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CONTINUOUS DEPENDENCE OF FIXED POINT SETS

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ABSTRACT. The stability of the fixed point sets of a uniformly convergent sequence of set valued contractions is proved under the assumption that the maps are defined on a closed bounded subset B of Hilbert space and take values in the family of nonempty closed convex subsets of B.

In [**1**] the convergence of a sequence of fixed points of a convergent sequence of set valued contractions was investigated in a metric space setting. By restricting the underlying space to be a Hilbert space we prove the convergence of the sequence of fixed point sets of a convergent sequence of set valued contractions. This also extends a similar result for point valued maps [**2**, Theorem (10.1.1)] to the set valued case.

Let A be a closed bounded subset of a Hilbert space H, d the norm of H, and D the Hausdorff metric on the closed subsets of A generated by d. We assume that the family of set valued maps F_k, $k=0, 1, \cdots$, satisfy

(1) $F_k(x)$ is a nonempty closed convex subset of A for each $x \in A$.

(2) Each F_k is a set valued contraction, i.e. there is a $K \in [0, 1)$ such that $D(F_k(x), F_k(y)) \leq Kd(x, y)$ for $x, y \in A$ and $k=0, 1, \cdots$.

(3) $\lim_{k\to\infty} D(F_k(x), F_0(x))=0$ uniformly for all $x \in A$.

THEOREM. *If the conditions* 1–3 *are satisfied then the fixed point sets of the sequence* $\{F_k\}$, $k=1, 2, \cdots$, *converge to the fixed point set of* F_0 *in the Hausdorff metric* D.

Before proving the theorem some lemmas on the closest point projection map associated with each F_k are required. For $k=0, 1, \cdots$, define the maps f_k by $f_k(x)=\{\text{unique closest point in } F_k(x) \text{ to } x\}$ for $x \in A$. The iterates of each f_k are denoted by f_k^n, $n=2, 3, \cdots$. The distance between any $x \in A$ and closed subset C of A will be $d(x, C)=\inf_{c\in C} d(x, c)$.

The following result was given in [**3**, Lemma 5] for a finite dimensional space, but the statement and proof are valid for any Hilbert space.

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LEMMA 1. *If E and F are closed convex subsets of H and e and f are the closest points in E and F to a point $v \in H$, then*

$$d(e,f) \leqq (h^2 + 4hl)^{1/2}, \quad \textit{where } h = D(E, F) \textit{ and } l = d(v, E).$$

For $s \geqq 0$ we define the continuous monotone increasing function $g(s)=(s^2+4sr)^{1/2}$, where r is the diameter of A.

LEMMA 2. *The maps f_k, $k=0, 1, \cdots$, are equicontinuous on A.*

PROOF. For any $k=0, 1, \cdots$, and $x, y \in A$ let q denote the closest point in $F_k(x)$ to y. Consider the inequality

$$d(f_k(x), f_k(y)) \leqq d(f_k(x), q) + d(q, f_k(y)). \tag{1}$$

The term $d(f_k(x), q)$ is bounded by $d(x, y)$, since projection onto a closed convex set is nonexpansive. Lemma 1 implies that $d(q, f_k(y))$ is bounded by $g(D(F_k(x), F_k(y)))$. Since F_k is a set valued contraction and g is monotone we have $g(D(F_k(x), F_k(y))) \leqq g(Kd(x, y))$. The inequality (1) can then be written as $d(f_k(x), f_k(y)) \leqq d(x, y) + g(Kd(x, y))$. The map g is continuous and $g(s) \to 0$ as $s \to 0$. Therefore, the latter inequality proves the lemma.

LEMMA 3. *The sequence of maps $\{f_k^n\}$, $k=1, 2, \cdots$, converges uniformly on A to f_0^n, for each n.*

PROOF. For $n=1$ the uniform convergence follows from,

$$d(f_k(x), f_0(x)) \leqq g(D(F_k(x), F_0(x)))$$

and the uniform convergence of the maps F_k to F_0. Make the induction assumption that f_k^{n-1}, $k=1, 2, \cdots$, converges uniformly on A to f_0^{n-1}. Given $\varepsilon > 0$ there is a $\delta > 0$ such that $u, v \in A$ and $d(u, v) < \delta$ implies $d(f_k(u), f_k(v)) < \varepsilon/2$, for all k, in view of the equicontinuity of the f_k. The uniform convergence of the sequences $\{f_k\}$ and $\{f_k^{n-1}\}$ to f_0 and f_0^{n-1} permits the choice of an integer N such that $k \geqq N$ implies

$$d(f_k^{n-1}(x), f_0^{n-1}(x)) < \delta \quad \text{and} \quad d(f_k(x), f_0(x)) < \varepsilon/2$$

for all $x \in A$. Considering the inequality

$$d(f_k^n(x), f_0^n(x)) \leqq d(f_k(f_k^{n-1}(x)), f_k(f_0^{n-1}(x))) + d(f_k(f_0^{n-1}(x)), f_0(f_0^{n-1}(x)))$$

the remarks above imply that for $k \geqq N$ the right side of the inequality is strictly less than ε. This proves uniform convergence for all n.

PROOF OF THE THEOREM. By a result of Nadler [**1**, Theorem 5] the sequence of iterates $\{f_k^n(x)\}$ converges to a fixed point of F_k for $k=0, 1, \cdots$, and all $x \in A$. If P_k denotes the fixed point set of f_k, then this same result contains the estimate

$$d(f_k^n(x), P_k) \leqq \sum_{i=n}^{\infty} (r + i)K^i. \tag{2}$$

Each P_k is a closed subset and can be written as

$$P_k = \left\{ y \in A : \lim_{n \to \infty} f_k^n(x) = y,\ x \in A \right\}.$$

Given $\varepsilon > 0$ choose any $x \in A$ and let $P_k(x) = \lim_{n\to\infty} f_k^n(x)$, $k = 0, 1, \cdots$. Consider the inequality

$$\begin{aligned} &d(P_k(x), P_0(x)) \\ &\quad \leqq d(P_k(x), f_k^n(x)) + d(f_k^n(x), f_0^n(x)) + d(f_0^n(x), P_0(x)). \end{aligned} \tag{3}$$

By the estimate (2) we can choose an integer N such that $d(f_k^N(x), P_k(x)) < \varepsilon/3$ for $k = 0, 1, \cdots$, and all $x \in A$. The uniform convergence of $\{f_k^N\}$ to f_0^N permits the choice of an integer M such that $k \geqq M$ implies $d(f_k^N(x), f_0^N(x)) < \varepsilon/3$ for all $x \in A$. Therefore, by (3), $d(P_k(x), P_0(x)) < \varepsilon$ for all $x \in A$. Since the points $P_k(x)$ range over P_k as x ranges over A, we have shown that $D(P_k, P_0) < \varepsilon$ for $k \geqq M$. This proves convergence of P_k to P_0 in the D metric.

References

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