let $m_j \geq t_j$ and $\langle m_j \rangle$ a subsequence of integers such that

$$|v(U_s) - v_{m_j}(U_s)| < \epsilon_j \quad \text{for every } s.$$

Define w_j as the element in $\mathbb{R}^n$ with $(w_j)_{i_s} = y_{i_s} + v_{m_j}(U_s) - v(U_s)$

$$(w_j)_i = (y_j)_i \quad \text{if } i \neq i_s \text{ for all } s.$$

By our construction $(w_j, \mathcal{U})$ is, for all large j, an objection of K against L to x_{m_j}. As $x_{m_j} \in M(v_{m_j})$, there is a counterobjection of L against K: $(z_j, \mathcal{W}_j)$. As usual, we can suppose that $\mathcal{W}_j = \mathcal{W}$, independently from j. As $\langle z_j \rangle$ is a bounded sequence, it has some cluster point z.

Passing to the limit in the Definition 3.1 it is easy to see that $(z, \mathcal{W})$ is a counterobjection to $(y, \mathcal{U})$. This finishes the proof. □

Remark 3.1. For the bargaining set $\mathcal{M}_1^{(i)}$ (cf. [10]) one can prove in a similar way that the corresponding multifunction is upper semicontinuous, making some obvious modifications, which make the proof even easier.

In [10] Maschler shows that inequalities determine the bargaining set $\mathcal{M}_1^{(i)}$. Using this fact here also another proof of the upper semicontinuity follows.

To analyze now the behaviour of the kernel, let us begin with some definitions:

Definition 3.3. Let S be a coalition and x a payoff vector. The excess of S with respect to x is defined by

$$e(S, x) = v(S) - \sum_{i \in S} x_i.$$

The surplus of i against j (with respect to x) is

$$s_{ij}(x) = \max \{e(S, x) : S \ni i, S \not\ni j\}$$

Definition 3.4. The kernel of a game $v \in T^n$ is $\{(x, \mathcal{I}) \in PC(v)$: for each $I \in \mathcal{I}$ there are not $i, j \in I$ such that $s_{ij}(x) > s_{ji}(x)$ and $x_j > v(\{j\})\}$.

Denote by K: $T^n \twoheadrightarrow \mathbb{R}^n$ the kernel multifunction, defined by $K(v) = \{x \in \mathbb{R}^n$: there is an $\mathcal{I}$: $(x, \mathcal{I})$ is in the kernel of $v\}$ for all $v \in T^n$.

Theorem 3.2. *The kernel multifunction K is upper semicountinuous.*

Proof. As in the previous theorem it is enough to show that $\limsup K(v_n) \subset K(v)$. Let $x_j \in K(v_{n_j})$ and $x_j \to x$, where n_j is a subsequence of $\mathbb{N}$. We must show that $x \in K(v)$.

This can be easily shown by contradiction, and it is left to the reader. □

We conclude now with the equal division core. In a coalition C, let $e(C) = \dfrac{v(C)}{|C|}$. Then

Definition 3.5. The equal division core of a game $v \in T^n$ is the set

$\{(x, \mathcal{I}) \in PC(v)$: there is no $C \subset N$ with $x_i < e(C)$ for all $i \in C\}$.

Denote by E: $T^n \twoheadrightarrow \mathbb{R}^n$ the equal division core multifunction, namely $EDC(v) = \{x \in \mathbb{R}^n$: there is an $\mathcal{I}$ such that $(x, \mathcal{I})$ is in the equal division core of $v\}$.

Theorem 3.3. *The equal division core multifunction is upper semicontinuous.*

Proof. This is an easy exercise (cf. [8]). □

About the lower semicontinuity of the three solution concepts considered in this section, we observe that no one of them has this property. For what the bargaining set and the kernel is concerned, we mention the example given in [16]. The following example shows the lack of lower semicontinuity of E.

