BIMONTHLY REPORT

(NASA-CR-120321) CONTINUOUS AND DISCRETE DESCRIBING FUNCTION ANALYSIS OF THE LST SYSTEM Bi-Monthly Report (Systems Research Lab., Champaign, Ill.) 59 p HC $6.00 CSCL 14B

N74-31907

Unclas G3/14 15992

RESEARCH STUDY ON STABILIZATION AND CONTROL

MODERN SAMPLED-DATA CONTROL THEORY

SYSTEMS RESEARCH LABORATORY

P.O. BOX 2277, STATION A
3206 VALLEY BROOK DRIVE
CHAMPAIGN, ILLINOIS 61820

PREPARED FOR GEORGE C. MARSHALL SPACE FLIGHT CENTER

HUNTSVILLE, ALABAMA

BI-MONTHLY REPORT

RESEARCH STUDY ON STABILIZATION AND CONTROL

– MODERN SAMPLED - DATA CONTROL THEORY

SUBTITLE: CONTINUOUS AND DISCRETE DESCRIBING FUNCTION ANALYSIS OF THE LST SYSTEM

November 1, 1973 NAS8-29853

BY B.C.KUO
G.SINGH
R.A.YACKEL

PREPARED FOR GEORGE C. MARSHALL SPACE FLIGHT CENTER

HUNTSVILLE, ALABAMA

CONTRACT NAS8-29853 DCN 1-2-40-23018

SYSTEMS RESEARCH LABORATORY

P. O. BOX 2277, STATION A
CHAMPAIGN, ILLINOIS 61820

5. A Describing Function of the CMG Nonlinearity Using the Analytical Torque Equation

A describing function of the CMG frictional nonlinearity was derived earlier using the straight-line approximated input-output relation between the frictional torque T_{GF} and the CMG angular displacement θ_G.

However, it is possible to derive a describing function for the CMG frictional torque using the analytical relation between T_{GF} and θ_G.

It has been established that the frictional nonlinearity of the CMG can be described by the square-law relation.

$$\frac{dT_{GF}}{d\theta_G} \approx \gamma(T_{GFI} - T_{GFO})^2 \tag{5-1}$$

where

$$T_{GFI} = T_{GF}\, SGN(\dot{\theta}_G) \tag{5-2}$$

T_{GFO} = saturation level of T_{GF}

γ = positive constant

Carrying out the integration on both sides of Eq. (5-1) yields

$$\theta_G + C_1 = \frac{-1}{\gamma(T^+_{GF} - T_{GFO})} \qquad \dot{\theta}_G \geq 0 \tag{5-3}$$

$$\theta_G + C_2 = \frac{-1}{\gamma(T^-_{GF} + T_{GFO})} \qquad \dot{\theta}_G \leq 0 \tag{5-4}$$

where C_1 and C_2 are constants of integration, and

$$T_{GF}^{+} = T_{GF} \qquad \dot{\theta}_G \geq 0 \tag{5-5}$$

$$T_{GF}^{-} = T_{GF} \qquad \dot{\theta}_G \leq 0 \tag{5-6}$$

Then, C_1 and C_2 are given by

$$C_1 = -\theta_{Gi} - \frac{1}{\gamma(T_{GFi}^{+} - T_{GFO})} \qquad \dot{\theta}_G \geq 0 \tag{5-7}$$

$$C_2 = -\theta_{Gi} - \frac{1}{\gamma(T_{GFi}^{-} + T_{GFO})} \qquad \dot{\theta}_G \leq 0 \tag{5-8}$$

where θ_{Gi} and T_{GFi} denote the initial values of θ_G and T_{GF}, respectively.

For a sinusoidal input, θ_G is represented by

$$\theta_G = A \cos \omega t \tag{5-9}$$

It is important to note that for the input of Eq. (5-9) $\theta_{Gi} = -A$ when $\dot{\theta}_G \geq 0$, and $\theta_{Gi} = A$ when $\dot{\theta}_G \leq 0$.

Solving for T_{GF}^{+} and T_{GF}^{-} from Eqs. (5-3) and (5-4), respectively, we have

$$T_{GF}^{+} = \frac{-1}{\gamma(A \cos \omega t + C_1)} + T_{GFO} \qquad \dot{\theta}_G \geq 0 \tag{5-10}$$

$$T_{GF}^{-} = \frac{-1}{\gamma(A \cos \omega t + C_2)} - T_{GFO} \qquad \dot{\theta}_G \leq 0 \tag{5-11}$$

with

$$C_1 = A - \frac{1}{\gamma(T_{GFi}^{+} - T_{GFO})} \tag{5-12}$$

$$C_2 = -A - \frac{1}{\gamma(T_{GFi}^{-} + T_{GFO})} \tag{5-13}$$

where

$$T_{GFi}^{+} = - T_{GFO}\left(-\frac{1}{a} + \sqrt{\frac{a^2+1}{a^2}}\right) \quad (5\text{-}14)$$

$$T_{GFi}^{-} = T_{GFO}\left(-\frac{1}{a} + \sqrt{\frac{a^2+1}{a^2}}\right) \quad (5\text{-}15)$$

$$a = 2\gamma A T_{GFO} \quad (5\text{-}16)$$

With the describing function method, the frictional torque T_{GF} may be approximated by the fundamental component of the Fourier series. The dc component is zero, since the input-output relation is symmetrical about the zero-torque axis.

Thus,

$$T_{GF} = A_1 \sin \omega t + B_1 \cos \omega t$$

$$= \sqrt{A_1^2 + B_1^2} \cos (\omega t - \phi) \quad (5\text{-}17)$$

$$\phi = \tan^{-1} \frac{A_1}{B_1} \quad (5\text{-}18)$$

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} T_{GF} \sin \omega t \, d\omega t = \frac{1}{\pi}\int_0^{\pi} T_{GF}^{-} \sin\omega t \, d\omega t$$

$$+ \frac{1}{\pi}\int_{\pi}^{2\pi} T_{GF}^{+} \sin\omega t \, d\omega t \quad (5\text{-}19)$$

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} T_{GF} \cos\omega t \, d\omega t = \frac{1}{\pi}\int_0^{\pi} T_{GF}^{-} \cos\omega t \, d\omega t$$

$$+ \frac{1}{\pi}\int_{\pi}^{2\pi} T_{GF}^{+} \cos\omega t \, d\omega t \quad (5\text{-}20)$$

Substitution of Eqs. (5-10) and (5-11) into Eq. (5-19) gives

$$A_1 = \frac{1}{\pi}\int_0^{\pi} \frac{-1}{\gamma(A\cos\omega t + C_2)}\sin n\omega t\ d\omega t - \frac{1}{\pi}\int_0^{\pi} T_{GFO}\sin n\omega t\ d\omega t$$

$$+ \frac{1}{\pi}\int_{\pi}^{2\pi} \frac{-1}{\gamma(A\cos\omega t + C_1)}\sin n\omega t\ d\omega t + \frac{1}{\pi}\int_{\pi}^{2\pi} T_{GFO}\sin n\omega t\ d\omega t \quad (5\text{-}21)$$

$$A_1 = \frac{1}{\pi A\gamma}\ell n\ (A\cos\omega t + C_2)\Big|_0^{\pi} - \frac{2T_{GFO}}{\pi}$$

$$+ \frac{1}{\pi A\gamma}\ell n\ (A\cos\omega t + C_1)\Big|_{\pi}^{2\pi} - \frac{2T_{GFO}}{\pi} \quad (5\text{-}22)$$

Thus

$$A_1 = \frac{1}{\pi A\gamma}\left[\ell n\left(\frac{C_2 - A}{C_2 + A}\right) + \ell n\left(\frac{C_1 + A}{C_1 - A}\right)\right] - \frac{4T_{GFO}}{\pi} \quad (5\text{-}23)$$

In arriving at the last expression it is noted that

$$C_2 < -A$$

$$C_1 > A$$

and $C_1 = -C_2$ over their respective ranges of $\dot{\theta}_G$. Equation (5-23) is simplified further to

$$A_1 = -\frac{4T_{GFO}}{\pi} + \frac{1}{\pi A\gamma}\ell n\left[\frac{(C_2 - A)(C_1 + A)}{(C_2 + A)(C_1 - A)}\right]$$

$$= -\frac{4}{\pi}T_{GFO} + \frac{2}{\pi A\gamma}\ell n\left[\frac{C_1 + A}{C_1 - A}\right] \quad (5\text{-}24)$$

Now substitution of Eqs. (5-10) and (5-11) into Eq. (5-20) yields

$$B_1 = \frac{1}{\pi}\int_0^{\pi} \frac{-1}{\gamma[A\cos\omega t + C_2]}\cos\omega t\ d\omega t - \frac{1}{\pi}\int_0^{\pi} T_{GFO}\cos\omega t\ d\omega t$$

$$+ \frac{1}{\pi}\int_{\pi}^{2\pi} \frac{-1}{\gamma[A\cos\omega t + C_1]}\cos\omega t\ d\omega t + \frac{1}{\pi}\int_{\pi}^{2\pi} T_{GFO}\cos\omega t\ d\omega t \quad (5\text{-}25)$$

Evaluating each of the integrals in the last equation, we have

$$\int_0^{\pi} T_{GFO} \cos\omega t \; d\omega t = \int_{\pi}^{2\pi} T_{GFO} \cos\omega t \; d\omega t = 0 \qquad (5\text{-}26)$$

Since $C_2 < -A$, C_2 is always negative, and $C_2^2 > A^2$, the first integral of B_1 becomes

$$I_1 = \frac{1}{\pi}\int_0^{\pi} \frac{-1}{\gamma(A\cos\omega t + C_2)} \cos\omega t \; d\omega t = \frac{1}{\pi\gamma}\left(-\frac{\omega t}{A}\right)_0^{\pi} + \frac{C_2}{\pi\gamma A}\int_0^{\pi} \frac{d\omega t}{A\cos\omega t + C_2} \qquad (5\text{-}27)$$

or

$$I_1 = -\frac{1}{\gamma A} + \frac{C_2}{\pi\gamma A}\left[\frac{2}{\sqrt{C_2^2 - A^2}} \tan^{-1}\frac{(C_2 - A)\tan(\omega t/2)}{\sqrt{C_2^2 - A^2}}\right]_0^{\pi}$$

$$= -\frac{1}{\gamma A} - \frac{C_2}{\gamma A\sqrt{C_2^2 - A^2}} \qquad (5\text{-}28)$$

where the fact that C_2 is negative has been used. Also, $\tan \pi/2$ is taken to be $+\infty$ since $\omega t/2$ expands from 0 to $\pi/2$.

Similarly, the third integral of B_1 in Eq. (5-25) is written

$$I_2 = \frac{1}{\pi}\int_{\pi}^{2\pi} \frac{-1}{\gamma(A\cos\omega t + C_1)} \cos\omega t \; d\omega t$$

$$= \frac{-1}{\gamma A} + \frac{C_1}{\pi\gamma A}\left[\frac{2}{\sqrt{C_1^2 - A^2}} \tan^{-1}\frac{(C_1 - A)\tan(\omega t/2)}{\sqrt{C_1^2 - A^2}}\right]_{\pi}^{2\pi}$$

$$= \frac{-1}{\gamma A} + \frac{C_1}{\gamma A\sqrt{C_1^2 - A^2}} \qquad (5\text{-}29)$$

In arriving at the last equation, we have recognized that $C_1 > A$ and have used that $\tan \pi/2 = -\infty$, since in this case $\omega t/2$ expands from $\pi/2$ to π.

Thus,

$$B_1 = I_1 + I_2 = -\frac{2}{\gamma A} - \frac{C_2}{\gamma A\sqrt{C_2^2-A^2}} + \frac{C_1}{\gamma A\sqrt{C_1^2-A^2}}$$

$$= \frac{2}{\gamma A}\left(\frac{C_1}{\sqrt{C_1^2-A^2}} - 1\right) \qquad (5\text{-}30)$$

The describing function in complex form is written as

$$N(A) = \frac{B_1 - jA_1}{A} \qquad (5\text{-}31)$$

where A_1 and B_1 are given by Eqs. (5-24) and (5-30), respectively. A digital computer program for the computation of N(A) and -1/N(A) is listed in Table 5-1. The constant A is represented by E in this program. The parameters of the nonlinearity are:

$T_{GFO} = 0.1$ ft-lb

$\gamma = 1.38 \times 10^5$

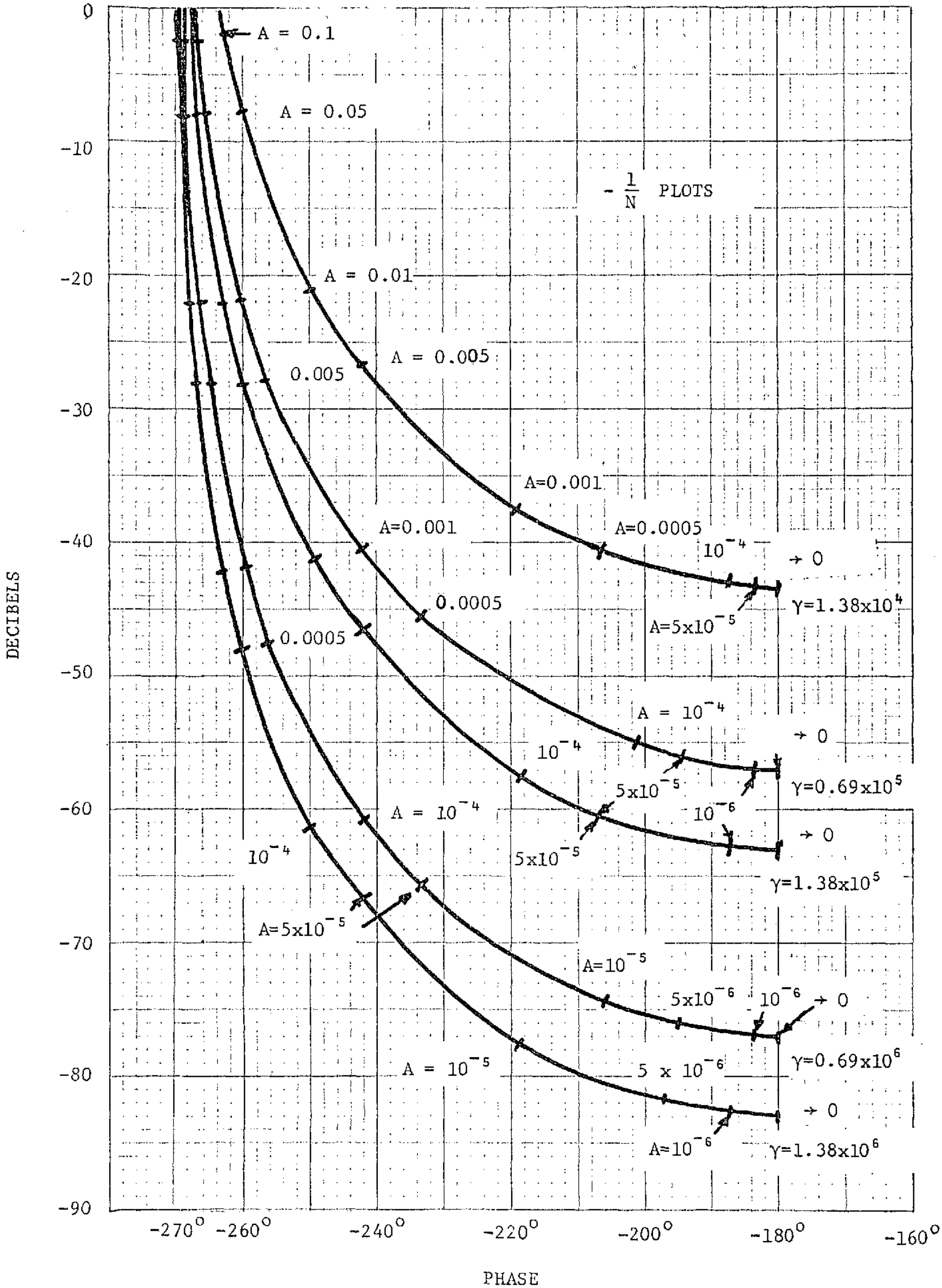
Figure 5-1 shows the magnitude (db) versus phase (degrees) plots of -1/N(A) for $\gamma = 1.38 \times 10^4$, 0.69×10^5, 1.38×10^5, 0.69×10^6, and 1.38×10^6, as the magnitude of A varies. Note that as A becomes large, the magnitude of -1/N(A) approaches infinity and the phase approaches -270 degrees. As A decreases, the magnitude of -1/N(A) decreases and the phase approaches -180 degrees. In the limit as $A \to 0$, -1/N(A) goes to $-1/\gamma T_{GFO}^2$.

Asymptotic Behavior of -1/N(A) for Very Small Values of A

Figure 5-1 shows that as A approaches zero, the magnitude of -1/N(A) in db approaches $20 \log_{10}[1/\gamma T_{GFO}^2]$ and the phase is -180 degrees. The asymptotic behavior of -1/N(A) for very small values of A is derived here analytically.

