Continuous Data Assimilation Using General Interpolant Observables

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Abstract

We present a new continuous data assimilation algorithm based on ideas that have been developed for designing finite-dimensional feedback controls for dissipative dynamical systems, in particular, in the context of the incompressible two-dimensional Navier–Stokes equations. These ideas are motivated by the fact that dissipative dynamical systems possess finite numbers of determining parameters (degrees of freedom) such as modes, nodes and local spatial averages which govern their long-term behavior. Therefore, our algorithm allows the use of any type of measurement data for which a general type of approximation interpolation operator exists. Under the assumption that the observational measurements are free of noise, our main result provides conditions, on the finite-dimensional spatial resolution of the collected data, sufficient to guarantee that the approximating solution, obtained by our algorithm from the measurement data, converges to the unknown reference solution over time. Our algorithm is also applicable in the context of signal synchronization in which one can recover, asymptotically in time, the solution (signal) of the underlying dissipative system that is corresponding to a continuously transmitted partial data.

Keywords: Determining modes, volume elements and nodes; continuous data assimilation; two-dimensional Navier–Stokes equations; signal synchronization.

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1 Introduction

The goal of continuous data assimilation, and signal synchronization, is to use low spatial resolution observational measurements, obtained continuously in time, to accurately find the corresponding reference solution from which future predictions can be made. The motivating application of continuous data assimilation is weather prediction. The classical method of continuous data assimilation, see Daley [12], is to insert observational measurements directly into a model as the latter is being integrated in time. We propose a new approach based on ideas from control theory, see Azouani and Titi [2]. A slightly similar approach in the context of stochastic differential equations, using the low Fourier modes as observables/measurements, appears in a recent work by Blömker, Law, Stuart and Zygalakis [4]. Rather than inserting the measurements directly into the model, i.e. into the nonlinear term, we introduce a feedback control term that forces the model toward the reference solution that is corresponding to the observations. This is motivated by the fact that it is often difficult to insert observational data into the model. For example, if the measured data is obtained as the values of the exact solutions at a discrete set of spatial nodal points, then it difficult to insert this data directly into the underlying equation, because it is not possible to obtain the exact values of the spatial derivatives. One should observe that in order to guarantee a unique corresponding reference solution one has to supply observational data with enough spatial resolution. This is the object of this paper.

While the classical method of continuous data assimilation is simple in concept, special care has to be taken concerning how the observations are inserted into a model in practice. For example, it is generally necessary to separate the fast and slow parts of a solution before inserting the observations into the model. The method proposed here does not require such a decomposition. Since the observations are not directly inserted into the model, we can rely on the dissipation already present in the dynamics to filter the observed data, i.e. the viscous term will suppress the "spill over" oscillations in the fine scales. The advantage of this approach is that it works for a general class of interpolant observables without modification.

Let $u(t)$ represent the state at time t of the dynamical system in which we are interested, and let $I_h(u(t))$ represent our observations of this system at a coarse spatial resolution of size h. Given observational measurements, $I_h(u(t))$, for $t \in [0, T]$, our goal is to construct an increasingly accurate initial condition from which predictions of $u(t)$, for $t > T$, can be made. We do this by constructing an approximate solution $v(t)$ that converges to $u(t)$ over time.

Suppose the time evolution of u is governed by a given evolution equation of the form

$$\frac{du}{dt} = F(u), \tag{1}$$

where the initial data, u_0, is missing. Our algorithm for constructing $v(t)$ from the observational measurements $I_h(u(t))$ for $t \in [0,T]$ is given by

$$\frac{dv}{dt} = F(v) - \mu I_h(v) + \mu I_h(u), \tag{2}$$

$$v(0) = v_0, \tag{3}$$

where μ is a positive relaxation parameter, which relaxes the coarse spatial scales of v toward those of the observed data, and v_0 is taken to be arbitrary. It is worth stressing that our algorithm is designed to work for general dissipative dynamical systems of the form (1). Such systems are known to have finitely many degrees of freedom in the form of determining parameters of the type $I_h(u)$, see, for example, Cockburn, Jones and Titi [10], Foias, Manley, Rosa and Temam [15], Foias and Prodi [16], Foias and Temam [17], [18], Jones and Titi [23], [24], and references therein. The incompressible two-dimensional Navier–Stokes equations provide a concrete example of a dissipative dynamical system of this type.

Note that the equations used in numerical weather forecasting are compressible three-dimensional equations involving variable density and a velocity equation that is coupled to a whole set of state variables. Our approach assumes the global existence of the underlying model, and the estimates we provide use the available estimates for the globally existing solutions. Therefore, we will not be able to prove any theorems regarding even the three-dimensional Navier Stokes equations using our techniques. Note, however, that Korn [25] shows the three-dimensional Lagrangian-averaged Navier–Stokes-α model of turbulence posses a finite number of determining observations and uses this fact to obtain results about the classical method of data assimilation. Therefore, it should be possible to obtain results for the new method of data assimilation presented here for solutions governed by the three-dimensional Lagrangian-averaged Navier–Stokes-α model. Similarly, the three-dimensional Leray-α model [9] and the three-dimensional primitive equations [8] are other systems known to be globally well-posed to which our algorithm and analysis should apply. Although our analysis doesn't apply to complex systems in which there is no proof of global existence, it would still be interesting to numerically test our algorithm in more realistic cases. In particular, there is significantly more work to do before the data assimilation algorithm presented here can be applied to numerical weather forecasting.

To demonstrate our approach and data assimilation algorithm we consider the incompressible two-dimensional Navier–Stokes equations. These equations serve as a paradigm because they are amenable to mathematical analysis while at the same time posses non-linear dynamical properties similar to the equations used in realistic weather models. Thus, our study of data assimilation for the incompressible two-dimensional Navier–Stokes equations should be viewed as a mathematical problem motivated by real-world applications. With this in mind we suppose that the evolution of u is governed by the two-dimensional Navier–Stokes system

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \tag{4}$$

$$\nabla \cdot u = 0, \tag{5}$$

in the physical domain Ω, with either no-slip Dirichlet, or periodic, boundary conditions with zero spatial average. Here $u(x,t)$ represents velocity of the fluid at time t at position x, $\nu > 0$ represents the kinematic viscosity, $p(x,t)$ is the pressure and $f(x,t)$ is a time dependent body force applied to the fluid.

In the case of no-slip Dirichlet boundary conditions we take $u = 0$ on $\partial\Omega$. The domain Ω is an open, bounded and connected set in $\mathbf{R}^2$ with C^2 boundary, such that $\partial\Omega$ can be represented locally as the graph of a C^2 function. In the case of periodic boundary conditions we require u and f to be $L-$periodic, in both x_1 and x_2 directions, with zero spatial averages over the fundamental periodic domain $\Omega = [0, L]^2$.

Continuous data assimilation, in the context of the incompressible two-dimensional Navier–Stokes equations, was first studied by Browning, Henshaw and Kreiss in [7], later by Henshaw, Kreiss and Yström in [22] and also by Olson and Titi in [27] and [28], motivated by the concept of finite number of determining modes which was introduced for the first time in [16] (see also [15], [27], and references therein). These studies treated the case of periodic boundary conditions, where the observations of the velocity field were given by the low Fourier modes with wave numbers k, such that $|k| \leq 1/h$. Since the low modes essentially represent the large spatial scales of the solution, the classical data assimilation algorithm works well for this type of observations. We treat more general observations of the velocity field. Observations of vorticity or of the stream function should be treatable using similar analysis; however, observations of the pressure field pose additional difficulties.

It is worth mentioning that the method of data assimilation studied here is consistent with some of the signal synchronization algorithms. Most recently, a similar idea has also been introduced in [14] to show that the long-time dynamics of the two-dimensional Navier–Stokes equations can be imbedded in an infinite-dimensional dynamical system that is induced by an ordinary differential equations, named *determining form*, which is governed by a globally Lipschitz vector field.

Let us denote by $H^m(\Omega)$ the Sobolev space of index m, and by $\dot{H}^m(\Omega)$ its subspace of functions with zero spatial averages. The method of constructing v, given by (2), allows the use of general interpolant observables, given by interpolants $I_h \colon H^1(\Omega) \to L^2(\Omega)$ ($I_h \colon \dot{H}^1(\Omega) \to \dot{L}^2(\Omega)$ in the periodic case) that are linear and satisfy the following approximation property:

$$\|\varphi - I_h(\varphi)\|_{L^2(\Omega)}^2 \leq \gamma_0 h^2 \|\varphi\|_{H^1(\Omega)}^2 \tag{6}$$

for every $\varphi \in H^1(\Omega)$. The orthogonal projection onto the low Fourier modes, with wave numbers k such that $|k| \leq 1/h$, mentioned above, is an example of such interpolant observable. However, there are many other interpolant observables which satisfy (6). Note that $\|\varphi - I_h(\varphi)\|_{L^2(\Omega)}^2 \to 0$ as $h \to 0$. This implies our observational measurements are noise free. The case of stochastically noisy data will be studied in a future work [3].

The term general interpolant observable and its associated interpolant operator I_h should be distinguished from the observation operators used in data assimilation. The former perform a classical interpolation of the state vector between resolutions, while the later map general observables that are related to the state vector by an operator or a functional from the space of observations to the space of the state vector. In the present context an operator that maps pressure observations to the velocity space would be an example of an observation operator but not of an interpolant. Such observation operators will not be considered here.