Example 3.1. Let $N = \{1, 2, 3\}$ and $v(i) = 0$ for all $i \in N$, and $v_k(1) = v_k(2) = 0$, $v_k(3) = \frac{1}{k}$ for all $k \in \mathbb{N}$, $v_k(1, 2) = 1 - k^{-1}$, $v(1, 2) = 1$. Let $v_k(N) = v(N) = 1$ for all $k \in \mathbb{N}$, and $v_k(1, 3) = v(1, 3) = v_k(2, 3) = v(2, 3) = 1$.

Then $E(v_k) \neq \phi$, $\left(\frac{1}{2}, \frac{1}{2}, 0\right) \in E(v)$ but it cannot be approximated by any sequence in $\langle E(v_k)\rangle$, and $v_k \to v$.

For similar results on E and others related to the stability of the regular configurations of the games see [8].

4. The Least Core and the Least Tax Core

The least core was introduced by Maschler, Peleg and Shapley [11]. For $v \in G^n$ let

$$\epsilon(v) = \min_{x \in I^*(v)} \max_{S \in 2^N - \{\phi, N\}} (v(S) - \sum_{i \in S} x_i),$$

Let $v_{\epsilon(v)}$ be the game, associated to v defined by:

$$v_{\epsilon(v)}(N) = v(N), \quad v_{\epsilon(v)}(\phi) = 0,$$

$$v_{\epsilon(v)}(S) = v(S) - \epsilon(v) \quad \text{if } S \neq N, S \neq \phi. \tag{9}$$

Definition 4.1. The least core of v, denoted by $LC(v)$, is the core of the game $v_{\epsilon(v)}$: $LC(v) = C(v_{\epsilon(v)})$.

We present a geometric interpretation of $v_{\epsilon(v)}$. Let $p \in B^n$ be the game defined by:

