Continuous Finite-Time Control for Robotic Manipulators with Terminal Sliding Modes

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Abstract - *A new continuous finite-time control scheme for trajectory-tracking problem of robotic manipulators is proposed using terminal sliding mode (TSM). The finite-time convergence property of TSM is applied in both the reaching phase and the sliding phase of the sliding mode control system. As a result, the closed-loop system is globally finite-time stable and the trajectory-tracking objective is achieved in finite time. The resulting control law is continuous therefore chattering-free. Furthermore, it overcomes the common singularity problem in TSM. Theoretical analysis shows that the proposed control strategy has stronger robustness and disturbance-attenuation ability compared with the conventional boundary-layer method. Simulation results are given to illustrate the effectiveness of the proposed algorithm.*

Keywords: finite-time stability, terminal sliding mode, fractional power, trajectory tracking

1 Introduction

Trajectory tracking control of robot manipulators is of practical significance, and as the most fundamental task in robot control, has been extensively studied in recent years[1]. Conventionally, most of the existing results are achieved by computed torque control or inverse-dynamics control[2], which is a special application of feedback linearization of nonlinear systems, leading to a linear time-invariant closed-loop system with asymptotic stability, which means that the system trajectories converge to the equilibrium as time goes to infinity. Some kinds of continuous nonsmooth feedback controllers have been developed for the finite-time stabilization problem of dynamical systems, which means that with the proposed feedback control laws, the closed-loop systems are finite-time convergent to the desired states besides being Lyapunov stable, such as finite-time control for the double integrator system[3] and homogeneous finite-time control using homogeneity with negative relative degree [4,5].

As a matter of fact, a kind of non-Lipschitz sliding mode - TSM also has finite-time convergent property[6,7] and has been applied to control robotic manipulators for finite-time stability[8-12]. Sliding mode control is a kind of robust nonlinear feedback control technique. The basic control strategy can be designed in two steps: the choice of a sliding manifold such that the corresponding zero dynamics exhibits the desired behavior; the determination of a control law, which is often discontinuous, capable of forcing system trajectory to reach the manifold in a finite time and remain on it, featuring the so-called sliding mode, in spite of possible matched disturbances and parameter uncertainties with the known upper and lower bounds. The standard sliding mode is a linear one with asymptotical stability. TSM is based on the properties of terminal attractor[13], which is a class of nonlinear differential equations with finite-time solution. Its main advantage consists in the ability to significantly reduce the transient time to finite time.

Although the finite-time-stabilizing problem of dynamic system has been studied by quite a few people from different perspectives, among the controllers there is a common point, that is the smooth parts of the controllers are constructed by the terms with fractional powers, which are referred as fractional power control[4,5]. Different from homogeneous finite-time control which is constructed with only positive fractional powers, the negative fractional powers emerging in the TSM control may arise the singularity problem around the origin, and some restrictions decided by the strict sliding modes have been added to the parameters of TSM to avoid the difficulty[6-11]. However, exact sliding mode is hardly guaranteed in practice, and even in simulation. Recently, a discontinuous non-singular TSM control scheme only with the items of positive fractional power has been developed while

maintaining the major advantages of the traditional TSM control such as stabilizing the system in finite time[12].

As pointed out above, sliding mode control is usually discontinuous on the sliding manifold for robustness. Due to dscontinuities, sliding mode control systems encounter a drawback of chattering, which is undesirable in practice, since it involves high control activities and further may excite undesirable high frequency dynamics. One conventional way to counter the chattering phenomenon is adding a boundary layer around the sliding manifold and use continuous control inside the boundary[2]. Thanks to the finite-time convergent property of TSM, we will use it to design a kind of continuous reaching law to achieve finite-time convergence of the state on the sliding manifold. Combining the continuous reaching law and the non-singular TSM, we develop a new kind of continuous finite-time controller for trajectory tracking of robotic manipulators. The resulting control can be viewed as a trade-off between discontinuous feedback and linear feedback. If the parameters are carefully selected, it may enjoy the benefits from these two classes of controllers such as robustness and chattering-elimination. Compared with conventional boundary layer method, the proposed approach has better robustness property and disturbance-attenuation ability.

2 Basic concepts

Definition 2. Consider a free system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{f}(0) = 0, \quad \boldsymbol{x} \in R^n \tag{1}$$

where $\boldsymbol{f} : D \to R^n$ is continuous on an open neighborhood D of the origin, the equilibrium point $\boldsymbol{x} = 0$ of the system is (locally) finite-time stable if it is Lyapunov stable and finite-time convergent in a neighborhood $U \subseteq D$. Here, the finite-time convergence means: for any initial condition $\boldsymbol{x}_0 \in U/\{0\}$, there is settling-time function $T(\boldsymbol{x}_0) : U/\{0\} \to (0, \infty)$ such that every solution $\boldsymbol{x}(t, \boldsymbol{x}_0)$ of the system (1) is defined with $\boldsymbol{x}(t, \boldsymbol{x}_0) \in U/\{0\}$ for $t \in [0, T(\boldsymbol{x}_0))$ and satisfies $\lim_{t \to T(\boldsymbol{x}_0)} \boldsymbol{x}(t, \boldsymbol{x}_0) = 0$ and $\boldsymbol{x}(t, \boldsymbol{x}_0) = 0$, if $t \geq T(\boldsymbol{x}_0)$. Moreover, if $U = D = R^n$, the origin is globally finite-time stable.

Definition 2. Consider a controlled system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}, \quad \boldsymbol{x} \in \boldsymbol{R}^n, \quad \boldsymbol{u} \in \boldsymbol{R}^m \tag{2}$$

with $\boldsymbol{g}(\boldsymbol{x}) \neq 0$. It is finite-time stabilizable if there is a feedback law $\boldsymbol{u}(\boldsymbol{x})$ such that $\boldsymbol{x} = 0$ is a (locally) finite-time stable equilibrium of the closed-loop system.

