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CONTINUOUS DEPENDENCE RESULTS FOR A PROBLEM IN PENETRATIVE CONVECTION

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Abstract. Continuous dependence inequalities are derived for a system of equations that models penetrative convection in a thermally conducting viscous fluid with a linear buoyancy law. Both the forward-in-time problem and the improperly posed backward-in-time problem are analyzed. These results indicate that solutions depend continuously on a parameter in the boundary data.

1. Introduction. The phenomenon of penetrative convection has attracted the attention of a number of investigators in diverse fields. Various models have been formulated to describe this process. A recent monograph by Straughan [11] reviews some of the mathematical aspects of these models, with an emphasis on the derivation of continuous dependence results for solutions of properly and improperly posed initial-boundary value problems. Most of these convection models consist of coupled systems of partial differential equations that include the Navier-Stokes equations. Consequently, a study of penetrative convection often rests on an analysis of the Navier-Stokes system. A number of investigations on the question of continuous dependence of solutions to this system on various types of data, both forward and backward in time, have appeared recently in the literature. These include the study of Franchi and Straughan [4] who establish continuous dependence on the body force for solutions of the system backward in time, the work of Song [9] on well-posed problems, and a paper by Ames and Payne [2] that deals with stabilizing the Navier-Stokes system backward in time against errors in the body force, coefficient of kinematic viscosity, and boundary data. The earliest study of the Navier-Stokes system backward in time has been attributed to Serrin [8] who established uniqueness of the solution to the equations defined on a bounded spatial domain. The task of stabilizing the final value problem against errors in the final data was first addressed by Knops and Payne [5]. In this paper, we continue to investigate the Navier-Stokes system but couple it with a convective heat equation. The resulting system models

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penetrative convection in a thermally conducting viscous fluid with a linear buoyancy law.

Our goal then is to derive inequalities indicating that a solution to the Navier-Stokes equations coupled with the convective heat equation depends continuously on a parameter specified in the boundary data. We shall consider both the forward-in-time problem and the improperly posed backward-in-time problem on bounded spatial domains. Let $\Omega \subseteq \mathbb{R}^n$ $(n = 2, 3)$ be such a domain with boundary Γ. If $\Omega \subset \mathbb{R}^2$, then we require Γ to be Lipschitz while for domains in $\mathbb{R}^3$, we assume Γ is a C^2 surface. We consider the system

$$v_{i,t} + v_j v_{i,j} = -p_{,i} + \Delta v_i + b_i T, \tag{1.1}$$

$$v_{i,i} = 0, \tag{1.2}$$

$$T_{,t} + v_i T_{,i} = \Delta T \tag{1.3}$$

defined on $\Omega \times (0, t_0)$. The auxiliary conditions we associate with these equations take the form of the boundary conditions

$$\left.\begin{aligned} v_i &= 0 \\ T_{,i} n_i + \kappa T &= 0 \end{aligned}\right\} \quad \text{on } \Gamma \times [0, t_0] \tag{1.4}$$

and the initial conditions

$$\left.\begin{aligned} v_i(x, 0) &= v_i^0(x) \\ T(x, 0) &= T_0(x) \end{aligned}\right\} \quad \text{for } x \in \Omega. \tag{1.5}$$

Here $v_i(x, t)$ is the ith $(i = 1, 2, 3)$ component of the fluid velocity, p is the modified pressure, $T(x, t)$ is the temperature, and $b_i(x, t)$ is the body force. Without loss of generality, we have assumed that the coefficient of viscosity and thermal diffusivity are unity. Standard indicial notation is used throughout this paper and differentiation is denoted by a comma. We shall restrict our attention to classical solutions of (1.1)–(1.5), which we assume to exist on the time interval $(0, t_0)$. In addition, we assume, for the problem in $\mathbb{R}^3$, that the data are sufficiently smooth and compatible so that the partial differential equation (1.3) is satisfied on $t = 0$.

In the next section we derive some supplementary inequalities that we need to establish our main results. Sections 3 and 4 are devoted to the forward-in-time problem defined for $\Omega \subseteq \mathbb{R}^3$ and $\Omega \subseteq \mathbb{R}^2$, respectively. We turn to the ill-posed backward-in-time problem in Sec. 5. Some of the bounds we need to establish continuous dependence on the boundary parameter for solutions of the forward-in-time problem are included in the appendix.

As is to be expected, the results for the forward problem in $\mathbb{R}^3$ are much less satisfactory than those in $\mathbb{R}^2$. To derive the continuous dependence result in $\mathbb{R}^3$ we require more smoothness and compatibility, and some restriction on the size of the data is required if the result is to hold on $(0, t_0)$. For the ill-posed backward-in-time problem, solutions must of course be more severely constrained.

2. Useful inequalities. In the subsequent sections we will make frequent use of an assortment of integral inequalities. We list here a number of those which will be needed in the later sections.

We remark first that on the interval $(0, t_0)$ where v_i is assumed to be bounded, T satisfies the maximum principle for parabolic equations, and it follows that

$$|T| \leq \max_{\Omega} |T_0|. \tag{2.1}$$

The maximum value of $|T|$ cannot occur on $\partial\Omega$ for $0 \leq t \leq t_0$, since if it did occur at a point P on $\partial\Omega$ at time $\hat{t}$ then if $T(P, \hat{t}) > 0$ it follows that $\frac{\partial T}{\partial n}(P, \hat{t}) < 0$, a contradiction. Similarly if $T(P, \hat{t}) < 0$ then $\frac{\partial T}{\partial n}(P, \hat{t}) > 0$, again a contradiction. Thus, the maximum value of $|T|$ must occur on $t = 0$.

We now derive two L_p inequalities for T. The second of these inequalities will be used with various values of p in our proofs in Secs. 3 and 4. We show, in particular, that

$$\int_{\Omega} T^{2p} dx + 2p\kappa \int_0^t \int_{\Gamma} T^{2p}\, d\sigma\, d\eta + 2p(2p-1) \int_0^t \int_{\Omega} T^{2p-2} T_{,i} T_{,i}\; dx\, d\eta = \int_{\Omega} T_0^{2p}\, dx \tag{2.2}$$

and

$$\int_{\Omega} T^{2p}\, dx \leq \left[\int_{\Omega} T_0^{2p}\, dx\right] \exp\{-2p^{-1}(2p-1)\mu(\kappa, p)t\}, \tag{2.3}$$

where

$$\mu(\kappa, p) = \inf_{\varphi \in \mathcal{D}} \frac{\int_{\Omega} \varphi_{,i}\, \varphi_{i}\, dx + \frac{p^2\kappa}{2p-1} \int_{\Gamma} \varphi^2 d\sigma}{\int_{\Omega} \varphi^2 dx}, \tag{2.4}$$

and $\mathcal{D}$ is the class of Dirichlet integrable functions having L_2 boundary integrals. Bounds for $\mu(k, p)$ have been given by Payne and Weinberger [7], Sperb [10], etc.

The inequalities (2.2) and (2.3) follow from the observation that

$$\frac{d}{dt} \int_{\Omega} T^{2p} dx = -2p\kappa \int_{\Gamma} T^{2p} d\sigma - 2p(2p-1) \int_{\Omega} T^{2p-2} T_{,i} T_{,i}\; dx. \tag{2.5}$$

We have used the differential equation for T and (1.2). An integration of (2.5) leads to (2.2) while the rewriting of (2.5) as

$$\frac{d}{dt} \int_{\Omega} T^{2p} dx = -2p\kappa \int_{\Gamma} [T^2]^p d\sigma - \frac{2(2p-1)}{p} \int_{\Omega} (T^p)_{,i} (T^p)_{,i}\; dx \tag{2.6}$$

and use of (2.4) leads to

$$\frac{d}{dt} \int_{\Omega} T^{2p} dx \leq -2p^{-1}(2p-1)\mu(\kappa, p) \int_{\Omega} T^{2p}\, dx. \tag{2.7}$$

An integration of (2.7) then yields (2.3).

We next derive a differential inequality for the L_2 integral of the velocity. Proceeding directly we have

$$\begin{aligned} \frac{d}{dt}\left[\int_{\Omega} v_i v_i\, dx\right] &= 2 \int_{\Omega} v_i\{-p_{,i} + \Delta v_i + b_i T - v_j v_{i,j}\}\, dx \\ &= -2 \int_{\Omega} v_{i,j} v_{i,j} dx + 2 \int_{\Omega} b_i v_i T\, dx \\ &\leq -2 \int_{\Omega} v_{i,j} v_{i,j} dx + 2[b_i b_i]^{1/2}_{\max} \left\{\int_{\Omega} v_i v_i\, dx \int_{\Omega} T^2 dx\right\}^{1/2}. \end{aligned} \tag{2.8}$$

By rescaling T if necessary we may assume that

$$b_i b_i < 1. \tag{2.9}$$

Thus for any positive constant γ_1 it follows that

$$\begin{aligned} \frac{d}{dt}\left[\int_\Omega v_i v_i\, dx\right] &\leq -2\int_\Omega v_{i,j}v_{i,j}\, dx + \gamma_1 \int_\Omega v_i v_i dx + \gamma_1^{-1}\int_\Omega T^2\, dx \\ &\leq -(2\lambda - \gamma_1)\int_\Omega v_i v_i\, dx + \gamma_1^{-1}\left[\int_\Omega T_0^2\, dx\right]\exp\{-\mu(\kappa,1)t\}. \end{aligned} \tag{2.10}$$

We have used (2.2) with $p = 1$ and introduced the constant λ which is defined as

$$\lambda = \inf \frac{\int_\Omega \psi_{i,j}\psi_{i,j}\, dx}{\int_\Omega \psi_i \psi_i\, dx} \tag{2.11}$$

where ψ_i lies in the class of divergence-free piecewise continuously differentiable functions that vanish on Γ. Velte [12] has shown that in $\mathbb{R}^2, \lambda$ is precisely the first eigenvalue in the corresponding plate buckling problem. In higher dimensions λ remains larger than the corresponding first eigenvalue of the Laplacian. So in the next section we use this first eigenvalue of the Laplacian, the so-called first clamped membrane eigenvalue, for our constant λ.