Table 5-1

```
L
      "   LST CONTINOUS DESCRIBING FUNCTION - EXACT CMG NONLINEARITY
      COMPLEX GV,GN*16
      REAL*8 P(20),PI,RAD,TO,GAMMA,ESTART,E,AA,R,TGFI,TGFN,TGFP,C1,C2
,A1,B1
      REAL*8 AZ
      PI=3.14159
      RAD=180./PI
      TO=.1
      S=10.
      GAMMA=S*1.38E5
      ESTART=1.E-13
      NP=2
      ND=15
      P(1)=1.
      P(2)=5.
      WRITE(6,100)
      WRITE(6,101)
      DO 1 J=1,ND
      DO 1 I=1,NP
      E=ESTART*P(I)*(10.**(J-1))
      AA=2.*GAMMA*E*TO
      R=(-1./AA)+DSQRT((AA*AA+1.)/(AA*AA))
      TGFI=R*TO
      TGFN=TGFI
      TGFP=-TGFI
      C1=E-1./(GAMMA*(TGFP-TO))
      C2=-E-1./(GAMMA*(TGFN+TO))
      A1=(-4.*TO/PI)+(1./(PI*GAMMA*E))*DLOG(((C1+E)*(C2-E))/((C1-E)*%
      (C2+E)))
      AZ=DLOG(((C1+E)*(C2-E))/((C1-E)*(C2+E)))
      A1=(-4.*TO/PI)+(AZ/(PI*GAMMA*E))
      B1=(-1./(GAMMA*E))*(2.+C2/DSQRT(C2*C2-E*E)-C1/DSQRT(C1*C1-E*E))
      A1=A1/E
      B1=B1/E
      GN=DCMPLX(B1,-A1)
      GV=-1./GN
      G1=REAL(GV)
      G2=AIMAG(GV)
      GMAG=CABS(GV)
      GDB=20.*ALOG10(GMAG)
      GPHASE=RAD*ATAN2(G2,G1)
      IF(GPHASE.GE.0.)GPHASE=GPHASE-360.
      WRITE(6,102)E,TGFI,GPHASE,GDB,GMAG
      1 CONTINUE
      100 FORMAT('   CONTINOUS DESCRIBING FUNCTION FOR CMG NONLINEARI
TY')
      101 FORMAT(/,8X,'E',11X,'TGFI',10X,'PHASE',10X,'DB',9X,'MAGNITU
DE'/)
      102 FORMAT(1P5E14.5)
      STOP
      END
C
```



- 1/N PLOTS
DECIBELS
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
PHASE
-270°
-260°
-240°
-220°
-200°
-180°
-160°
A = 0.1
A = 0.05
A = 0.01
A = 0.005
0.005
A=0.001
A=0.0005
A=0.001
0.0005
0.0005
A = 10^-4
10^-4
A = 10^-4
10^-4
A=5x10^-5
5x10^-5
5x10^-5
A=5x10^-5
10^-6
A=10^-5
5x10^-6
10^-6
A = 10^-5
5 x 10^-6
A=10^-6
→ 0
γ=1.38x10^4
γ=0.69x10^5
γ=1.38x10^5
γ=0.69x10^6
γ=1.38x10^6

Figure 5-1

Dividing both sides of Eq. (5-24) by A and expanding the logarithmic term into a power series, we have

$$\frac{A_1}{A} = -\frac{4}{\pi A}T_{GFO} + \frac{4}{\pi A^2\gamma}\left[\frac{A}{C_1} + \frac{A^3}{C_1^3} + \frac{A^5}{C_1^5} + \dots\right]$$

$$= -\frac{4}{\pi A}T_{GFO} + \frac{4}{\gamma\pi}\left[\frac{1}{C_1 A} + \frac{A}{C_1^3} + \frac{A^3}{C_1^5} + \dots\right] \qquad (5\text{-}32)$$

Taking the limit on both sides of Eq. (5-32) as $A \to 0$, and using the fact that

$$\lim_{A\to 0} C_1 = \frac{1}{\gamma T_{GFO}} \qquad (5\text{-}33)$$

we have

$$\lim_{A\to 0}\left(\frac{A_1}{A}\right) = \lim_{A\to 0}\left[-\frac{4}{\pi A}T_{GFO} + \frac{4}{\pi\gamma}\left(\frac{1}{C_1 A} + \frac{A}{C_1^3} + \frac{A^3}{C_1^5} + \dots\right)\right]$$

$$= \lim_{A\to 0}\left[-\frac{4}{\pi A}T_{GFO} + \frac{4}{\pi\gamma}\frac{\gamma T_{GFO}}{A} + \frac{4A\gamma^2 T_{GFO}{}^3}{\pi} + \dots\right]$$

Substituting Eq. (5-33) into the last equation we have

$$\lim_{A\to 1}\left(\frac{A_1}{A}\right) = \lim_{A\to 0}\left(\frac{4A}{\pi}[\gamma^2 T_{GFO}{}^3 + A^2\gamma^4 T_{GFO}{}^5 + \ldots]\right) = 0 \qquad (5\text{-}34)$$

Dividing both sides of Eq. (5-30) by A and taking the limit as A approaches zero, we have

$$\lim_{A\to 0}\left(\frac{B_1}{A}\right) = \lim_{A\to 0}\frac{2}{\gamma A^2}\left(\frac{C_1}{\sqrt{C_1^2 - A^2}} - 1\right)$$

$$= \lim_{A\to 0}\frac{2}{\gamma A^2}\left(\frac{1}{\sqrt{1-(A/C_1)^2}} - 1\right) \qquad (5\text{-}35)$$

Expanding $[1 - (A/C_1)^2]^{-1/2}$ into a power series, and using only the first two terms, Eq. (5-35) becomes

$$\lim_{A\to 0}\left(\frac{B_1}{A}\right) = \lim_{A\to 0}\frac{2}{\gamma A^2}\left(\frac{1}{2}\left(\frac{A}{C_1}\right)^2\right)$$

$$= \lim_{A\to 0}\frac{1}{\gamma C_1^2} = \gamma T_{GFO}^2 \qquad (5\text{-}36)$$

Thus,

$$\lim_{A\to 0} 1/N(A) = \lim_{A\to 0}\frac{1}{\frac{B_1}{A} - j\frac{A_1}{A}} = \frac{1}{\gamma T_{GFO}{}^2} \qquad (5\text{-}37)$$

As shown in Fig. 5-1, the gain-phase plot of $-1/N(A)$ as A approaches zero is a point which lies on the -180 deg line with a magnitude of $20 \log_{10}[1/\gamma T_{GFO}{}^2]$.

Asymptotic Behavior of -1/N(A) For Very Large Values of A

For very large values of A, the value of C_1 becomes

$$\lim_{A\to\infty} C_1 = \lim_{A\to\infty}\left[A + \frac{2}{\gamma T_{GFO}}\right] \tag{5-38}$$

Then

$$\lim_{A\to\infty} \frac{A_1}{A} = \lim_{A\to\infty} \left\{ -\frac{4}{\pi A} T_{GFO} + \frac{2}{\pi\gamma}\left(\frac{1}{C_1 A} + \frac{A}{C_1^3} + \frac{A^3}{C_1^5} + \ldots\right) \right\}$$

$$= \lim_{A\to\infty} \left(-\frac{4}{\pi A} T_{GFO}\right) = -0 \tag{5-39}$$

Similarly we can show that

$$\lim_{A\to\infty} \frac{B_1}{A} = +0 \tag{5-40}$$

Thus,

$$\lim_{A\to\infty}\left[-1/N(A)\right] = -j/0 = \infty\underline{\lfloor -270} \tag{5-41}$$

As shown in Figure 5-1, the gain-phase plots of -1/N(A) approach $\infty\underline{\lfloor -270^\circ}$ as $A \to \infty$ for all values of γ and T_{GFO}.

6. Computer Simulation of the Simplified LST System with the Analytical Torque Expressions

A computer simulation of the LST system is presented here to corroborate the results of the describing function analysis of the last chapter. Since the describing function analysis has been carried out with the analytical torque expressions for the CMG frictional nonlinearity, the simulation model of the nonlinearity also has the same characteristics. This model of the nonlinearity is implemented by using the expressions for T_{GF} in Eqs. (2-38) and (2-39) with initial conditions for θ_G and T_{GF} being redefined each time a sign change in $\dot{\theta}_G$ occurs.

The simplified LST system is represented by the block diagram of Figure 1-7. The linear transfer function which the nonlinear element NL sees is given by

$$G(s) = \frac{J_V s^2}{J_G J_V s^4 + J_V K_p s^3 + J_V K_I s^2 + K_I H K_1 s + K_I H K_0}$$

Two sets of numerical values are considered as follows:

	System 1	System 2
J_V	10^5	10^5
J_G	2.1	3.7
K_p	216.	280.
K_I	9700.	10000
H	600	200
K_1	1371.02	3000
K_0	5758.35	20000

The frequency-domain plots of G(s) for both systems are given in Figure 6-1 in db versus phase coordinates. Figure 6-1 also contains the -1/N curves of Figure 5-1 for $\gamma = 1.38 \times 10^5$, 1.38×10^6 and 1.38×10^7.

With $\gamma = 1.38 \times 10^7$, the -1/N curve intersects the G(s) curves of the two systems at two points each. Among these the stable points for sustained oscillations are the ones on the left at the higher frequencies. The approximate magnitudes and frequencies of the oscillations are 6×10^{-6} rad and 4.4 rad/sec, respectively, for system 1, and 2×10^{-5} rad and 5.6 rad/sec, respectively, for system 2. The curves in Figure 6.1 also show that for γ considerably smaller than 1.38×10^7, both systems will exhibit a stable response, although for certain values of γ system 2 will show sustained oscillations while system 1 is stable.

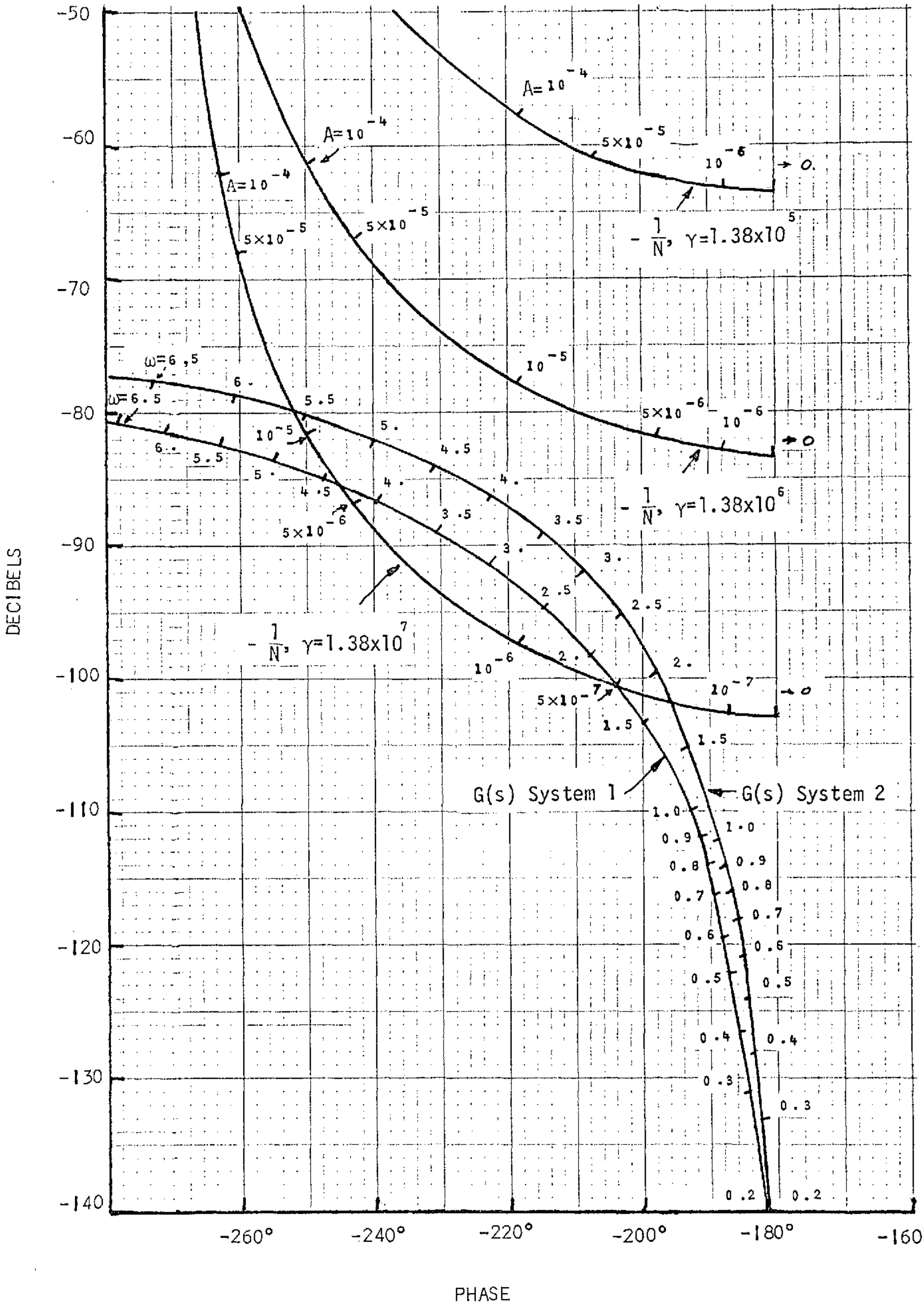
For the computer simulation, the input to the LST system, χ, is set to zero, along with all the initial states, except for the vehicle position θ_V. The initial value of θ_V is set at 5×10^{-5} rad, which is chosen so that the input signal to the nonlinearity, θ_G, would be large enough to cause the torque to saturate, while at the same time the limiting value of the input signal is not exceeded.

The following quantities are plotted from the simulation runs:

θ_V = vehicle position (rad)

ω_V = vehicle velocity (rad/sec)

θ_G = Gimbal position (rad)



DECIBELS
PHASE
-50
-60
-70
-80
-90
-100
-110
-120
-130
-140
-260°
-240°
-220°
-200°
-180°
-160°
$-\frac{1}{N}$, γ=1.38x10^5
$-\frac{1}{N}$, γ=1.38x10^6
$-\frac{1}{N}$, γ=1.38x10^7
G(s) System 1
G(s) System 2

Figure 6-1

ω_G = Gimbal velocity (rad/sec)

T_{GF} = Torque output of the nonlinearity (ft-lb)

Error = Error input command (rad/sec) to the CMG

$= \chi - K_0\theta_V - K_1\omega_V$

Figures 6-2 and 6-3 show the plots of the above listed quantities for system 1 with $\gamma = 1.38 \times 10^7$. It may be noted from the plot of T_{GF} in Figure 6-3 that the system has a sustained oscillation. This oscillation is not seen on the other plots because of the large initial transients. Figure 6-4 through 6-5 show the continuation of Figures 6-2 and 6-3 with proper scales. Figures 6-6 and 6-7 show the response plots for system 2 with $\gamma = 1.38 \times 10^7$, and Figures 6-8 and 6-9 show the continuation of these plots with proper scales. The frequencies and magnitudes of oscillations obtained with the two systems are quite close to the predicted values. The small discrepancy is attributed to the discretization of the nonlinearity implementation on the digital computer.

Figures 6-10 and 6-11 show the response plots for system 1 with $\gamma = 0.69 \times 10^7$ and their continuations are shown in Figures 6-12 and 6-13, respectively. Figures 6-14 and 6-15 show the response plots for system 1 with $\gamma = 1.38 \times 10^5$. As predicted, the system is stable for both of the lower γ values.