One physically relevant example of an interpolant which satisfies condition (6) are the volume elements studied in [23] and [24] (see also Foias and Titi [19]). In this case

$$I_h(\varphi)(x) = \sum_{j=1}^{N} \bar{\varphi}_j \Big(\chi_{Q_j}(x) - \frac{h^2}{L^2} \Big) \quad \text{where} \quad \bar{\varphi}_j = \frac{1}{h^2} \int_{Q_j} \varphi(x)\, dx,$$

and the domain $\Omega = [0, L]^2$, for the periodic boundary conditions case, has been divided into N equal squares Q_j, with sides $h = L/\sqrt{N}$. Volume elements generalize to any domain Ω on which the Bramble–Hilbert lemma holds. An elementary discussion of this lemma in the context of finite element methods appears in Brenner and Scott [5].

In addition, we also consider interpolant observables given by linear interpolants $I_h\colon H^2(\Omega) \to L^2(\Omega)$, that satisfy the following approximation property:

$$\|\varphi - I_h(\varphi)\|_{L^2(\Omega)}^2 \le \gamma_1 h^2 \|\varphi\|_{H^1(\Omega)}^2 + \gamma_2 h^4 \|\varphi\|_{H^2(\Omega)}^2, \tag{7}$$

for every $\varphi \in H^2(\Omega)$. An example of this type of interpolant is given by measurements at a discrete set of nodal points in Ω. Specifically, let $h > 0$ be given, and let $\Omega = \cup_{j=1}^{N_h} Q_j$, where Q_j are disjoint subsets such that $\operatorname{diam} Q_j \le h$, for $j = 1, 2, \ldots, N_h$, and let $x_j \in Q_j$ be arbitrary points. Then set, for example,

$$I_h(\varphi)(x) = \sum_{k=1}^{N_h} \varphi(x_k) \chi_{Q_j}(x),$$

in the no-slip Dirichlet boundary conditions case. However, in the case of periodic boundary conditions we divide, as before, the domain $\Omega = [0, L]^2$ into N identical cubes, $\{Q_j\}_{j=1}^{N}$, with sides $h = L/\sqrt{N}$, and take the interpolant operator that is induced by the nodal values, $I_h\colon \dot{H}^2(\Omega) \to \dot{L}^2(\Omega)$, to be

$$I_h(\varphi)(x) = \sum_{k=1}^{N_h} \varphi(x_k) \Big(\chi_{Q_j}(x) - \frac{h^2}{L^2} \Big). \tag{8}$$

Notice that, by construction, the spatial average of $I_h(\varphi)$, given in (8), is zero. Following ideas in [24] (see also [18]) we will show in Appendix A that the interpolant operators, $I_h(\varphi)$, defined by (8), satisfy the approximation property

(7). Further details concerning smooth interpolant observables and operators that satisfy (7), which are induced by nodal values, are included in Appendix A. These smoother observables, and their analytic properties, are needed for the study of a similar data assimilation algorithm with stochastic noisy data [3].

Our paper is organized as follows. First, we recall the functional setting of the two-dimensional Navier–Stokes equations necessary for our analysis and then use this setting to formulate our new method of continuous data assimilation. After this we proceed to the task of finding conditions on h and μ under which the approximate solution obtained by this algorithm converges to the reference solution over time. From a physical point of view, the spatial resolution h of the observational measurements is difficult and costly to change, whereas the relaxation parameter μ is an easily changed mathematical constant. Our main results, therefore, focus on finding bounds on h for which there exists a μ that guarantees the success of our algorithm. We also prove a number of propositions that provide estimates on μ. Section 3 treats the case of smooth, bounded domains with no-slip Dirichlet boundary conditions, while section 4 treats the case of periodic boundary conditions. Our main results may be stated as follows:

Theorem 1. *Let Ω be an open, bounded and connected set in $\mathbf{R}^2$ with C^2 boundary, and let u be a solution to equations (4)–(5) with no-slip Dirichlet boundary conditions. Assume that I_h satisfies (6), with h small enough such that*

$$1/h^2 \geq c_1 \lambda_1 G^2,$$

where c_1 is a constant given in (36). Then there exists $\mu > 0$, given explicitly in Proposition 1, such that $\|v-u\|_{L^2(\Omega)} \to 0$ exponentially, as $t \to \infty$.

Here G denotes the Grashof number

$$G = \frac{1}{\nu^2 \lambda_1} \limsup_{t\to\infty} \|f(t)\|_{L^2} \tag{9}$$

where λ_1 is the smallest eigenvalue of the Stokes operator subject to homogeneous Dirichlet boundary conditions. Let us remark, again, that the constant c_1 depends only on γ_0, given in (6), and the shape, but not the size, of the domain Ω. In particular, c_1 is given by (36) where the constant c is chosen so the bound (16) on the non-linear term holds. Moreover, μ may be chosen equal to $5c^2G^2\nu\lambda_1$ as indicated in Proposition 1, below.

Results similar to Theorem 1 hold when I_h satisfies (7), however, we omit the proof of this result in the case of no-slip Dirichlet boundary conditions and instead proceed directly to the the case of periodic boundary conditions where sharper estimates may be obtained. In particular, we prove

Theorem 2. *Let $\Omega = [0,L]^2$ and let u be a solution to equations (4)–(5) with periodic boundary conditions. Let I_h satisfy either (6) or (7), with h small enough such that*

$$1/h^2 \geq c_2 \lambda_1 G\big(1 + \log(1+G)\big),$$

where c_2 is a constant given in (39). Then there exists $\mu > 0$, given explicitly in Proposition 2, such that $\|v-u\|_{H^1(\Omega)} \to 0$ exponentially, as $t \to \infty$.

Let us remark that c_2 depends only on γ_0 in the case I_h satisfies (6) and only on γ_1 and γ_2 in the case that I_h satisfies (7). Also, μ may be chosen as $3c_2\nu\lambda_1 G(1+\log(1+G))/c_0$. In particular, μ is given in Proposition 2 and the constant c_2 is defined in (39) as an increasing function of c_0 and c, where c_0 is the constant appearing in either (24) or (25) and c is chosen large enough so that the bounds in both (22) and (37) hold.

Note that the estimate on the length scale h in Theorem 2 can be compared to previous results reported in [27]. Let $\tilde{v}(t)$ be the approximate solution obtained by the method of continuous data assimilation introduced in [27] for the interpolant observable $I_h(u)$ given by projection onto the Fourier modes with wave numbers $|k| < 1/h$. In [27] it was shown, that for small values of h, such that $1/h^2 \sim \lambda_1 G$, $\|u(t) - \tilde{v}(t)\|_{H^1(\Omega)} \to 0$ exponentially fast, as $t \to \infty$. Up to a logarithmic correction term, Theorem 2 states similar estimates on h for the new algorithm which covers a much wider class of interpolant observables.

The final section of this paper discusses numerical simulations, which are in progress, related works, and closes with a few concluding remarks. We supplement this paper with an appendix in which we introduce smooth interpolant operators, that are induced by nodal values, and which satisfy inequality (7).

2 Preliminaries

This section reviews the functional setting of the two-dimensional Navier–Stokes equations with no-slip and periodic boundary conditions, recalls some facts that will be used in the remainder of the paper and then gives an explicit formulation of our new method for continuous data assimilation in this context. Following Constantin and Foias [11], Foias, Manley, Rosa and Temam [15], Robinson [29] and Temam [30], we begin by defining a suitable domain Ω and space $\mathcal{V}$ of smooth functions which satisfy each type of boundary conditions.

No-slip Dirichlet Boundary Conditions. Let Ω be an open, bounded and connected domain with C^2 boundary. Define $\mathcal{V}$ to be set of all C^∞ vector fields from Ω to $\mathbf{R}^2$ that are divergence free and compactly supported.

Periodic Boundary Conditions. Let $\Omega = [0, L]^2$ for some fixed $L > 0$. Define $\mathcal{V}$ to be the set of all L-periodic trigonometric polynomials from $\mathbf{R}^2$ to $\mathbf{R}^2$ that are divergence free and have zero averages.

Given $\mathcal{V}$ corresponding to either type of boundary conditions let H be the closure of $\mathcal{V}$ in $L^2(\Omega; \mathbf{R}^2)$ and V be the closure of $\mathcal{V}$ in $H^1(\Omega; \mathbf{R}^2)$. The spaces H and V are Hilbert spaces with inner products

$$(u, v) = \int_\Omega u(x) \cdot v(x)\, dx \qquad \text{and} \qquad ((u, v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}\, dx,$$

respectively. Denote the norms of H and V by

$$|u| = \sqrt{(u, u)} \qquad \text{and} \qquad \|u\| = \sqrt{((u, u))},$$

and the dual of V by V^* with the pairing $\langle u, v\rangle$ where $u \in V^*$ and $v \in V$.

Define the Leray projector P_σ as the orthogonal projection from $L^2(\Omega; \mathbf{R}^2)$ onto H, and define the Stokes operator $A\colon V \to V^*$, and the bilinear term $B\colon V \times V \to V^*$ to be the continuous extensions of the operators given by

$$Au = -P_\sigma \Delta u \qquad \text{and} \qquad B(u, v) = P_\sigma(u \cdot \nabla v),$$

respectively, for any smooth solenoidal vector fields u and v in $\mathcal{V}$.