The inequality (2.10) is of the form

$$\frac{d\Psi}{dt} \leq -k_1\Psi + k_2 e^{-\mu(\kappa,1)t}, \tag{2.12}$$

which integrates to give (suppressing the arguments of μ)

$$\Psi(t) \leq \Psi(0)e^{-k_1 t} + \frac{k_2}{k_1 - \mu}[e^{-\mu t} - e^{-k_1 t}] \tag{2.13}$$

or

$$\int_\Omega v_i v_i\, dx \leq \int_\Omega v_i^0 v_i^0\, dx\ e^{-(2\lambda-\gamma_1)t} + \frac{\left(\int_\Omega T_0^2 dx\right)}{\gamma_1[2\lambda - \gamma_1 - \mu]}[e^{-\mu t} - e^{-(2\lambda-\gamma_1)t}]. \tag{2.14}$$

Choosing

$$\gamma_1 = 2\lambda - \mu(\kappa, 1) \tag{2.15}$$

we are led to the decay bound

$$\int_\Omega v_i v_i\, dx \leq \int_\Omega v_i^0 v_i^0\, dx\ e^{-\mu(\kappa,1)t} + \frac{t}{[2\lambda - \mu(\kappa,1)]}\int_\Omega T_0^2\, dx\ e^{-\mu(\kappa,1)t}. \tag{2.16}$$

Returning to (2.8) we observe that

$$\begin{aligned} \frac{d}{dt}\left[\int_\Omega v_i v_i\, dx\right] =& -2\int_\Omega v_{i,j}v_{i,j}\, dx + 2\left\{\int_\Omega T_0^2\, dx\ e^{-\mu(\kappa,1)t}\right\}^{1/2} \\ &\times \left\{\int_\Omega v_i^0 v_i^0\, dx\ e^{-\mu(\kappa,1)t} + \frac{t}{[2\lambda - \mu(\kappa,1)]}\int_\Omega T_0^2\, dx\ e^{-\mu(\kappa,1)t}\right\}^{1/2} \end{aligned} \tag{2.17}$$

or upon integration

$$\begin{aligned} 2\int_0^t \int_\Omega v_{i,j} v_{i,j}\, dx\, d\eta + \int_\Omega v_i v_i\, dx \\ \leq \int_\Omega v_i^0 v_i^0\, dx + K_1 \int_0^t \eta e^{-\mu(\kappa,1)\eta}\, d\eta + K_2 \int_0^t e^{-\mu(\kappa,1)\eta}\, d\eta \end{aligned} \tag{2.18}$$

for computable K_1 and K_2. It follows then that for a computable K (which depends on the data but is independent of t_0)

$$2\int_0^t \int_\Omega v_{i,j} v_{i,j}\, dx\, d\eta + \int_\Omega v_i v_i\, dx \leq K. \tag{2.19}$$

In the next two sections we also require a bound for the quantity $\int_0^t \int_\Gamma e^{-\beta(t-\eta)} T^2\, d\sigma\, d\eta$ for specific values of the positive constant β. To find such a bound we first note that

$$\frac{1}{2}\int_\Gamma T^2\, d\sigma + \int_\Omega T_{,i} T_{,i}\, dx = -\int_\Omega T \Delta T\, dx = -\int_\Omega T T_{,t}\, dx, \tag{2.20}$$

from which we have

$$\begin{aligned} \int_0^t \int_\Gamma e^{-\beta(t-\eta)} T^2\, d\sigma\, d\eta &\leq -2\int_0^t \int_\Omega e^{-\beta(t-\eta)} T T_{,\eta}\, dx\, d\eta \\ &\leq \left[\int_\Omega T_0^2\, dx\right] e^{-\beta t} + \beta \int_0^t \int_\Omega e^{-\beta(t-\eta)} T^2\, dx\, d\eta. \end{aligned} \tag{2.21}$$

Using (2.3) in the last term we obtain

$$\int_0^t \int_\Gamma e^{-\beta(t-\eta)} T^2\, d\sigma\, d\eta \leq \frac{2\mu(\kappa,1)}{[2\mu(\kappa,1) - \beta]} \left[\int_\Omega T_0^2\, dx\right] e^{-\beta t}, \tag{2.22}$$

the bound that we will be using in the next two sections.

Finally, we list two Sobolev inequalities which will prove useful. For Dirichlet integrable functions, φ, defined on Ω and vanishing on Γ we have

$$\int_\Omega \varphi^4\, dx \leq \Lambda^2 \left(\int_\Omega \varphi^2\, dx\right)\left(\int_\Omega \varphi_{,i} \varphi_{,i}\, dx\right) \quad \text{in } \mathbb{R}^2 \tag{2.23}$$

and

$$\int_\Omega \varphi^4\, dx \leq \nu^2 \left(\int_\Omega \varphi^2\, dx\right)^{1/2} \left(\int_\Omega \varphi_{,i} \varphi_{,i}\, dx\right)^{3/2} \quad \text{in } \mathbb{R}^3. \tag{2.24}$$

For specific values of Λ and ν, see [1]. It has been known for some time that $\Lambda \leq 2^{-1/2}$ and $\nu \leq 3^{-3/4}$, [8].

Bounds for various other norms of v_i and T will be derived in the Appendix.

3. The forward-in-time problem in $\mathbb{R}^3$. Let (v_i^*, T^*) be the solution of (1.1)–(1.3) with $p = p^*$ and satisfying

$$\left.\begin{aligned} v_i^* &= 0 \\ T,_i^* n_i + \kappa^* T^* &= 0 \end{aligned}\right\} \quad \text{on } \Gamma \times [0, t_0] \tag{3.1}$$

as well as

$$\left.\begin{aligned} v_i^*(x,0) &= v_i^0(x) \\ T^*(x,0) &= T_0(x) \end{aligned}\right\} \quad \text{for } x \in \Omega. \tag{3.2}$$

Our goal is to show that if κ^* is close to κ then the perturbation (u_i, θ) defined as

$$\begin{aligned} u_i &= v_i^* - v_i, \\ \theta &= T^* - T \end{aligned} \tag{3.3}$$

will be small in some appropriate measure. Specifically, we derive an inequality of the form

$$\int_\Omega [a u_i u_i + \theta^2]\, dx \le (\kappa^* - \kappa)^2 F(t) \tag{3.4}$$

for some positive constant a and a function $F(t)$.

We first note that u_i and θ satisfy

$$u_{i,t} + v_j^* u_{i,j} + u_j v_{i,j} = -P_{,i} + \Delta u_i + b_i \theta \tag{3.5}$$

$$u_{i,i} = 0 \qquad \text{in } \Omega \times (0, t_0) \tag{3.6}$$

$$\theta_{,t} + v_i^* \theta_{,i} + u_i T_{,i} = \Delta\theta \tag{3.7}$$

with

$$\left.\begin{aligned} u_i &= 0 \\ \theta_{,i} n_i + \kappa\theta &= -\alpha T^* \end{aligned}\right\} \quad \text{on } \Gamma \times [0, t_0] \tag{3.8}$$

and

$$\theta(x,0) = u_i(x,0) = 0, \quad x \in \Omega. \tag{3.9}$$

Here $P = p^* - p$ and

$$\alpha = \kappa^* - \kappa. \tag{3.10}$$

For some positive constant a, to be determined later, we now define the function $\Phi(t)$ given by

$$\Phi(t) = \int_\Omega [\theta^2 + a u_i u_i]\, dx. \tag{3.11}$$

We will derive a first-order differential inequality for $\Phi(t)$ which will yield a result of the type (3.4).

Differentiating (3.11) we have

$$\begin{aligned} \frac{d\Phi}{dt} &= 2\int_\Omega [\theta\{\Delta\theta - v_i^*\theta_{,i} - u_i T_{,i}\} + a u_i\{-P_{,i} + \Delta u_i + b_i\theta - v_j^* u_{i,j} - u_j v_{i,j}\}]\, dx \\ &= -2\int_\Omega \theta_{,i}\theta_{,i}\, dx - 2\kappa\int_\Gamma \theta^2\, d\sigma - 2\alpha\int_\Gamma T^*\theta\, d\sigma + 2\int_\Omega u_i\theta_{,i} T\, dx \\ &\quad - 2a\left\{\int_\Omega u_{i,j}u_{i,j}\, dx + \int_\Omega u_i u_j v_{i,j}\, dx - \int_\Omega b_i\theta u_i\, dx\right\}. \end{aligned} \tag{3.12}$$