System 1 $\gamma = 1.38 \times 10^7$

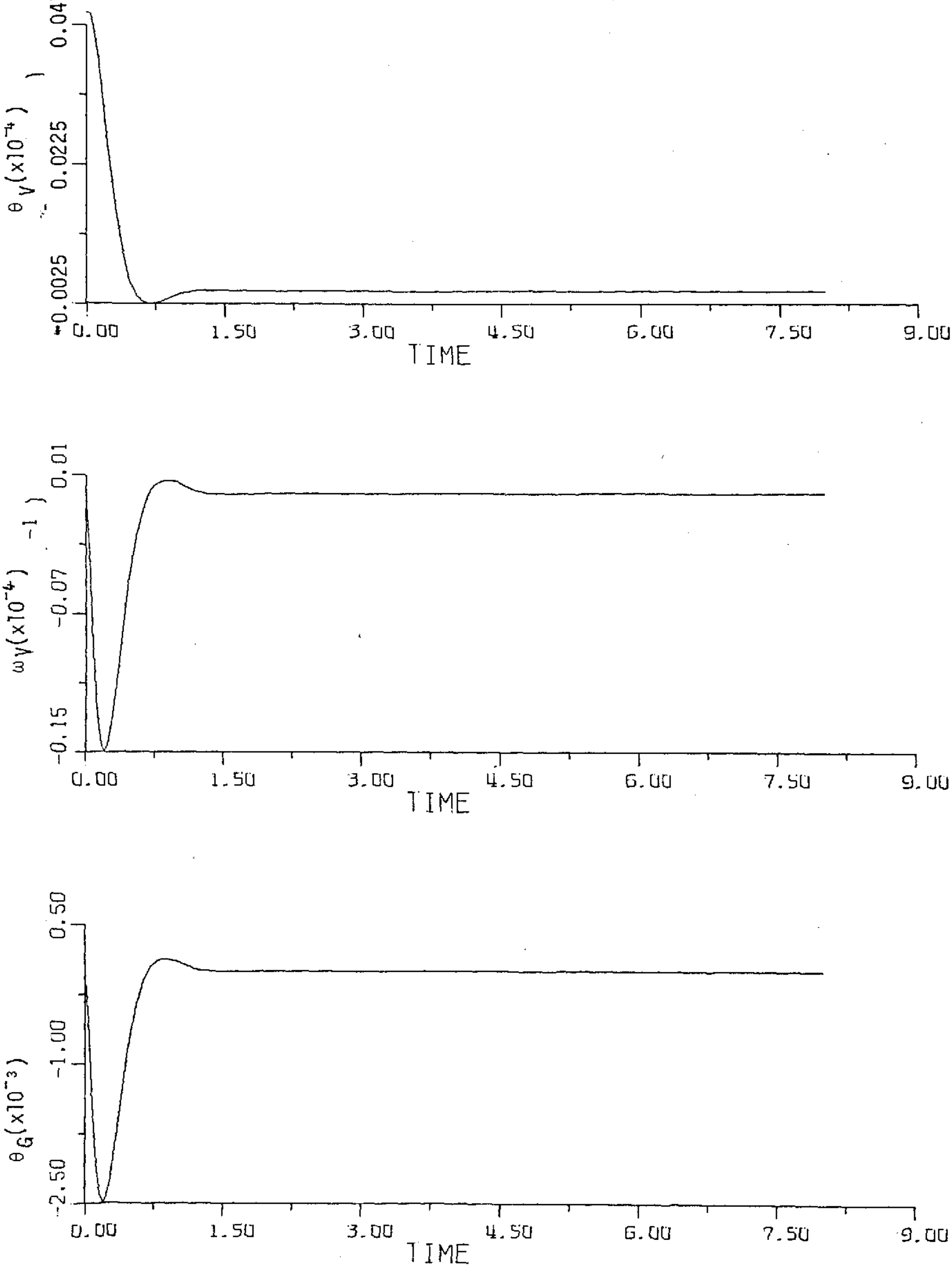


Figure 6-2

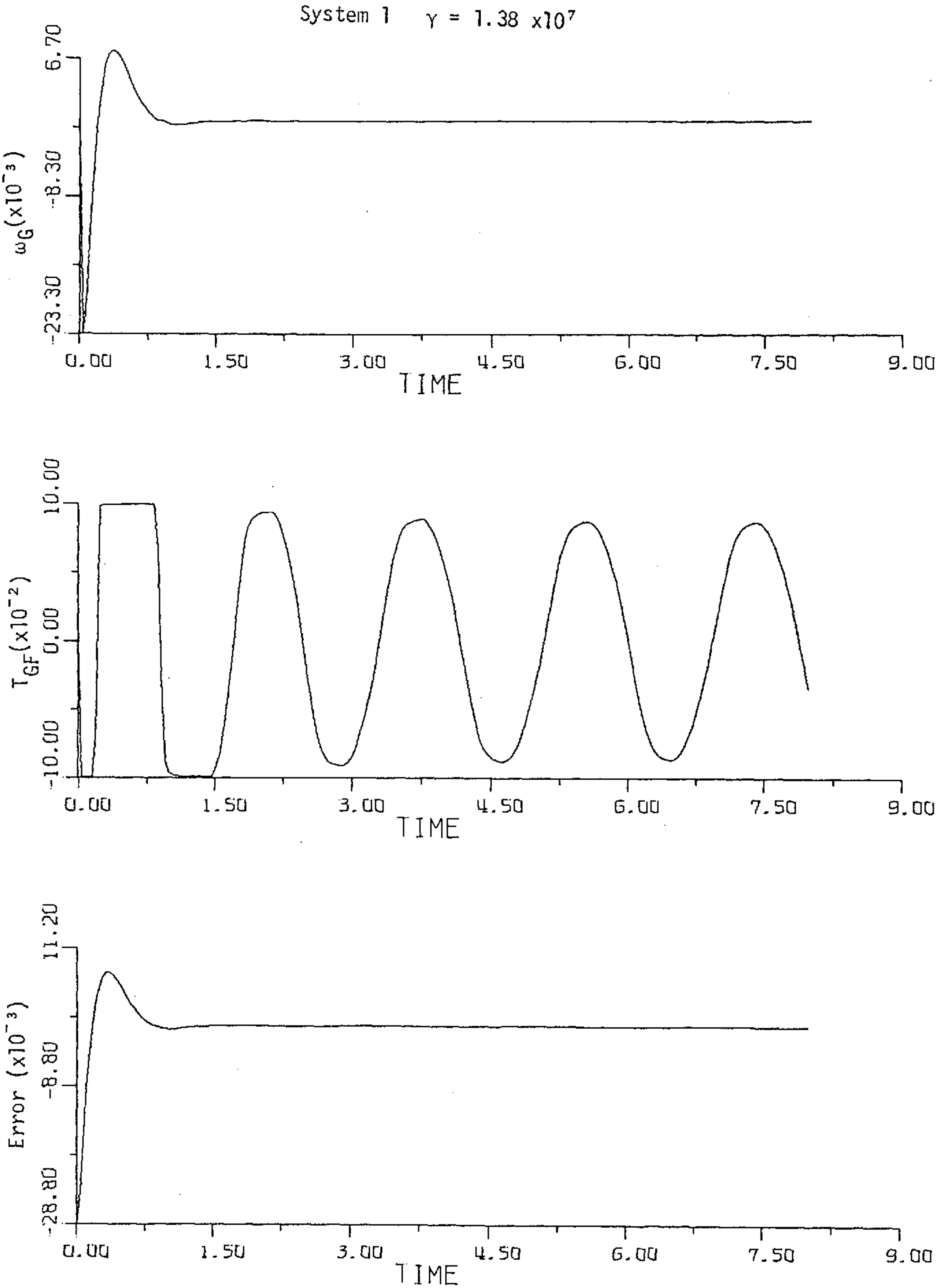

System 1 γ = 1.38 x10⁷
6.70
-8.30
-23.30
ωG(x10⁻³)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
10.00
0.00
-10.00
TGF(x10⁻²)
TIME
11.20
-8.80
-28.80
Error (x10⁻³)
TIME

Figure 6-3

System 1 $\gamma = 1.38 \times 10^{7}$

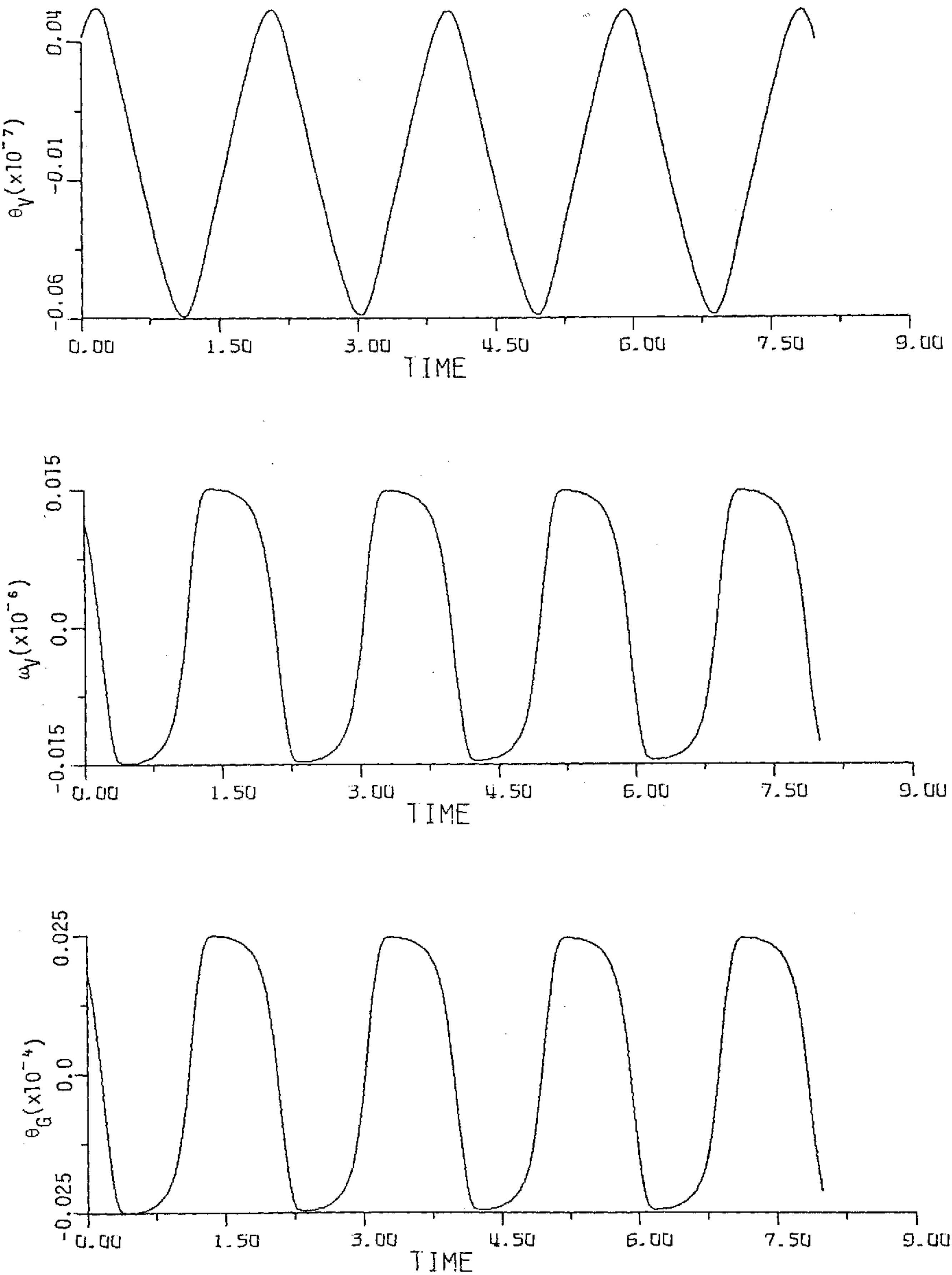


Figure 6-4

System 1 $\gamma = 1.38 \times 10^{7}$

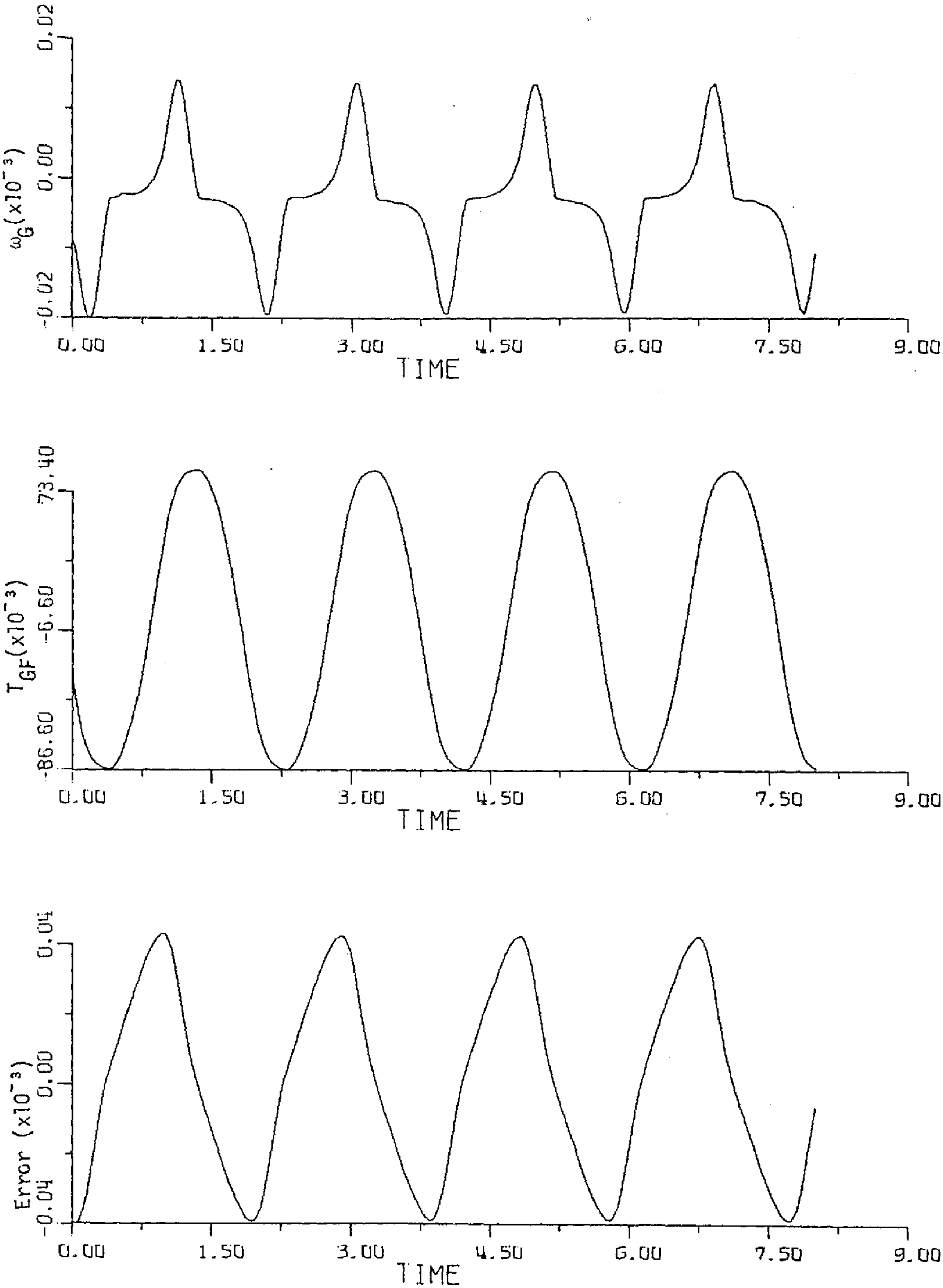


Figure 6-5

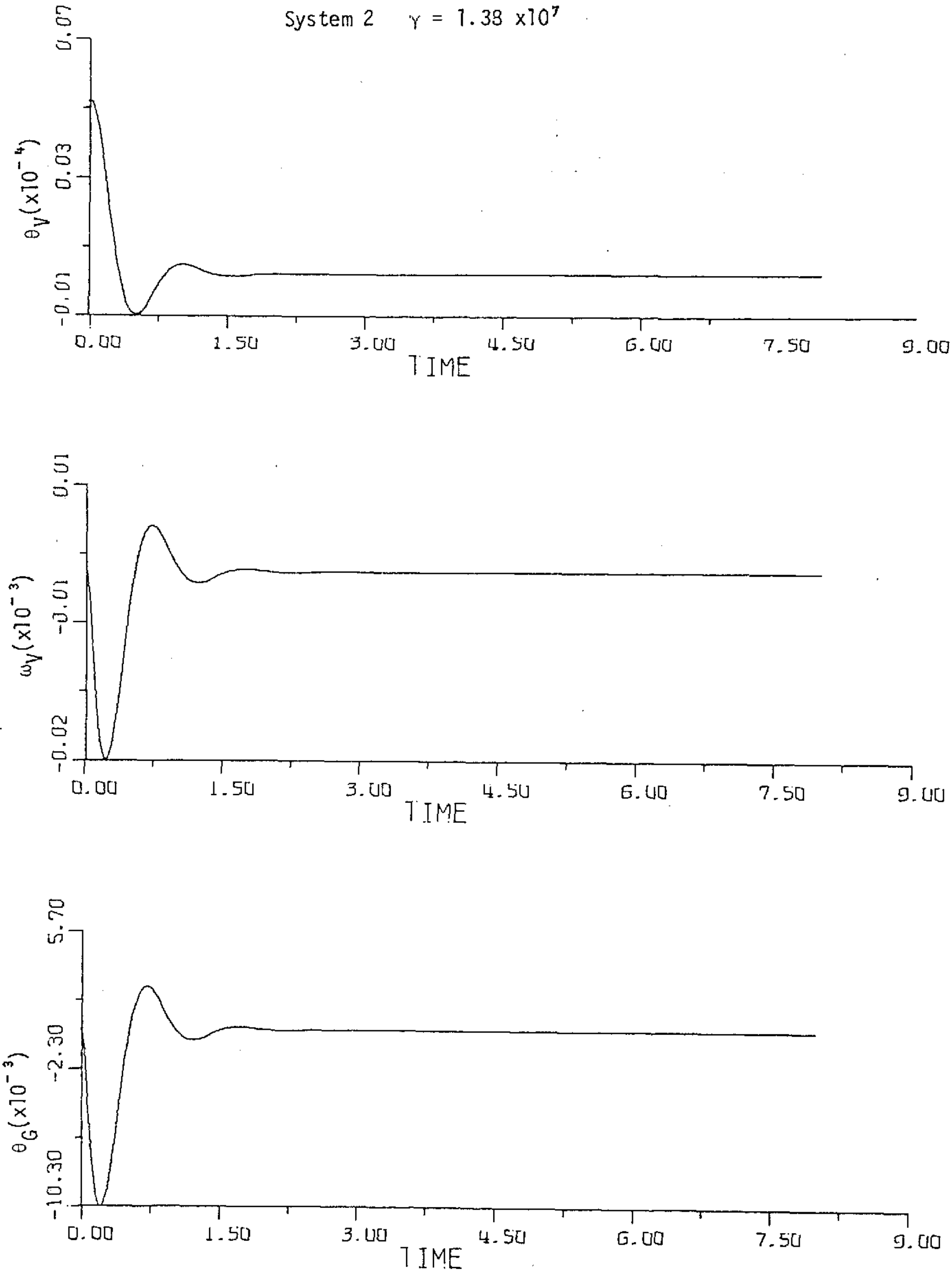

System 2 γ = 1.38 x10^7
0.07
0.03
-0.01
θy(x10^-4)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
0.01
-0.01
-0.02
ωy(x10^-3)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
5.70
-2.30
-10.30
θG(x10^-3)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 6-6