$$p(S) = -1 \quad \text{if } S \neq N, S \neq \phi, \quad p(\phi) = p(N) = 0.$$

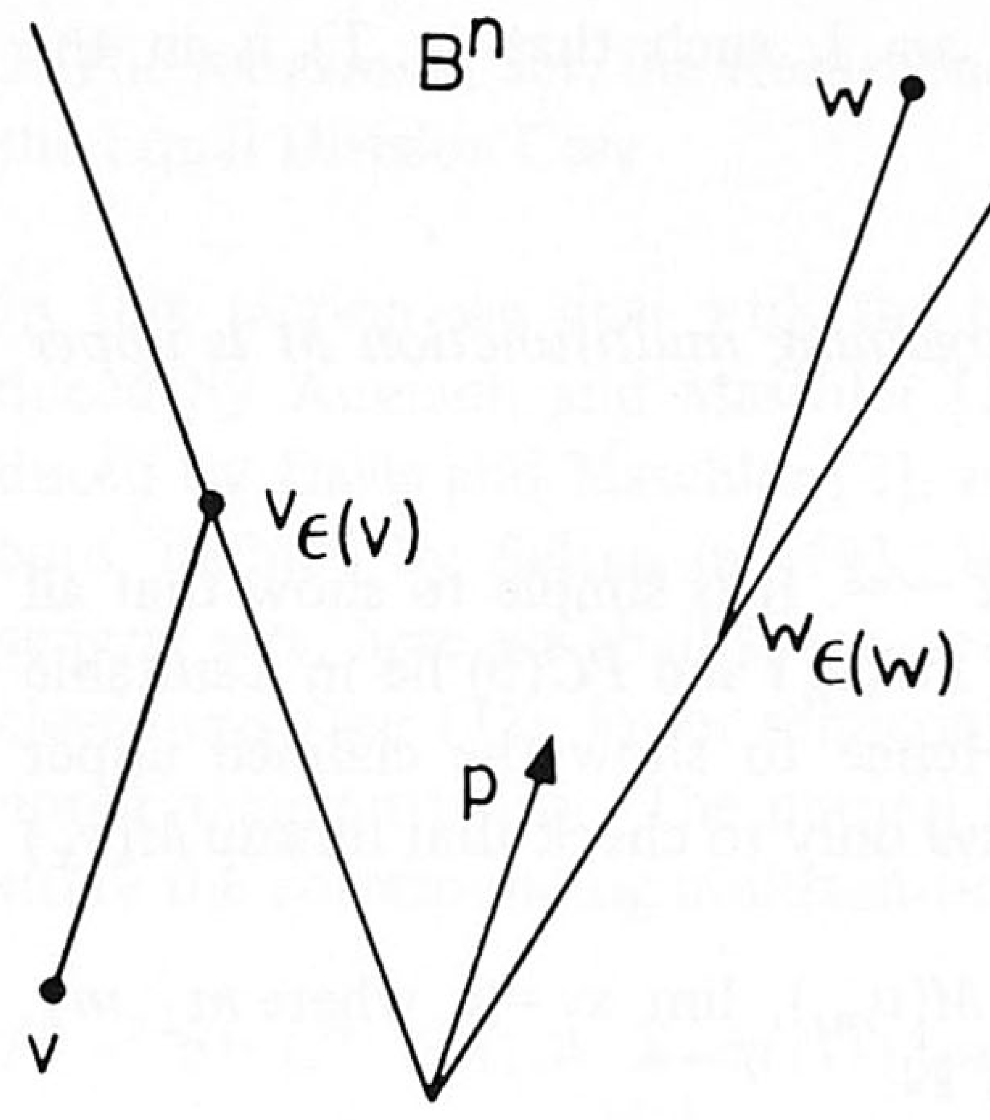

Fig. 1. Geometric description of the least core

Then for $v \in G^n - B^n (w \in B^n)$ the game $v_{\epsilon(v)}(w_{\epsilon(w)})$ is the unique game of the half-line

$$\{v + \epsilon p : \epsilon \in [0, +\infty)\}(\{w + \epsilon p : \epsilon \in (-\infty, 0]\})$$

where the half-line enters (leaves) the cone B^n (see Fig. 1). Before proving the continuity of LC: $G^n \twoheadrightarrow \mathbb{R}^n$ we state (in a form suitable to our aims) a wellknown lemma about multifunctions [see [3] p. 23].

Lemma 4.1. *Let X and Y be metric spaces. Let $F : X \twoheadrightarrow Y$ be a compact valued and upper semicontinuous multifunction and let $G : X \twoheadrightarrow Y$ be a closed multifunction. Let $(F \cap G)(x) := F(x) \cap G(x) \neq \phi$ for each $x \in X$. Then $F \cap G$ is a (compact valued and) upper semicontinuous multifunction.*

For each $v \in G^n$ let $\alpha(v) := \max_{S \in 2^N} (v(S) - \frac{1}{n}|S| v(N))$ and $\gamma(v) := \max_{i \in N} \frac{1}{n} v(N) - v(\{i\}) + 1$.

Note that $v + \alpha(v)p \in B^n$ because $\frac{1}{n}(v(N), v(N), \ldots, v(N))$ is a core element of that game. Furthermore $v - \gamma(v)p \notin B^n$ because $\sum_{i=1}^{n} (v - \gamma(v)p)\{i\}) \geqslant v(N) + n > v(N) = (v - \gamma(v)p)(N)$ which implies that $\phi = I(v - \gamma(v)p) \supset C(v - \gamma(v)p)$. Let $F : G^n \twoheadrightarrow G^n$ be the multifunction with $F(v) = [v - \gamma(v)p, \; v + \alpha(v)p]$ for $v \in G^n$, where the image $F(v)$ of v consists of the line segment with the end point $v - \gamma(v)p$ outside B^n and the end point $v + \alpha(v)p$ inside B^n. Since $\alpha : G^n \to \mathbb{R}$ and $\gamma : G^n \to \mathbb{R}$ are continuous, the multifunction F is compact-valued and upper semicontinuous. Applying Lemma 4.1 with G^n in the role of X and Y and $G(v) = B^n - \text{int}(B^n)$ for each v, and noting that $F \cap G(v)$ consists of the unique point $v_{\epsilon(v)}$, yields the first part of

Theorem 4.1.

(i) The map $v \mapsto v_{\epsilon(v)}$ is continuous.

(ii) $LC : G^n \twoheadrightarrow \mathbb{R}^n$ is a continuous multifunction.