We have used (3.5)–(3.10). Let us consider some of the terms separately. For instance, from the Schwarz and Hölder inequalities and (2.24) we have

$$
\begin{aligned}
\left|\int_\Omega u_i\theta_{,i} T\,dx\right| &\le \left[\int_\Omega \theta_{,i}\theta_{,i}\,dx\right]^{1/2}\left[\int_\Omega u_iu_iT^2\,dx\right]^{1/2}\\
&\le \left[\int_\Omega \theta_{,i}\theta_{,i}\,dx\right]^{1/2}\left[\int_\Omega |u_iu_i|^{3/2}\,dx\right]^{1/3}\left[\int_\Omega T^6\,dx\right]^{1/6}\\
&\le \left[\int_\Omega \theta_{,i}\theta_{,i}\,dx\right]^{1/2}\left[\int_\Omega u_iu_i\,dx\int_\Omega [u_iu_i]^2\,dx\right]^{1/6}\left[\int_\Omega T^6\,dx\right]^{1/6}\\
&\le \nu^{1/3}\left[\int_\Omega \theta_{,i}\theta_{,i}\,dx\right]^{1/2}\left[\int_\Omega u_iu_i\,dx\int_\Omega u_{i,j}u_{i,j}\,dx\right]^{1/4}\left[\int_\Omega T^6\,dx\right]^{1/6}.
\end{aligned}
\tag{3.13}
$$

Thus

$$
\begin{aligned}
&\left|\int_\Omega u_i\theta_{,i} T\,dx\right|\\
&\le \nu^{1/3}\left[\int_\Omega \theta_{,i}\theta_{,i}\,dx\right]^{1/2}\left[\int_\Omega u_iu_i\,dx\int_\Omega u_{i,j}u_{i,j}\,dx\left\{\int_\Omega T^6\,dx\right\}^{2/3}\right]^{1/4}\\
&\le \nu^{1/3}\frac{\gamma_1^{-1}}{2}\int_\Omega \theta_{,i}\theta_{,i}\,dx+\frac{\nu^{1/3}\gamma_1}{2}\left[\int_\Omega u_iu_i\,dx\int_\Omega u_{i,j}u_{i,j}\,dx\left(\int_\Omega T^6\,dx\right)^{2/3}\right]^{1/2}\\
&\le \nu^{1/3}\frac{\gamma_1^{-1}}{2}\int_\Omega \theta_{,i}\theta_{,i}\,dx+\nu^{1/3}\frac{\gamma_1\gamma_2}{4}\int_\Omega u_{i,j}u_{i,j}\,dx\\
&\quad+\frac{\nu^{1/3}\gamma_1\gamma_2^{-1}}{4}\left(\int_\Omega T^6\,dx\right)^{2/3}\int_\Omega u_iu_i\,dx,
\end{aligned}
\tag{3.14}
$$

for arbitrary positive constants γ_1 and γ_2. We will use inequality (2.3) to bound $\int_\Omega T^6\,dx$, but first let us bound other terms in (3.12). Clearly

$$
\begin{aligned}
\left|\int_\Omega u_iu_jv_{i,j}\,dx\right| &\le \left[\int_\Omega [u_iu_i]^2\,dx\int_\Omega v_{k,j}v_{k,j}\,dx\right]^{1/2}\\
&\le \nu\left[\int_\Omega u_iu_i\,dx\right]^{1/4}\left[\int_\Omega u_{k,j}u_{k,j}\,dx\right]^{3/4}\left[\int_\Omega v_{l,m}v_{l,m}\,dx\right]^{1/2}\\
&\le \frac{\nu}{4}\left\{\gamma_3^3\int_\Omega u_iu_i\,dx\left[\int_\Omega v_{k,j}v_{k,j}\,dx\right]^2+3\gamma_3^{-1}\int_\Omega u_{i,j}u_{i,j}\,dx\right\}
\end{aligned}
\tag{3.15}
$$

for arbitrary positive γ_3. We have used Young's inequality in the last step. We note also that

$$
\left|\alpha\int_\Gamma T^*\theta\,d\sigma\right| \le \frac{\alpha^2}{2}\gamma_4\int_\Gamma [T^*]^2\,d\sigma+\frac{\gamma_4^{-1}}{2}\int_\Gamma \theta^2\,d\sigma \tag{3.16}
$$

and

$$\left|\int_\Omega b_i\theta u_i\,dx\right| \le \frac{\gamma_5}{2}\int_\Omega \theta^2\,dx + \frac{\gamma_5^{-1}}{2}\int_\Gamma u_iu_i\,dx \tag{3.17}$$

where we have made use of (2.9).

We now insert (3.14)–(3.17) into (3.12) to obtain

$$\begin{aligned}\frac{d\Phi}{dt} \le &-(2-\nu^{1/3}\gamma_1^{-1})\int_\Omega \theta_{,i}\theta_{,i}\,dx - (2\kappa-\gamma_4^{-1})\int_\Gamma \theta^2\,d\sigma + a\gamma_5\int_\Omega\theta^2\,dx\\ &+\alpha^2\gamma_4\int_\Gamma [T^*]^2\,d\sigma\\ &-\left(2a - \frac{\nu^{1/3}\gamma_1\gamma_2}{2} - \frac{3\nu\gamma_3^{-1}a}{2}\right)\int_\Omega u_{i,j}u_{i,j}\,dx\\ &+\left\{\gamma_5^{-1}a + \frac{\nu\gamma_3^3}{2}a\left[\int_\Omega v_{i,j}v_{i,j}\,dx\right]^2 + \frac{\nu^{1/3}}{2}\gamma_1\gamma_2^{-1}\int_\Omega [T^6\,dx]^{2/3}\right\}\int_\Omega u_iu_i\,dx.\end{aligned} \tag{3.18}$$

If we make the specific choices

$$\gamma_1=\nu^{1/3};\qquad \gamma_2=a\nu^{-2/3};\qquad \gamma_3=3\mu;\qquad \gamma_4=\kappa^{-1} \tag{3.19}$$

and make use of (2.4) and (2.11) we find

$$\begin{aligned}\frac{d\Phi}{dt} \le &-[\mu(\kappa,1)-a\gamma_5]\int_\Omega\theta^2\,dx + \alpha^2\kappa^{-1}\int_\Gamma [T^*]^2\,d\sigma - a[\lambda-\gamma_5^{-1}]\int_\Omega u_iu_i\,dx\\ &+\frac{27\nu^4a}{2}\left[\int_\Omega v_{i,j}v_{i,j}\,dx\right]^2\int_\Omega u_iu_i\,dx + \frac{\nu^{4/3}}{2a}\left[\int_\Omega T^6\,dx\right]^{3/2}\int_\Omega u_iu_i\,dx.\end{aligned} \tag{3.20}$$

We next choose

$$\gamma_5 = -\frac{(\lambda-\mu)+\sqrt{(\lambda-\mu)^2+4a}}{2a} \tag{3.21}$$

and conclude that

$$\begin{aligned}\frac{d\Phi}{dt} \le &-\frac{1}{2}[(\lambda+\mu)-\{(\lambda-\mu)^2+4a\}^{1/2}]\Phi\\ &+\left[\frac{27\nu^4}{2}\left\{\int_\Omega v_{i,j}v_{i,j}\,dx\right\}^2 + \frac{\nu^{4/3}}{2a^2}\left\{\int_\Omega T^6\,dx\right\}^{2/3}\right]\Phi + \alpha^2\kappa^{-1}\int_\Gamma [T^*]^2\,d\sigma,\end{aligned} \tag{3.22}$$

where $\mu=\mu(\kappa,1)$. We now choose the constant a so that the quantity in square brackets in the first term on the right is positive. Any value of a less than $\lambda\mu$ will suffice. We observe that as a decreases the coefficient of $\{\int_\Omega T^6\,dx\}^{2/3}$ increases. We could, for instance, choose $a=\frac{\lambda\mu}{2}$, but in any case we are led to an inequality of the form

$$\begin{aligned}\frac{d\Phi}{dt} \le &-A_1\Phi + A_2\left[\int_\Omega v_{i,j}v_{i,j}\,dx\right]^2\Phi\\ &+A_3\left[\int_\Omega T^6\,dx\right]^{2/3}\Phi + \alpha^2\kappa^{-1}\int_\Gamma [T^*]^2\,d\sigma.\end{aligned} \tag{3.23}$$

We have already bounded some of the terms on the right, but we need a bound for $\int_\Omega v_{i,j}v_{i,j}\,dx$. With sufficient smoothness and compatibility, a bound for this expression is given by (A.22) in the Appendix, where it is shown that for sufficiently small data or on the interval $(0, \min\{t_0, t_1\})$ (where the definition of t_1 is given by (A.20))

$$\int_\Omega v_{i,j} v_{i,j}\, dx \le H_2(t) e^{-\frac{1}{2}\mu(\kappa,1)t}. \tag{3.24}$$

The insertion of (3.24) into (3.23) then leads to an inequality of the form

$$\begin{aligned} \frac{d\Phi}{dt} &\le -A_1\Phi + A_2 H_2^2(t) e^{-\mu(\kappa,1)t}\Phi \\ &\quad + A_3 B_{10} e^{-\frac{20}{9}\mu(\kappa,3)t}\Phi + \alpha^2\kappa^{-1}\int_\Gamma [T^*]^2\, d\sigma. \end{aligned} \tag{3.25}$$

The expression for $H_2(t)$ is given by (A.23). For sufficiently small data, $Q_1(t)$ may be replaced by $Q_1(0)$ and (3.25) integrates to give

$$\Phi(t) \le M_2\alpha^2 \int_0^t e^{-A_1(t-\eta)} \left\{ \int_\Gamma [T^*]^2\, d\sigma \right\} d\eta. \tag{3.26}$$

Otherwise, for $0 \le t < \min\{t_0, t_1\}$ we obtain an inequality of type (3.26) in which M_2 is now a function of t that tends to infinity as $t \to t_1$. The continuous dependence inequality is finally made explicit by bounding the integral on the right of (3.26) in terms of data. This bound follows from (2.21) with $\beta = A_1$. Thus we obtain

$$\Phi(t) \le \frac{2M_2\mu(\kappa,1)\alpha^2}{[2\mu(\kappa,1) - A_1]} \int_\Omega T_0^2\, dx\, e^{-A_1 t} \tag{3.27}$$

is valid for $0 \le t < \min\{t_0, t_1\}$, which establishes continuous dependence on the parameter κ.