System 2 $\gamma = 1.38 \times 10^7$

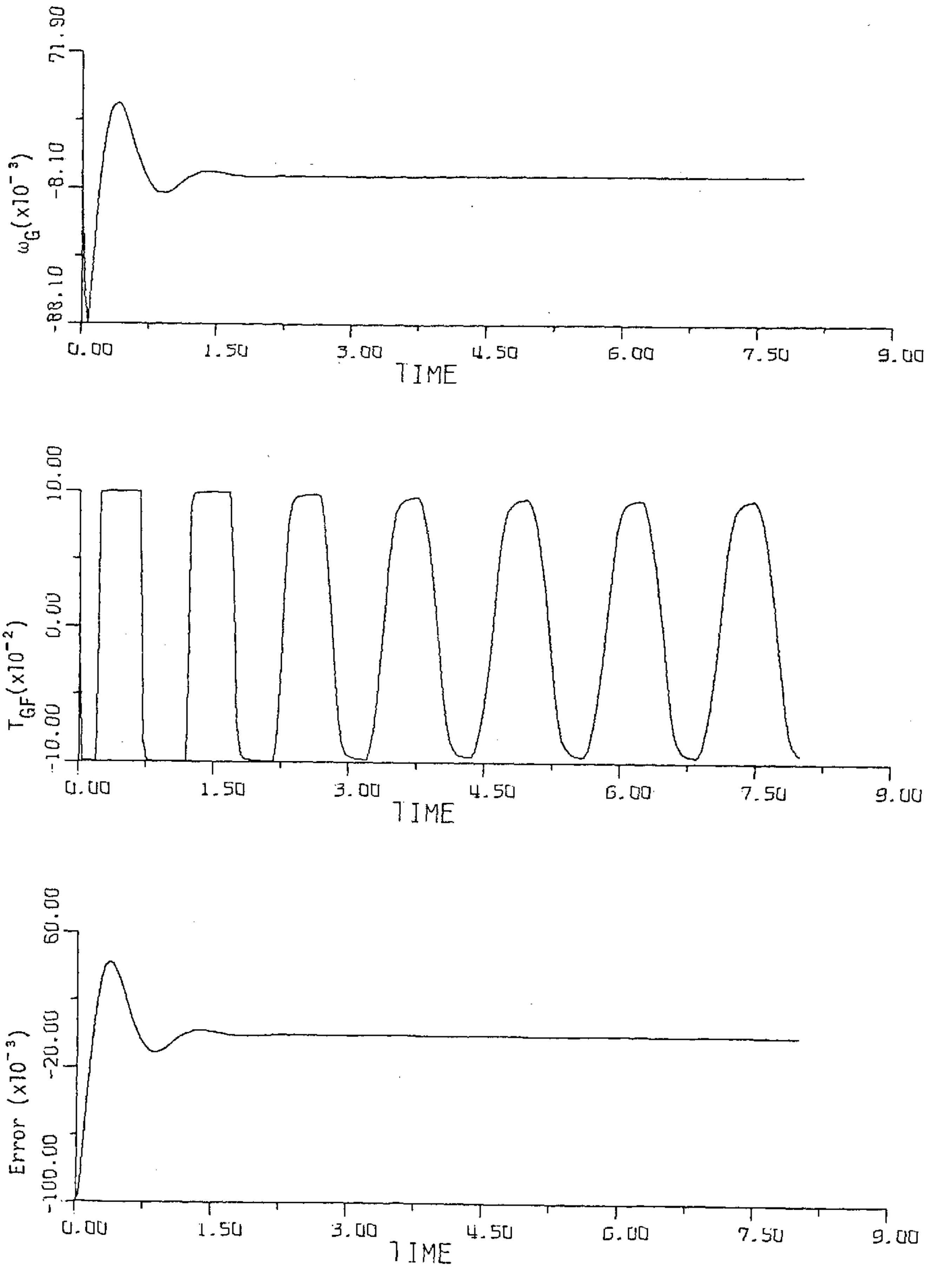


Figure 6-7

System 2 $\gamma = 1.38 \times 10^7$

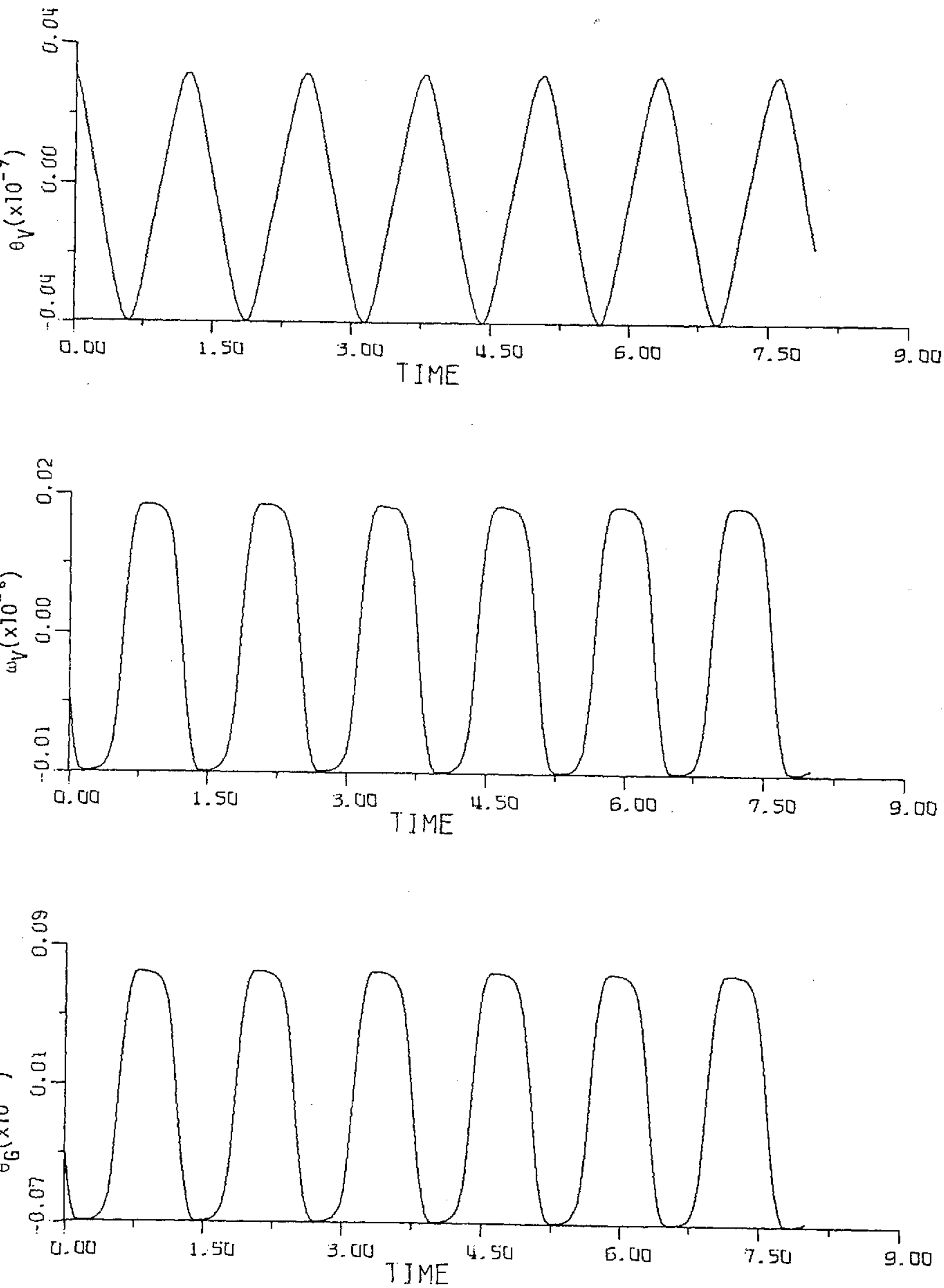


Figure 6-8

System 2 $\gamma = 1.38 \times 10^7$

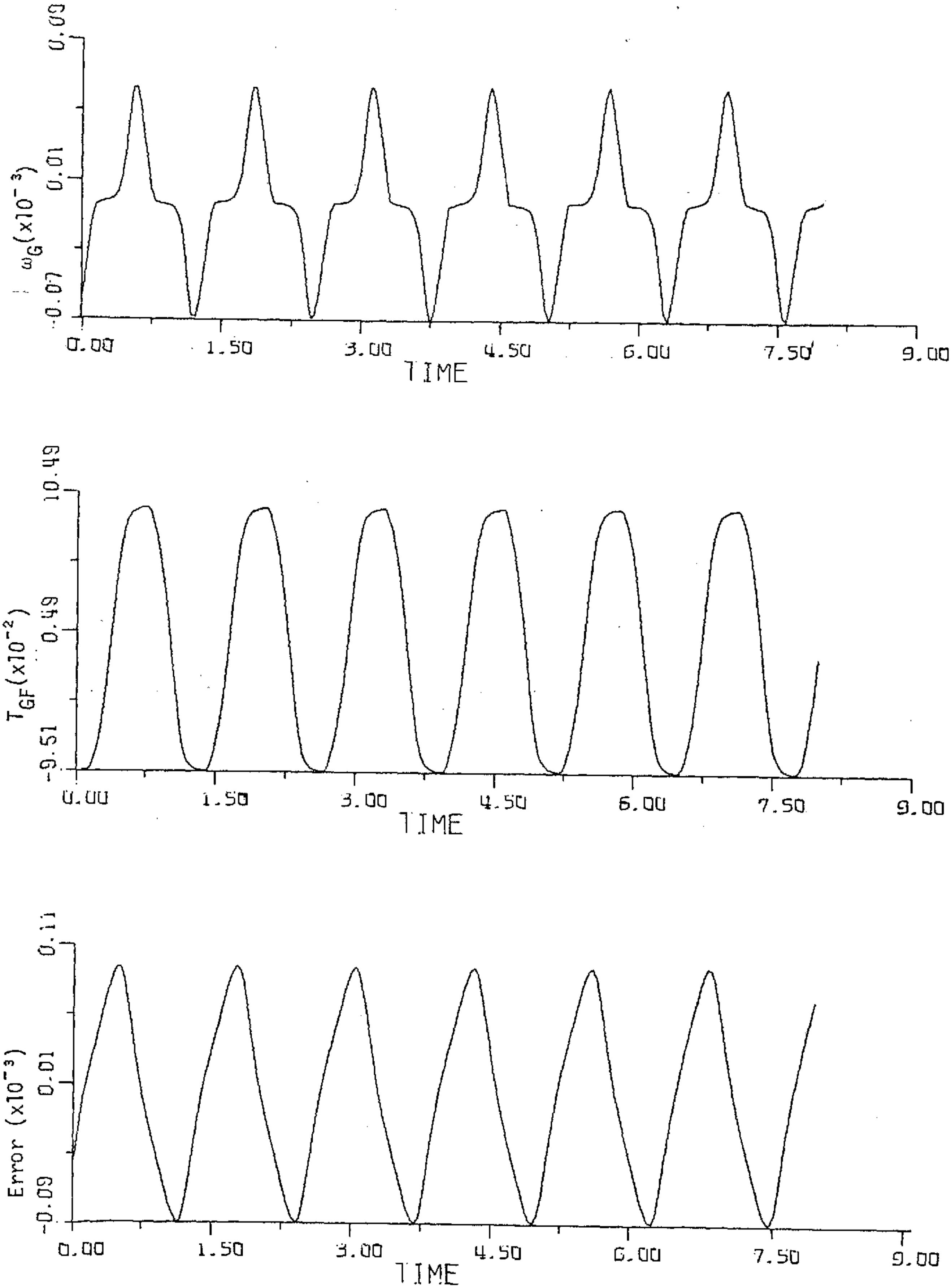
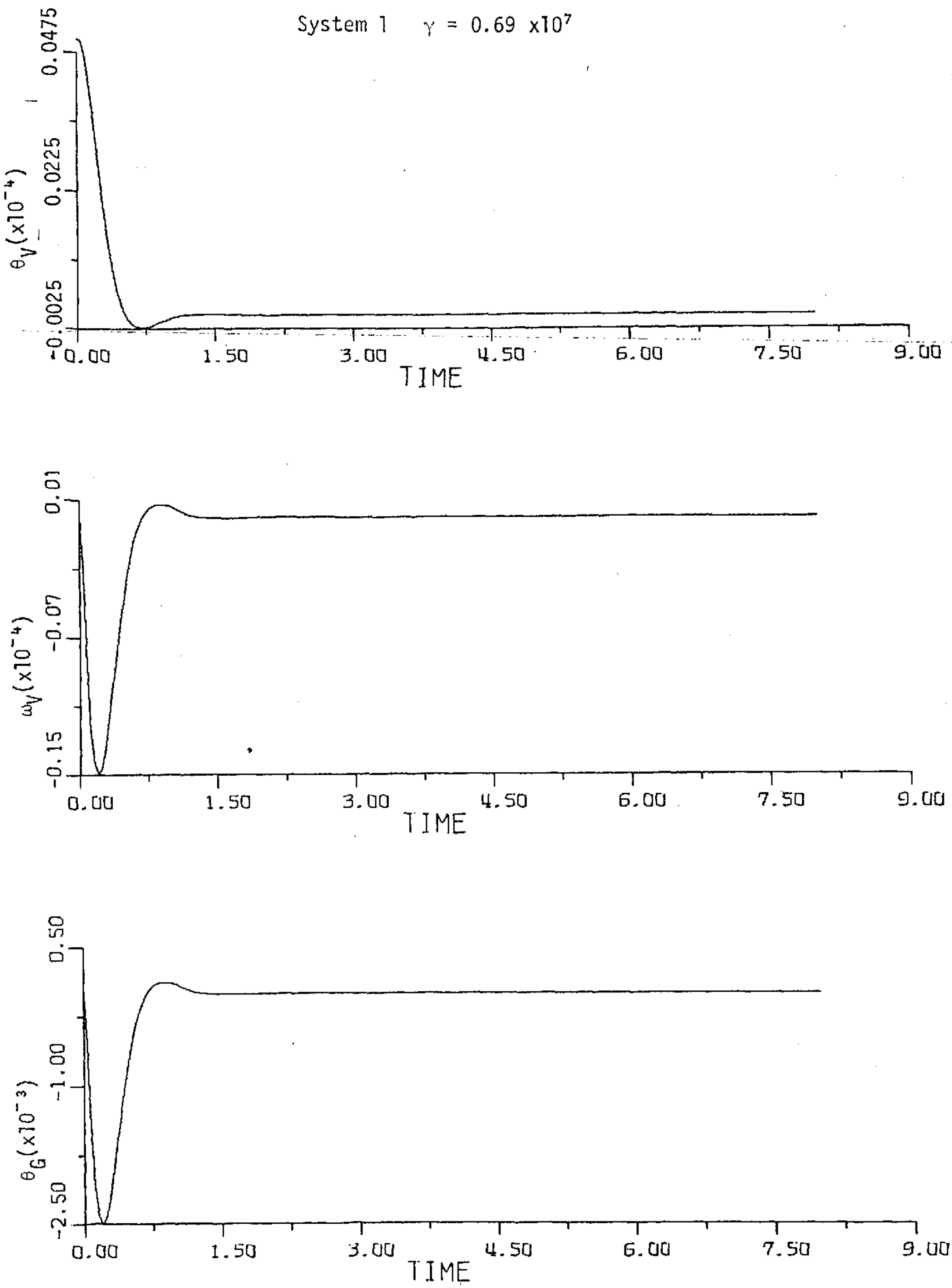


Figure 6-9



System 1 γ = 0.69 x10⁷
θy(x10⁻⁴)
0.0475
0.0225
-0.0025
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
ωy(x10⁻⁴)
0.01
-0.07
-0.15
TIME
θG(x10⁻³)
0.50
-1.00
-2.50
TIME