Lemma 1. Consider the nonlinear system described in (1), suppose there is a C^1 (continuously differentiable) function $V(\boldsymbol{x})$ defined in a neighborhood $D \subset R^n$ of the origin, and there are real numbers $\beta > 0$ and $0 < \gamma < 1$, such that $V(\boldsymbol{x}) > 0$ on D and

$$\dot{V}(\boldsymbol{x}) + \beta V^{\gamma}(\boldsymbol{x}) \leq 0 \tag{3}$$

(along the trajectory) on D. Then the origin of the system is finite-time stable. Moreover, the settling time, depending on the initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$, is given by

$$T(\boldsymbol{x}_0) \leq \frac{1}{\beta(1-\gamma)} V^{1-\gamma}(\boldsymbol{x}_0) \tag{4}$$

for $\boldsymbol{x}_0$ in some open neighborhood of the origin. If $\hat{U} = R^n$ and $V(\boldsymbol{x})$ is also radially unbounded, the origin is globally finite-time stable.

Definition 3. The TSM and fast TSM can be described by the following first-order nonlinear differential equations

$$\dot{x} + \beta |x|^{\gamma} sign(x) = 0 \quad \dot{x} - \alpha x - \beta |x|^{\gamma} sign(x) = 0 \tag{5}$$

where $x \in R, \alpha, \beta > 0, \quad 0 < \gamma < 1$.

Remark 1. The expression (5) is a little different from the previously reported TSM and fast TSM[6-12].

$$s = \dot{x} + \beta x^{q/p} = 0, \; s = \dot{x} + \alpha x + \beta x^{q/p} = 0 \tag{6}$$

where $\alpha, \beta > 0, \quad p > q > 0$ are integers, p is odd. This is because of the fact that for $x < 0$, the fractional power q/p may lead to the item $x^{q/p} \notin R$, which means $\dot{x} \notin R$ contradicting with the system we are considering. The equation (5) should be the exact expression of TSM in spite that we have been suggesting only real solution for (6) is considered because this suggestion has been involved in (5).

Remark 2. We can easily express the so-called non-singular TSM[12] in the new form TSM[6-12].

$$s = x + \beta |\dot{x}|^{\gamma} sign(\dot{x}) = 0, \quad \beta > 0, 1 < \gamma < 2 \tag{7}$$

Theorem 1. The equilibrium point $x = 0$ of the continuous non-Lipschitz differential equations (5) is globally finite-time stable, i.e., for any given initial condition $x(0) = x_0$, the system state converges to $x = 0$ in finite time

$$T(x_0) = \frac{1}{\beta(1-\gamma)} |x_0|^{1-\gamma}$$

$$T(x_0) = \frac{1}{\alpha(1-\gamma)} \ln \frac{\alpha |x_0|^{1-\gamma} + \beta}{\beta} \tag{8}$$

respectively and stay there forever.

Theorem 1 can be easily proved with the definition 1 of finite-time stability. Furthermore, another extended Lyapunov function description of finite-time stability of Lemma 1 can be described with the form of fast TSM as

$$\dot{V}(\boldsymbol{x}) + \alpha V(\boldsymbol{x}) + \beta V^{\gamma}(\boldsymbol{x}) \leq 0 \tag{9}$$

and the settling time can be given by

$$T(\boldsymbol{x}_0) \le \frac{1}{\alpha(1-\gamma)} \ln \frac{\alpha V^{1-\gamma}(\boldsymbol{x}_0)+\beta}{\beta} \tag{10}$$

It is evident that the inequalities (9) and (10) means exponential stability plus finite-time stability means faster finite-time stability.

3 Finite-time controller design

In the absence of friction, the dynamics of a serial *n*-link rigid robotic manipulator can be written as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{\tau} \tag{10}$$

where $\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}} \in R^n$ denote the vectors of joint angular position, velocity and acceleration respectively. $\boldsymbol{\tau} \in R^n$ is the vector of applied joint torque, $\boldsymbol{M}(\boldsymbol{q}) \in R^{n\times n}$ is the positive definite inertia matrix, $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \in R^n$ is the vector of centripetal and Coriolis torques, $\mathbf{G}(\mathbf{q}) \in R^n$ is the vector of gravitational torques.

The trajectory tracking control of robot manipulators can be formulated as follows: Let $\boldsymbol{q}_d \in R^n$ be a given twice differentiable desired trajectory, and define the tracking error as $\tilde{\boldsymbol{q}} = \boldsymbol{q} - \boldsymbol{q}_d$. The control objective is to find a feedback control law $u(q,\dot{q})$ such that the manipulator output q tracks the desired trajectory $\boldsymbol{q}_d$, the tracking error converges to zero in finite time.

The following notions are introduced for simplicity and used in the analysis and design of TSM controller.

$$\begin{aligned} sig(\boldsymbol{x})^{\gamma} &= \left[|x_1|^{\gamma_1} sign(x_1), \cdots, |x_n|^{\gamma_n} sign(x_n)\right]^T \in R^n \\ \boldsymbol{x}^{\gamma} &= \left[x_1^{\gamma_1}, \cdots, x_n^{\gamma_n}\right]^T \in R^n \\ |\boldsymbol{x}| &= \left[|x_1|, \cdots, |x_n|\right]^T \in R^n \end{aligned} \tag{12}$$

where $\boldsymbol{x} \in R^n$.Then the TSM can be defined as

$$\boldsymbol{s} = \tilde{\boldsymbol{q}} + \boldsymbol{\beta} sig(\dot{\tilde{\boldsymbol{q}}})^{\gamma} = 0 \tag{13}$$

with $\boldsymbol{s} = [s_1,\cdots,s_n]^T \in R^n$, $\boldsymbol{\beta} = diag(\beta_1,\cdots,\beta_n)$ and $1 < \gamma_i < 2, \quad i = 1,2,\cdots,n$.