We have established the following result:

THEOREM 1. Let (v_i, T) and (v_i^*, T^*) be solutions of (1.1)–(1.3) subject to conditions (1.4)–(1.5) and (3.1), (3.2) respectively. Then if Γ is smooth, the data are compatible on $\Gamma \times \{0\}$ and a lies in the interval $(0, \lambda\mu(\kappa, 1))$, it follows that for $t < \min\{t_0, t_1\}$ with t_1 defined by (A.20)

$$\int_\Omega [(T - T^*)^2 + a(v_i - v_i^*)(v_i - v_i^*)]\, dx \le P(t, t_1, T_0, v^0)(\kappa - \kappa^*)^2 e^{-A_1 t}. \tag{3.28}$$

Here $P(t, t_1, T_0, v^0)$ is a computable quantity that tends to infinity as $t \to t_1$, and

$$A_1 = \tfrac{1}{2}[(\lambda + \mu(\kappa,1)) - \{(\lambda - \mu(\kappa,1))^2 + 4a\}^{1/2}]. \tag{3.29}$$

It is clear that this result is far from optimal because of the more or less arbitrary choices made for the constants arising in the derivation. Different choices for values of the constants could lead to a larger interval of continuous dependence. We have not pursued the question of the optimal choice for these constants. We should point out that (3.27) holds for t_0 in the indicated range as long as the base flow remains bounded on that interval.

4. The forward-in-time problem in $\mathbb{R}^2$. The derivation of the stability inequality is much easier in $\mathbb{R}^2$ due to the use of the Sobolev inequality (2.23) instead of (2.24). We proceed as in the three-dimensional case through (3.12), but instead of (3.13) we have in $\mathbb{R}^2$

$$\begin{aligned}
&\left|\int_\Omega u_i\theta_{,i} T\,dx\right| \\
&\le \left[\int_\Omega \theta_{,i}\theta_{,i}\,dx\right]^{1/2}\left\{\int_\Omega [u_iu_i]^2\,dx\right\}^{1/4}\left\{\int_\Omega T^4\,dx\right\}^{1/4} \\
&\le \Lambda^{1/2}\left[\int_\Omega \theta_{,i}\theta_{,i}\,dx\right]^{1/2}\left[\int_\Omega u_iu_i\,dx\int_\Omega u_{j,k}u_{j,k}\,dx\int_\Omega T^4\,dx\right]^{1/4} \qquad (4.1)\\
&\le \frac{\Lambda^{1/2}}{2}\left\{\delta_1^{-1}\int_\Omega \theta_{,i}\theta_{,i}\,dx+\delta_1\left[\int_\Omega u_iu_i\,dx\int_\Omega u_{j,k}u_{j,k}\,dx\int_\Omega T^4\,dx\right]^{1/2}\right\} \\
&\le \frac{\Lambda^{1/2}}{2}\left\{\delta_1^{-1}\int_\Omega \theta_{,i}\theta_{,i}\,dx+\frac{\delta_1\delta_2}{2}\int_\Omega u_{i,j}u_{i,j}\,dx+\frac{\delta_1\delta_2^{-1}}{2}\int_\Omega u_iu_i\,dx\int_\Omega T^4\,dx\right\}.
\end{aligned}$$

Also in $\mathbb{R}^2$

$$\begin{aligned}
\left|\int_\Omega u_iu_jv_{i,j}\,dx\right| &\le \left[\int_\Omega [u_iu_i]^2\,dx\int_\Omega v_{i,j}v_{i,j}\,dx\right\}^{1/2} \\
&\le \Lambda\left\{\int_\Omega u_iu_i\,dx\int_\Omega u_{j,k}u_{j,k}\,dx\int_\Omega v_{l,m}v_{l,m}\,dx\right\}^{1/2} \qquad (4.2)\\
&\le \frac{\Lambda\delta_3}{2}\int_\Omega u_iu_i\,dx\int_\Omega v_{l,m}v_{l,m}\,dx+\frac{\Lambda\delta_3^{-1}}{2}\int_\Omega u_{i,j}u_{i,j}\,dx.
\end{aligned}$$

Inequalities (3.16) and (3.17) are both valid in $\mathbb{R}^2$. Assembling terms we have

$$\begin{aligned}
\frac{d\Phi}{dt} \le &-(2-\Lambda^{1/2}\delta_1^{-1})\int_\Omega \theta_{,i}\theta_{,i}+(2\kappa-\gamma_4^{-1})\int_\Gamma \theta^2\,d\sigma+a\gamma_5\int_\Omega \theta^2\,dx+\alpha^2\gamma_4\int_\Omega [T^*]^2\,d\sigma \\
&-\left(2a-\frac{\Lambda^{1/2}\delta_1\delta_2}{2}\Lambda\delta_3^{-1}a\right)\int_\Omega u_{i,j}u_{i,j}\,dx \\
&+\left\{\gamma_5^{-1}a+\Lambda\delta_3 a\int_\Omega v_{i,j}v_{i,j}\,dx+\frac{\Lambda^{1/2}\delta_1}{2}\delta_2^{-1}\int_\Omega T^4\,dx\right\}\int_\Omega u_iu_i\,dx.
\end{aligned} \qquad (4.3)$$

We now make the choices

$$\delta_1=\Lambda^{1/2};\qquad \delta_2=\Lambda^{-1}a;\qquad \delta_3=2\Lambda;\qquad \gamma_4=\kappa^{-1} \qquad (4.4)$$

and are led to

$$\begin{aligned}
\frac{d\Phi}{dt} \le &-[(\mu(\kappa,1)-a\gamma_5]\int_\Omega \theta^2\,dx+\alpha^2\kappa^{-1}\int_\Gamma [T^*]^2\,d\sigma-a[\lambda-\gamma_5^{-1}]\int_\Omega u_iu_i\,dx \\
&+2\Lambda^2a\int_\Omega v_{i,j}v_{i,j}\,dx\int_\Omega u_ku_k\,dx+\frac{\Lambda^2}{2a}\int_\Omega T^4\,dx\int_\Omega u_iu_i\,dx.
\end{aligned}$$

We again choose γ_5 as in (3.21) and find

$$\frac{d\Phi}{dt} \le -\frac{1}{2}[(\lambda+\mu) - \{(\lambda-\mu)^2 + 4a\}^{1/2}\Phi + \left[2\Lambda^2 \int_\Omega v_{i,j}v_{i,j}\,dx + \frac{\Lambda^2}{2a^2}\int_\Omega T^4\,dx\right]\Phi + \alpha^2\kappa^{-1}\int_\Gamma (T^*)^2\,d\sigma. \tag{4.5}$$

Again we write the inequality as

$$\frac{d\Phi}{dt} \le -\widetilde{A}_1\Phi + \widetilde{A}_2\left(\int_\Omega v_{i,j}v_{i,j}\,dx\right)\Phi + \widetilde{A}_3\left(\int_\Omega T^4\,dx\right)\Phi + \alpha^2\kappa^{-1}\int_\Gamma (T^*)\,d\sigma, \tag{4.6}$$

which integrates to give

$$\Phi(t) \le \alpha^2\kappa^{-1}\left[\int_0^t \exp\left\{-\widetilde{A}_1(t-\eta) + \widetilde{A}_2\int_\eta^t\int_\Omega v_{i,j}v_{i,j}\,dx\,d\rho + \widetilde{A}_3\int_\eta^t\int_\Omega T^4\,dx\,d\rho\right\}d\eta\right]\int_\Gamma (T^*)^2\,d\sigma. \tag{4.7}$$

Integrating (2.17) from η to t and using (2.14) to bound $\int_\Omega v_iv_i\,dx$ at time η we find

$$\int_\eta^t\int_\Omega v_{i,j}v_{i,j}\,dx\,d\eta \le (B_6 + B_7\eta)e^{-\mu(\kappa,1)\eta}. \tag{4.8}$$

Also from (2.3)

$$\int_\eta^t\int_\Omega T^4\,dx\,d\eta \le \left[\int_\Omega T_0^4\,dx\right][3\mu(\kappa,2)]^{-1}e^{-3\mu(\kappa,2)\eta}. \tag{4.9}$$

Inserting (4.8) and (4.9) into (4.7) we find

$$\begin{aligned}\Phi(t) &\le \alpha^2\kappa^{-1}\int_0^t\left[\exp\{-\widetilde{A}_1(t-\eta) + \widetilde{A}_2(B_6+B_7\eta)e^{-\mu(\kappa,1)\eta} + \widetilde{A}_3B_8e^{-3\mu(\kappa,2)\eta}\}\int_\Gamma [T^*]^2\,d\sigma\right]d\eta \\ &\le \alpha^2 M_3\int_0^t\int_\Gamma e^{-\widetilde{A}_1(t-\eta)}[T^*]^2\,d\sigma\,d\eta.\end{aligned} \tag{4.10}$$

The bound for the last integral is carried out just as in the derivation of the bound for the right-hand side of (3.26) in $\mathbb{R}^3$. Thus we find

$$\Phi(t) \le \left\{\frac{2M_3\mu(\kappa,1)}{[2\mu(\kappa,1)-\widetilde{A}_1]}\right\}\alpha^2\left\{\int_\Omega T_0^2\,dx\right\}e^{-\widetilde{A}_1 t}. \tag{4.11}$$

We have proved

THEOREM 2. *If (v_i, T) and (v_i^*, T^*) are two solutions of the system (1.1)–(1.3) with $\Omega \subseteq \mathbb{R}^2$ that satisfy the same auxiliary conditions except that (3.1) is satisfied by T^*, then for a constant a and function $F(t)$ depending on the data,*

$$\int_\Omega [(T-T^*)^2 + a(v_i - v_i^*)(v_i - v_i^*)]\,dx \le (\kappa-\kappa^*)^2F(t). \tag{4.12}$$

It is interesting that in $\mathbb{R}^2$ our stability result has involved neither bounds for $\int_\Omega v_{i,j}v_{i,j}\,dx$ nor $\int_\Omega v_{i,t}v_{i,t}\,dx$. Consequently, it has not been necessary to assume that the differential equation is satisfied on $t = 0$. We note further that in $\mathbb{R}^2$ we have not had to impose smallness assumptions on the initial data in order to obtain a stability inequality that is valid for all t. This means that in $\mathbb{R}^2$ we may permit t_0 to go to infinity. These results are similar to those obtained by Song [9] for the Navier-Stokes equations.