Denote the domain of A by $\mathcal{D}(A) = \big\{\, u \in V : Au \in H \,\big\}$. The linear operator A is self-adjoint and positive definite with compact inverse $A^{-1}\colon H \to H$. Thus, there exists a complete orthonormal set of eigenfunctions w_i in H such that $Aw_i = \lambda_i w_i$ where $0 < \lambda_i \leq \lambda_{i+1}$ for $i \in \mathbf{N}$. Writing λ_1 as the smallest eigenvalue of A we have the following Poincaré inequalities:

$$\text{if } u \in V \text{ then } \lambda_1 |u|^2 \leq \|u\|^2, \tag{10}$$

$$\text{if } u \in \mathcal{D}(A) \text{ then } \lambda_1 \|u\|^2 \leq |Au|^2. \tag{11}$$

Note that for $u \in H$, $|u| = \|u\|_{L^2(\Omega)}$ and for $u \in V$ the Poincaré inequality implies $\|u\|$ is equivalent to $\|u\|_{H^1(\Omega)}$.

The bilinear term B has the algebraic property that

$$\big\langle B(u, v), w\big\rangle = -\big\langle B(u, w), v\big\rangle, \tag{12}$$

for $u, v, w \in V$, and consequently the orthogonality property that

$$\big\langle B(u, w), w\big\rangle = 0. \tag{13}$$

Here the pairing $\langle \cdot, \cdot\rangle$ denotes the dual action of V^* on V. Details may be found, e.g., in [11], [15], [29] and [30].

In the case of periodic boundary conditions the bilinear term possesses the additional orthogonality property

$$\big(B(w, w), Aw\big) = 0, \qquad \text{for every} \qquad w \in \mathcal{D}(A); \tag{14}$$

and consequently one has

$$\big(B(u, w), Aw\big) + \big(B(w, u), Aw\big) = -\big(B(w, w), Au\big). \tag{15}$$

Note that the bilinear term satisfies a number of inequalities which hold for either no-slip or periodic boundary conditions. These are

$$\big|\big\langle B(u, v), w\big\rangle\big| \leq c|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2}, \tag{16}$$

for every $u, v, w \in V$,

$$\big|\big(B(u, v), w\big)\big| \leq c|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w| \tag{17}$$

for every $u \in V$, $v \in \mathcal{D}(A)$ and $w \in H$, and

$$\big|\big(B(u, v), w\big)\big| \leq c|u|^{1/2}|Au|^{1/2}\|v\||w|, \tag{18}$$

for every $u \in \mathcal{D}(A)$ and $v, w \in V$, where c is a dimensionless constant depending only on the shape, but not the size, of Ω. These inequalities may be obtained from the Hölder's inequality, the Sobolev inequalities and Ladyzhenskaya's inequality, see, e.g., [11], [15], [29] and [30].

We write the incompressible two-dimensional Navier–Stokes equations with the above notation in functional form as

$$\frac{du}{dt} + \nu A u + B(u,u) = f \tag{19}$$

with initial condition $u(0) = u_0$. We have assumed $f \in H$ so that $P_\sigma f = f$. As shown in [11], [15], [29] and [30] these equations are well-posed; and possess a compact finite-dimensional global attractor, when f is time-independent. Specifically, we have

Theorem 3 (Existence and Uniqueness of Strong Solutions)**.** *Suppose $u_0 \in V$ and $f \in L^\infty\big((0,\infty),H\big)$. Then the initial value problem (19) has a unique solution that satisfies*

$$u \in C\big([0,T];V\big) \cap L^2\big((0,T);D(A)\big) \quad \text{and} \quad \frac{du}{dt} \in L^2\big((0,T);H\big),$$

for any $T > 0$.

We now give bounds on solutions u of (19) that will be used in our later analysis. With the exception of inequality (22) due to Dascaliuc, Foias and Jolly [13] these estimates appear in any the references listed above.

Theorem 4. *Fix $T > 0$, and let G be the Grashof number given in (9). Suppose that u is the solution of (19), corresponding to the initial value u_0, then there exists a time t_0, which depends on u_0, such that for all $t \geq t_0$ we have:*

$$|u(t)|^2 \leq 2\nu^2 G^2 \qquad \text{and} \qquad \int_t^{t+T} \|u(\tau)\|^2 d\tau \leq 2\big(1 + T\nu\lambda_1\big)\nu G^2. \tag{20}$$

In the case of periodic boundary conditions we also have:

$$\|u(t)\|^2 \leq 2\nu^2\lambda_1 G^2, \qquad \int_t^{t+T} |Au(\tau)|^2 d\tau \leq 2(1 + T\nu\lambda_1)\nu\lambda_1 G^2; \tag{21}$$

furthermore, if $f \in H$ is time-independent then

$$|Au(t)|^2 \leq c\nu^2\lambda_1^2(1+G)^4. \tag{22}$$

We now write the continuous data assimilation equations (2) for the incompressible two-dimensional Navier–Stokes equations. Let u be a strong solution of (4)–(5), or equivalently (19), as given by Theorem 3, and let I_h be an interpolation operator satisfying (24) or (25). Suppose that u is to be recovered from the observational measurements $I_h(u(t))$, that have been continuously recorded for times t in $[0,T]$. Then, the approximating solution v with initial condition $v_0 \in V$, chosen arbitrarily, shall be given by

$$\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla) v + \nabla q = f + \mu(I_h(u) - I_h(v)),$$

$$\nabla \cdot v = 0,$$

on the interval $[0,T]$. Observe that in the periodic setting we demand, through construction, that the spatial average of $I_h(\varphi)$ is zero, for every φ in the relevant domain of I_h. This is done for technical reason in order to guarantee that the spatial average of v is preserved, and hence can be chosen to be constant zero. Using the above functional setting the above system is equivalent to

$$\frac{dv}{dt} + \nu A v + B(v,v) = f + \mu P_\sigma (I_h(u) - I_h(v)), \tag{23}$$

on the interval $[0,T]$. Furthermore, inequalities (6) and (7) imply

$$|P_\sigma(w - I_h(w))|^2 \leq |w - I_h(w)|^2 \leq c_0 h^2 \|w\|^2, \tag{24}$$

for every $w \in V$, where $c_0 = \gamma_0$, and respectively

$$|P_\sigma(w - I_h(w))|^2 \leq |w - I_h(w)|^2 \leq \frac{1}{2} c_0 h^2 \|w\|^2 + \frac{1}{4} c_0^2 h^4 |Aw|^2, \tag{25}$$

for every $w \in \mathcal{D}(A)$,where c_0 depends only on γ_1 and γ_2.

If we knew u_0 exactly, then we could take $v_0 = u_0$ and the resulting solution v would be identical to u for all time; this is due to the uniqueness of the solutions of (23) (see Theorem 5, below). However, if we knew u_0 exactly, there would be no need for continuous data assimilation in the first place and one could integrate (19) directly with the initial value u_0. Intuitively speaking it makes sense to take $v_0 = P_\sigma I_h(u(0))$, which is the initial observation of the solution u. However, v_0 chosen in this way may not be an element of V. The main point of the data assimilation method given in (23) is to avoid the difficulties which come from the direct insertion of observational measurements into the approximate solution. A choice for v_0 in agreement with this philosophy is $v_0 = 0$. In fact, our results to hold equally well when v_0 is chosen to be any element of V. In either case we obtain an approximating solution v constructed using only the observations of the solution $I_h(u)$ and the known values of ν and f.

We now show the data assimilation equations (23) are well-posed. When I_h satisfies (24) we show well-posedness for both no-slip Dirichlet and periodic boundary conditions. When I_h satisfies (25) we will deal here, for simplicity, with only the case of periodic boundary conditions.

Theorem 5. *Suppose I_h satisfies (24) and $\mu c_0 h^2 \leq \nu$, where c_0 is the constant appearing in (24). Then the continuous data assimilation equations (23) possess unique strong solutions that satisfy*

$$v \in C\big([0,T]; V\big) \cap L^2\big((0,T); D(A)\big) \quad \textit{and} \quad \frac{dv}{dt} \in L^2\big((0,T); H\big), \tag{26}$$

for any $T > 0$. Furthermore, this solution depends continuously on the initial data v_0 in the V norm.

Proof. Define $g = f + \mu P_\sigma I_h(u)$. Theorem 3 implies $u \in C\big([0,T];V\big)$. Consequently

$$|P_\sigma I_h(u)| \le |u - I_h(u)| + |u| \le \big(c_0^{1/2} h + \lambda_1^{-1/2}\big)\|u\|$$

implies that $P_\sigma I_h(u) \in C\big([0,T];H\big)$. Hence $g \in C\big([0,T];H\big)$. This means there is a constant M such that $|g|^2 < M$ for every $t \in [0,T]$.

We now show the existence of solutions v to (23) using the Galerkin method. The proof follows the same ideas as the proof of Theorem 3. Let P_n be the n-th Galerkin projector and v^n be the solution to the finite-dimensional Galerkin truncation

$$\begin{cases} \dfrac{dv^n}{dt} + \nu A v^n + P_n B(v^n, v^n) = P_n g - \mu P_n I_h(v^n) \\ v^n(0) = P_n v_0. \end{cases} \tag{27}$$

First, we observe that (27) is a finite system of ODEs, which has short time existence and uniqueness. We focus on the maximal interval of existence, $[0, T_n)$, and show uniform bounds for v^n, which are independent of n. This in turn will imply the global existence for (27). Thus, our aim is to find bounds on v^n which are uniform in n. This will then show global existence of solutions to (23).