Figure 6-10

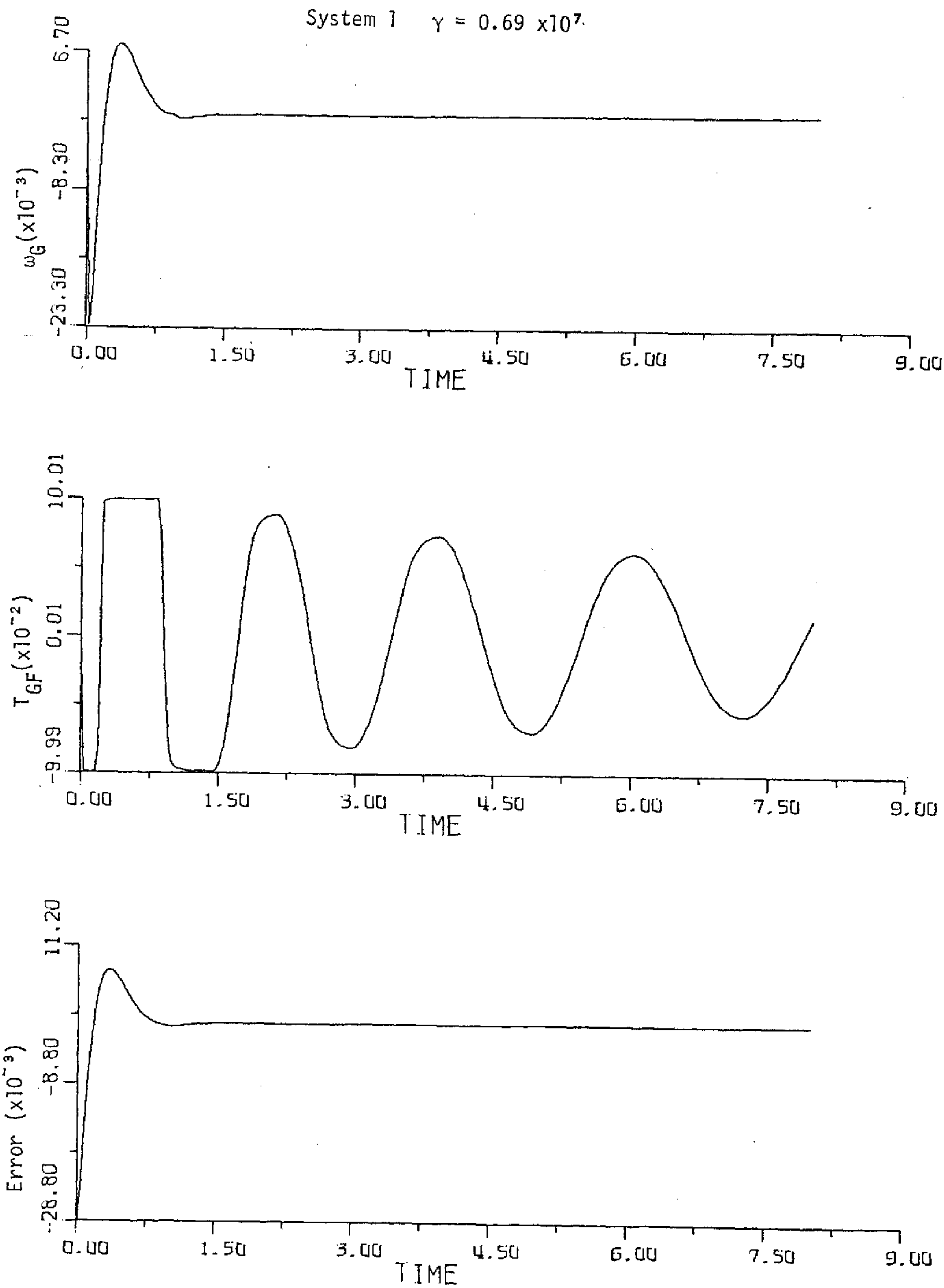

System 1 γ = 0.69 x10⁷
6.70
-8.30
-23.30
$\omega_G(x10^{-3})$
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
10.01
0.01
-9.99
$T_{GF}(x10^{-2})$
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
11.20
-8.80
-28.80
Error (x10⁻³)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 6-11

System 1 $\gamma = 0.69 \times 10^7$

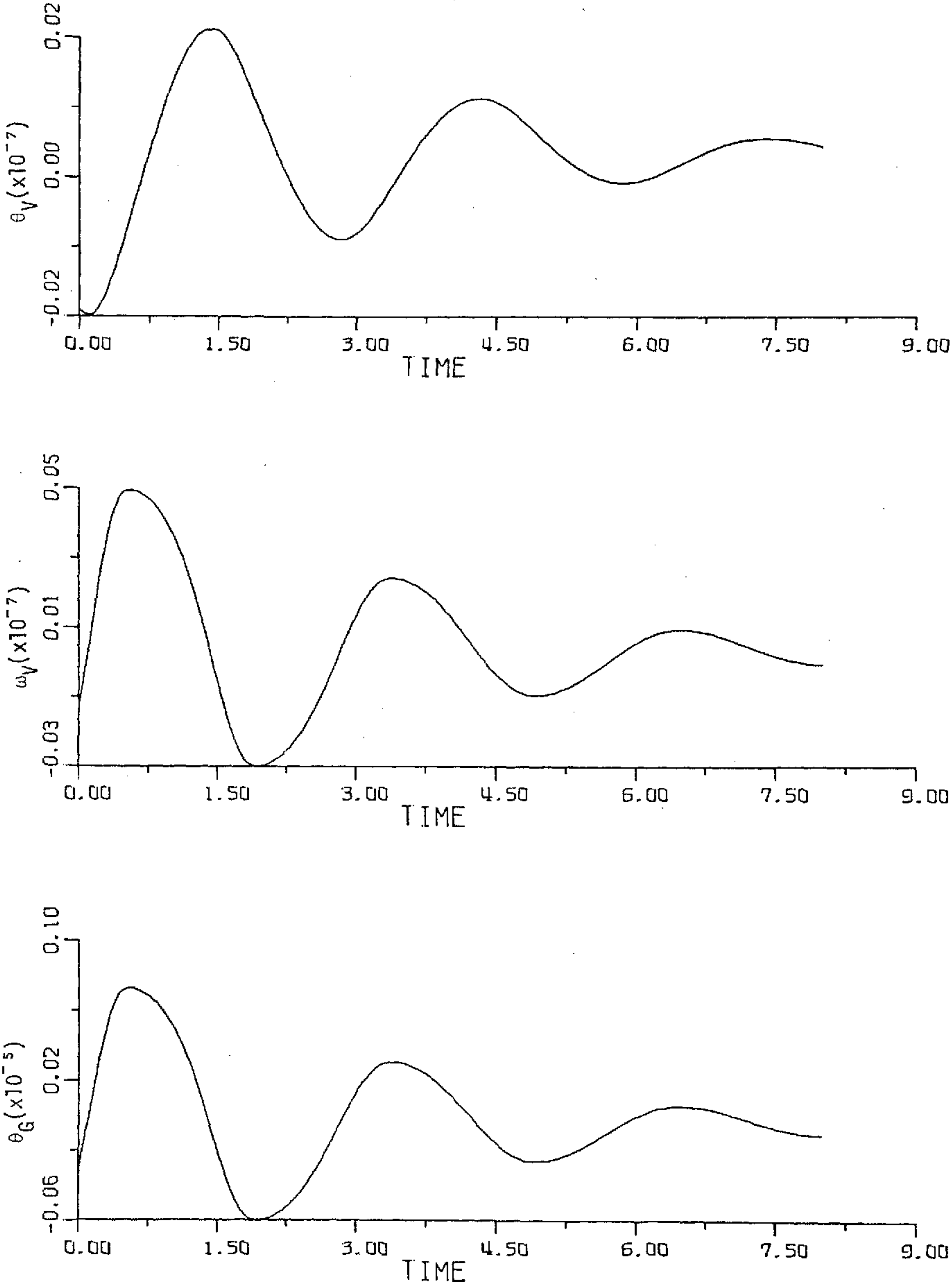
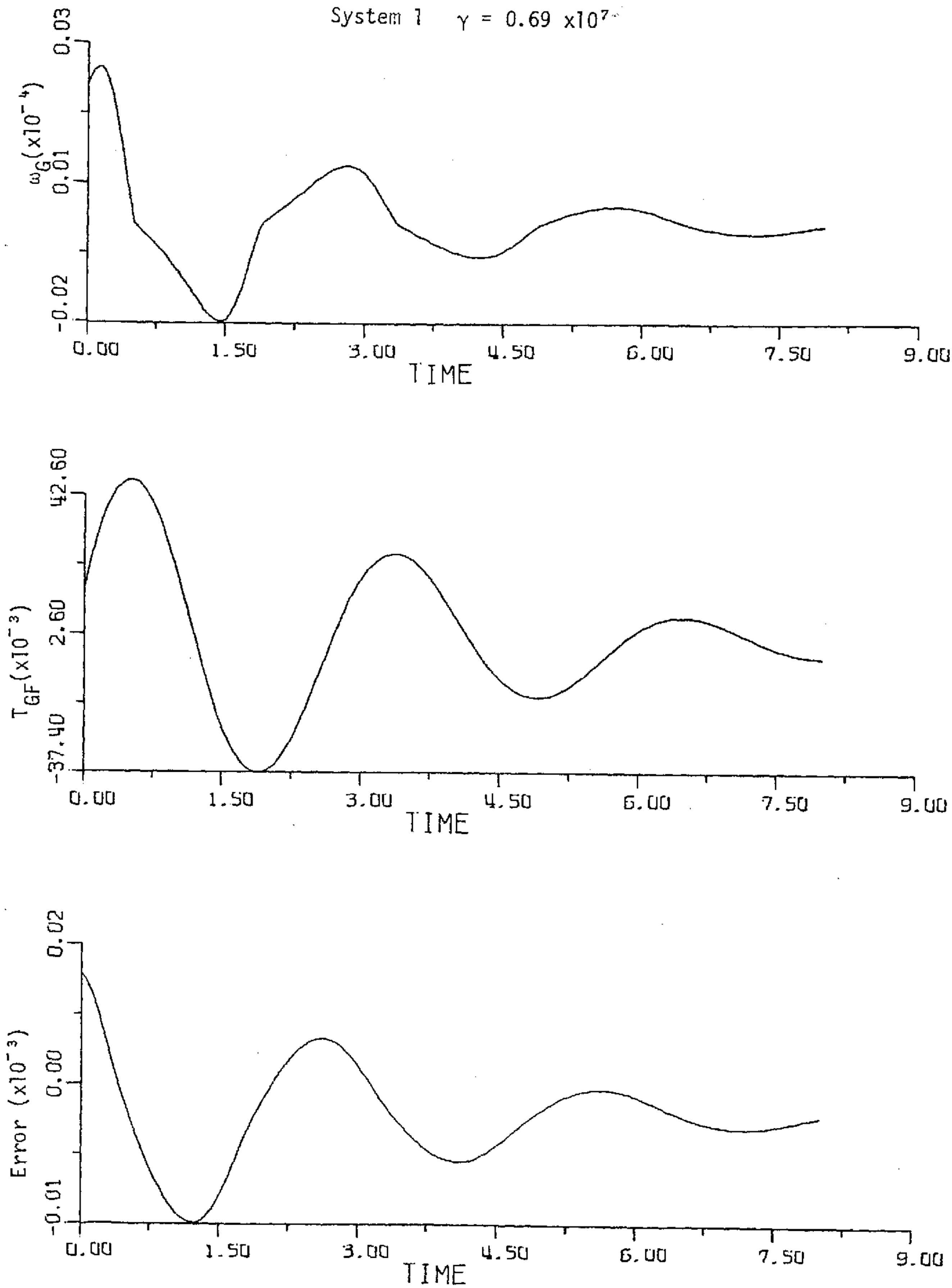


Figure 6-12



System 1 γ = 0.69 x10^7
ω_G(x10^-4)
0.03
0.01
-0.02
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
T_GF(x10^-3)
42.60
2.60
-37.40
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
Error (x10^-3)
0.02
0.00
-0.01
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 6-13

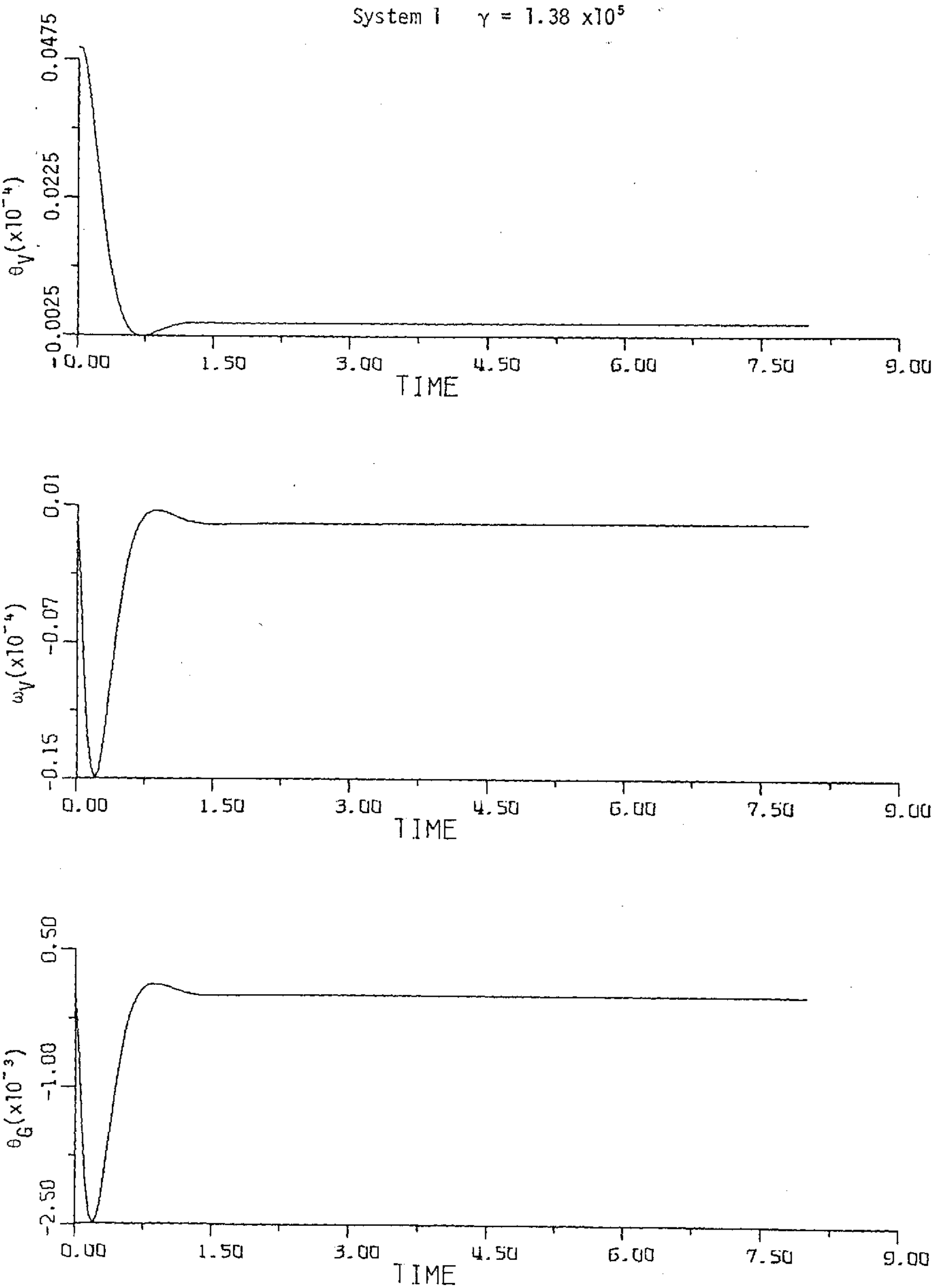

System 1 $\gamma = 1.38 \times 10^5$
$\theta_y (\times 10^{-4})$
0.0475
0.0225
0.0025
-0.0025
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
$\omega_y (\times 10^{-4})$
0.01
-0.07
-0.15
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
$\theta_G (\times 10^{-3})$
0.50
-1.00
-2.50
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 6-14