Proof. Part *(i)* is already proved and part *(ii)* follows from *(i)* and the fact mentioned in Sect. 2 that the core multifunction on B^n is continuous. □

The least tax core for games with non empty imputation set was introduced by Tijs and Driessen [19]. For each game $v \in I^n$, denote by v^b the corresponding bargaining game defined by:

$$v^b(N) = v(N), \quad v^b(\phi) = 0,$$
$$v^b(S) = \sum_{i \in S} v(\{i\}) \quad \text{if } S \neq N, S \neq \phi. \tag{10}$$

Remark that $v^b \in B^n$ because the vector $z \in \mathbb{R}^n$ with

$$z_i = v(\{i\}) + \frac{1}{n}\left(v(N) - \sum_{i=1}^{n} v(\{i\})\right)$$

for each $i \in N$ lies in the core of v^b.

Definition 4.2. For every game $v \in I^n$ we define the corresponding tax game T_v as the game belonging to the line segment $[v, v^b]$ and to B^n, which is nearest to v.

The core of T_v is called the least tax core of v and it is denoted by $LTC(v)$: $LTC(v) = C(T_v)$.

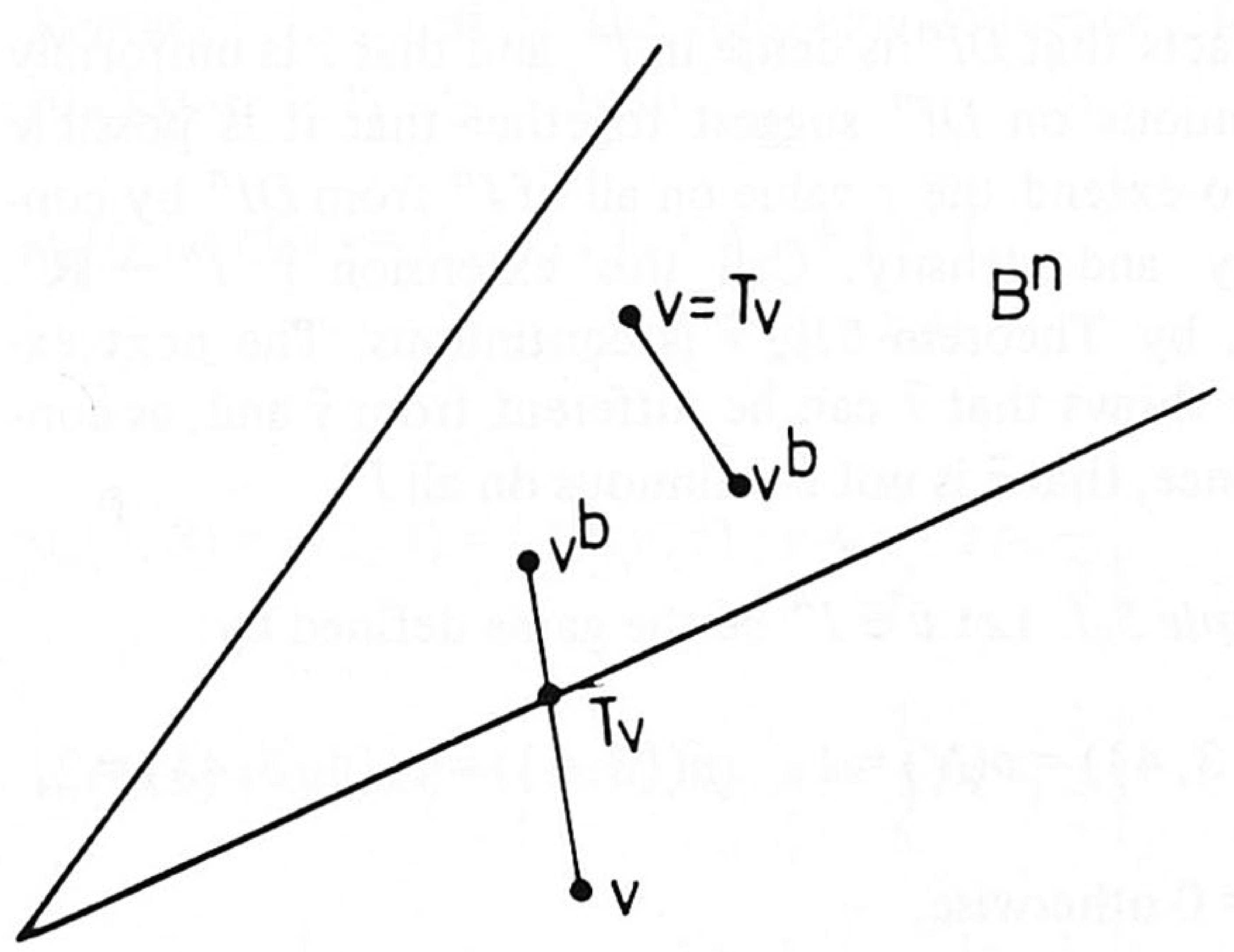

Fig. 2. Geometric description of the least tax core

Remark 4.1. It is straightforward to show that $v^b \in \text{int}(B^n)$ if and only if $v(N) > \sum_{i=1}^{n} v(\{i\})$.

Theorem 4.2. *The multifunction $LTC : I^n \twoheadrightarrow \mathbb{R}^n$ is continuous.*

Proof. Let $H : I^n \twoheadrightarrow I^n$ be the multifunction defined as:

$$H(v) = \begin{cases} v & \text{if } v \in \text{int}(B^n) \\ [v, v^b] \cap bd(B^n) & \text{if } v \notin \text{int}(B^n) \end{cases} \tag{11}$$

We shall prove the following facts:

(a) H is upper semicontinuous at every point of I^n

(b) $H(v)$ is not single valued only in the case that $v \in bd(B^n)$, $v^b \in bd(B^n)$ and $v \neq v^b$

(c) If $v \in bd(B^n)$, $v^b \in bd(B^n)$, then $LTC(v)$ is a singleton