$$\begin{aligned} \dot{\boldsymbol{s}} = \ddot{\tilde{\boldsymbol{q}}} + \boldsymbol{\beta} diag\left(\gamma_1 |\dot{\tilde{q}}_1|^{\gamma_1-1}, \cdots, \gamma_n |\dot{\tilde{q}}_n|^{\gamma_n-1}\right) \\ \left(\boldsymbol{M}(\boldsymbol{q})^{-1}(\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q})) - \ddot{\boldsymbol{q}}_d\right) \end{aligned} \tag{14}$$

The conventional TSM control can be designed as a discontinuous control law according to a discontinuous reaching law such as

$$\dot{\boldsymbol{s}} = -\boldsymbol{k} sign(\boldsymbol{s}) \tag{15}$$

where $\boldsymbol{k} = diag(k_1,\cdots,k_n), k_i > 0, i = 1,\cdots,n$ and $sign(\boldsymbol{s}) = [sign(s_1),\cdots,sign(s_n)]^T$. A discontinuous TSM control can be designed as

$$\boldsymbol{\tau} = \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) - \boldsymbol{M}(\boldsymbol{q})\left(\boldsymbol{k} sign(\boldsymbol{s}) - \ddot{\boldsymbol{q}}_d + \boldsymbol{\beta}^{-1}\boldsymbol{\gamma}^{-1}|\dot{\tilde{\boldsymbol{q}}}|^{2-\gamma}\right) \tag{16}$$

which is similar with the reference [12].

Retaining the property of finite-time reaching of TSM but eliminating discontinuities, we propose a kind of continuous fast-TSM-type reaching condition as

$$\dot{\boldsymbol{s}} = -\boldsymbol{k}_1\boldsymbol{s} - \boldsymbol{k}_2 sig(\boldsymbol{s})^{\rho} \tag{17}$$

The inverse dynamics controller is designed as

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) - \boldsymbol{M}(\boldsymbol{q}) \\ &\left(\boldsymbol{k}_1\boldsymbol{s} + \boldsymbol{k}_2 sig(\boldsymbol{s})^{\rho} - \ddot{\boldsymbol{q}}_d + \boldsymbol{\beta}^{-1}\boldsymbol{\gamma}^{-1}|\dot{\tilde{\boldsymbol{q}}}|^{2-\gamma}\right) \end{aligned} \tag{18}$$

Then Replacing the control $\boldsymbol{\tau}$ in (14) with (18) yields

$$\dot{\boldsymbol{s}} = -\boldsymbol{\beta} diag\left(\gamma_1 |\dot{\tilde{q}}_1|^{\gamma_1-1}, \cdots, \gamma_n |\dot{\tilde{q}}_n|^{\gamma_n-1}\right)\left(\boldsymbol{k}_1\boldsymbol{s} + \boldsymbol{k}_2 sig(\boldsymbol{s})^{\rho}\right) \tag{19}$$

Thus, if $\dot{\tilde{\boldsymbol{q}}} \ne 0$, the equation (19) satisfies the reaching condition (17), and the system will reach TSM $\boldsymbol{s} = 0$ in finite time. Here we note that submanifold $\dot{\tilde{\boldsymbol{q}}} = 0$ might hinder reachability of TSM when $\boldsymbol{s} \ne 0$ but $\dot{\boldsymbol{s}} = 0$. Indeed, the closed-looped system of robotic manipulator (11) under the control law (18) with $\dot{\tilde{\boldsymbol{q}}} = 0$ can be written in the form of

$$\ddot{\tilde{\boldsymbol{q}}} = -\boldsymbol{k}_1\boldsymbol{s} - \boldsymbol{k}_2 sig(\boldsymbol{s})^{\rho} \tag{20}$$

Therefore, if $\dot{\tilde{\boldsymbol{q}}} = 0$ and $\boldsymbol{s} \ne 0$, $\ddot{\tilde{\boldsymbol{q}}} > 0$ when $\boldsymbol{s} < 0$, $\ddot{\tilde{\boldsymbol{q}}} < 0$ when $\boldsymbol{s} > 0$, and $\ddot{\tilde{\boldsymbol{q}}} = 0$ only when $\boldsymbol{s} = 0$. It means that with respect to a small time interval δ, at the next instant $\dot{\tilde{\boldsymbol{q}}} = \ddot{\tilde{\boldsymbol{q}}}\delta \ne 0$ in the reaching phase where $\boldsymbol{s} \ne 0$. Therefore we can conclude that TSM still can be reached in finite time and then stay in it thereafter. Once TSM is reached, the system will move along the TSM till converge to the equilibrium point $\tilde{\boldsymbol{q}} = 0$ stably in finite time.

Remark 3. The control law (18) is continuous therefore chattering-free and does not involve any negative-fractional power therefore singularity-free.

Remark 4. Please note even for the certain system as (1), the conventional TSM control design still need the discontinuous control as (16) to guarantee the finite-time reaching to TSM. Here we achieve the same objective with the continuous control. The commonly used boundary-layer method can only guarantee the finite-time reaching to the boundary layer, inside which the control is linear, only asymptotical convergence can be obtained.

Remark 5. With fast finite-time convergence property of fast TSM, $\ddot{\tilde{\boldsymbol{q}}}$ in (20) can be kept in a big value no matter how far or near to the TSM. This property can further avoid system state stuck in the neighbourhood of

$\dot{\tilde{\boldsymbol{q}}} = 0$ in the reaching phase. Furthermore, The nonlinear item $\left|\dot{\tilde{q}}_i\right|^{\gamma_i - 1}$ of the reaching condition (19) can increase the convergent rate to TSM around the neighbourhood of $\dot{\tilde{q}}_i = 0$ because the fractional power $0 < \gamma_i - 1 < 1$ makes $\left|\dot{\tilde{q}}_i\right|^{\gamma_i - 1}$ amplify $\dot{\tilde{q}}_i$ when $\dot{\tilde{q}}_i < 1$. For example, if we choose $\dot{\tilde{q}}_i = 0.00001$, $\gamma = 1.2$, $\left|\dot{\tilde{q}}_i\right|^{\gamma_i - 1} = 0.1$. This property can accelerate the system to escape from the neighbourhood of $\dot{\tilde{q}}_i = 0$ in the reaching phase. On the other hand, it can keep system state in TSM more strongly around the equilibrium point in the sliding phase.