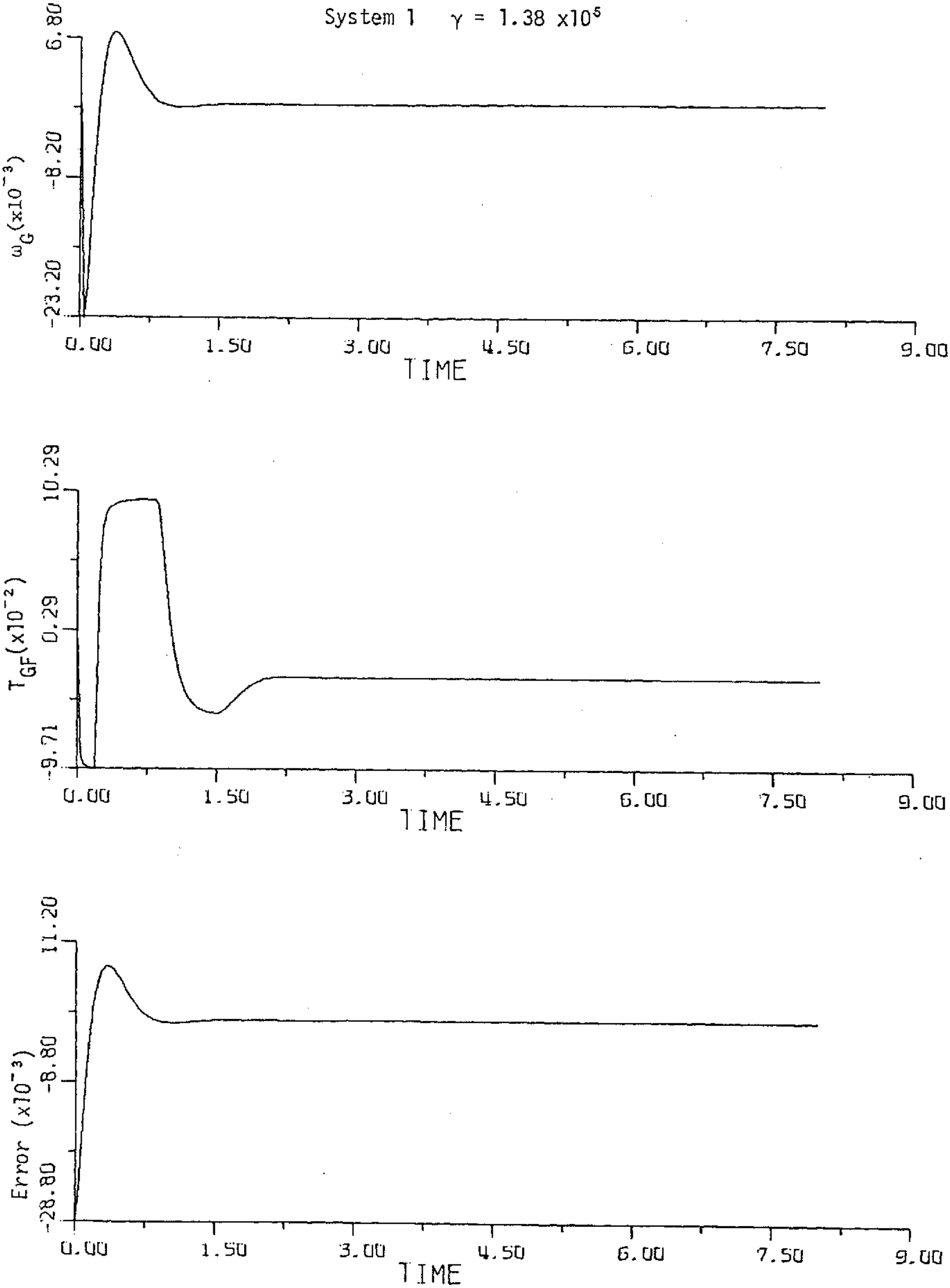

System 1 γ = 1.38 x10⁵
6.80
-8.20
-23.20
ω_G(x10⁻³)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
10.29
0.29
-9.71
T_{GF}(x10⁻²)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
11.20
-8.80
-28.80
Error (x10⁻³)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 6-15

7. Transfer Functions of the Sampled-Data LST System

Since the actual LST system has sample-and-hold between the vehicle controller and the CMG controller, the system should be modelled as a sampled-data control system. Figure 7-1 shows the block diagram of the simplified LST system with sampled data. Since it is necessary to isolate the CMG nonlinearity from the linear dynamics for analytical purposes, a sample-and-hold is inserted in front of the nonlinearity as an approximation.

Referring to Figure 7-1, the following equations are written using e^*, $\theta_G{}^*$ and $\theta_V{}^*$ as outputs.

$$e^* = (G_1\chi)^* + N^*\left[\frac{G_{ho}G_1G_6G_7}{\Delta_0}\right]^* \theta_G{}^* - \left[\frac{G_{ho}G_1G_2G_3G_6G_7}{\Delta_0}\right]^* e^* \qquad (7\text{-}1)$$

$$\theta_G{}^* = -N^*\left[\frac{G_{ho}G_6}{s\Delta_0}\right]^*\theta_G{}^* + \left[\frac{G_{ho}G_2G_3G_6}{s\Delta_0}\right]^* e^* \qquad (7\text{-}2)$$

$$\theta_V{}^* = \left[\frac{G_{ho}G_2G_3G_6G_7}{\Delta_0}\right]^* e^* - N^*\left[\frac{G_{ho}G_6G_7}{\Delta_0}\right]^* \theta_G{}^* \qquad (7\text{-}3)$$

where the symbol * denotes the z-transform operation, and N^* represents the discrete describing function of the CMG frictional nonlinearity;

$$\Delta_0 = 1 + G_3G_6 \qquad (7\text{-}4)$$

Equations (7-1) through (7-3) are portrayed by the sampled signal flow graph of Figure 7-2. Applying Mason's gain formula to this flow graph yields the determinant of the graph as

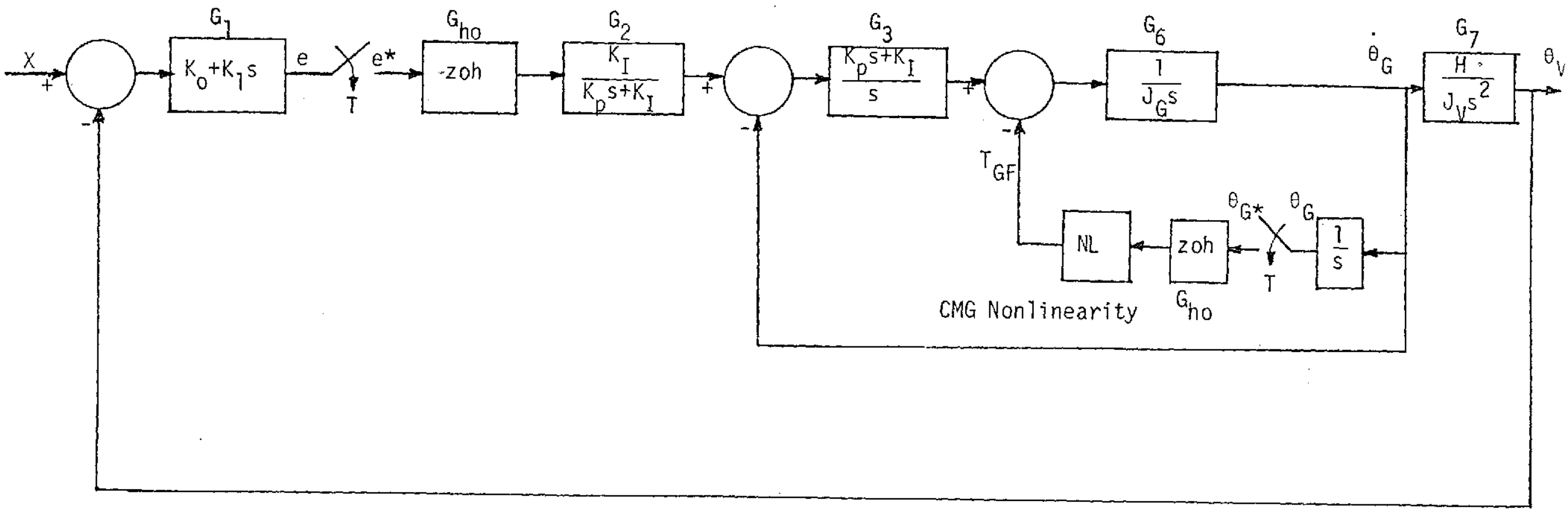


Figure 7-1. A block diagram of the simplified LST control system with sampled data

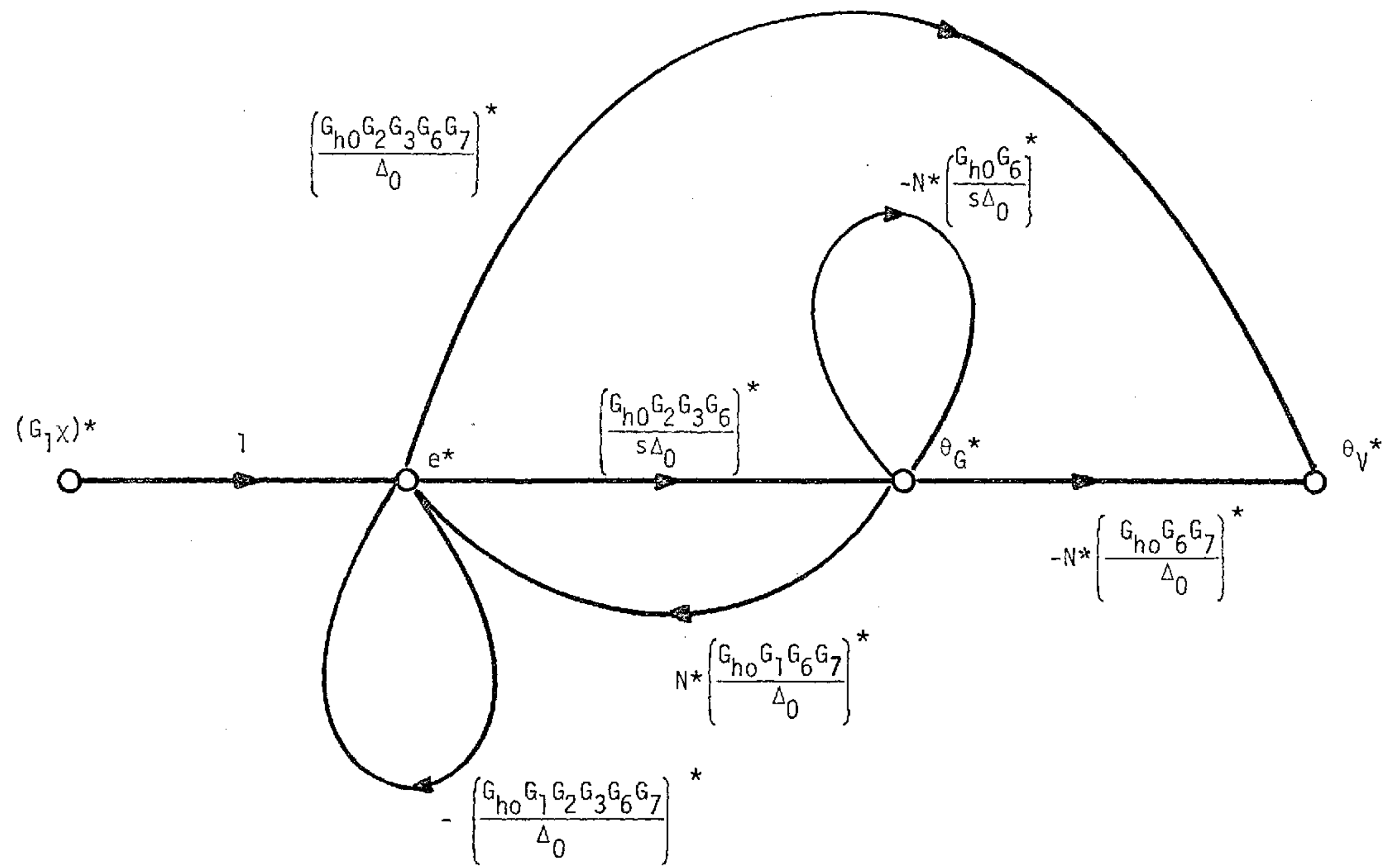


Figure 7-2. The Sampled Signal Flow Graph for the Equivalent System.

$$\Delta = 1 + N^*\left\{\left[\frac{G_{ho}G_6}{s\Delta_0}\right]^* - \left[\frac{G_{ho}G_6G_7G_1}{\Delta_0}\right]^*\left[\frac{G_{ho}G_2G_3G_6}{s\Delta_0}\right]^*\right\}$$

$$+ N^*\left[\frac{G_{ho}G_1G_2G_3G_6G_7}{\Delta_0}\right]^*\left[\frac{G_{ho}G_6}{s\Delta_0}\right]^* + \left[\frac{G_{ho}G_1G_2G_3G_6G_7}{\Delta_0}\right]^* \quad (7\text{-}5)$$

The last equation is put into the form of (with $N(z) = N^*$)

$$1 + N(z)G(z) = 0 \quad (7\text{-}6)$$

where

$$G(z) = \frac{A_1(z)-A_2(z)A_3(z)+A_4(z)A_1(z)}{1+A_4(z)} \quad (7\text{-}7)$$

$$A_1(z) = (1-z^{-1})\mathfrak{Z}\left\{\frac{G_6}{s^2\Delta_0}\right\} \quad (7\text{-}8)$$

$$A_2(z) = (1-z^{-1})\mathfrak{Z}\left\{\frac{G_6G_7G_1}{s\Delta_0}\right\} \quad (7\text{-}9)$$

$$A_3(z) = (1-z^{-1})\mathfrak{Z}\left\{\frac{G_2G_3G_6}{s^2\Delta_0}\right\} \quad (7\text{-}10)$$

$$A_4(z) = (1-z^{-1})\mathfrak{Z}\left\{\frac{G_1G_2G_3G_6G_7}{s\Delta_0}\right\} \quad (7\text{-}11)$$

Substitution of the system transfer functions into the above expressions yields

$$\Delta_0 = \frac{J_Gs^2+K_ps+K_I}{J_Gs^2} \quad (7\text{-}12)$$

$$A_1(z) = (1-z^{-1})\mathfrak{Z}\left\{\frac{1}{J_Gs^3+K_ps^2+K_Is}\right\} \quad (7\text{-}13)$$

$$A_2(z) = (1-z^{-1})\mathfrak{Z}\left\{\frac{H(K_0+K_1s)}{J_Vs^2(J_Gs^2+K_ps+K_I)}\right\} \quad (7\text{-}14)$$

$$A_3(z) = (1-z^{-1}) \mathcal{Z} \left[\frac{K_I}{s^2(J_G s^2 + K_p s + K_I)} \right] \tag{7-15}$$

$$A_4(z) = (1-z^{-1}) \mathcal{Z} \left[\frac{K_I H(K_0 + K_1 s)}{J_v s^3 (J_G s^2 + K_p s + K_I)} \right] \tag{7-16}$$

The following system parameters are used for System 1:

$H = 600$ ft-lb-sec

$J_G = 2.1$ ft-lb-sec^2

$K_0 = 5.75835 \times 10^3$

$K_1 = 1.37102 \times 10^3$

$K_p = 216$ ft-lb/rad/sec

$K_I = 9700$ ft-lb/rad

$J_v = 10^5$ ft-lb-sec^2

Taking the z-transforms of the functions inside the brackets in Eqs. (7-13) through (7-16), we have the following results:

$$A_1(z) = A_{13} + A_{14} \frac{z-1}{z-e^{aT}} + A_{15} \frac{z-1}{z-e^{\bar{a}T}} \tag{7-17}$$

$a = -51.429993 - j44.42923$

$\bar{a} = -51.429993 + j44.42923$

$A_{13} = 1.0309376 \times 10^{-4}$

$A_{14} = -5.15469 \times 10^{-5} - j5.9669183 \times 10^{-5}$

$A_{15} = -5.15469 \times 10^{-5} + j5.9669183 \times 10^{-5}$

$$A_2(z) = A_{22} \frac{T}{z-1} + A_{23} + A_{24} \frac{z-1}{z-e^{aT}} + A_{25} \frac{z-1}{z-e^{\bar{a}T}} \tag{7-18}$$

$A_{22} = 3.5616776 \times 10^{-3}$

$A_{23} = 7.6870434 \times 10^{-4}$

$A_{24} = -3.8435217 \times 10^{-4} - j4.8499764 \times 10^{-4}$

$A_{25} - 3.8435217 \times 10^{-4} + j4.8499764 \times 10^{-4}$

$$A_3(z) = A_{32} \frac{T}{z-1} + A_{33} + A_{34} \frac{z-1}{z-e^{aT}} + A_{35} \frac{z-1}{z-e^{aT}} \qquad (7\text{-}19)$$

$A_{32} = 1.0$

$A_{33} = -2.2268891 \times 10^{-2}$

$A_{34} = 1.1134446 \times 10^{-2} + j1.6350537 \times 10^{-3}$

$A_{35} = 1.1134446 \times 10^{-2} - j1.6350537 \times 10^{-3}$

$$A_4(z) = A_{41} \frac{T^2(z+1)}{2(z-1)^2} + A_{42} \frac{T}{z-1} + A_{43} + A_{44} \frac{z-1}{z-e^{aT}} + A_{45} \frac{z-1}{z-e^{aT}} \qquad (7\text{-}20)$$

$A_{41} = 34.54837$

$A_{42} = 7.4564409$

$A_{43} = -0.17352653$

$A_{44} = 8.6763263 \times 10^{-2} + j1.6520832 \times 10^{-2}$

$A_{45} = 8.6763263 \times 10^{-2} - j1.6520832 \times 10^{-2}$

The following system parameters are used for System 2. The same expressions for $A_1(z)$, $A_2(z)$, $A_3(z)$ and $A_4(z)$ are preserved.