5. Backward-in-time problem. Instead of considering the original equations backward in time, we will replace t by $-t$ and study the resulting system forward in time. The governing perturbation equations can then be written

$$-u_{i,t} + v_j^* u_{i,j} + u_j v_{i,j} = -P_{,i} + \Delta u_i + b_i\theta \tag{5.1}$$

$$u_{i,i} = 0 \qquad \text{in } \Omega \times (0, t_0) \tag{5.2}$$

$$-\theta_{,t} + v_i^*\theta_{,i} + u_i T_{,i} = \Delta\theta \tag{5.3}$$

with

$$u_i = 0 \tag{5.4}$$

$$\text{on } \Gamma \times [0, t_0]$$

$$\theta_{,i} n_i + \kappa\theta = -\alpha T^* \tag{5.5}$$

and

$$\theta(x,0) = u_i(x,0) = 0, \quad x \in \Omega. \tag{5.6}$$

Here $u_i = v_i^* - v_i$, $\theta = T^* - T$, $P = p^* - p$, and $\alpha = \kappa^* - \kappa$. We assume, without loss, that

$$|b_i| \leq 1. \tag{5.7}$$

Continuous dependence of solutions on the parameter α can be obtained using logarithmic convexity arguments (see Payne [6]). We thus introduce the functional

$$\Phi(t) = \int_0^t \int_\Omega (t-\eta)^2 (u_i u_i + \theta^2)\,dx\,d\eta + \alpha^2 Q^2 \tag{5.8}$$

where Q^2 is a constant data term to be chosen. We now show that there exist constants a_1 and a_2 such that

$$\Phi\Phi'' - (\Phi')^2 \geq -a_1\Phi\Phi' - a_2\Phi^2. \tag{5.9}$$

To this end, we differentiate (5.8) with respect to t to obtain

$$\Phi'(t) = 2\int_0^t \int_\Omega (t-\eta)(u_i u_i + \theta^2)\,dx\,d\eta \tag{5.10}$$

$$= 2\int_0^t \int_\Omega (t-\eta)^2 (u_i u_{i,\eta} + \theta\theta_{,\eta})\,dx\,d\eta. \tag{5.11}$$

Substitution of the differential equations (5.1), (5.2), and (5.3) and integration by parts results in

$$\begin{aligned}\Phi'(t) = {} & 2\int_0^t \int_\Omega (t-\eta)^2 u_i u_j v_{i,j}\,dx\,d\eta - 2\int_0^t \int_\Omega (t-\eta)^2 b_i u_i \theta\,dx\,d\eta \\ & + 2\int_0^t \int_\Omega (t-\eta)^2 u_{i,j} u_{i,j}\,dx\,d\eta + 2\int_0^t \int_\Omega (t-\eta)^2 \theta u_i T_{,i}\,dx\,d\eta \\ & + 2\int_0^t \int_\Omega (t-\eta)^2 \theta_{,i}\theta_{,i}\,dx\,d\eta + 2\int_0^t \int_\Gamma (t-\eta)^2 (\kappa\theta^2 + \alpha\theta T^*)\,d\sigma\,d\eta.\end{aligned} \tag{5.12}$$

Differentiating (5.12), we are led to

$$\begin{aligned}\Phi''(t) = {} & 4\int_0^t \int_\Omega (t-\eta) u_i u_j v_{i,j}\,dx\,d\eta - 4\int_0^t \int_\Omega (t-\eta) b_i u_i \theta\,dx\,d\eta \\ & - 4\int_0^t \int_\Omega (t-\eta)^2 u_{i,\eta} \Delta u_i\,dx\,d\eta + 4\int_0^t \int_\Omega (t-\eta) \theta u_i T_{,i}\,dx\,d\eta \\ & - 4\int_0^t \int_\Omega (t-\eta)^2 \theta_{,\eta}\,\Delta\theta\,dx\,d\eta + 4\int_0^t \int_\Gamma (t-\eta) (\kappa\theta^2 + \alpha\theta T^*)\,d\sigma\,d\eta.\end{aligned} \tag{5.13}$$

A second use of the differential equations allows us to write (5.13) as

$$\begin{aligned}\Phi''(t) = {} & 4\int_0^t \int_\Omega (t-\eta)^2 (\psi_i\psi_i + \chi^2)\,dx\,d\eta + 4\int_0^t \int_\Omega (t-\eta) u_i u_j v_{i,j}\,dx\,d\eta \\ & - 4\int_0^t \int_\Omega (t-\eta) b_i u_i \theta\,dx\,d\eta - \int_0^t \int_\Omega (t-\eta)^2 v_j^* u_{i,j} v_l^* u_{i,l}\,dx\,d\eta \\ & - 4\int_0^t \int_\Omega (t-\eta)^2 u_{i,\eta} u_j v_{i,j}\,dx\,d\eta + 4\int_0^t \int_\Omega (t-\eta)^2 u_{i,\eta} b_i \theta\,dx\,d\eta \\ & - \int_0^t \int_\Omega (t-\eta)^2 v_i^* \theta_{,i}\, v_l^* \theta_{,l}\,dx\,d\eta - 4\int_0^t \int_\Omega (t-\eta)^2 u_i T_{,i}\,\theta_{,\eta}\,dx\,d\eta \\ & + 4\int_0^t \int_\Omega (t-\eta) \theta u_i T_{,i}\,dx\,d\eta + 4\int_0^t \int_\Gamma (t-\eta) \alpha\theta T^*\,d\sigma\,d\eta \\ & - 4\int_0^t \int_\Gamma (t-\eta)^2 \alpha T^* \theta_{,\eta}\,d\sigma\,d\eta\end{aligned} \tag{5.14}$$

where

$$\psi_i = u_{i,t} - \tfrac{1}{2} v_j^* u_{i,j}, \qquad \chi = \theta_t - \tfrac{1}{2} v_i^* \theta_{,i}\,. \tag{5.15}$$

Observing that Φ' can be written as

$$\Phi' = 2\int_0^t \int_\Omega (t-\eta)^2 (u_i\psi_i + \theta\chi)\,dx\,d\eta, \tag{5.16}$$

we next form $\Phi\Phi'' - (\Phi')^2$ to obtain

$$\begin{aligned}\Phi\Phi'' - (\Phi')^2 \geq{}& 4S^2 + 4\Phi \int_0^t \int_\Omega (t-\eta) u_i u_j v_{i,j}\, dx\, d\eta - 4\Phi \int_0^t \int_\Omega (t-\eta) b_i u_i \theta\, dx\, d\eta \\ & - \Phi \int_0^t \int_\Omega (t-\eta)^2 v_j^* u_{i,j} v_l^* u_{i,l}\, dx\, d\eta - 4\Phi \int_0^t \int_\Omega (t-\eta)^2 u_{i,\eta} u_j v_{i,j}\, dx\, d\eta \\ & + 4\Phi \int_0^t \int_\Omega (t-\eta)^2 u_{i,\eta} b_i \theta\, dx\, d\eta - \Phi \int_0^t \int_\Omega (t-\eta)^2 v_i^* \theta_{,i}\, v_l^* \theta_{,l}\, dx\, d\eta \\ & - 4\Phi \int_0^t \int_\Omega (t-\eta)^2 u_i T_{,i}\, \theta_{,\eta}\, dx\, d\eta + 4\Phi \int_0^t \int_\Omega (t-\eta) \theta u_i T_{,i}\, dx\, d\eta \\ & + 4\Phi \left[\int_0^t \int_\Gamma (t-\eta) \alpha\theta T^*\, d\sigma\, d\eta - \int_0^t \int_\Gamma (t-\eta)^2 \alpha T^* \theta_{,\eta}\, d\sigma\, d\eta \right] \\ ={}& 4S^2 + \sum_{n=1}^{8} I_n + 4\Phi \int_0^t \int_\Gamma (t-\eta)\alpha\theta T^*\, d\sigma\, d\eta - \int_0^t \int_\Gamma (t-\eta)^2 \alpha T^* \theta_{,\eta}\, d\sigma\, d\eta \end{aligned} \tag{5.17}$$

where

$$\begin{aligned} S^2 = {}& \left[\int_0^t \int_\Omega (t-\eta)^2 (\psi_i\psi_i + \chi^2)\, dx\, d\eta \right] \left[\int_0^t \int_\Omega (t-\eta)^2 (u_i u_i + \theta^2)\, dx\, d\eta \right] \\ & - \left(\int_0^t \int_\Omega (t-\eta)^2 (u_i \psi_i + \theta\chi)\, dx\, d\eta \right)^2 . \end{aligned} \tag{5.18}$$

Clearly, S^2 is nonnegative by Schwarz's inequality. It also follows from (5.18) that

$$\left[\int_0^t \int_\Omega (t-\eta)^2 (\psi_i\psi_i + \chi^2)\, dx\, d\eta \right]^{1/2} \left[\int_0^t \int_\Omega (t-\eta)^2 (u_i u_i + \theta^2)\, dx\, d\eta \right]^{1/2} \leq S + \frac{1}{2}\Phi'. \tag{5.19}$$

We now proceed to bound each of the integrals on the right-hand side of (5.17) in terms of Φ^2 and $\Phi\Phi'$. In order to accomplish this, we assume that, uniformly in $\Omega \times [0, t_0)$,

$$|v^*|,\ |\nabla v|,\ |\nabla T| \leq M \tag{5.20}$$

for a positive constant M. Such an assumption restricts the class of solutions we shall consider in the sequel.