(d) $LTC(v) = C \circ H(v)$ for every v.

(a) H is upper semicontinuous everywhere: this is proved by showing that $H_{|B^n}$ and $H_{|I_n - \text{int}(B_n)}$ are upper semicontinuous.

In the first case, if $v \in \text{int}(B_n)$, the claim is trivial and if $v \in bd(B^n)$ the claim easily follows from the fact that $v \in H(v)$. In the second case we can apply a similar argument as we used to prove theorem 4.1 (choosing $F(v) = [v, v^b]$).

(b) The only non trivial case is when $v \notin B^n$, $v^b \in bd(B^n)$. The last condition implies that $C(v^b) = \{(v(1), \ldots, v(n))\}$: see Remark 4.1. As $v \notin B^n$ there is at least one coalition S such that $v(S) > \sum_{i \in S} v(\{i\})$. This means that for all $w \in [v, v^b)$:

$$w(S) > \sum_{i \in S} w(\{i\}) = \sum_{i \in S} v(\{i\}), \quad \text{hence } [v, v^b) \notin B^n.$$

(c) In this case it is straightforward to verify that for all $w \in [v, v^b]$: $C(w) = \{(v(1), \ldots, v(n))\}$, from which we can conclude that $LTC(v)$ is a singleton.

(d) easily follows from (b), (c) and (11). We are now able to finish the proof. Namely a multifunction which is upper semicontinuous and single valued is also lower semicontinuous. Hence H is lower semicontinuous (and then continuous) for all the games

$$v \notin E = \{w \in I^n : w \in bd(B^n), w^b \in bd(B^n), w \neq w^b\}.$$

For games $v \in E$ (c) shows that LTC is lower semicontinuous. This finishes the proof. □

5. The τ-Value

The τ-value was introduced by Tijs [17] for quasi-balanced games (see the definition below) and an axiomatic characterization was given in [18]. An extension to the family of all games with non empty imputation set was described by Tijs and Driessen [19].