Begin by taking inner products of (27) with v^n to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|v^n|^2 + \nu\|v^n\|^2 &= (g, v^n) - \mu\big(I_h(v^n), v^n\big) \\ &= (g, v^n) + \mu\big(v^n - I_h(v^n), v^n\big) - \mu|v^n|^2 \\ &\le \frac{1}{2\mu}|g|^2 + \frac{\mu}{2}|v^n|^2 + \frac{\mu}{2}\big|P_\sigma(v^n - I_h(v^n))\big|^2 + \frac{\mu}{2}|v^n|^2 - \mu|v^n|^2 \\ &\le \frac{1}{2\mu}|g|^2 + \frac{\mu c_0 h^2}{2}\|v^n\|^2, \end{aligned}$$

where we used (24) in the above estimates. By hypothesis, the size of the cube, h, is chosen to be small enough such that $\mu c_0 h^2 \le \nu$. Therefore,

$$\frac{d}{dt}|v^n|^2 + \nu\|v^n\|^2 \le \frac{1}{\mu}|g|^2, \tag{28}$$

and consequently

$$\frac{d}{dt}|v^n|^2 + \nu\lambda_1|v^n|^2 \le \frac{1}{\mu}M, \quad \text{for every} \quad t \in [0, T_n). \tag{29}$$

Multiplying (29) by $e^{\nu\lambda_1 t}$ and integrating yields

$$|v^n(t)|^2 \le |v_0|^2 e^{-\nu\lambda_1 t} + \frac{M}{\mu\nu\lambda_1}\Big(1 - e^{-\nu\lambda_1 t}\Big) \le \rho_H^2, \quad \text{for every} \quad t \in [0, T_n),$$

where

$$\rho_H^2 = |v_0|^2 + \frac{M}{\mu\nu\lambda_1}.$$

As this bound holds uniformly in n for T_n arbitrarily large, we have global existence on the interval $[0,T]$, for all $T \geq 0$. Now, integrating (28) yields

$$|v^n(t)|^2 - |v_0|^2 + \nu \int_0^t \|v^n\|^2 \leq \frac{t}{\mu} M.$$

It follows that

$$\int_0^t \|v^n(\tau)\| d\tau \leq \sigma_V^2, \quad \text{for every} \quad t \in [0,T],$$

where

$$\sigma_V^2 = \frac{1}{\nu}|v_0|^2 + \frac{T}{\mu\nu} M.$$

Now, take inner products of (27) with Av^n to obtain

$$\frac{1}{2}\frac{d}{dt}\|v^n\|^2 + \nu|Av^n|^2 + \big(B(v^n,v^n),Av^n\big) = (g,Av^n) - \mu(I_h(v^n),Av^n).$$

Inequality (17) implies

$$\begin{aligned} \big|\big(B(v^n,v^n),Av^n\big)\big| &\leq c|v^n|^{1/2}\|v^n\||Av^n|^{3/2} \\ &\leq \frac{1}{4}\left(\frac{6^{3/4}}{\nu^{3/4}} c|v^n|^{1/2}\|v^n\|\right)^4 + \frac{3}{4}\left(\frac{\nu^{3/4}}{6^{3/4}}|Av^n|^{3/2}\right)^{4/3} \\ &\leq \frac{54c^4}{\nu^3}|v^n|^2\|v^n\|^4 + \frac{\nu}{8}|Av^n|^2. \end{aligned}$$

Furthermore,

$$\big|(g,Av^n)\big| \leq |g||Av^n| \leq \frac{2}{\nu}|g|^2 + \frac{\nu}{8}|Av^n|^2$$

and by (24) along with the assumption that $\mu c_0 h^2 \leq \nu$ we obtain

$$\begin{aligned} -\mu(I_h(v^n),Av^n) &= \mu(v^n - I_h(v^n),Av^n) - \mu\|v^n\|^2 \\ &\leq \frac{\mu^2}{\nu}\big|P_\sigma(v^n - I_h(v^n))\big|^2 + \frac{\nu}{4}|Av^n|^2 - \mu\|v^n\|^2 \\ &\leq \frac{\mu^2 c_0 h^2}{\nu}\|v^n\|^2 + \frac{\nu}{4}|Av^n|^2 - \mu\|v^n\|^2 \leq \frac{\nu}{4}|Av^n|^2. \end{aligned} \tag{30}$$

Therefore,

$$\frac{d}{dt}\|v^n\|^2 + \nu|Av^n|^2 \leq \frac{108c^4}{\nu^3}|v^n|^2\|v^n\|^4 + \frac{4}{\nu}|g|^2, \tag{31}$$

and consequently

$$\frac{d}{dt}\|v^n\|^2 - \frac{108c^4}{\nu^3}|v^n|^2\|v^n\|^4 \leq \frac{4}{\nu}|g|^2 \leq \frac{4}{\nu}M, \tag{32}$$

for every $t \in [0,T]$. Define

$$\psi^n(t) = \exp\Big\{ -\frac{108c^4}{\nu^3}\int_0^t |v^n|^2\|v^n\|^2 \Big\}. \tag{33}$$

Since

$$\int_0^t |v^n|^2\|v^n\|^2 \le \rho_H^2 \int_0^t \|v^n\|^2 \le \rho_H^2\sigma_V^2 < \infty, \quad \text{for every} \quad t \in [0,T],$$

we have that $\psi^n(t) > 0$ for every $t \in [0,T]$. Multiplying (29) by $\psi^n(t)$ and integrating yields

$$\|v^n(t)\|^2 \le \frac{1}{\psi^n(t)}\Big\{ \|v_0\|^2 + \frac{4}{\nu}M\int_0^t \psi^n(s)ds \Big\} \le \rho_V^2, \quad \text{for all} \quad t \in [0,T],$$

where

$$\rho_V^2 = \frac{1}{\psi^n(T)}\Big\{ \|v_0\|^2 + \frac{4T}{\nu}M \Big\}.$$

Now, integrating (31) yields

$$\|v^n(t)\|^2 - \|v_0\|^2 + \nu\int_0^t |Av^n|^2 \le \frac{108c^4}{\nu^3}\int_0^t \Big(|v^n|^2\|v^n\|^4 + \frac{4}{\nu}|g|^2 \Big) \le \sigma_{\mathcal{D}(A)}^2,$$

for every $t \in [0,T]$, where

$$\sigma_{\mathcal{D}(A)}^2 = \frac{108c^4T}{\nu^3}\ \Big\{ \rho_H^2\rho_V^4 + \frac{4}{\nu}M \Big\}.$$

The bounds ρ_V and $\sigma_{\mathcal{D}(A)}$ are uniform in n. Uniform bounds on $|dv/dt|$ then proceed in exactly the same way as for the two-dimensional Navier–Stokes equations. Since the estimates on the Galerkin solutions are uniform in n, Aubin's compactness theorem [1] allows one to extract subsequences in such a way that the limit v satisfies (23) and (26).

Next, we show that such solutions are unique and depend continuously on the initial data. Let v_1 and v_2 be two solutions for (23) both satisfying the conditions in (26). Choose K large enough such that $\|v_1\|^2 \le K$ and $\|v_2\|^2 \le K$ for almost every $t \in [0,T]$. Let $\delta = v_1 - v_2$. Then δ satisfies

$$\frac{d\delta}{dt} + \nu A\delta + B(v_1,\delta) + B(\delta,v_2) = -\mu P_\sigma I_h(\delta).$$

Taking inner product with $A\delta$ yields

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 + \nu|A\delta|^2 + \big(B(v_1,\delta),A\delta\big) + \big(B(\delta,v_2),A\delta\big) = -\mu(I_h(\delta),A\delta).$$

Here we used the fact that

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 = \Big(\frac{d\delta}{dt},A\delta\Big),$$

which can be justified by Lemma 1.2 in Chapter 3 of Temam [31] or Theorem 7.2 in Robinson [29] which is due to Lions–Magenes [26]. Estimate the right-hand side of this equation as in (30) to obtain

$$-\mu(I_h(\delta), A\delta) \leq \frac{\mu^2 c_0 h^2}{2\nu}\|\delta\|^2 + \frac{\nu}{2}|A\delta|^2 - \mu\|\delta\|^2 \leq \frac{\nu}{2}|A\delta|^2.$$

Here we have again used (24) and the hypothesis that $\mu c_0 h^2 \leq \nu$. It follows that

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 + \frac{\nu}{2}|A\delta|^2 \leq \big|(B(v_1,\delta), A\delta)\big| + \big|(B(\delta, v_2), A\delta)\big|. \tag{34}$$

The proof of uniqueness and continuity now proceeds as for the incompressible two-dimensional Navier–Stokes equations. In particular, estimate the nonlinear terms on the right-hand side of (34) using (17) and (18) as

$$\big|(B(v_1,\delta), A\delta)\big| \leq c|v_1|^{1/2}\|v_1\|^{1/2}\|\delta\|^{1/2}|A\delta|^{3/2} \leq \frac{27c^4K^2}{4\nu^3\lambda_1}\|\delta\|^2 + \frac{\nu}{4}|A\delta|^2,$$

and

$$\big|(B(\delta, v_2), A\delta)\big| \leq c|\delta|^{1/2}\|v_2\||A\delta|^{3/2} \leq \frac{27c^4K^2}{4\nu^3\lambda_1}\|\delta\|^2 + \frac{\nu}{4}|A\delta|^2.$$

Therefore,

$$\frac{d}{dt}\|\delta\|^2 \leq \frac{27c^4K^2}{2\nu^3\lambda_1}\|\delta\|^2, \quad \text{for all} \quad t \in [0,T].$$

Integrating yields

$$\|\delta(t)\|^2 \leq \|\delta_0\|^2 \exp\left\{\frac{27c^4K^2}{2\nu^3\lambda_1}t\right\}.$$

Thus, the solutions v to (23), which satisfy (26), also satisfy $v \in \mathcal{C}([0,T],V)$, and depend continuously on the initial data in the V norm. □

Theorem 6. *In the case of periodic boundary conditions suppose that I_h satisfies (25), and $\mu c_0 h^2 \leq \nu$, where c_0 is the constant appearing in (25). Then the continuous data assimilation equations (23) possess unique strong solutions that satisfy (26), for any $T > 0$. Furthermore, this solution is in $C\big([0,T],V\big)$ and depends continuously on the initial data v_0 in the V norm.*

Proof. The proof is similar to the proof of Theorem 5 but makes use of the identity (14) to obtain estimates on $\|v\|$ and $\int_0^t |Av|^2$ directly. □

The algorithm given by equation (23) for constructing the approximate solution v contains two parameters h and μ. The first parameter h has dimensions of length and corresponds to the resolution of the observational measurements represented by $I_h(u)$. Smaller values of h correspond to spatially more accurate resolved measurements. The relaxation parameter μ controls the rate at which the approximating solution v is forced toward the observable part of the reference solution u. Larger values of μ cause $I_h(v)$ to faster track $I_h(u)$. It is the parameter μ which distinguishes (23) from the previous methods of continuous data assimilation studied in [7], [22], [27] and [28].