4 Robustness analysis

Generally, in practical robot systems, the perturbations in system parameters and external disturbances are inevitable. In this case, the parameter matrices in the model (1) can be divided as bounded external disturbance.

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q}) &= \boldsymbol{M}_0(\boldsymbol{q}) + \boldsymbol{\delta M}(\boldsymbol{q}) \\ \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \boldsymbol{C}_0(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{\delta C}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ \boldsymbol{G}(\boldsymbol{q}) &= \boldsymbol{G}_0(\boldsymbol{q}) + \boldsymbol{\delta G}(\boldsymbol{q}) \end{aligned} \tag{21}$$

where $\boldsymbol{M}_0(\boldsymbol{q})$, $\boldsymbol{C}_0(\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\boldsymbol{G}_0(\boldsymbol{q})$ are the nominal parts and are assumed to be known exactly, $\boldsymbol{\delta M}(\boldsymbol{q})$, $\boldsymbol{\delta C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\boldsymbol{\delta G}(\boldsymbol{q})$ represent the perturbations in the system matrixes. Then, the dynamical model of robotic manipulator can be rewritten as

$$\boldsymbol{M}_0(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}_0(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}_0(\boldsymbol{q}) + \boldsymbol{F}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) = \boldsymbol{\tau} + \boldsymbol{\tau}_d \tag{22}$$

$\boldsymbol{F}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) = \boldsymbol{\delta M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{\delta C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{\delta G}(\boldsymbol{q}) \in R^n$ is the lumped system uncertainties and is assumed to be bounded by positive known function $\boldsymbol{\omega} = [\omega_1, \cdots, \omega_n]^T$, i.e., $\left|\boldsymbol{F}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\right| < \boldsymbol{\omega}$, and $\left|\boldsymbol{\tau}_d\right| < \boldsymbol{D}, \boldsymbol{\tau}_d \in R^n$ is the bounded external disturbance.

Actually, the proposed algorithm is also robust with respect to the bounded system uncertainties and external disturbance. In this case, the derivative (14) becomes

$$\begin{aligned} \dot{\boldsymbol{s}} &= \dot{\tilde{\boldsymbol{q}}} + \boldsymbol{\beta} diag\left(\gamma_1\left|\dot{\tilde{q}}_1\right|^{\gamma_1 - 1}, \cdots, \gamma_n\left|\dot{\tilde{q}}_n\right|^{\gamma_n - 1}\right) \\ &\left(\boldsymbol{M}_0(\boldsymbol{q})^{-1}(\boldsymbol{\tau} + \boldsymbol{\tau}_d - \boldsymbol{C}_0(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{G}_0(\boldsymbol{q}) - \boldsymbol{F}) - \ddot{\boldsymbol{q}}_d\right) \end{aligned} \tag{23}$$

With the similar controller as (18) and the nominal system functions, we have

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{C}_0(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}_0(\boldsymbol{q}) - \boldsymbol{M}_0(\boldsymbol{q}) \\ &\left(\boldsymbol{k}_1\boldsymbol{s} + \boldsymbol{k}_2 sig(\boldsymbol{s})^{\boldsymbol{\rho}} - \ddot{\boldsymbol{q}}_d + \boldsymbol{\beta}^{-1}\boldsymbol{\gamma}^{-1}\left|\dot{\tilde{\boldsymbol{q}}}\right|^{2-\boldsymbol{\gamma}}\right) \end{aligned} \tag{24}$$

Applying (24) to (23) yields

$$\begin{aligned} \dot{\boldsymbol{s}} &= -\boldsymbol{\beta} diag\left(\gamma_1\left|\dot{\tilde{q}}_1\right|^{\gamma_1 - 1}, \cdots, \gamma_n\left|\dot{\tilde{q}}_n\right|^{\gamma_n - 1}\right) \\ &\left(\boldsymbol{k}_1\boldsymbol{s} + \boldsymbol{k}_2 sig(\boldsymbol{s})^{\boldsymbol{\rho}} - \boldsymbol{M}_0(\boldsymbol{q})^{-1}(\boldsymbol{\tau}_d - \boldsymbol{F})\right) \end{aligned} \tag{25}$$

Furthermore, we can change (25) into the following two forms

$$\begin{aligned} \dot{\boldsymbol{s}} &= -\boldsymbol{\beta} diag\left(\gamma_1\left|\dot{\tilde{q}}_1\right|^{\gamma_1 - 1}, \cdots, \gamma_n\left|\dot{\tilde{q}}_n\right|^{\gamma_n - 1}\right) \\ &\left(\left(\boldsymbol{k}_1 - diag\left(\frac{\boldsymbol{M}'_{0i}(\boldsymbol{q})(\boldsymbol{\tau}_d - \boldsymbol{F})}{s_1}\right)\right)\boldsymbol{s} + \boldsymbol{k}_2 sig(\boldsymbol{s})^{\boldsymbol{\rho}}\right) \end{aligned} \tag{26}$$

$$\begin{aligned} \dot{\boldsymbol{s}} &= -\boldsymbol{\beta} diag\left(\gamma_1\left|\dot{\tilde{q}}_1\right|^{\gamma_1 - 1}, \cdots, \gamma_n\left|\dot{\tilde{q}}_n\right|^{\gamma_n - 1}\right) \\ &\left(\boldsymbol{k}_1\boldsymbol{s} + \left(\boldsymbol{k}_2 - diag\left(\frac{\boldsymbol{M}'_{0i}(\boldsymbol{q})(\boldsymbol{\tau}_d - \boldsymbol{F})}{\left|s_1\right|^{\rho_1} sign(s_1)}\right)\right) sig(\boldsymbol{s})^{\boldsymbol{\rho}}\right) \end{aligned} \tag{27}$$

$\boldsymbol{M}_0(\boldsymbol{q})^{-1} = \left[\boldsymbol{M}'_{01}(\boldsymbol{q}), \cdots, \boldsymbol{M}'_{0n}(\boldsymbol{q})\right]^T, \boldsymbol{M}'_{0i}(\boldsymbol{q}) \in R^n$.