For $v \in G^n$, the upper vector $b^v = (b_1^v, \ldots, b_n^v)$ and the lower vector $a^v = (a_1^v, \ldots, a_n^v)$ are defined as follows. For each $i \in N$:

$$b_i^v = v(N) - v(N - \{i\}), \quad a_i^v = \max_{S \ni i} (v(S) - \sum_{k \in S - \{i\}} b_k^v)$$

Let

$$Q^n = \left\{ v \in G^n : a^v \leqslant b^v, \sum_{i=1}^n a_i^v \leqslant v(N) \leqslant \sum_{i=1}^n b_i^v \right\}$$

Q^n is called the set of quasi-balanced games.

It is easy to show that Q^n is a full dimensional closed convex cone in G^n, including the cone B^n. Furthermore $Q^n \subset I^n$, because $v \in Q^n$ implies

$$v(N) \geqslant \sum_{i=1}^n a_i^v \geqslant \sum_{i=1}^n v(\{i\}) \quad \text{(see [17])}.$$

Definition 5.1. For a game $v \in Q^n$, the τ-value $\tau(v)$ is defined as the unique element belonging to the line segment $[a^v, b^v]$ and to the preimputation set $I^*(v)$.

It is easy to prove the continuity of the τ-value:

Theorem 5.1. *$\tau : Q^n \to \mathbb{R}^n$ is continuous.*

Proof. Apply Lemma 3.1 with $F(v) = [a^v, b^v]$ and $G(v) = I^*(v)$.

Then τ is upper semicontinuous (as a multifunction) hence a continuous function. □

Among the other properties fulfilled by the τ-value, we mention the so called dummy player property. For a game v let

$$D^v = \{i \in N : v(S \cup \{i\}) = v(S) + v(\{i\})\}$$
$$\text{for all } S \subset N - \{i\}) \tag{13}$$

be the set of the dummy players of v.

It is easy to verify that, if i is a dummy player of $v \in Q^n$, then $\tau_i(v) = v(\{i\})$. The following extension $\bar{\tau} : I^n \to \mathbb{R}^n$ of the τ value was described in [19]: the definition in two steps is motivated by the reason of preserving, for $\bar{\tau}$, the dummy player property:

Definition 5.2. Let $v \in I^n$ and consider the set of the dummy players D^v. For every $i \in D^v$ let $\bar{\tau}_i(v) = v(\{i\})$.

If $D^v = N$, then $\bar{\tau}$ is well-defined. If not, consider the game $v^f : 2^{N - D^v} \to \mathbb{R}^{|N - D^v|}$ which is obtained from v, ignoring the dummy players. Without loss of generality, suppose $N - D^v = M = \{1, \ldots, m\}$, with $0 < m \leqslant n$. Consider now the game $q(v)$ belonging to $[v^f, v_b^f] \cap Q^m$ which is nearest to v^f. (Here v_b^f is the bargaining game, related to v^f, as defined in Sect. 4.) As $q(v)$ is quasi-balanced, its τ value is well defined. Finally let $\bar{\tau}(v) = (\bar{\tau}_1(v), \ldots, \bar{\tau}_n(v))$ be such that $\bar{\tau}_i(v) = v(\{i\})$ if $i \in D^v$ and $\bar{\tau}_i(v) = \tau_i(q(v))$ otherwise.

Observe that the set $DI^n = \{v \in I^n : D^v = \phi\}$ of all dummy-free games is open and dense in I^n. We can prove:

Theorem 5.2. *$\bar{\tau} : DI^n \to \mathbb{R}^n$ is continuous everywhere.*

Proof. (Outline) Define $H : DI^n \twoheadrightarrow Q^n$ as

$$H(v) = \begin{cases} \{v\} & \text{if } v \in \text{int}\,(Q^n) \\ [v, v^b] \cap bd(Q^n) & \text{if } v \notin \text{int}\,(Q^n) \end{cases}$$

As in the proof of Theorem 4.1 we can prove that:

(a) $\bar{\tau} = \tau \circ H$

(b) H is u.s.c.