The condition that $\mu c_0 h^2 \leq \nu$, given in Theorem 5, places a restriction on the size of μh^2 compared to the viscosity ν, sufficient to ensure the data assimilation equations are well-posed. This restriction is due to the fact that the the interpolant operator μI_h might generate large gradients and spatial oscillations ("spill over" to the fine scales) that need to be controlled (suppressed) by the viscosity term. Notice that in the case when $I_h = P_{m_h}$, where P_{m_h} is the orthogonal projection onto the linear sub-space spanned by the Fourier modes with wave numbers $|k| \leq m_h = \frac{1}{h}$, such oscillations are not generated, since $-(\mu I_h(v), v) = -\mu |P_{m_h} v|^2$ and $-(\mu I_h(v), Av) = -\mu \|P_{m_h} v\|^2$. Consequently, there is no restriction on μh^2 and μ can be taken arbitrary large. In the limit when $\mu \to \infty$ one obtains exactly the same algorithm introduced in [27] (see also [21]). In particular, one has $P_{m_h} v = P_{m_h} u$, and all that one needs to do is to solve for $q = (I - P_{m_h})v$, for which an explicit evolution equation is presented in [27].

Next, we give further conditions on h and μ which guarantee that the difference between the approximating solution v and the reference solution u converges to zero as $t \to \infty$.

3 No-slip Dirichlet Boundary Conditions Case

In this section we prove Theorem 1. We first recall the following generalized Gronwall inequality proved in Jones and Titi [23] (see also [15]).

Lemma 1 (Uniform Gronwall Inequality)**.** *Let α be a locally integrable real valued function on $(0,\infty)$ satisfying for some $0 < T < \infty$ the following conditions:*

$$\liminf_{t\to\infty} \int_t^{t+T} \alpha(s)ds > 0 \quad \textit{and} \quad \limsup_{t\to\infty} \int_t^{t+T} \max(-\alpha(s), 0)ds < \infty.$$

Suppose Y is an absolutely continuous non-negative function on $(0,\infty)$ such that $dY/dt + \alpha(t)Y \leq 0$ a.e. on $(0,\infty)$. Then $Y(t) \to 0$ exponentially as $t \to \infty$.

We now state and prove a lemma leading to our main result.

Proposition 1. *Let Ω be an open, bounded and connected set in $\mathbf{R}^2$ with C^2 boundary, and let u be a solution of the incompressible two-dimensional Navier–Stokes equations (19) on Ω with no-slip Dirichlet boundary conditions. Let v be the approximating solution given by equations (23). Then $|u - v| \to 0$, as $t \to \infty$, provided $\mu c_0 h^2 \leq \nu$ and $\mu \geq 5c^2 G^2 \nu \lambda_1$.*

Proof. Consider the time evolution of $w = u - v$. Since

$$B(u,u) - B(v,v) = B(u,w) + B(w,v) = B(u,w) + B(w,u) - B(w,w)$$

and

$$I_h(u) - I_h(v) = I_h(w),$$

subtracting equation (23) from equation (19) yields

$$\frac{dw}{dt} + \nu Aw + B(u,w) + B(w,u) - B(w,w) = -\mu P_\sigma I_h(w). \tag{35}$$

Taking the inner product of (35) with w, and using again (24), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 + \big(B(w,u),w\big) &= -\mu(I_h(w),w)\\ &= \mu(w - I_h(w),w) - \mu|w|^2\\ &\le \frac{\mu}{2}\big|P_\sigma(w - I_h(w))\big|^2 + \frac{\mu}{2}|w|^2 - \mu|w|^2\\ &\le \frac{\mu c_0 h^2}{2}\|w\|^2 - \frac{\mu}{2}|w|^2 \le \frac{\nu}{2}\|w\|^2 - \frac{\mu}{2}|w|^2.\end{aligned}$$

Since (16) implies

$$\big|(B(w,u),w)\big| \le c\|u\||w|\|w\| \le \frac{c^2}{2\nu}\|u\|^2|w|^2 + \frac{\nu}{2}\|w\|^2,$$

we obtain

$$\frac{d}{dt}|w|^2 + \Big(\mu - \frac{c^2}{\nu}\|u\|^2\Big)|w|^2 \le 0.$$

Denote

$$\alpha(t) = \mu - \frac{c^2}{\nu}\|u\|^2.$$

Taking $T = (\nu\lambda_1)^{-1}$ in Theorem 4 we have for $t \ge t_0$ that

$$\int_t^{t+T} \|v\|^2 \le 2(1 + T\nu\lambda_1)\nu G^2 = 4\nu G^2.$$

Thus

$$\liminf_{t\to\infty} \int_t^{t+T} \alpha(s)ds \ge \frac{\mu}{\nu\lambda_1} - 4c^2G^2 \ge c^2G^2 > 0,$$

and by Lemma 1 it follows that $|w| \to 0$, exponentially, as $t \to \infty$. ☐

Proof of Theorem 1. The hypothesis of Proposition 1 require that

$$\mu c_0 h^2 \le \nu \qquad \text{and} \qquad \mu \ge 5c^2G^2\nu\lambda_1.$$

Therefore

$$\frac{1}{h^2} \ge \frac{\mu c_0}{\nu} \ge c_1 G^2 \lambda_1 \tag{36}$$

where $c_1 = 5c_0c^2$. ☐

4 Periodic Boundary Conditions Case

In this section we prove Theorem 2. We begin with an elementary inequality which will be referred to in the sequel.

Lemma 2. *Let* $\phi(r) = r - \beta(1 + \log r)$ *where* $\beta > 0$*. Then*

$$\min\{\phi(r) : r \geq 1\} \geq -\beta \log \beta.$$

Proof. Note first that

$$\phi(1) = 1 - \beta \qquad \text{and} \qquad \lim_{r\to\infty} \phi(r) = \infty.$$

The derivative $\phi'(r) = 1 - \beta/r$ is zero if and only if $r = \beta$. Therefore

$$\min\{\phi(r) : r \geq 1\} = \begin{cases} 1 - \beta & \text{if } 0 < \beta \leq 1 \\ -\beta \log \beta & \text{if } 1 < \beta. \end{cases}$$

Observe that over the interval $0 < \beta \leq 1$ we have $1 - \beta \geq -\beta \log \beta$, which concludes our proof. □

We now state and prove a lemma leading to the proof of Theorem 2.

Proposition 2. *Let* $\Omega = [0, L]^2$*, for some fixed* $L > 0$*. Let* u *be a solution of the incompressible two-dimensional Navier–Stokes equations (19) on* Ω *equipped with periodic boundary conditions. Let* v *be the approximating solution given by equations (23), where* I_h *satisfies (24). Then* $\|u - v\| \to 0$*, as* $t \to \infty$*, provided* $\mu c_0 h^2 \leq \nu$ *and* $\mu \geq 3\nu\lambda_1\big(2c \log 2c^{3/2} + 4c \log(1 + G)\big)G$*.*

Proof. The proof makes use of the orthogonality properties (14) and (15) along with the Brézis–Gallouet inequality [6] which may be written as

$$\|u\|_{L^\infty(\Omega)} \leq c\|u\| \left\{1 + \log \frac{|Au|^2}{\lambda_1 \|u\|^2}\right\}, \tag{37}$$

which will allow us to obtain sharper estimates than for the case of no-slip boundary conditions (see also [32] for similar, and other, logarithmic estimates for the nonlinear term of the NSE).