Rewriting (5.12) and applying Schwarz's inequality as well as (5.20), and the arithmetic-geometric mean inequality, we find that

$$\begin{aligned} & -\int_0^t \int_\Omega (t-\eta)^2 (u_{i,j} u_{i,j} + \theta_{,i}\, \theta_{,i}\,)\, dx\, d\eta \\ & \qquad \geq -\frac{1}{2}\Phi' - \frac{1}{2}(3M+1)\Phi + \int_0^t \int_\Gamma (t-\eta)^2 [\kappa\theta^2 + \alpha\theta T^*]\, d\sigma\, d\eta. \end{aligned} \tag{5.21}$$

The following bounds also follow from (5.20) and the arithmetic-geometric mean inequality:

$$I_1 \geq -4\Phi M \int_0^t \int_\Omega (t-\eta) u_i u_i \, dx \, d\eta \geq -2M\Phi\Phi', \tag{5.22}$$

$$I_2 \geq -2\Phi \int_0^t \int_\Omega (t-\eta)(u_i u_i + \theta^2) \, dx \, d\eta \geq -\Phi\Phi', \tag{5.23}$$

$$I_3 \geq -4\Phi M^2 \int_0^t \int_\Omega (t-\eta)^2 u_{i,j} u_{i,j} \, dx \, d\eta, \tag{5.24}$$

$$I_6 \geq -4\Phi M^2 \int_0^t \int_\Omega (t-\eta)^2 \theta_{,i} \theta_{,i} \, dx \, d\eta, \tag{5.25}$$

$$I_8 \geq -4\Phi M \int_0^t \int_\Omega (t-\eta)(u_i u_i + \theta^2) \, dx \, d\eta \geq -2M\Phi\Phi'. \tag{5.26}$$

To bound I_4, we observe that

$$\begin{aligned} -4\Phi \int_0^t \int_\Omega (t-\eta)^2 u_{i,\eta} u_j v_{i,j} \, dx \, d\eta = & -4\Phi \int_0^t \int_\Omega (t-\eta)^2 \left(u_{i,\eta} - \frac{1}{2} v_k^* u_{i,k} \right) u_j v_{i,j} \, dx \, d\eta \\ & - 2\Phi \int_0^t \int_\Omega (t-\eta)^2 v_k^* u_{i,k} u_j v_{i,j} \end{aligned} \tag{5.27}$$

so that

$$\begin{aligned} I_4 \geq & -4M\Phi \left(\int_0^t \int_\Omega (t-\eta)^2 \psi_i \psi_i \, dx \, d\eta \right)^{1/2} \left(\int_0^t \int_\Omega (t-\eta)^2 u_i u_i \, dx \, d\eta \right)^{1/2} \\ & - M^2 \Phi \left[\int_0^t \int_\Omega (t-\eta)^2 u_{i,j} u_{i,j} \, dx \, d\eta + \int_0^t \int_\Omega (t-\eta)^2 u_i u_i \, dx \, d\eta \right] \end{aligned} \tag{5.28}$$

after using (5.20) and applying the arithmetic-geometric mean and Schwarz's inequalities. Similarly, we can bound I_5 and I_7 to obtain

$$\begin{aligned} I_5 \geq & -4\Phi \left(\int_0^t \int_\Omega (t-\eta)^2 \psi_i \psi_i \, dx \, d\eta \right)^{1/2} \left(\int_0^t \int_\Omega (t-\eta)^2 \theta^2 \, dx \, d\eta \right)^{1/2} \\ & - M\Phi \left[\int_0^t \int_\Omega (t-\eta)^2 \theta^2 \, dx \, d\eta + \int_0^t \int_\Omega (t-\eta)^2 u_{i,j} u_{i,j} \, dx \, d\eta \right] \end{aligned} \tag{5.29}$$

and

$$\begin{aligned} I_7 \geq & -4\Phi M \left(\int_0^t \int_\Omega (t-\eta)^2 \chi^2 \, dx \, d\eta \right)^{1/2} \left(\int_0^t \int_\Omega (t-\eta)^2 u_i u_i \, dx \, d\eta \right)^{1/2} \\ & - M^2 \Phi \left[\int_0^t \int_\Omega (t-\eta)^2 \theta_{,i} \theta_{,i} \, dx \, d\eta + \int_0^t \int_\Omega (t-\eta)^2 u_i u_i \, dx \, d\eta \right]. \end{aligned} \tag{5.30}$$

Rewriting (5.12), we note it follows from (5.20) that

$$\begin{aligned} -\int_0^t \int_\Omega (t-\eta)^2 (u_{i,j} u_{i,j} + \theta_{,i} \theta_{,i}) \, dx \, d\eta \geq & -\frac{1}{2}\Phi' - \frac{1}{2}(3M+1)\Phi \\ & + \int_0^t \int_\Gamma (t-\eta)^2 (\kappa\theta^2 + \alpha\theta T^*) \, d\sigma \, d\eta. \end{aligned} \tag{5.31}$$

Combining the bounds for I_n and using (5.19) and (5.31), we derive

$$\begin{aligned}\Phi\Phi'' - (\Phi')^2 \geq 4S^2 &- 4(2M+1)\Phi S - C_1\Phi^2 - C_2\Phi\Phi' \\ &+ 4\Phi\left[\int_0^t\int_\Gamma (t-\eta)\alpha\theta T^*\,d\sigma\,d\eta - \int_0^t\int_\Gamma (t-\eta)^2\alpha T^*\theta_{,\eta}\,d\sigma\,d\eta\right] \\ &+ (5M^2+M)\Phi\int_0^t\int_\Gamma (t-\eta)^2(\kappa\theta^2+\alpha\theta T^*)\,d\sigma\,d\eta\end{aligned} \tag{5.32}$$

where

$$C_1 = \tfrac{1}{2}(6M^2+M)(3M+1) + M(2M+1), \tag{5.33}$$

$$C_2 = \tfrac{1}{2}(6M^2+17M+6). \tag{5.34}$$

We now complete the square on the first two terms on the right side of (5.32) and discard the nonnegative part to obtain

$$\begin{aligned}\Phi\Phi'' - (\Phi')^2 \geq &- C_3\Phi^2 - C_2\Phi\Phi' + 4\Phi\left[\int_0^t\int_\Gamma (t-\eta)\alpha\theta T^*\,d\sigma\,d\eta\right. \\ &\left.- \int_0^t\int_\Gamma (t-\eta)^2\alpha T^*\theta_{,\eta}\,d\sigma\,d\eta\right] \\ &+ (5M^2+M)\Phi\int_0^t\int_\Gamma (t-\eta)^2(\kappa\theta^2+\alpha\theta T^*)\,d\sigma\,d\eta\end{aligned} \tag{5.35}$$

with $C_3 = C_1 + (2M+1)^2$.

It remains for us to estimate the boundary terms in (5.35). Integration and use of the Schwarz inequality lead to the bounds

$$\begin{aligned}J_1 &= 4\Phi\left[\int_0^t\int_\Gamma (t-\eta)\alpha\theta T^*\,d\sigma\,d\eta - \int_0^t\int_\Gamma (t-\eta)^2\alpha T^*\theta_{,\eta}\,d\sigma\,d\eta\right] \\ &\geq -4\Phi\alpha\left(\int_0^t\int_\Gamma (t-\eta)^2\theta^2\,d\sigma\,d\eta\right)^{1/2}\left(\int_0^t\int_\Gamma T^2\,d\sigma\,d\eta\right)^{1/2} \\ &\quad - 4\Phi\alpha\left(\int_0^t\int_\Gamma (t-\eta)^2\theta^2\,d\sigma\,d\eta\right)^{1/2}\left(\int_0^t\int_\Gamma (t-\eta)^2 T_{,\eta}^2\,d\sigma\,d\eta\right)^{1/2}\end{aligned} \tag{5.36}$$

and

$$\begin{aligned}J_2 &= (5M^2+M)\Phi\int_0^t\int_\Gamma (t-\eta)^2(\kappa\theta^2+\alpha\theta T^*)\,d\sigma\,d\eta \\ &\geq (5M^2+M)\Phi(\kappa+\alpha)\int_0^t\int_\Gamma (t-\eta)^2\theta^2\,d\sigma\,d\eta \\ &\quad - (5M^2+M)\Phi\alpha\left(\int_0^t\int_\Gamma (t-\eta)^2\theta^2\,d\sigma\,d\eta\right)^{1/2}\left(\int_0^t\int_\Gamma (t-\eta)^2 T^2\,d\sigma\,d\eta\right)^{1/2}.\end{aligned} \tag{5.37}$$