$\boldsymbol{M}_0(\boldsymbol{q})$ is a positive definite inertia matrix, so we have

$$\boldsymbol{M}'_{0i}(\boldsymbol{q})(\boldsymbol{\tau}_d - \boldsymbol{F}) \le \boldsymbol{M}'_{0i}(\boldsymbol{q})(\boldsymbol{D} + \boldsymbol{\omega}) \tag{28}$$

Therefore, if we choose

$$\boldsymbol{k}_1 = diag\left(\frac{\boldsymbol{M}'_i(\boldsymbol{q})(\boldsymbol{D} + \boldsymbol{\omega})}{\left|s_1\right|}\right) + \boldsymbol{\eta}_1 \tag{29}$$

$$\boldsymbol{k}_2 = diag\left(\frac{\boldsymbol{M}'_i(\boldsymbol{q})(\boldsymbol{D} + \boldsymbol{\omega})}{\left|s_1\right|^{\rho_1}}\right) + \boldsymbol{\eta}_2 \tag{30}$$

with $\boldsymbol{\eta}_1 = diag(\eta_{11}, \cdots, \eta_{1n}), \boldsymbol{\eta}_2 = diag(\eta_{21}, \cdots, \eta_{2n})$, $\eta_{1i}, \eta_{2i} > 0, i = 1, \cdots, n$, The equations (25) and (26) can be respectively rewritten as

$$\begin{aligned} \dot{\boldsymbol{s}} &= -\boldsymbol{\beta} diag\left(\gamma_1\left|\dot{\tilde{q}}_1\right|^{\gamma_1 - 1}, \cdots, \gamma_n\left|\dot{\tilde{q}}_n\right|^{\gamma_n - 1}\right) \\ &\left(\boldsymbol{k}'_1\boldsymbol{s} + \boldsymbol{k}_2 sig(\boldsymbol{s})^{\boldsymbol{\rho}}\right) \end{aligned} \tag{31}$$

$$\begin{aligned} \dot{\boldsymbol{s}} &= -\boldsymbol{\beta} diag\left(\gamma_1\left|\dot{\tilde{q}}_1\right|^{\gamma_1 - 1}, \cdots, \gamma_n\left|\dot{\tilde{q}}_n\right|^{\gamma_n - 1}\right) \\ &\left(\boldsymbol{k}_1\boldsymbol{s} + \boldsymbol{k}'_2 sig(\boldsymbol{s})^{\boldsymbol{\rho}}\right) \end{aligned} \tag{32}$$

where $\boldsymbol{k}'_1 = \boldsymbol{k}_1 - diag\left(\boldsymbol{M}'_{0i}(\boldsymbol{q})(\boldsymbol{\tau}_d - \boldsymbol{F})/s_1\right) \ge \boldsymbol{\eta}_1$, $\boldsymbol{k}'_2 = \boldsymbol{k}_2 - diag\left(\boldsymbol{M}'_{0i}(\boldsymbol{q})(\boldsymbol{\tau}_d - \boldsymbol{F})/\left(\left|s_i\right|^{\rho_1} sign(s_i)\right)\right) \ge \boldsymbol{\eta}_2$

Therefore, if $\dot{\tilde{\boldsymbol{q}}} \ne 0$, the equations (31) and (32) are still kept in the form of reaching condition (17), which means that the finite-time convergent property is still held if we

choose the gains $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ as (29) and (30). Meanwhile, the gains guarantee the system trajectory will finite-time converge to the regions

$$\boldsymbol{\Delta} = \min\{\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2\} \tag{33}$$

$$\boldsymbol{\Delta}_1 = \left\{ |\boldsymbol{s}| \le \frac{\boldsymbol{M}^{-1}(\boldsymbol{q})(\boldsymbol{D}+\boldsymbol{\omega})}{\boldsymbol{k}_1 - \boldsymbol{\eta}_1} \right\} \tag{34}$$

$$\boldsymbol{\Delta}_2 = \left\{ |\boldsymbol{s}| \le \left(\frac{\boldsymbol{M}^{-1}(\boldsymbol{q})(\boldsymbol{D}+\boldsymbol{\omega})}{\boldsymbol{k}_2 - \boldsymbol{\eta}_2} \right)^{\frac{1}{\rho}} \right\} \tag{35}$$

where $\boldsymbol{\Delta} = [\Delta_1, \cdots, \Delta_n]^T$, $\Delta_i > 0, i = 1, \cdots, n$. $\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2$ is the results from the chosen gains $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ respectively. With the chosen $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, the finite-time convergent property is always held as (31) and (32), so the system will convergence to the smaller one as (33).

Remark 6. In the region (33), the neighborhood $\boldsymbol{\Delta}_1$ is a result for linear control with power one such as inside the conventional boundary layer, and $\boldsymbol{\Delta}_2$ is a result for TSM control with the fractional power $\boldsymbol{\rho}$. If we choose $\boldsymbol{k}_1 = \boldsymbol{k}_2$ big enough and $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2$ such that $\boldsymbol{M}^{-1}(\boldsymbol{q})(\boldsymbol{D}+\boldsymbol{\omega}) / (\boldsymbol{k}_i - \boldsymbol{\eta}_i) < 1, i = 1,2$, $\boldsymbol{\Delta}_2$ can be reduced greatly with $\boldsymbol{\Delta}_2 << \boldsymbol{\Delta}_1$. This means TSM control has stronger robustness and disturbance rejection ability.

Here we face the similar problem as section 3, that is $\dot{\tilde{\boldsymbol{q}}} = 0$ might hinder the reachability of TSM outside the region (33). Certainly, we can also prove that it is impossible with the similar way as follows.