Take the inner product of equation (35) with Aw and and use the orthogonality relations (14) and (15) to obtain

$$\frac{1}{2}\frac{d\|w\|^2}{dt} + \nu|Aw|^2 = \big(B(w, w), Au\big) - \mu(I_h(w), Aw).$$

Using (37) and the hypothesis $\mu c_0 h^2 \leq \nu$ we have

$$\big|\big(B(w, w), Au\big)\big| \leq c\|w\|^2 \left\{1 + \log \frac{|Aw|^2}{\lambda_1 \|w\|^2}\right\} |Au|,$$

and thanks to (24) as in (30) we have

$$-\mu(I_h(w), Aw) \le \frac{\mu^2 c_0 h^2}{2\nu}\|w\|^2 + \frac{\nu}{2}|Aw|^2 - \mu\|w\|^2 \le \frac{\nu}{2}|Aw|^2 - \frac{\mu}{2}\|w\|^2.$$

Therefore,

$$\frac{d\|w\|^2}{dt} + \nu|Aw|^2 \le \left(2c|Au|\left\{1 + \log\frac{|Aw|^2}{\lambda_1\|w\|^2}\right\} - \mu\right)\|w\|^2,$$

or

$$\frac{d\|w\|^2}{dt} + \left(\nu\lambda_1\frac{|Aw|^2}{\lambda_1\|w\|^2} - 2c|Au|\left\{1 + \log\frac{|Aw|^2}{\lambda_1\|w\|^2}\right\} + \mu\right)\|w\|^2 \le 0.$$

Now setting

$$\beta = \frac{2c|Au|}{\nu\lambda_1} \qquad \text{and} \qquad r = \frac{|Aw|^2}{\lambda_1\|w\|^2}$$

in Lemma 2, and noting that $r \ge 1$, by Poincaré's inequality (11), we obtain

$$\frac{d\|w\|^2}{dt} + \left\{\mu - 2c|Au|\log\frac{2c|Au|}{\nu\lambda_1}\right\}\|w\|^2 \le 0.$$

By (22) we estimate

$$2c\log\frac{2c|Au|}{\nu\lambda_1} \le c_3 + c_4\log(1+G) =: J, \tag{38}$$

where $c_3 = 2c\log 2c^{3/2}$ and $c_4 = 4c$. It follows that

$$\frac{d\|w\|^2}{dt} + \left\{\mu - J|Au|\right\}\|w\|^2 \le 0,$$

and by virtue of Young's inequality we have

$$\frac{d\|w\|^2}{dt} + \frac{1}{2}\left\{\mu - \frac{J^2}{\mu}|Au|^2\right\}\|w\|^2 \le 0.$$

Denote

$$\alpha(t) = \frac{1}{2}\left\{\mu - \frac{J^2}{\mu}|Au(t)|^2\right\}.$$

Taking $T = (\nu\lambda_1)^{-1}$ in Theorem 4 we have for $t \ge t_0$ that

$$\int_t^{t+T} |Au|^2 \le 2(1 + T\nu\lambda_1)\nu\lambda_1 G^2 = 4\nu\lambda_1 G^2.$$

By hypothesis $\mu \ge 3\nu\lambda_1 JG$. Thus,

$$\liminf_{t\to\infty}\int_t^{t+T}\alpha(s)ds \ge \frac{\mu}{2\nu\lambda_1} - \frac{2\nu\lambda_1}{\mu}J^2G^2 \ge \frac{5}{6}JG > 0,$$

and consequently $\|w\| \to 0$ exponentially, as $t \to \infty$. $\square$

Proposition 3. *Let $\Omega = [0, L]^2$, for some fixed $L > 0$. Let u be a solution of the incompressible two-dimensional Navier–Stokes equations (19) on Ω, equipped with periodic boundary conditions. Let v be the approximating solution given by equations (23) where I_h satisfies (25). Then $\|u - v\| \to 0$, as $t \to \infty$, provided $\mu c_0 h^2 \leq \nu$ and $\mu \geq 3\nu\lambda_1\big(2c\log 2c^{3/2} + 8c\log(1+G)\big)G$.*

Proof. The proof is the same as the proof of Proposition 2 except that we use (25) so that the estimate for $-\mu\big(I_h(w), Aw\big)$ has to be modified as

$$\begin{aligned} -\mu(I_h(w), Aw) &\leq \frac{\mu^2 c_0 h^2}{2\nu}\|w\|^2 + \frac{\mu^2 c_0^2 h^4}{4\nu}|Aw|^2 + \frac{\nu}{4}|Aw|^2 - \mu\|w\|^2 \\ &\leq \frac{\nu}{2}|Aw|^2 - \frac{\mu}{2}\|w\|^2. \end{aligned}$$

Then, the rest of the proof follows without change. ☐

Proof of Theorem 2. The hypothesis of Proposition 2 or Proposition 3 require that

$$\mu c_0 h^2 \leq \nu \qquad \text{and} \qquad \mu \geq 3\nu\lambda_1 J G.$$

Therefore,

$$\frac{1}{h^2} \geq \frac{\mu c_0}{\nu} \geq c_2\lambda_1 G\big(1 + \log(1+G)\big), \tag{39}$$

where $c_2 = 3\max\{c_3, c_4\}$. ☐

5 Conclusions

As shown in this paper, the algorithm given by (2), for constructing $v(t)$ from the observations $I_h(u(t))$, yields an approximation for $u(t)$ such that

$$\|u(t) - v(t)\|_{L^2(\Omega)} \to 0 \qquad \text{exponentially, as } t \to \infty, \tag{40}$$

provided the observations have fine enough spatial resolution. This result has the following consequence. To accurately predict $u(t)$ for time T into the future it is sufficient to have observational data $I_h(u(t))$ accumulated over an interval of time linearly proportional to T in the immediate past.

In particular, suppose it is desired to predict $u(t)$ with accuracy $\epsilon > 0$ on the interval $[t_1, t_1 + T^*]$, where t_1 is the present time and $T^* > 0$ determines how far into the future to make the prediction. Let h be small enough and μ large enough so that Theorem 1 implies (40). Thus, there is $\alpha > 0$ and a constant $C > 0$ such that

$$\|u(t) - v(t)\|_{L^2(\Omega)} \leq Ce^{-\alpha t} \qquad \text{for all} \qquad t \geq 0.$$

Now use $v(t_1)$ as the initial condition from which to make a future prediction.

Let w be a solution to (19) with initial condition $w(t_1) = v(t_1)$. Known results on continuous dependence on initial conditions, see, for example, [11], [21], [29] or [30], imply there is $\beta > 0$ such that

$$\|w(t) - u(t)\|_{L^2(\Omega)} \leq \|w(t_1) - u(t_1)\|_{L^2(\Omega)} e^{\beta(t - t_1)} \qquad \text{for} \qquad t \geq t_1.$$

Therefore

$$\|w(t) - u(t)\| \leq Ce^{-\alpha t_1 + \beta T} < \epsilon \qquad \text{for} \qquad t \in [t_1, t_1 + T]$$

provided $\alpha t_1 \geq \beta T + \ln(C/\epsilon)$. Thus $w(t)$ predicts $u(t)$ with accuracy ϵ on the interval $[t_1, t_1 + T]$.

Work is currently underway to numerically test Theorem 2 in the case of determining finite volume elements and nodes. A particular focus is how to tune the parameter μ. If μ is very large, then the effects of "spill over" into the fine scales may become significant, whereas if μ is small, then convergence of the approximate solution may be slow or not happen at all. Preliminary numerical simulations performed by Gesho [20] for the two-dimensional incompressible Navier–Stokes equations confirm that the continuous data assimilation algorithm given by equation (2) works directly, without additional filtering, for observational measurements at a discrete set of nodal points, where I_h is given by (8). Although our analytical estimates are comparable to previous results on data assimilation using Fourier modes, as with those results, sharper analysis appears to be required for sharp bounds on h. In particular, as with previous computational work (cf. [21],[27] and [28]) the approximating solution $v(t)$ converges to the reference solution $u(t)$ under much less stringent conditions than required by our theory.

The main advantage of introducing a control term that forces the approximate solution toward the reference solution is that we can rely on the viscous dissipation, already present in the dynamics, to filter the observational data (that is, to suppress the spatial oscillations, i.e. the "spill over" into the fine scales, that are generated by the coarse-mesh stabilizing term $\mu I_h(v)$). In addition to working for a general class of interpolant observables this technique also allows processing of observational data which contains stochastic noise. In particular, the same algorithm can be used to obtain an approximation $v(t)$ that converges (in some sense) to the reference solution $u(t)$, to within an error of the order of μ times the variance of the noise in the measurements. This work [3] is in progress.

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A Estimates for Nodal Interpolants

This appendix contains inequalities and estimates for interpolant operators that will be used for observables obtained from nodal measurements of the velocity field.

Consider a function $u \in H^2_{\text{per}}(\Omega)$, where $\Omega = [0, L]^2$ is a basic domain of periodicity. Let $\sqrt{N}$ be a positive integer and partition Ω into N squares with sides of length $h = L/\sqrt{N}$. Let $\mathcal{J} = \{\, 1, 2, \dots, \sqrt{N}\,\}^2$ and for each $\alpha \in \mathcal{J}$ define the semi-open square

$$Q_\alpha = [(j-1)h, jh) \times [(k-1)h, kh), \qquad \text{where} \qquad \alpha = (i, j) \in \mathcal{J}.$$

Moreover, for $\varphi \in L^1(\Omega)$ we denote by:

$$\langle \varphi \rangle = \frac{1}{L^2} \int_\Omega \varphi(x)\, dx.$$

Fix nodal points $x_\alpha \in Q_\alpha$, and suppose we are given the nodal values $u(x_\alpha)$, for every $\alpha \in \mathcal{J}$. Based on these nodal values, we define two interpolant operators, I_h and $\tilde{I}_h$, which we will show that they satisfy the approximation estimate (7). Specifically, define

$$\mathcal{I}_h(u)(x) = \sum_{\alpha \in \mathcal{J}} u(x_\alpha)\psi_\alpha(x), \tag{41}$$

and

$$I_h(u)(x) = \mathcal{I}_h(u)(x) - \langle \mathcal{I}_h(u) \rangle = \sum_{\alpha \in \mathcal{J}} u(x_\alpha)\Big(\psi_\alpha(x) - \langle \psi_\alpha \rangle\Big), \tag{42}$$

where

$$\psi_\alpha(x) = \sum_{(j,k) \in \mathbf{Z}^2} \chi_{Q_\alpha}(x_1 + jL, x_2 + kL),$$

is the L-periodic characteristic function of the semi-open square Q_α. Next, we define

$$\tilde{\mathcal{I}}_h(u)(x) = \sum_{\alpha \in \mathcal{J}} u(x_\alpha)\tilde{\psi}_\alpha(x), \tag{43}$$

and

$$\tilde{I}_h(u)(x) = \tilde{\mathcal{I}}_h(u)(x) - \langle \tilde{\mathcal{I}}_h(u) \rangle = \sum_{\alpha \in \mathcal{J}} u(x_\alpha)\Big(\tilde{\psi}_\alpha(x) - \langle \tilde{\psi}_\alpha \rangle\Big), \tag{44}$$

where

$$\tilde{\psi}_\alpha(x) = (\rho_\epsilon * \psi_\alpha)(x)$$

is a mollified version of ψ_α by the mollifier $\rho_\epsilon(x) = \epsilon^{-2}\rho(x/\epsilon)$. Here we take

$$\rho(\xi) = \begin{cases} K_0 \exp\Big(\dfrac{-1}{1 - |\xi|^2}\Big) & \text{for } |\xi| < 1 \\[2ex] 0 & \text{for } |\xi| \geq 1, \end{cases}$$

and

$$(K_0)^{-1} = \int_{|\xi|<1} \exp\Big(\frac{-1}{1-|\xi|^2}\Big)\, d\xi.$$

The mollification parameter ϵ will be chosen $\epsilon = \frac{h}{10}$.