$H = 200$ ft-lb-sec

$J_G = 3.7$ ft-lb-sec^2

$K_0 = 2 \times 10^4$

$K_1 = 3 \times 10^3$

$K_p = 280$ ft-lb/rad/sec

$K_I = 10^4$ ft-lb/rad

$J_V = 10^5$ ft-lb-sec^2

The corresponding coefficients in Eqs. (7-17) through (7-20) are

$$a = -37.83783 - j35.651077$$
$$\bar{a} = -37.83783 + j35.651077$$
$$A_{13} = 9.9999976 \times 10^{-5}$$
$$A_{14} = -4.9999973 \times 10^{-5} - j5.3066848 \times 10^{-5}$$
$$A_{15} = -4.9999973 \times 10^{-5} + j5.3066848 \times 10^{-5}$$
$$A_{22} = 3.9999932 \times 10^{-3}$$
$$A_{23} = 4.8799929 \times 10^{-4}$$
$$A_{24} = -2.43997 \times 10^{-4} - j3.150655 \times 10^{-4}$$
$$A_{25} = -2.43997 \times 10^{-4} + j3.150655 \times 10^{-4}$$
$$A_{32} = 9.9999982 \times 10^{-1}$$
$$A_{33} = -2.8 \times 10^{-2}$$
$$A_{34} = 1.4000002 \times 10^{-2} + j8.3391555 \times 10^{-4}$$
$$A_{35} = 1.4000002 \times 10^{-2} - j8.3391555 \times 10^{-4}$$
$$A_{41} = 39.999985$$
$$A_{42} = 4.8800011$$
$$A_{43} = -1.5144014 \times 10^{-1}$$
$$A_{44} = 7.5720072 \times 10^{-2} + j1.1923421 \times 10^{-2}$$
$$A_{45} = 7.5720072 \times 10^{-2} - j1.1923421 \times 10^{-2}$$

It can be shown that if T approaches zero, the z-transfer function G(z) in Eq. (7-7) reverts to that of the continuous transfer function G(s) of Eq. (1-16).

8. The Discrete Describing Function of the CMG Frictional Nonlinearity

In order to study the condition of self-sustained oscillations of the LST system with sampled data, it is necessary to evaluate the discrete describing function of the CMG frictional nonlinearity, N(z).

The first step in the derivation of N(z) involves the interchanging of the positions of the nonlinearity and the zero-order hold in Figure 8-1a. This step is justified since the nonlinearity is amplitude dependent only, so that the signal of T_{GF} is not affected by this interchange. Figure 8-1b illustrates the transposition between NL and zoh.

The second step involves the assumption that θ_G is sinusoidal; that is,

$$\theta_G(t) = A\cos(\omega t + \phi) \qquad (8\text{-}1)$$

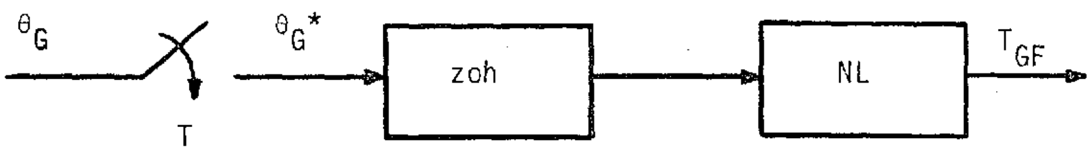
where A, ω, and φ denote the amplitude, the frequency in radians, and the phase in degrees of the sinusoid, respectively.

The z-transform of $\theta_G(t)$ is

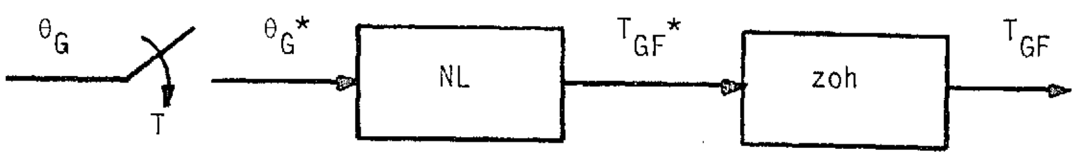
$$\theta_G(z) = \sum_{k=0}^{\infty} A\cos\left(\frac{2\pi k}{N} + \phi\right)z^{-k} \qquad (8\text{-}2)$$

or in closed form,

$$\theta_G(z) = \frac{Az\left[\left(z - \cos\frac{2\pi}{N}\right)\cos\phi - \sin\frac{2\pi}{N}\sin\phi\right]}{z^2 - 2z\cos\frac{2\pi}{N} + 1} \qquad (8\text{-}3)$$



(a)



(b)

Figure 8-1

An important consideration is that because of the periodic nature of the sampler, $\theta_G(t)$, $\theta_G*(t)$, and $T_{GF}*(t)$ are all periodic functions of period NT, where N is a positive integer ≥ 2. Therefore, $\omega = 2\pi/NT$, and $\omega T = 2\pi/N$.

The output of the nonlinearity in Figure 8-1b is denoted by $T_{GF}^*(t)$, and its z-transform is $T_{GF}(z)$. The discrete describing function (DDF) of the nonlinearity is defined as

$$N(z) = \frac{T_{GF}(z)}{\theta_G(z)} \tag{8-4}$$

It turns out that the discrete describing function (DDF) for $N = 2$ must be derived separately, and a general expression for $N(z)$ can be obtained for all $N \geq 3$.

<u>The DDF for N = 2</u>

Let $T_{GF}(kT)$ denote the value of $T_{GF}^*(t)$ at $t = kT$. For $N = 2$, the signal $T_{GF}^*(t)$ is a periodic function with a period of 2T. The z-transform of $T_{GF}^*(t)$ is written

$$T_{GF}(z) = T_{GF}(0)(1 + z^{-2} + z^{-4} + \ldots) + T_{GF}(T)(z^{-1} + z^{-3} + \ldots)$$

$$= \frac{T_{GF}(0)z^2 + T_{GF}(T)z}{z^2 - 1} \tag{8-5}$$

For the CMG frictional nonlinearity, it has been established in chapter 2 that

$$T_{GF}(t) = T_{GF0}\frac{\frac{R}{R+1} - \frac{a}{2}[1 - \cos(\omega t + \phi)]}{\frac{1}{R+1} + \frac{a}{2}[1 - \cos(\omega t + \phi)]} \qquad \dot{\theta}_G \leq 0 \tag{8-6}$$

$$T_{GF}(t) = T_{GFO}\frac{\frac{R}{R-1} + \frac{a}{2}[1 - \cos(\omega t + \phi)]}{\frac{1}{R-1} + \frac{a}{2}[1 - \cos(\omega t + \phi)]} \qquad \dot{\theta} \geq 0 \quad (8\text{-}7)$$

Let us introduce the following notation:

$$T_{GF}^{-}(kT) = T_{GF}(t)\Big|_{t=kT} \qquad \dot{\theta}_G \leq 0 \qquad (8\text{-}8)$$

$$T_{GF}^{+}(kT) = T_{GF}(t)\Big|_{t=kT} \qquad \dot{\theta}_G \geq 0 \qquad (8\text{-}9)$$

We have,

$$T_{GF}^{-}(kT) = T_{GFO}\frac{\frac{R}{R+1} - \frac{a}{2}[1 - \cos(\frac{2\pi k}{N} + \phi)]}{\frac{1}{R+1} + \frac{a}{2}[1 - \cos(\frac{2\pi k}{N} + \phi)]} \quad \dot{\theta}_G \leq 0 \quad (8\text{-}10)$$

$$T_{GF}^{+}(kT) = T_{GFO}\frac{\frac{R}{R-1} + \frac{a}{2}[1 - \cos(\frac{2\pi k}{N} + \phi)]}{\frac{1}{R-1} + \frac{a}{2}[1 - \cos(\frac{2\pi k}{N} + \phi)]} \quad \dot{\theta}_G \geq 0 \quad (8\text{-}11)$$

For N = 2,

$$\begin{aligned} T_{GF}(0) &= T_{GF}^{-}(0) & 0 \leq \phi < \pi \\ &= T_{GF}^{+}(0) & \pi \leq \phi < 2\pi \\ T_{GF}(T) &= T_{GF}^{+}(T) & \pi \leq \phi < 2\pi \\ &= T_{GF}^{-}(T) & 0 \leq \phi < \pi \end{aligned} \qquad (8\text{-}12)$$

Substituting N = 2 into Eq. (8-3), we have

$$\theta_G(z) = \frac{Az\cos\phi}{z + 1} \tag{8-13}$$

Using Eqs. (8-5) and (8-13), the DDF for N = 2 is determined,

$$N(z) = \frac{T_{GF}(0)z + T_{GF}(T)}{A(z - 1)\cos\phi} \tag{8-14}$$

Also, for N = 2, z = -1, the last equation becomes

$$N(z) = \frac{T_{GF}(0) - T_{GF}(T)}{2A\cos\phi} \tag{8-15}$$

For stability analysis, we define

$$F(z) = -\frac{1}{N(z)} = \frac{2A\cos\phi}{T_{GF}(T) - T_{GF}(0)} \tag{8-16}$$

The DDF for N $\geq$ 3

In general, the z-transform of the output of the nonlinearity may be written as

$$T_{GF}(z) = \sum_{m=0}^{\infty} \sum_{k=0}^{N-1} T_{GF}(kT)z^{-k-mN}$$

$$= \frac{\sum_{k=0}^{N-1} T_{GF}(kT)z^{N-k}}{z^N - 1} \tag{8-17}$$

Using Eq. (8-2) for $\theta_G(z)$, the discrete describing function $N(z)$ is written

$$N(z) = \frac{T_{GF}(z)}{\theta_G(z)} = \frac{\sum_{k=0}^{N-1} T_{GF}(kT)z^{N-k}}{(z^N - 1)\sum_{k=0}^{\infty} A\cos(\frac{2\pi k}{N} + \phi)z^{-k}} \qquad (8\text{-}18)$$

The denominator of $N(z)$ may be simplified as follows:

$$(z^N - 1)\sum_{k=0}^{\infty} A\cos(\frac{2\pi k}{N} + \phi)z^{-k} = A\sum_{k=0}^{\infty} z^{N-k}\cos(\frac{2\pi k}{N} + \phi) - A\sum_{k=0}^{\infty} z^{-k}\cos(\frac{2\pi k}{N} + \phi)$$

$$= A\sum_{k=0}^{N-1}\cos(\frac{2\pi k}{N} + \phi)z^{N-k-1} \qquad (8\text{-}19)$$

Thus,

$$N(z) = \frac{\sum_{k=0}^{N-1} T_{GF}(kT)z^{N-k-1}}{A\sum_{k=0}^{N-1}\cos(\frac{2\pi k}{N} + \phi)z^{N-k-1}} \qquad (N \geq 3) \qquad (8\text{-}20)$$

As an alternative we may expand $z^N - 1$ as

$$z^N - 1 = \prod_{k=0}^{N-1}(z - e^{j2\pi k/N}) \qquad (8\text{-}21)$$

Then, using Eq. (8-3) for $\theta_G(z)$, $N(z)$ is written

$$N(z) = \frac{\sum_{k=0}^{N-1} T_{GF}(kT) z^{N-k}[z^2 - 2z\cos\frac{2\pi}{N} + 1]}{\prod_{k=0}^{N-1}(z - e^{j2\pi k/N}) Az[(z - \cos\frac{2\pi}{N})\cos\phi - \sin\frac{2\pi}{N}\sin\phi]} \qquad (8\text{-}22)$$

For N = 3, N(z) is simplified to

$$N(z) = \frac{T_{GF}(0)z^2 + T_{GF}(T)z + T_{GF}(2T)}{A(z-1)[(z+0.5)\cos\phi - 0.866\sin\phi]} \qquad (8\text{-}23)$$

For N > 3,

$$N(z) = \frac{\sum_{k=0}^{N-1} T_{GF}(kT) z^{N-k-1}}{A(z-1)\prod_{k=2}^{N-2}(z - e^{j2\pi k/N})[(z - \cos\frac{2\pi}{N})\cos\phi - \sin\frac{2\pi}{N}\sin\phi]} \qquad (8\text{-}24)$$

where in general,

$$\begin{aligned} T_{GF}(kT) &= T_{GF}^{-}(kT) \qquad 0 < \frac{2\pi k}{N} + \phi < \pi \\ &= T_{GF}^{+}(kT) \qquad \pi < \frac{2\pi k}{N} + \phi < 2\pi \end{aligned} \qquad (8\text{-}25)$$

where $0 \leq (2\pi k/N + \phi) \leq 2\pi$ must be satisfied by appropriate conversion of the angle $2\pi k/N + \phi$.

For stability studies the critical regions of $F(z) = -1/N(z)$ should be constructed for N = 2, 3, ..., with ϕ varied from 0^o to 360^o, and A from 0 to infinity.

The following theorems on the properties of $-1/N(z)$ are useful for simplifying the task of the construction of the critical regions.

Theorem 8-1

For any integral N, the magnitude and phase of $-1/N(z)$ repeat for every $\phi = 2\pi/N$ radians.

Proof: The negative inverse of the discrete describing function is written

$$F(z) = -\frac{1}{N(z)} = \frac{-A[(z - \cos\frac{2\pi}{N})\cos\phi - \sin\frac{2\pi}{N}\sin\phi](z^N - 1)}{\left\{\sum_{k=0}^{N-1} T_{GF}(kT)z^{N-k-1}\right\}(z^2 - 2z\cos\frac{2\pi}{N} + 1)} \quad (8\text{-}26)$$

Let

$$F(z) = \frac{F_1(z)F_2(z)}{F_3(z)F_4(z)} \quad (8\text{-}27)$$

where

$$F_1(z) = -A[(z - \cos\frac{2\pi}{N})\cos\phi - \sin\frac{2\pi}{N}\sin\phi] \quad (8\text{-}28)$$

$$F_2(z) = z^N - 1 \quad (8\text{-}29)$$

$$F_3(z) = \sum_{k=0}^{N-1} T_{GF}(kT)z^{N-k-1} \quad (8\text{-}30)$$

$$F_4(z) = (z^2 - 2z\cos\frac{2\pi}{N} + 1) \quad (8\text{-}31)$$

Let

$$F(z)_N = [F(z)]_{\phi = \phi + 2\pi/N} \tag{8-32}$$

$$F_1(z)_N = [F_1(z)]_{\phi = \phi + 2\pi/N} \tag{8-33}$$

$$F_3(z)_N = [F_3(z)]_{\phi = \phi + 2\pi/N} \tag{8-34}$$

Then,

$$\frac{|F(z)|}{|F(z)_N|} = \frac{|F_1(z)|}{|F_3(z)|}\frac{|F_3(z)_N|}{|F_1(z)_N|} \tag{8-35}$$

and

$$\text{Arg}[F(z)] - \text{Arg}[F(z)_N] = \text{Arg}[F_1(z)] - \text{Arg}[F_3(z)] - \text{Arg}[F_1(z)_N] + \text{Arg}[F_3(z)_N] \tag{8-36}$$

Also,

$$|F_1(z)| = \left|A\sin\frac{2\pi}{N}\right| \tag{8-37}$$

$$|F_1(z)_N| = \left|A\sin\frac{2\pi}{N}\right| = |F_1(z)| \tag{8-38}$$

Let us express $F_3(z)_N$ as

$$F_3(z)_N = \sum_{k=0}^{N-1} T_{GF}(kT)_N z^{N-k-1} \tag{8-39}$$

where

$$T_{GF}(kT)_N = [T_{GF}(kT)]_\phi = \phi + 2\pi/N \tag{8-40}$$

It can be shown that for any integral N,

$$F_3(z)_N = zF_3(z) = \sum_{k=0}^{N-1} T_{GF}(kT) z^{N-k} \tag{8-41}$$

Then, Eq. (8-35) becomes

$$\frac{|F(z)|}{|F(z)_N|} = \frac{|F_3(z)_N|}{|F_3(z)|} = \frac{|zF_3(z)|}{|F_3(z)|} = 1 \tag{8-42}$$

The argument of $F_1(z)$ is

$$\mathrm{Arg}[F_1(z)] = \phi - \pi/2 \tag{8-43}$$

Then

$$\mathrm{Arg}[F_1(z)_N] = \phi - \pi/2 + 2\pi/N \tag{8-44}$$

Thus, Eq. (8-36) becomes

$$\mathrm{Arg}[F(z)] - \mathrm{Arg}[F(z)_N] = -\frac{2\pi}{N} - \mathrm{Arg}[F_3(z)] + \mathrm{Arg}[F_3(z)_N]$$

$$= -\frac{2\pi}{N} - \mathrm{Arg}[F_3(z)] + \mathrm{Arg}[zF_3(z)]$$

$$= -\frac{2\pi}{N} - \mathrm{Arg}[F_3(z)] + \frac{2\pi}{N} + \mathrm{Arg}[F_3(z)]$$

$$= 0 \qquad (8\text{-}45)$$

Q.E.D.

As an illustrative example of Theorem 8-1, let us consider the case of $N = 4$.