Let us now assume that

$$\int_0^t\int_\Gamma T^2\,d\sigma\,d\eta + \int_0^t\int_\Gamma (t-\eta)^2 T_{,\eta}^2\,d\sigma\,d\eta \leq M^2. \tag{5.38}$$

Hence,

$$J_1 \geq -8\alpha\Phi M\left(\int_0^t\int_\Gamma(t-\eta)^2\theta^2\,d\sigma\,d\eta\right)^{1/2} \geq -4\Phi\frac{\alpha^2M^2}{\mu_1}+\mu_1\int_0^t\int_\Gamma(t-\eta)^2\theta^2\,d\sigma\,d\eta \tag{5.39}$$

for a positive constant μ_1. Also, we have

$$\begin{aligned} J_2 \geq{}& (5M^2+M)\Phi(\kappa+\alpha)\int_0^t\int_\Gamma(t-\eta)^2\theta^2\,d\sigma\,d\eta \\ &-\phi\left[\frac{\alpha^2M^4(5M+1)^2}{2\mu_2}+\frac{\mu_2}{2}\int_0^t\int_\Gamma(t-\eta)^2\theta^2\,d\sigma\,d\eta\right] \end{aligned} \tag{5.40}$$

upon applying the arithmetic-geometric mean inequality. To complete our bounds on J_1 and J_2, we now need to find a bound for $\int_0^t\int_\Gamma(t-\eta)^2\theta^2\,d\sigma\,d\eta$. We do this by considering the identity

$$0=\int_0^t\int_\Omega(t-\eta)^2\theta\{-\theta_{,\eta}+v_i^*\theta_{,i}+u_iT_{,i}-\Delta\theta\}\,dx\,d\eta. \tag{5.41}$$

Integrating by parts using the boundary conditions (5.4)–(5.5) and rearranging the expression leads to the inequality

$$\begin{aligned} (\kappa+\alpha)\int_0^t\int_\Gamma(t-\eta)^2\theta^2\,d\sigma\,d\eta \leq{}& -\alpha\int_0^t\int_\Omega(t-\eta)^2\theta T\,d\sigma\,d\eta+\int_0^t\int_\Omega(t-\eta)^2\theta^2\,dx\,d\eta \\ &-\int_0^t\int_\Omega(t-\eta)^2\theta u_iT_{,i}\,dx\,d\eta. \end{aligned} \tag{5.42}$$

If we apply Schwarz's inequality to (5.42) and use the assumption (5.20) as well as the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} &(\kappa+\alpha)\int_0^t\int_\Gamma(t-\eta)^2\theta^2\,d\sigma\,d\eta \\ &\leq\frac{\gamma}{2}\int_0^t\int_\Gamma(t-\eta)^2\theta^2\,d\sigma\,d\eta+\frac{\alpha^2}{2\gamma}\int_0^t\int_\Gamma(t-\eta)^2T^2\,d\sigma\,d\eta \\ &\quad+\int_0^t\int_\Omega(t-\eta)\theta^2\,dx\,d\eta+\frac{M}{2}\int_0^t\int_\Omega(t-\eta)^2[\theta^2+u_iu_i]\,dx\,d\eta. \end{aligned}$$

Recalling the definition of Φ, the expression (5.10), and the assumption (5.38) about the boundary data, we then conclude that

$$\left(\kappa+\alpha-\frac{\gamma}{2}\right)\int_0^t\int_\Gamma(t-\eta)^2\theta^2\,d\sigma\,d\eta \leq \frac{1}{2\gamma}\alpha^2M^2+\Phi'+\frac{1}{2}M\Phi. \tag{5.43}$$

Letting $\kappa+\alpha-\frac{\gamma}{2}=\mu$, we choose γ so that $\mu>0$ and (5.43) is the desired estimate. Consequently, if we choose $\mu_1=1$ and $\mu_2=2M(5M+1)(\kappa+\alpha)$ and define

$$Q^2=\left(\frac{1}{2\gamma}+\frac{1}{\mu}\right)M^2+M^3(5M+1), \tag{5.44}$$

then

$$J_1 \geq -2(M+4)\Phi^2 - 4\Phi\Phi', \tag{5.45}$$

$$J_2 \geq -\frac{1}{\kappa^*}\Phi^2. \tag{5.46}$$

It then follows from (5.35) that there are positive constants a_1 and a_2 so that inequality (5.9) is satisfied. The change of variable $\xi = e^{-\alpha_1 t}$ transforms (5.9) into

$$\frac{d^2}{d\xi^2}[\ln \Phi \xi^{-a_2/a_1^2}] \geq 0$$

and hence it follows from Jensen's inequality that

$$\Phi(t) \leq e^{(-a_2/a_1)t}[\Phi(0)]^{\delta(t)}[\Phi(t_0)e^{(a_2 t_0/a_1)}]^{1-\delta(t)} \tag{5.47}$$

where

$$0 < \delta(t) = \frac{e^{-a_1 t} - e^{-a_1 t_0}}{1 - e^{-a_1 t_0}} < 1. \tag{5.48}$$

Since

$$\Phi(0) = \alpha^2 Q^2$$

and

$$\Phi(t_0) = \int_0^{t_0}\int_\Omega (t_0-\eta)^2(u_i u_i + \theta^2)\,dx\,d\eta + \alpha^2 Q^2,$$

inequality (5.47) will be the desired continuous dependence result provided we assume

$$\Phi(t_0)e^{a_2 t_0/a_1} \leq K^2. \tag{5.49}$$

We thus conclude from (5.47) that

$$\Phi(t) \leq CK^{2[1-\delta(t)]}\alpha^{2\delta(t)} \tag{5.50}$$

for $t \in [0, t_0)$ and a computable constant C. Hence solutions belonging to the appropriate class of functions depend Hölder continuously on the parameter α in the measure Φ.

In summary, we have established the following result.

THEOREM 3. Let (v_i, T) and (v_i^*, T^*) be two solutions of the system (1.1)–(1.3) backward in time that satisfy the same initial conditions (1.5) and boundary conditions on v_i but have different parameters κ and κ^* in condition (1.4). If v^*, v, and T belong to the classes of functions satisfying

$$|v^*|, |\nabla v|, |\nabla T| \leq M$$

and

$$\int_0^t\int_\Gamma T^2\,d\sigma\,d\eta + \int_0^t\int_\Gamma (t-\eta)T_{,\eta}^2\,d\sigma\,d\eta \leq M^2$$

as well as the bound (5.49), then there exists a computable constant B such that

$$\int_0^t\int_\Omega (t-\eta)^2[(v_i - v_i^*)(v_i - v_i^*) + (T-T^*)^2]\,dx\,d\eta \leq B(\kappa-\kappa^*)^{\delta(t)}$$

for $0 < \delta(t) < 1$, $t \in [0, t_0)$.

Appendix. Our task here is to establish the inequality (3.24), but before starting the proof of this result let us first derive the following bound for the L_2 integral of $T_{,t}$, which will be needed in the derivation of (3.24):

$$\int_{\Omega(t)} T_{,t}^2\, dx \le \int_{\Omega(0)} T_{,t}^2\, dx\, e^{-\mu(\kappa,1)t} + |T_0|_m^2 \int_0^t \int_{\Omega(\eta)} e^{-\mu(2\kappa,1)(t-\eta)} v_{i,t}v_{i,t}\, dx\, d\eta, \quad (A.1)$$

where the symbol $\Omega(t)$ is used to indicate that integration is taken over Ω at time t, and $|T_0|_m$ is the maximum value of $|T_0|$ in Ω.

To establish (A.1) we observe that

$$\begin{aligned} \frac{d}{dt}\int_\Omega T_{,t}^2\, dx &= 2\int_\Omega T_{,t}\{\Delta T - T_{,i}\, v_i\}_{,t}\, dx \\ &= -2\kappa \int_\Gamma T_{,t}^2\, d\sigma - 2\int_\Omega T_{,it}\, T_{,it}\, dx \\ &\quad + 2|T_0|_m \left[\int_\Omega T_{,it}\, T_{,it}\, dx \int_\Omega v_{i,t}v_{i,t}\, dx\right]^{1/2}. \end{aligned} \quad (A.2)$$

Thus

$$\frac{d}{dt}\int_\Omega T_{,t}^2\, dx \le -2\kappa\int_\Gamma T_{,t}^2\, d\sigma - (2-\gamma_9)\int_\Omega T_{,it}\, T_{,it}\, dx + \gamma_9^{-1}|T_0|_m^2 \int_\Omega v_{i,t}v_{,i,t}\, dx. \quad (A.3)$$

Choosing $\gamma_9 = 1$ we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega T_{,t}^2\, dx &\le -2\kappa\int_\Gamma T_{,t}^2\, d\sigma - \int_\Omega T_{,it}\, T_{,it}\, dx + |T_0|_m^2\int_\Omega v_{i,t}v_{i,t}\, dx \\ &\le -\mu(2\kappa,1)\int_\Omega T_{,t}^2\, dx + |T_0|_m^2 \int_\Omega v_{i,t}v_{i,t}\, dx. \end{aligned} \quad (A.4)$$

An integration of (A.4) leads directly to (A.1).

In general, the first integral term on the right of (A.1) is not data. However, if Γ is sufficiently smooth and the data are compatible on the boundary Γ at $t = 0$ so that the T equation is satisfied on $t = 0$, we have

$$\int_{\Omega(0)} T_{,t}^2\, dx = \int_\Omega [\Delta T_0 - v_i^0 T_{0,i}]^2\, dx, \quad (A.5)$$

which is a computable data term.