Indeed, the closed-looped system of robotic manipulator (22) under the control law (24) with $\dot{\tilde{\boldsymbol{q}}} = 0$ can be written in the form of

$$\ddot{\tilde{\boldsymbol{q}}} = -\boldsymbol{k}_1 \boldsymbol{s} - \boldsymbol{k}_2 sig(\boldsymbol{s})^{\rho} + \boldsymbol{M}_0(\boldsymbol{q})^{-1}(\boldsymbol{\tau}_d - \boldsymbol{F}) \tag{36}$$

which can be understood as the following two forms

$$\begin{aligned} \ddot{\tilde{\boldsymbol{q}}} &= -\left(\boldsymbol{k}_1 - diag\left(\frac{\boldsymbol{M}'_{0i}(\boldsymbol{q})(\boldsymbol{\tau}_d - \boldsymbol{F})}{s_1} \right) \right) \boldsymbol{s} - \boldsymbol{k}_2 sig(\boldsymbol{s})^{\rho} \\ &= -\boldsymbol{k}'_1 \boldsymbol{s} - \boldsymbol{k}_2 sig(\boldsymbol{s})^{\rho} \end{aligned} \tag{37}$$

$$\begin{aligned} \ddot{\tilde{\boldsymbol{q}}} &= -\boldsymbol{k}_1 \boldsymbol{s} - \left(\boldsymbol{k}_2 - diag\left(\frac{\boldsymbol{M}'_{01}(\boldsymbol{q})(\boldsymbol{\tau}_d - \boldsymbol{F})}{|s_1|^{\rho_1} sign(s_1)} \right) \right) sig(\boldsymbol{s})^{\rho} \\ &= -\boldsymbol{k}_1 \boldsymbol{s} - \boldsymbol{k}'_2 sig(\boldsymbol{s})^{\rho} \end{aligned} \tag{38}$$

Similar analysis as section 3 can easily conclude that the region (33) can also be reached in finite time. After the system enters the region (33), the system dynamics can be described as

$$\boldsymbol{s} = \tilde{\boldsymbol{q}} + \boldsymbol{\beta} sig(\dot{\tilde{\boldsymbol{q}}})^{\gamma} = \boldsymbol{\varphi}, \quad |\boldsymbol{\varphi}| \le \boldsymbol{\Delta} \tag{39}$$

where $\boldsymbol{\varphi} = [\varphi_1, \cdots, \varphi_n]^T$. Different from the finite-time convergence to TSM $\boldsymbol{s} = 0$ in section 3, where the system converges to the origin along TSM in finite time, here TSM $\boldsymbol{s}$ only converges to a bounded neighborhood of $\boldsymbol{s} = 0$ in finite time. In order to further explore the system dynamics in the neighborhood $\boldsymbol{\Delta}$, we change the dynamics (39) to this form

$$\begin{aligned} &\tilde{\boldsymbol{q}} + \left(\boldsymbol{\beta} - diag\left(\frac{\varphi_i}{|\dot{\tilde{q}}_i|^{\gamma_i} sign(\dot{\tilde{q}}_i)} \right) \right) sig(\dot{\tilde{\boldsymbol{q}}})^{\gamma} \\ &= \tilde{\boldsymbol{q}} + \boldsymbol{\beta}' sig(\dot{\tilde{\boldsymbol{q}}})^{\gamma} = 0 \end{aligned} \tag{40}$$

If $\boldsymbol{\beta} = diag\left(\Delta_1 / |\dot{\tilde{q}}_1|^{\gamma_1}, \cdots, \Delta_n / |\dot{\tilde{q}}_n|^{\gamma_n} \right) + \boldsymbol{\eta}_3$ and $\boldsymbol{\eta}_3 = diag(\eta_{31}, \cdots, \eta_{3n}), \eta_{3i} > 0, i = 1, \cdots, n$, the equation (40) is still kept in the form of TSM, which also means the system will converge to the region in finite time.

$$\boldsymbol{\Delta}_{\dot{\tilde{\boldsymbol{q}}}} = \left\{ |\dot{\tilde{\boldsymbol{q}}}| \le \left((\boldsymbol{\beta} - \boldsymbol{\eta}_3)^{-1} \boldsymbol{\Delta} \right)^{\frac{1}{\gamma}} \right\} \tag{41}$$

Furthermore, we can have

$$\boldsymbol{\Delta}_{\tilde{\boldsymbol{q}}} = \left\{ |\tilde{\boldsymbol{q}}| \le \left| \boldsymbol{\beta} sig(\dot{\tilde{\boldsymbol{q}}})^{\gamma} \right| + |\boldsymbol{\varphi}| = \left(\boldsymbol{\beta}(\boldsymbol{\beta} - \boldsymbol{\eta}_3)^{-1} + \boldsymbol{I} \right) \boldsymbol{\Delta} \right\} \tag{42}$$

5 Simulations

Consider a two-link robotic manipulator moving in a plane of the form (11) with

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix}, \boldsymbol{C} = 0, \boldsymbol{G}(\boldsymbol{q}) = g \begin{bmatrix} 0 \\ m_2 q_2 \end{bmatrix}$$

where $m_1 = 18.8$ and $m_2 = 13.2$ (Hong et al., 2002). In this example, the robotic manipulator starts at the initial position $\boldsymbol{q}_0 = [3\ 3]^T$ and initial velocity $\dot{\boldsymbol{q}}_0 = [0\ 0]^T$. The control objective is to drive the manipulator joints to track desired trajectory $\boldsymbol{q}_d = [0.6\sin(1.5t) \quad \sin 2t]^T$ in finite time.