Observe that $\langle I_h \rangle = \langle \tilde{I}_h \rangle = 0$, and that $\tilde{\mathcal{I}}_h(x)$ and $\tilde{I}_h(x)$ are C^∞ periodic functions.

We now state as Proposition 4 the estimate that was proved by Jones and Titi as inequality (6.2) in [24].

Proposition 4. *Let Q be a square with sides of length $\ell > 0$, and $\varphi \in H^2(Q)$. Then for every $x, y \in Q$ one has*

$$|\varphi(x) - \varphi(y)| \le 2\bigg(4\|\nabla\varphi\|^2_{L^2(Q)} + \ell^2 \Big\|\frac{\partial^2 \varphi}{\partial x_1 \partial x_2}\Big\|^2_{L^2(Q)}\bigg)^{1/2}.$$

We now use Proposition 4 to obtain estimate (7) concerning the accuracy of the interpolant operators $\mathcal{I}_h$ and I_h. Namely, we have

Proposition 5. *Suppose $u \in H^2_{\text{per}}(\Omega)$, and let $\mathcal{I}_h(u)$ and $I_h(u)$ be as in (41) and (42), respectively. Then*

(i) $\big\|u - \mathcal{I}_h(u)\big\|_{L^2(\Omega)} \le 4h\big\|\nabla u\big\|_{L^2(\Omega)} + 2h^2\Big\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big\|_{L^2(\Omega)}.$

(ii) $\big\|(u - \langle u \rangle) - I_h(u)\big\|_{L^2(\Omega)} \le 8h\big\|\nabla u\big\|_{L^2(\Omega)} + 4h^2\Big\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big\|_{L^2(\Omega)}.$

Moreover, if $\langle u \rangle = 0$, then there exists a constant $c > 0$ such that we can replace the term $\Big\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big\|_{L^2(\Omega)}$, in the above estimates, by $c\big\|\Delta u\big\|_{L^2(\Omega)}$.

Proof. First observe that

$$\sum_{\alpha \in \mathcal{J}} \psi_\alpha(x) = 1 \qquad \text{for every} \qquad x \in \mathbf{R}^2.$$

Therefore,

$$\begin{aligned}\|u - \mathcal{I}_h(u)\|^2_{L^2(\Omega)} &= \int_\Omega \Big|u(x) - \sum_{\alpha\in\mathcal{J}} u(x_\alpha)\psi_\alpha(x)\Big|^2 dx \\ &= \int_\Omega \Big|\sum_{\alpha\in\mathcal{J}} \big(u(x) - u(x_\alpha)\big)\psi_\alpha(x)\Big|^2 dx \\ &= \int_\Omega \sum_{\alpha,\beta\in\mathcal{J}} \big(u(x) - u(x_\alpha)\big)\cdot\big(u(x) - u(x_\beta)\big)\psi_\alpha(x)\psi_\beta(x)\, dx.\end{aligned}$$

Since

$$\psi_\alpha(x)\psi_\beta(x) = \begin{cases} 0 & \text{if } \alpha \ne \beta \\ \psi_\alpha(x) & \text{if } \alpha = \beta, \end{cases}$$

the above gives

$$\|u - \mathcal{I}_h(u)\|^2_{L^2(\Omega)} = \int_\Omega \sum_{\alpha \in \mathcal{J}} \big|u(x) - u(x_\alpha)\big|^2 \psi^2_\alpha(x)\, dx.$$

Applying Proposition 4 to the square $\overline{Q}_\alpha$ we obtain

$$|u(x) - u(x_\alpha)|^2 \le 4\bigg(4\|\nabla u\|^2_{L^2(Q_\alpha)} + h^2 \Big\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big\|^2_{L^2(Q_\alpha)}\bigg), \text{ for every}\, x \in \overline{Q}_\alpha.$$

Hence

$$\begin{aligned}\|u - \mathcal{I}_h(u)\|^2_{L^2(\Omega)} &\le \sum_{\alpha \in J} 4h^2 \bigg(4\|\nabla u\|^2_{L^2(Q_\alpha)} + h^2 \Big\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big\|^2_{L^2(Q_\alpha)}\bigg) \\ &\le 16h^2 \|\nabla u\|^2_{L^2(\Omega)} + 4h^4 \Big\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big\|^2_{L^2(\Omega)};\end{aligned}$$

which proves (i).

Next, we focus on proving (ii). By virtue of the Cauchy-Schwarz inequality we observe that

$$\big\|\langle u\rangle - \langle \mathcal{I}_h(u)\rangle\big\|_{L^2(\Omega)} \le \big\|u - \mathcal{I}_h(u)\big\|_{L^2(\Omega)}.$$

Therefore, (ii) follows from combining the triangle inequality together with the above observation, (41), (42) and part (i).

Finally, we recall the fact that for $\langle u\rangle = 0$ one has $\big\|u\big\|_{H^2(\Omega)} \le c\big\|\Delta u\big\|_{L^2(\Omega)}$, which concludes the proof. ☐

We now provide similar estimates for the C^∞ periodic interpolants $\tilde{\mathcal{I}}_h$ and $\tilde{I}_h$. In order to do this we make the assumptions that $N \ge 9$ and $\epsilon = \frac{h}{10}$. Define

$$\tilde{Q}_\alpha = [(j-2)h, (j+1)h] \times [(k-2)h, (k+1)h], \quad \text{where} \quad \alpha = (k, j) \in \mathcal{J}.$$

Since $\epsilon < h/2$ we obtain that

$$\mathcal{U}_\alpha = Q_\alpha + B(0, \epsilon) = \{\, x + y : x \in Q_\alpha \text{ and } |y| < \epsilon \,\} \subseteq \tilde{Q}_\alpha, \;\text{ for }\; \alpha \in \mathcal{J},$$

and

$$\mathcal{C}_\alpha = \mathcal{U}_\alpha \setminus \bigcup_{\beta \ne \alpha} \mathcal{U}_\beta \ne \emptyset, \quad \text{for} \quad \alpha \in \mathcal{J}.$$

The following two propositions now follow immediately from the definition of $\tilde{\psi}_\alpha$ and the fact that $\epsilon = \frac{h}{10}$.

Proposition 6. *The functions $\tilde{\psi}_\alpha$, for $\alpha \in \mathcal{J}$, form a smooth partition of unity satisfying*

(i) $0 \le \tilde{\psi}_\alpha(x) \le 1$, *and* $\operatorname{supp}(\tilde{\psi}_\alpha) \subseteq \big(\mathcal{U}_\alpha + (L\mathbf{Z})^2\big)$,

(ii) $\tilde{\psi}_\alpha(x) = 1$, *for all* $x \in \big(\mathcal{C}_\alpha + \big(L\mathbf{Z}\big)^2\big)$, *and* $\sum_{\alpha\in\mathcal{J}} \tilde{\psi}_\alpha(x) = 1$, *for all* $x \in \mathbf{R}^2$,

(iii) $\langle\tilde{\psi}_\alpha\rangle = \big(\frac{h}{L}\big)^2$ *and* $\frac{4}{5}h \le \|\tilde{\psi}_\alpha\|_{L^2(\Omega)} \le \frac{6}{5}h$,

(iv) $\mathrm{supp}(\nabla\tilde{\psi}_\alpha) \subseteq \big(\big(\mathcal{U}_\alpha \setminus \mathcal{C}_\alpha\big) + L\mathbf{Z}^2\big)$,

(v) $|\nabla\tilde{\psi}_\alpha(x)| \le ch^{-1}$, *and* $|\frac{\partial^2}{\partial x_i \partial x_j}\tilde{\psi}_\alpha(x)| \le ch^{-2}$, *for all* $x \in \mathbf{R}^2$,

(vi) $\|\nabla\tilde{\psi}_\alpha\|_{L^2(\Omega)} \le c$.

Proposition 7. *Let* $\mathcal{K} = \{\, 1-\sqrt{N}, -1, 0, 1, -1+\sqrt{N}\,\}^2$. *The functions* $\tilde{\psi}_\alpha$ *are nearly orthogonal in the following sense:*

(i) $\displaystyle\int_\Omega \tilde{\psi}_\alpha(x)\tilde{\psi}_\beta(x)\,dx = \int_\Omega \big(\nabla\tilde{\psi}_\alpha(x)\big)\cdot\big(\nabla\tilde{\psi}_\beta(x)\big)\,dx = 0$ *for all* $\alpha, \beta \in \mathcal{J}$ *with* $\beta - \alpha \notin \mathcal{K}$.