We are now ready to prove (3.24). We first note that

$$\begin{aligned} \int_\Omega v_{i,j}v_{i,j}\, dx &= -\int_\Omega v_i\{v_{i,t} + v_j v_{i,j} + p_{,i} - b_i T\}\, dx \\ &= -\int_\Omega v_i v_{i,t}\, dx + \int_\Omega b_i v_i T\, dx \\ &\le \left[\int_\Omega v_i v_i\, dx\right]^{1/2} \left\{\left[\int_\Omega v_{i,t}v_{i,t}\, dx\right]^{1/2} + \left[\int_\Omega T^2\, dx\right]^{1/2}\right\}. \end{aligned} \quad (A.6)$$

A bound for $\int_\Omega v_i v_i\, dx$ is given by (2.16) and a bound for $\int_\Omega T^2\, dx$ by (2.3). It remains to bound the term $\int_\Omega v_{i,t}v_{i,t}\, dx$.

Now

$$\begin{aligned}
\frac{d}{dt}\int_\Omega v_{i,t}v_{i,t}\,dx &= 2\int_\Omega v_{i,t}\{\Delta v_i - p,_i + b_iT - v_jv_{i,j}\},_t\,dx \\
&= -2\int_\Omega v_{i,jt}v_{i,jt}\,dx + 2\int_\Omega b_iv_{i,t}T\,dx - 2\int_\Omega v_{i,t}v_{j,t}v_{i,j}\,dx \\
&\le -2\int_\Omega v_{i,jt}v_{i,jt}\,dx + 2\left[\int_\Omega T^2\,dx\int_\Omega v_{i,t}v_{i,t}\,dx\right]^{1/2} \\
&\quad + 2\left\{\int_\Omega v_{i,j}v_{i,j}\,dx\int_\Omega [v_{k,t}v_{k,t}]^2\,dx\right\}^{1/2} \\
&\le -2\int_\Omega v_{i,jt}v_{i,jt}\,dx + 2\left[\int_\Omega T^2\,dx\int_\Omega v_{i,t}v_{i,t}\,dx\right]^{1/2} \\
&\quad + 2\nu\left\{\left[\int_\Omega v_{i,j}v_{i,j}\,dx\right]^2\int_\Omega v_{k,t}v_{k,t}\,dx\right\}^{1/4}\left\{\int_\Omega v_{i,jt}v_{i,jt}\,dx\right\}^{3/4}.
\end{aligned} \tag{A.7}$$

Using the arithmetic-geometric mean and Young inequalities we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega v_{i,t}v_{i,t}\,dx \le -\left(2-\frac{3}{2}\nu\gamma_6\right)\int_\Omega v_{i,jt}v_{i,jt}\,dx \\
+ \gamma_7\int_\Omega v_{i,t}v_{i,t}\,dx + \gamma_7^{-1}\int_\Omega T,_t^2\,dx \\
+ \frac{\nu}{2}\gamma_6^{-1}\left[\int_\Omega v_{i,j}v_{i,j}\,dx\right]^2\left[\int_\Omega v_{i,t}v_{i,t}\,dx\right].
\end{aligned} \tag{A.8}$$

We introduce the notation

$$R(t) = \int_\Omega v_{i,t}v_{i,t}\,dx \tag{A.9}$$

and make use of (A.6) to obtain for $\frac{3}{2}\nu\gamma_6 < 2$

$$\begin{aligned}
\frac{dR}{dt} &\le \left\{\gamma_7 - \left(2-\frac{3}{2}\nu\gamma_6\right)\lambda\right\}R + \gamma_7^{-1}\int_\Omega T,_t^2\,dx \\
&\quad + \frac{\nu}{2}\gamma_6^{-1}\left[\int_\Omega v_iv_i\,dx\right]\left\{R^{1/2} + \left[\int_\Omega T^2\,dx\right]^{1/2}\right\}^2 R \\
&\le -\left\{2\lambda - \frac{3}{2}\gamma_6\nu\lambda - \gamma_7\right\}R + \gamma_7^{-1}\int_\Omega T,_t^2\,dx \\
&\quad + \frac{\nu}{2}\gamma_6^{-1}\left[\int_\Omega v_iv_i\,dx\right]\left\{(1+\gamma_8)R^2 + \gamma_8^{-1}(1+\gamma_8)\int_\Omega T^2\,dx\,R\right\}.
\end{aligned} \tag{A.10}$$

We have used (2.11) and the fact that $v_{i,t}$ vanishes on Γ. Choosing

$$\gamma_6 = 2[3\nu]^{-1}, \qquad \gamma_7 = \lambda - \tfrac{1}{2}\mu(\kappa,1), \tag{A.11}$$

we have, using (A.1),

$$\frac{dR}{dt} \le -\frac{1}{2}\mu(\kappa,1)R + \left[\lambda - \frac{1}{2}\mu(\kappa,1)\right]^{-1}\left\{B_1 e^{-\mu(\kappa,1)t} + |T_0|^2_m \int_0^t e^{-\mu(2\kappa,1)(t-\eta)}R(\eta)\,d\eta\right\} \tag{A.12}$$

$$+ (B_2 + B_3 t)e^{-\mu(\kappa,1)t}R^2 + (B_4 + B_5 t)e^{-2\mu(\kappa,1)t}R,$$

where the constants B_i are computable data terms obtained from (2.3), (2.16), and (A.5).

We consider first the special case

$$|T_0|^2_{\max} \le \tfrac{1}{2}\mu(\kappa,1)\mu(2\kappa,1)[\lambda - \tfrac{1}{2}\mu(\kappa,1)]. \tag{A.13}$$

In this case we have, setting

$$S(t) = R(t) + \frac{1}{2}\mu(\kappa,1)\int_0^t e^{-\mu(2\kappa,1)(t-\eta)}R(\eta)\,d\eta + Q_0, \tag{A.14}$$

where Q_0 is some sufficiently large data term,

$$\frac{dS}{dt} \le (B_2 + B_3 t)e^{-\mu(\kappa,1)t}S^2. \tag{A.15}$$

This integrates to give

$$\begin{aligned}\frac{1}{S(0)} &\le \frac{1}{S(t)} + \int_0^t (B_2 + B_3\eta)e^{-\mu(\kappa,1)\eta}\,d\eta \\ &= \frac{1}{S(t)} + [\mu(\kappa,1)]^{-2}\Big\{\mu(\kappa,1)B_2(1 - e^{-\mu(\kappa,1)t}) + B_3(1 - e^{-\mu(\kappa,1)t} \\ &\qquad -\mu(\kappa,1)te^{-\mu(\kappa,1)t})\Big\} \\ &= \frac{1}{S(t)}Q_1(t).\end{aligned} \tag{A.16}$$

Thus, provided $Q_1(t) < [S(0)]^{-1}$, we have

$$S(t) \le \left[\frac{1}{S(0)} - Q_1(t)\right]^{-1}. \tag{A.17}$$

Clearly, if

$$\mu(\kappa,1)B_2 + B_3 < [\mu(\kappa,1)]^2[S(0)]^{-1}, \tag{A.18}$$

i.e., if

$$[\mu(\kappa,1)B_2 + B_3][R(0) + Q_0] < \mu(\kappa,1)^2 \tag{A.19}$$

then (A.17) will hold for all time. This means that for sufficiently small data (A.17) will hold for all $t > 0$ (i.e., t_0 can be taken arbitrarily large). If (A.19) does not hold then (A.17) is valid only for some finite time interval, i.e., for $t < t_1$, where

$$Q_1(t_1)S(0) = 1. \tag{A.20}$$

If (A.13) is satisfied then for any fixed time interval $(0, t_0)$ for finite t_0 we may choose Q_0 so large that (A.15) is satisfied. However, when (A.13) does not hold, Q_0 will behave like $(B_2 + B_3 t_0)^{-1}e^{\mu(\kappa,1)t_0}$ and hence will go to infinity as $t_0 \to \infty$. For fixed t_0 then (A.17) holds provided $Q_1(t) < [S(0)]^{-1}$. Recalling that t_1 is the value of t at which $Q_1(t) = [S(0)]^{-1}$ then (A.17) holds for $0 \le t < \min\{t_0, t_1\}$.

Returning now to (A.6) we have

$$\int_\Omega v_{i,j}v_{i,j}\,dx \le \{(B_6+B_7t)R(t)\}^{1/2}e^{-\frac{1}{2}\mu(\kappa,1)t} + (B_8+B_9t)^{1/2}e^{-\frac{3}{2}\mu(\kappa,1)t}. \tag{A.21}$$

Thus for sufficiently small data or on the interval $(0,\min\{t_0,t_1\})$

$$\int_\Omega v_{i,j}v_{i,j}\,dx \le (B_6+B_7t)^{1/2}\left[\frac{1}{S(0)}-Q_1(t)\right]^{-1/2}e^{-\frac{1}{2}\mu(\kappa,1)t} + (B_8+B_9t)^{1/2}e^{-\frac{3}{2}\mu(\kappa,1)t} \tag{A.22}$$

$$\le H_2(t)e^{-\frac{1}{2}\mu(\kappa,1)t}$$

where

$$H_2(t) = [B_6+B_7t]^{1/2}\left[\frac{1}{S(0)}-Q_1(t)\right]^{-1/2} + (B_8+B_9t)^{1/2}e^{-\mu(\kappa,1)t}. \tag{A.23}$$

This establishes (3.24).

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