If we choose the TSM as (13), the conventional discontinuous finite-time TSM control, TSM control with the conventional boundary layer and the continuous finite-time TSM control proposed in this paper for this example can be respectively designed as

$$\boldsymbol{\tau} = \boldsymbol{G}(\boldsymbol{q}) - \boldsymbol{M}(\boldsymbol{q}) \left(\boldsymbol{k} sign(\boldsymbol{s}) - \ddot{\boldsymbol{q}}_d + \boldsymbol{\beta}^{-1} \boldsymbol{\gamma}^{-1} |\dot{\tilde{\boldsymbol{q}}}|^{2-\gamma} \right) \tag{43}$$

$$\boldsymbol{\tau} = \boldsymbol{G}(\boldsymbol{q}) - \boldsymbol{M}(\boldsymbol{q})\left(\boldsymbol{k}sat\left(\frac{\boldsymbol{s}}{\boldsymbol{\varepsilon}}\right) - \ddot{\boldsymbol{q}}_d + \boldsymbol{\beta}^{-1}\boldsymbol{\gamma}^{-1}\left|\dot{\tilde{\boldsymbol{q}}}\right|^{2-\gamma}\right) \quad (44)$$

$$\boldsymbol{\tau} = \boldsymbol{G}(\boldsymbol{q}) - \boldsymbol{M}(\boldsymbol{q}) \left(\boldsymbol{k}_1\boldsymbol{s} + \boldsymbol{k}_2 sig(\boldsymbol{s})^{\rho} - \ddot{\boldsymbol{q}}_d + \boldsymbol{\beta}^{-1}\boldsymbol{\gamma}^{-1}\left|\dot{\tilde{\boldsymbol{q}}}\right|^{2-\gamma}\right) \quad (45)$$

The simulation results for the controller (43) with the parameters as $\boldsymbol{k} = diag(10,10)$, $\boldsymbol{\beta} = diag(0.5,0.5)$ and $\boldsymbol{\gamma} = [1.5\ 1.5]^T$ are shown in figure 1. The intensive chattering emerges in the control signal after the system enters the sliding phase in spite that the perfect tracking is acquired in finite time. In order to eliminate the chattering, we adopt the controller (44) with the boundary layer where $\boldsymbol{\varepsilon} = diag(1,1)$, and the simulation results in figure 2 show the chattering is really eliminated but approaching to the TSM is asymptotic.

For the controller (45), we choose $\boldsymbol{k}_1 = diag(10,10)$, $\boldsymbol{k}_2 = diag(30,30)$, $\boldsymbol{\beta} = diag(0.5,0.5)$, the paramters $\boldsymbol{\gamma} = [1.5\ 1.5]^T$ and $\boldsymbol{\rho} = [0.2\ 0.2]^T$. The simulation results shown in the figure 3 demonstrate the merits of both the controllers (43) and (44): perfect tracking, i.e., tracking errors $\tilde{\boldsymbol{q}}$ and $\dot{\tilde{\boldsymbol{q}}}$ reach TSM $\boldsymbol{s} = 0$ in finite-time and then converge to $\tilde{\boldsymbol{q}} = 0$ and $\dot{\tilde{\boldsymbol{q}}} = 0$ along $\boldsymbol{s} = 0$ in finite time, and chattering-free. Furthermore, the respective shortcomings, i.e., the chattering and asymptotical approaching to TSM, disappeared. Meanwhile, the control law is singularity-free.

In order to exhibit the robustness and disturbance-attenuation property of the proposed algorithm, we assume that the control objective is to attain the tracking accuracy as $\left|\tilde{\boldsymbol{q}}\right| \le [0.001,\ 0.001]^T$ and $\left|\dot{\tilde{\boldsymbol{q}}}\right| \le [0.01,\ 0.001]^T$ in finite time in spite of the uncertainties and disturbances in (25) as

$$\boldsymbol{M}_0(\boldsymbol{q})^{-1}(\boldsymbol{\tau}_d - \boldsymbol{F}) = [3\sin(t)\ 3\sin(t)]^T \le [3\ 3]^T \quad (46)$$

If $\boldsymbol{\beta} = diag(0.5,0.5)$ and $\boldsymbol{\eta}_3 = diag(0.1,0.1)$ in (41), according to the required tracking accuracy of $\tilde{\boldsymbol{q}}$, the neighborhood $\boldsymbol{\Delta} \le [0.00044,\ 0.00044]^T$ of TSM is needed to be reached in a finite time. Furthermore, according to the required tracking accuracy of $\dot{\tilde{\boldsymbol{q}}}$, $\boldsymbol{\Delta}$ and (41), we are required to choose $\boldsymbol{\gamma} \le [1.48,\ 1.48]^T$. In order to assure the required neighborhood $\boldsymbol{\Delta}$, according to the equations (33), (34) and (35), we can choose the designed parameters as $\boldsymbol{k}_1 = \boldsymbol{k}_2 = diag(16,16)$, $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2 = diag(1,1)$ and $\boldsymbol{\rho} = [0.2\ 0.2]^T$, then we achieve the target as

$$\boldsymbol{\Delta}_1 = \{|\boldsymbol{s}| \le 0.2\},\ \boldsymbol{\Delta}_2 = \{|\boldsymbol{s}| \le 0.00032\},\ \boldsymbol{\Delta} = \boldsymbol{\Delta}_2 \quad (47)$$

Here it is clearly demonstrated that with the same control gain for the same uncertainties and disturbances, the fractional-power control has better robustness and disturbance-attenuation ability than the linear one. The simulation results with the chosen parameters as above are shown in figure 4. Here the region $\boldsymbol{\Delta}$ in (47) is reached at $t = 0.385s$, and then in the small neighborhood of TSM $\boldsymbol{s} = 0$, the control objectives for the angular position tracking $\tilde{\boldsymbol{q}}$ and the angular velocity tracking $\dot{\tilde{\boldsymbol{q}}}$ are achieved at $t = 2.56s$ and $t = 2.6s$ respectively.

6 Conclusions

We have developed singularity-free continuous TSM controllers for trajectory tracking of robotic manipulators with finite-time convergence. The new form of TSM can be used not only to design sliding mode for finite-time convergence to the equilibrium, but also to design continuous TSM control law to drive system states to reach TSM in finite time. By properly choosing the fractional powers, the proposed TSM controllers can enjoy stronger robustness property and chattering attenuation with finite time stability. The effectiveness of the developed algorithms is validated by simulation results. One of challenging works is to generate the results to the general *n*-order uncertain nonlinear systems.