Then the theorem follows by remembering that $\bar{\tau}$ is single valued. □

The facts that DI^n is dense in I^n, and that $\bar{\tau}$ is uniformly continuous on DI^n suggest together that it is possible also to extend the τ value on all of I^n from DI^n by continuity and density. Call this extension $\hat{\tau} : I^n \to \mathbb{R}^n$. Then, by Theorem 5.1, $\hat{\tau}$ is continuous. The next example shows that $\bar{\tau}$ can be different from $\hat{\tau}$ and, as consequence, that $\bar{\tau}$ is not continuous on all I^n.

Example 5.1. Let $v \in I^4$ be the game defined by:

$$v(\{2,3,4\}) = v(N) = 1, \quad v(\{3,4\}) = v(\{1,3,4\}) = 2,$$

$v(S) = 0$ otherwise.

$$v \in I^n - Q^n \quad \text{as} \quad b_1^v = 0 < 1 = a_1^v.$$

A straightforward calculation shows that

$$\bar{\tau}(v) = \left(0, 0, \frac{1}{2}, \frac{1}{2}\right), \quad \hat{\tau}(v) = \left(\frac{1}{5}, 0, \frac{2}{5}, \frac{2}{5}\right).$$

Observe in particular that the dummy player 1 gets more than zero ($= v(1)$) in $\hat{\tau}$.

6. Non Side Payment Games

In this final section we recall quickly stability properties of the λ-transfer value and of the Harsanyi solution and we state the simple results concerning the core of non side payment games. For other results related to the core see e.g. [6].

In [9] we show that the λ-transfer multifunction $S : \Gamma^n \twoheadrightarrow \mathbb{R}^n$ is not upper semicontinuous, while the Harsanyi multifunction $H : \Gamma^n \twoheadrightarrow \mathbb{R}^n$ satisfies this stability property. Here Γ^n is the set of the (non side payment) superadditive games. If we restrict our attention to the set $\tilde{\Gamma}_n$ of compactly generated games, equipped with Hausdorff's convergence, then S is upper semicontinuous. Both of these solution multifunctions lack lower semicontinuity.

Let us now state the properties of the core.

Definition 6.1. We say that $x \in v(N)$ is in the core of the game v, denoted by $x \in C(v)$, if for each $S \subset N$ there is no $y \in v(S)$ such that $y > x^S$ (i.e. $y_i > x_i^S$ for all $i \in S$).

Theorem 6.1. *The core multifunction $C : V \twoheadrightarrow \mathbb{R}^n$ is upper semicontinuous.*

Proof. A simple argument (by contradiction) shows that C has a closed graph. Then we conclude to upper semicontinuity by the same compactness argument used in Theorem 3.1. □

Remark 6.1. Consider the following sequence of 3-person games $v, v_1, v_2, \ldots$ with

$$v_n(N) = v(N) = \{(x, y, z) : x + y + z \leqslant 1,$$
$$x, y, z \geqslant 0\} + \mathbb{R}_-^{(1,2,3)}.$$

$$v_n(2,3) = v(2,3) = \left\{ (0, y, z) : y \leqslant \frac{1}{2}, z \leqslant \frac{1}{2} \right\}$$

$$v_n(1,3) = v(1,3) = \left\{ (x, 0, z) : x \leqslant \frac{1}{2}, z \leqslant \frac{1}{2} \right\}$$

$$v_n(1,2) = \left\{ (x, y, 0) : x \leqslant \frac{1}{2} + \frac{1}{n}, y \leqslant \frac{1}{2} - \frac{1}{n} \right\},$$

$$v(1,2) = \left\{ (x, y, 0) : x \leqslant \frac{1}{2}, y \leqslant \frac{1}{2} \right\}$$

It is easy to show (see (1) and (2)) that $v_n \to v$ (even in Hausdorff's sense) and that $x = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ belongs to $C(v)$.

But no sequence exists in $C(v_n)$ ($\neq \phi$ for all $n \in \mathbb{N}$) converging to x, as it can be shown for instance by contradiction.

To conclude, we mention the work [5], dedicated to the analysis of the continuity properties of several solution concepts proposed in the literature for bargaining problems.

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