(ii) $\displaystyle\Big|\int_\Omega \tilde{\psi}_\alpha(x)\tilde{\psi}_\beta(x)\,dx\Big| \le (h+2\epsilon)^2 = \frac{36}{25}h^2$, *for all* $\alpha, \beta \in \mathcal{J}$ *with* $\beta - \alpha \in \mathcal{K}$.

(iii) $\displaystyle\Big|\int_\Omega \big(\nabla\tilde{\psi}_\alpha(x)\big)\cdot\big(\nabla\tilde{\psi}_\beta(x)\big)\,dx\Big| \le c$, *for all* $\alpha, \beta \in \mathcal{J}$ *with* $\beta - \alpha \in \mathcal{K}$.

We are now ready to prove estimates concerning the accuracy of the interpolant operators $\tilde{\mathcal{I}}_h$ and $\tilde{I}_h$, that are the analog to those of Proposition 5.

Proposition 8. *Suppose* $u \in H^2_{\mathrm{per}}(\Omega)$, *and let* $\tilde{\mathcal{I}}_h(u)$ *and* $\tilde{I}_h(u)$ *be as in (43) and (44), respectively. Then there exists a constant* $c > 0$ *such that*

(i) $\|u - \tilde{\mathcal{I}}_h(u)\|_{L^2(\Omega)} \le c\Big(h\|\nabla u\|_{L^2(\Omega)} + h^2\Big\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big\|_{L^2(\Omega)}\Big)$.

(ii) $\|\big(u - \langle u\rangle\big) - \tilde{I}_h(u)\|_{L^2(\Omega)} \le c\Big(h\|\nabla u\|_{L^2(\Omega)} + h^2\Big\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big\|_{L^2(\Omega)}\Big)$.

(iii) $\|\nabla\tilde{I}_h(u)\|_{L^2(\Omega)} = \|\nabla\tilde{\mathcal{I}}_h(u)\|_{L^2(\Omega)} \le c\Big(\|\nabla u\|_{L^2(\Omega)} + h\Big\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big\|_{L^2(\Omega)}\Big)$.

(iv)
$$\begin{aligned}\|\nabla(u - \tilde{I}_h(u))\|_{L^2(\Omega)} &= \|\nabla(u - \tilde{\mathcal{I}}_h(u))\|_{L^2(\Omega)} \\ &\le c\Big(\|\nabla u\|_{L^2(\Omega)} + h\Big\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big\|_{L^2(\Omega)}\Big).\end{aligned}$$

Moreover, if $\langle u\rangle = 0$, *then there exists a constant* $c > 0$ *such that we can replace the term* $\Big\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big\|_{L^2(\Omega)}$, *in the above estimates, by* $c\|\Delta u\|_{L^2(\Omega)}$.

Proof. In what follows we will use some of the properties stated in Proposition 6 and Proposition 7.

$$\begin{aligned}\|u-\tilde{\mathcal{I}}_h(u)\|^2_{L^2(\Omega)} &= \int_\Omega \Big|u(x)-\sum_{\alpha\in\mathcal{J}} u(x_\alpha)\tilde{\psi}_\alpha(x)\Big|^2 dx \\ &= \int_\Omega \Big|\sum_{\alpha\in\mathcal{J}} \big(u(x)-u(x_\alpha)\big)\tilde{\psi}_\alpha(x)\Big|^2 dx \\ &= \int_\Omega \sum_{\alpha,\beta\in\mathcal{J}} \big(u(x)-u(x_\alpha)\big)\cdot\big(u(x)-u(x_\beta)\big)\tilde{\psi}_\alpha(x)\tilde{\psi}_\beta(x)\,dx.\end{aligned}$$

Since $\tilde{\psi}_\alpha(x)\tilde{\psi}_\beta(x)=0$ for $\alpha-\beta\notin\mathcal{K}$ (see Proposition 6) we have

$$\begin{aligned}&\|u-\tilde{\mathcal{I}}_h(u)\|^2_{L^2(\Omega)} \\ &\le \int_\Omega \sum_{\gamma\in\mathcal{K}}\sum_{\alpha\in\mathcal{J}} |u(x)-u(x_\alpha)||u(x)-u(x_{\alpha+\gamma})||\tilde{\psi}_\alpha(x)\tilde{\psi}_{\alpha+\gamma}(x)\,dx \\ &\le \sum_{\gamma\in\mathcal{K}}\Big(\sum_{\alpha\in\mathcal{J}}\int_\Omega |u(x)-u(x_\alpha)|^2\Big(\tilde{\psi}_\alpha(x)\Big)^2 dx\Big)^{1/2} \\ &\qquad\qquad \times\Big(\sum_{\alpha\in\mathcal{J}}\int_\Omega |u(x)-u(x_{\alpha+\gamma})|^2\Big(\tilde{\psi}_{\alpha+\gamma}(x)\Big)^2 dx\Big)^{1/2} \\ &= 9\sum_{\alpha\in\mathcal{J}}\int_\Omega |u(x)-u(x_\alpha)|^2\Big(\tilde{\psi}_\alpha(x)\Big)^2 dx \le 9\sum_{\alpha\in\mathcal{J}}\int_{\mathcal{U}_\alpha} |u(x)-u(x_\alpha)|^2 dx \\ &\le 9\sum_{\alpha\in\mathcal{J}}\int_{\tilde{Q}_\alpha} |u(x)-u(x_\alpha)|^2 dx.\end{aligned}$$

Applying Proposition 4 to each of the squares $\tilde{Q}_\alpha$, for $\alpha\in\mathcal{J}$, the above implies

$$\begin{aligned}\|u-\tilde{I}_h(u)\|^2_{L^2(\Omega)} &\le 9\sum_{\alpha\in\mathcal{J}} 36h^2\Big(4\|\nabla u\|^2_{L^2(\tilde{Q}_\alpha)}+h^2\Big\|\frac{\partial^2 u}{\partial x_1\partial x_2}\Big\|^2_{L^2(\tilde{Q}_\alpha)}\Big) \\ &= 81\sum_{\alpha\in\mathcal{J}} 36h^2\Big(4\|\nabla u\|^2_{L^2(Q_\alpha)}+h^2\Big\|\frac{\partial^2 u}{\partial x_1\partial x_2}\Big\|^2_{L^2(Q_\alpha)}\Big) \\ &= \gamma_1 h^2\|\nabla u\|^2_{L^2(\Omega)}+\gamma_2 h^4\Big\|\frac{\partial u}{\partial x_1\partial x_2}\Big\|^2_{L^2(\Omega)}.\end{aligned}$$

where $\gamma_1=11664$ and $\gamma_2=2916$. By this we conclude (i).

The proof of (ii) follows from (i) by following the same lines as the proof of part (ii) of Proposition 5.

Next, we focus on the proof of (iii). To this end we implement some of the steps used in the proof of part (i), above, and the properties stated in Proposition 6.

$$
\begin{aligned}
\|\nabla\tilde{\mathcal{I}}_h(u)\|^2_{L^2(\Omega)} &= \|\sum_{\alpha\in\mathcal{J}} u(x_\alpha)\nabla\tilde{\psi}_\alpha(\cdot)\|^2_{L^2(\Omega)} \\
&= \|\sum_{\alpha\in\mathcal{J}} u(x_\alpha)\nabla\tilde{\psi}_\alpha(\cdot) - u(\cdot)\nabla\big(\sum_{\alpha\in\mathcal{J}}\tilde{\psi}_\alpha(\cdot)\big)\|^2_{L^2(\Omega)} \\
&= \|\sum_{\alpha\in\mathcal{J}} (u(x_\alpha)-u(\cdot))\nabla\tilde{\psi}_\alpha(\cdot)\|^2_{L^2(\Omega)} \\
&\le c\sum_{\alpha\in\mathcal{J}} \|(u(x_\alpha)-u(\cdot))\nabla\tilde{\psi}_\alpha(\cdot)\|^2_{L^2(\tilde{Q}_\alpha)} \\
&\le \frac{c}{h^2}\sum_{\alpha\in\mathcal{J}} \|(u(x_\alpha)-u(\cdot))\|^2_{L^2(\tilde{Q}_\alpha)}.
\end{aligned}
$$

Applying Proposition 4 to each of the squares $\tilde{Q}_\alpha$, for $\alpha\in\mathcal{J}$, in the above estimate to obtain

$$
\begin{aligned}
\|\nabla\tilde{\mathcal{I}}_h(u)\|^2_{L^2(\Omega)} &\le c\Big(\sum_{\alpha\in\mathcal{J}}\|\nabla u\|^2_{L^2(\tilde{Q}_\alpha)} + h^2\sum_{\alpha\in\mathcal{J}}\Big\|\frac{\partial^2 u}{\partial x_1\partial x_2}\Big\|^2_{L^2(\tilde{Q}_\alpha)}\Big) \\
&\le 9c\Big(\sum_{\alpha\in\mathcal{J}}\|\nabla u\|^2_{L^2(Q_\alpha)} + h^2\sum_{\alpha\in\mathcal{J}}\Big\|\frac{\partial^2 u}{\partial x_1\partial x_2}\Big\|^2_{L^2(Q_\alpha)}\Big) \\
&= c\Big(\|\nabla u\|^2_{L^2(\Omega)} + h^2\Big\|\frac{\partial^2 u}{\partial x_1\partial x_2}\Big\|^2_{L^2(\Omega)}\Big),
\end{aligned}
$$

which concludes the proof of point (iii).

Point (iv) is an obvious consequence of (iii). The rest of the proof is similar to Proposition 5. □

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