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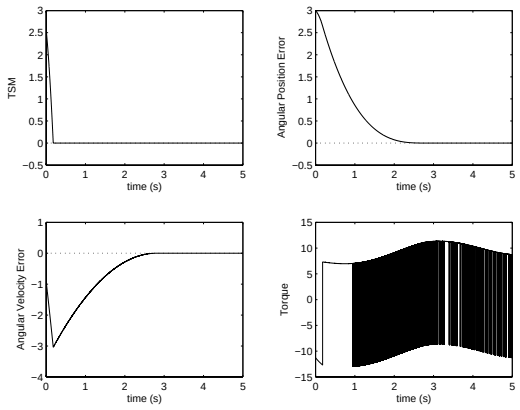

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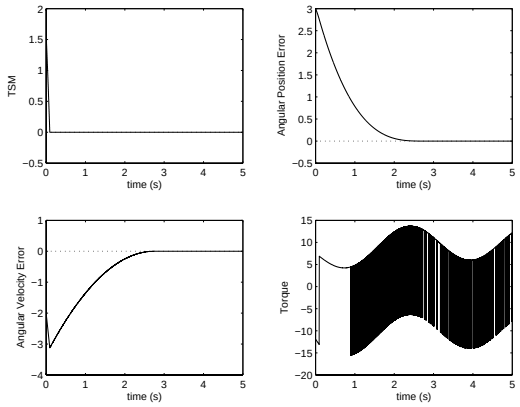

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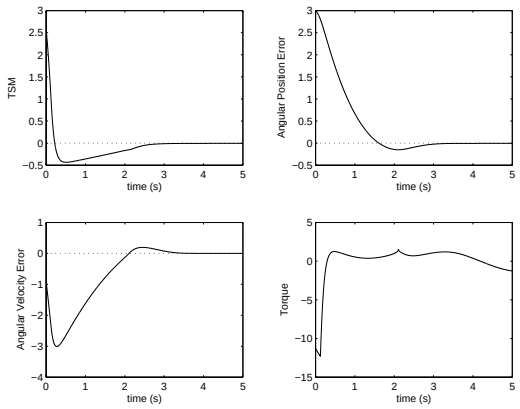
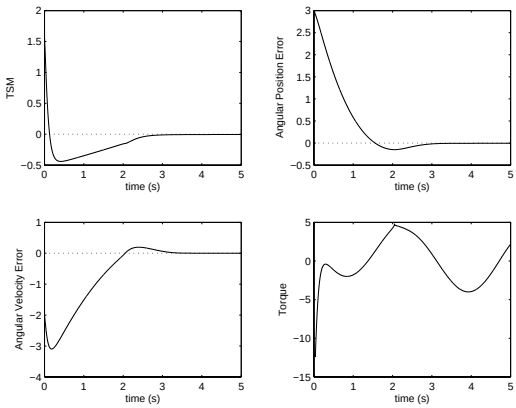
Fig.1 The discontinuous finite-time TSM control

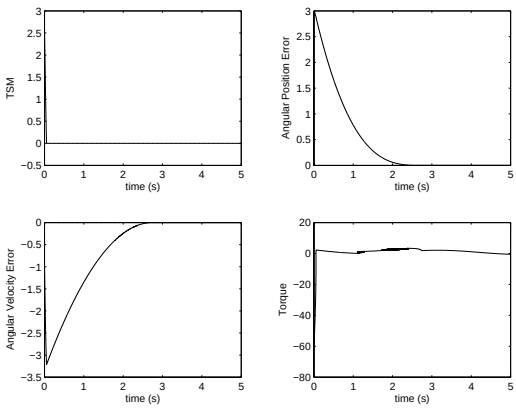


a) The joint 1



b) The joint 2

Fig. 2 TSM control with the boundary layer



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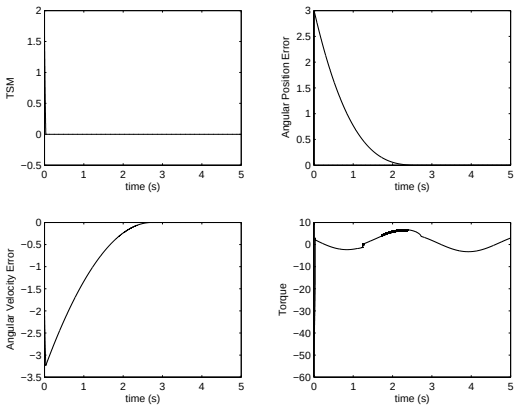
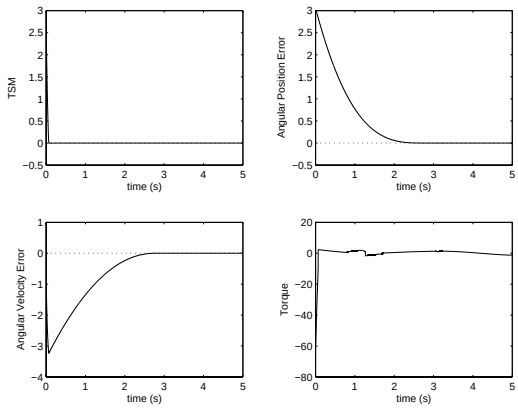


b) The joint 2

Fig.3 The continuous finite-time TSM control



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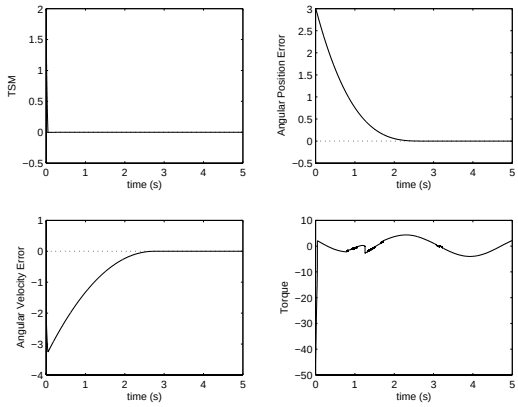

b) The joint 2

Fig.4 The continuous finite-time TSM control with uncertainties