Let $\frac{a}{2} = \gamma T_{GF0} A = K$

Then

$$T_{GF}^{-}(kT) = \frac{R + K(R + 1)[\cos(\phi + 2\pi k/N) - 1]}{1 - K(R + 1)[\cos(\phi + 2\pi k/N) - 1]} T_{GF0} \qquad (8\text{-}46)$$

$$T_{GF}^{+}(kT) = \frac{R - K(R - 1)[\cos(\phi + 2\pi k/N) - 1]}{1 - K(R - 1)[\cos\ \phi + 2\pi k/N) - 1]} T_{GF0} \qquad (8\text{-}47)$$

For $0 < \phi < \pi$,

$$F_3(z) = T_{GF}^{-}(0)z^3 + T_{GF}^{-}(T)z^2 + T_{GF}^{+}(2T)z + T_{GF}^{+}(3T) \qquad (8\text{-}48)$$

$$F_3(z)_N = T_{GF}^{-}(0)_N z^3 + T_{GF}^{+}(T)_N z^2 + T_{GF}^{+}(2T)_N z + T_{GF}^{-}(3T)_N \qquad (8\text{-}49)$$

It is easy to see that

$$T_{GF}^{-}(0)_N = T_{GF}^{-}(T)$$

$$T_{GF}^{+}(T)_N = T_{GF}^{+}(2T)$$

$$T_{GF}^{+}(2T)_N = T_{GF}^{+}(3T) \tag{8-50}$$

$$T_{GF}^{-}(3T)_N = T_{GF}^{-}(0)$$

which proves that $F_3(z)_N = zF_3(z)$.

Similar results are obtained for $\pi < \phi < 2\pi$.

Theorem 8-2

For odd N ($N \geq 3$), the magnitude and phase of $-1/N(z)$ repeat for every $\phi = \pi/N$.

Proof: Let $F(z)_N$, $F_1(z)_N$, $F_3(z)_N$ now be defined as $F(z)$, $F_1(z)$, and $F_3(z)$ with ϕ replaced by $\phi + \pi/N$, respectively. Then,

$$\mathrm{Arg}[F_1(z)_N] = \phi - \frac{\pi}{2} + \frac{\pi}{N} \tag{8-51}$$

$$\mathrm{Arg}[F(z)] - \mathrm{Arg}[F(z)_N] = -\frac{\pi}{N} + \mathrm{Arg}[F_3(z)_N] - \mathrm{Arg}[F_3(z)] \tag{8-52}$$

Using the same notation as in Theorem 8-1, it can be shown that for odd $N \geq 3$,

$$F_3(z)_N = -z^{-(N-1)/2}F_3(z) \tag{8-53}$$

Thus,

$$\text{Arg}[F_3(z)_N] = \pi - \frac{(N-1)}{2}\frac{2\pi}{N} + \text{Arg}[F_3(z)]$$

$$= \frac{\pi}{N} + \text{Arg}[F_3(z)] \tag{8-54}$$

Again,

$$\frac{|F(z)|}{|F(z)_N|} = \frac{|F_3(z)_N|}{|F_3(z)|} = 1 \tag{8-55}$$

and

$$\text{Arg}[F(z)] - \text{Arg}[F(z)_N] = -\frac{\pi}{N} + \text{Arg}[F_3(z)_N] - \text{Arg}[F_3(z)]$$

$$= 0 \tag{8-56}$$

Q.E.D.

As an illustrative example of Theorem 8-2, consider the case $N = 3$.

For $0 < \phi < \pi$,

$$F_3(z) = T_{GF}^{-}(0)z^2 + T_{GF}^{-}(T)z + T_{GF}^{+}(2T) \tag{8-57}$$

$$F_3(z)_N = T_{GF}^{-}(0)_N z^2 + T_{GF}^{+}(T)_N z + T_{GF}^{+}(2T)_N \tag{8-58}$$

It can readily be shown that

$$T_{GF}^{-}(0)_N = -T_{GF}^{+}(2T)$$

$$T_{GF}^{+}(T)_N = -T_{GF}^{-}(0) \qquad (8\text{-}59)$$

$$T_{GF}^{+}(2T)_N = -T_{GF}^{-}(T)$$

To carry out one of the identities above,

$$T_{GF}^{-}(T)_N = T_{GF0}\frac{R - K(R - 1)[\cos(\phi + \pi) - 1]}{1 - K(R - 1)[\cos(\phi + \pi) - 1]} = \frac{R + K(R - 1)(\cos\phi + 1)}{1 + K(R - 1)(\cos\phi + 1)}T_{GF0} \qquad (8\text{-}60)$$

$$T_{GF}^{-}(0) = T_{GF0}\frac{R + K(R + 1)(\cos\phi - 1)}{1 - K(R + 1)(\cos\phi - 1)} \qquad (8\text{-}61)$$

Using the relation, $R^2 + KR - 1 = 0$, we have

$$T_{GF}^{-}(T)_N = -T_{GF}^{-}(0) \qquad (8\text{-}62)$$

Thus,

$$F_3(z)_N = -z^{-1}F_3(z) \qquad (8\text{-}63)$$

The significance of the last two theorems is that the critical regions of $-1/N(z)$ need be computed only for $0^0 \leq \phi \leq \pi/N$ for odd N, and $0^0 \leq \phi \leq 2\pi/N$ for even N.

Theorem 8-3.

Asympotic Behavior of -1/N(z) as A approaches infinity.

(a) $\lim_{A\to\infty} |-1/N(z)| = \infty$ (8-64)

(b) For even $N \geq 4$, $0 \leq \phi \leq 2\pi/N$.

$$\lim_{A\to\infty} \mathrm{Arg}[-1/N(z)] = (\frac{1}{2} - \frac{1}{N})\pi + \phi \quad (8\text{-}65)$$

For odd $N > 3$, $0 \leq \phi \leq \pi/N$.

$$\lim_{A\to\infty} \mathrm{Arg}[-1/N(z)] = (1 - \frac{1}{N})\frac{\pi}{2} + \phi \quad (8\text{-}66)$$

(c) For $N = 2$ $\quad 0 \leq \phi \leq \pi$

$$\lim_{A\to\infty} \mathrm{Arg}[-1/N(z)] = 0^0 \qquad 0 \leq \phi < \pi/2$$

$$= \pi \qquad \pi/2 \leq \phi < \pi \quad (8\text{-}67)$$

(d) For $N = 3$ $\quad 0 \leq \phi \leq \pi/3$

$$\lim_{A\to\infty} \mathrm{Arg}[-1/N(z)] = -\frac{5\pi}{3} + \phi \quad (8\text{-}68)$$

Proof:

We can easily show that

$$\lim_{A\to\infty} T_{GF}^{-}(kT) = -T_{GF0} \qquad \dot{\theta}_G \leq 0 \quad (8\text{-}69)$$

$$\lim_{A\to\infty} T_{GF}^{+}(kT) = T_{GF0} \qquad \dot{\theta}_G \geq 0 \quad (8\text{-}70)$$

The magnitude of $-1/N(z)$ is directly proportional to A as A approaches infinity; thus (a) is proved.

<u>For N = 2</u> $0 \leq \phi \leq \pi/2$

$$F(z) = -1/N(z) = \frac{2A\cos\phi}{T_{GF}(T) - T_{GF}(0)} \tag{8-71}$$

Thus,

$$\lim_{A\to\infty} \text{Arg}[F(z)] = \lim_{A\to\infty} \text{Arg}\left(\frac{2A\cos\phi}{T_{GF}(T) - T_{GF}(0)}\right)$$

$$= \text{Arg}\,\frac{2A\cos\phi}{2T_{GF0}} = 0^{0} \qquad 0 \leq \phi < \pi/2$$

$$= \pi \qquad \pi/2 \leq \phi < \pi \tag{8-72}$$

This proves item (c).

<u>For N = 3</u> $0 \leq \phi \leq \pi/3$

$$F(z) = \frac{-A[(z + 0.5)\cos\phi - 0.866\sin\phi](z - 1)}{T_{GF}(0)z^{2} + T_{GF}(T)z + T_{GF}(2T)} \tag{8-73}$$

$$\lim_{A\to\infty} \text{Arg}[F(z)] = -\frac{5\pi}{3} - \lim_{A\to\infty} \text{Arg}[T_{GF}(0)z^{2} + T_{GF}(T)z + T_{GF}(2T)] + \phi$$

$$= -\frac{5\pi}{3} - \text{Arg}[-z^{2} - z + 1] + \phi$$

$$= -\frac{5\pi}{3} + \phi \tag{8-74}$$

This proves item (d).

For $N \geq 4$ and even $\quad 0 \leq \phi \leq 2\pi/N$

Using Eqs. (8-27) through (8-31), we have

$$\lim_{A\to\infty} \mathrm{Arg}[F(z)] = \lim_{A\to\infty} \left\{\mathrm{Arg}[F_1(z)] + \mathrm{Arg}[F_2(z)] - \mathrm{Arg}[F_3(z)] - \mathrm{Arg}[F_4(z)]\right\}$$

$$= \phi - \frac{\pi}{2} - \lim_{A\to\infty} \left\{\mathrm{Arg}[F_3(z)]\right\} + \mathrm{Arg}[F_2(z)] - \mathrm{Arg}[F_4(z)] \tag{8-75}$$

$$\mathrm{Arg}[F_2(z)] - \mathrm{Arg}[F_4(z)] = \mathrm{Arg}[(z - 1)\prod_{k=2}^{N-2}(z - e^{j2\pi k/N})]$$

$$= \mathrm{Arg}[e^{j\pi/N}(e^{j\pi/N} - e^{-j\pi/N})\prod_{k=2}^{N-2} e^{j2\pi/N}(1 - e^{j2\pi/N(k-1)})] \tag{8-76}$$

$$\mathrm{Arg}[F_2(z)] - \mathrm{Arg}[F_4(z)] = \mathrm{Arg}[e^{j\pi/N}(e^{j\pi/N} - e^{-j\pi/N})(e^{j2\pi/N})^{N-3}\prod_{k=1}^{N-3}(1 - e^{j2\pi k/N})]$$

$$= \frac{\pi}{N} + \frac{\pi}{2} + (N - 3)(\frac{2\pi}{N}) + \mathrm{Arg}\prod_{k=1}^{N-3} e^{j\pi k/N}(e^{-j\pi k/N} - e^{+j\pi k/N})$$

$$= \frac{\pi}{N} + (N - 3)(\frac{2\pi}{N}) + (N - 4)(-\frac{\pi}{2}) + \sum_{k=1}^{N-3}\frac{k\pi}{N} \tag{8-77}$$

$$\lim_{A\to\infty} \mathrm{Arg}[F_3(z)] = \lim_{A\to\infty} \mathrm{Arg}\sum_{k=0}^{N-1} T_{GF}(kT)z^{N-k-1}$$

$$= \text{Arg} \sum_{k=0}^{N/2-1} - T_{GFO} z^{N-k-1} + \text{Arg} \sum_{k=N/2}^{N-1} T_{GFO} z^{N-k-1}$$

$$= \text{Arg} \sum_{k=0}^{N/2-1} - T_{GFO} z^{N-k-1} = \frac{(\frac{N}{2} - 1)\pi}{N} = \frac{\pi}{2} - \frac{\pi}{N} \qquad (8\text{-}78)$$

Thus,

$$\lim_{A\to\infty} \text{Arg}[F(z)] = \phi - \frac{\pi}{2} + \frac{\pi}{N} + (N - 4)(-\frac{\pi}{2}) + (N - 3)(\frac{2\pi}{N}) + \sum_{k=1}^{N-3} \frac{k\pi}{N} - \frac{\pi}{2} + \frac{\pi}{N}$$

$$= (3 - \frac{N}{2} - \frac{4}{N} + \sum_{k=1}^{N-3} \frac{k}{N})\pi + \phi$$

$$= (\frac{1}{2} - \frac{1}{N})\pi + \phi \qquad (8\text{-}79)$$

<u>For N > 3 and odd</u> $0 \leq \phi \leq \pi/N$

For this case,

$$\lim_{A\to\infty} \text{Arg}[F_3(z)] = \text{Arg} \sum_{k=0}^{(N-1)/2} - T_{GFO} z^{N-k-1} + \text{Arg} \sum_{k=(N+1)/2}^{N-1} T_{GFO} z^{N-k-1}$$

$$= \frac{(N - 3)\pi}{2N} \qquad (8\text{-}80)$$

Thus,

$$\lim_{A\to\infty} \mathrm{Arg}[F(z)] = \phi - \frac{\pi}{2} + \frac{\pi}{N} + (N - 4)(- \frac{\pi}{2}) + (N - 3)(\frac{2\pi}{N}) + \sum_{k=1}^{N-3} \frac{k\pi}{N} - \frac{(N - 3)\pi}{2N}$$

$$= (3 - \frac{N}{2} - \frac{7}{2N} + \sum_{k=1}^{N-3} \frac{k}{N})\pi + \phi$$

$$= (1 - \frac{1}{N})\frac{\pi}{2} + \phi \tag{8-81}$$

and (b) is proved. Q.E.D.

Theorem 8-4.

Asymptotic Behavior of -1/N(z) as A approaches zero.

$$\lim_{A\to 0} F(z) = - \frac{1}{\gamma T_{GF0}^{2}} \tag{8-82}$$

for all ϕ and all N.

Proof: From Eq. (8-20),

$$F(z) = - \frac{1}{N(z)} = \frac{-A \sum_{k=0}^{N-1} \cos(\frac{2k\pi}{N} + \phi) z^{N-k-1}}{\sum_{k=0}^{N-1} T_{GF}(kT) z^{N-k-1}} \tag{8-83}$$

$$\lim_{A\to 0} F(z) = \frac{- \sum_{k=0}^{N-1} \cos(\frac{2k\pi}{N} + \phi) z^{N-k-1}}{\lim_{A\to 0} \sum_{k=0}^{N-1} \frac{T_{GF}(kT)}{A} z^{N-k-1}}$$

$$= \frac{-\sum_{k=0}^{N-1} \cos(\frac{2k\pi}{N} + \phi) z^{N-k-1}}{\sum_{k=0}^{N-1} \lim_{A\to 0} \frac{T_{GF}(kT)}{A} z^{N-k-1}} \quad (8\text{-}84)$$

Therefore, the problem is that of finding $\lim_{A\to 0} \frac{T_{GF}(kT)}{A}$.

First, let $T_{GF}(kT) = T_{GF}^{-}(kT)$. Then

$$\lim_{A\to 0} \frac{T_{GF}^{-}(kT)}{A} = \lim_{A\to 0} T_{GF0} \frac{\frac{R}{A(R+1)} - \gamma T_{GF0} + \gamma T_{GF0}\cos(\frac{2\pi k}{N} + \phi)}{\gamma T_{GF0}A - \gamma T_{GF0}A\cos(\frac{2\pi k}{N} + \phi) + \frac{1}{R+1}}$$

$$\cong T_{GF0} \lim_{A\to 0} \frac{R}{A(R+1)} + \gamma T_{GF0}^{2}[\cos(\frac{2\pi k}{N} + \phi) - 1] \quad (8\text{-}85)$$

where the fact that $\lim_{A\to 0} [1/(R+1)] = 1$ has been used.

$$\lim_{A\to 0} \frac{R}{A(R+1)} \cong \lim_{A\to 0} \frac{R}{A} = \lim_{A\to 0} \frac{1}{A}\left(- \frac{1}{2\gamma T_{GF0}A} + \sqrt{\frac{4\gamma^2 T_{GF0}^{2}A^2 + 1}{4\gamma^2 T_{GF0}^{2}A^2}}\right) \quad (8\text{-}86)$$

Or,

$$\lim_{A\to 0} \frac{R}{A(R+1)} = \lim_{A\to 0} \frac{1}{A}\left(- \frac{1}{2\gamma T_{GF0}A} + \frac{1}{2\gamma T_{GF0}A}\sqrt{1 + 4\gamma^2 T_{GF0}^{2}A^2}\right) \quad (8\text{-}87)$$

Expanding $\sqrt{1 + 4\gamma^2 T_{GFO}{}^2 A^2}$ into a power series, and using only the first two terms, we have,

$$\lim_{A\to 0} \frac{R}{A(R+1)} = \lim_{A\to 0} \frac{1}{A}\left(\frac{4\gamma^2 T_{GFO}{}^2 A^2}{4\gamma T_{GFO} A}\right) = \gamma T_{GFO} \tag{8-88}$$

Thus,

$$\lim_{A\to 0} \frac{T_{GF}{}^-(kT)}{A} = \gamma T_{GFO}{}^2 + \gamma T_{GFO}{}^2[\cos(\frac{2\pi k}{N} + \phi) - 1]$$

$$= \gamma T_{GFO}{}^2 \cos(\frac{2\pi k}{N} + \phi) \tag{8-89}$$

Similarly, it can be shown that

$$\lim_{A\to 0} \frac{T_{GF}{}^+(kT)}{A} = \lim_{A\to 0} \frac{T_{GF}{}^-(kT)}{A} = \gamma T_{GFO}{}^2 \cos(\frac{2\pi k}{N} + \phi) \tag{8-90}$$

Now,

$$\lim_{A\to 0} F(z) = \frac{-\sum_{k=0}^{N-1} \cos(\frac{2\pi k}{N} + \phi) z^{N-k-1}}{\gamma T_{GFO}{}^2 \sum_{k=0}^{N-1} \cos(\frac{2\pi k}{N} + \phi) z^{N-k-1}} = -\frac{1}{\gamma T_{GFO}{}^2} \tag{8-91}$$

for all ϕ and all N. Q.E.D.