CONTINUOUS IMAGES OF CLOSED SETS IN GENERALIZED BAIRE SPACES

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ABSTRACT. Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. Given a cardinal μ, we equip the set ${}^{\kappa}\mu$ consisting of all functions from κ to μ with the topology whose basic open sets consist of all extensions of partial functions of cardinality less than κ. We prove results that allow us to separate several classes of subsets of ${}^{\kappa}\kappa$ that consist of continuous images of closed subsets of spaces of the form ${}^{\kappa}\mu$. Important examples of such results are the following: (i) there is a closed subset of ${}^{\kappa}\kappa$ that is not a continuous image of ${}^{\kappa}\kappa$; (ii) there is an injective continuous image of ${}^{\kappa}\kappa$ that is not κ-Borel (i.e. that is not contained in the smallest algebra of sets on ${}^{\kappa}\kappa$ that contains all open subsets and is closed under κ-unions); (iii) the statement "*every continuous image of ${}^{\kappa}\kappa$ is an injective continuous image of a closed subset of ${}^{\kappa}\kappa$*" is independent of the axioms of ZFC; and (iv) the axioms of ZFC do not prove that the assumption "$2^{\kappa} > \kappa^{+}$" implies the statement "*every closed subset of ${}^{\kappa}\kappa$ is a continuous image of ${}^{\kappa}(\kappa^{+})$*" or its negation.

CONTENTS

1. INTRODUCTION

Let κ be an infinite regular cardinal. Given a cardinal μ, we equip the set ${}^{\kappa}\mu$ consisting of all functions $x : \kappa \longrightarrow \mu$ with the topology whose basic open sets are of the form

$$N_s \ = \ \{x \in {}^{\kappa}\mu \mid s \subseteq x\},$$

where s is an element of the set ${}^{<\kappa}\mu$ of all functions $t : \alpha \longrightarrow \mu$ with $\alpha < \kappa$. We let $\mathbf{\Sigma}^0_1(\kappa)$ denote the class of all open subsets of ${}^{\kappa}\kappa$ and we use $B(\kappa)$ to denote the class of all κ-Borel subsets of ${}^{\kappa}\kappa$, i.e. the class of all subsets that are contained in

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the smallest algebra of sets on ${}^\kappa\kappa$ that contains all open subsets and is closed under κ-unions. A subset of ${}^\kappa\kappa$ is a $\mathbf{\Sigma}^1_1$-subset if it is equal to the projection of a closed subset of ${}^\kappa\kappa \times {}^\kappa\kappa$ and it is a $\mathbf{\Delta}^1_1$-subset if both the set itself and its complement are $\mathbf{\Sigma}^1_1$-subsets. Since the graph of a continuous function $f : C \longrightarrow {}^\kappa\kappa$ with $C \subseteq {}^\kappa\kappa$ closed is again a closed subset of ${}^\kappa\kappa \times {}^\kappa\kappa$ and the spaces ${}^\kappa\kappa$ and ${}^\kappa\kappa \times {}^\kappa\kappa$ are homeomorphic, it follows that a subset A of ${}^\kappa\kappa$ is a $\mathbf{\Sigma}^1_1$-subset if and only if it is equal to the continuous image of a closed subset of ${}^\kappa\kappa$ (in the sense that there is a closed subset C of ${}^\kappa\kappa$ and a function $f : C \longrightarrow {}^\kappa\kappa$ such that $A = \mathrm{ran}(f)$ and f is continuous with respect to the subspace topology on C).

If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, then it is well-known (see, for example, [Lüc12, Section 2]) that the class of $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ is equal to the class of subsets of ${}^\kappa\kappa$ that are definable over the structure $(\mathrm{H}(\kappa^+), \in)$ by a Σ_1-formula with parameters. This shows that many interesting and important subsets of ${}^\kappa\kappa$ are equal to continuous images of closed subsets of spaces of the form ${}^\kappa\mu$. We present two examples of prominent $\mathbf{\Sigma}^1_1$-subsets that are contained in smaller classes of continuous images.

Example 1.1. *The club filter*

$$\mathrm{Club}_\kappa \ = \ \{x \in {}^\kappa\kappa \mid \exists C \subseteq \kappa \textit{ club } \forall \alpha \in C \ x(\alpha) = 0\}$$

is a continuous image of the space ${}^\kappa\kappa$*. Let T denote the set of all pairs (s,t) in ${}^\gamma\kappa \times {}^\gamma 2$ such that $\gamma < \kappa$ is a limit ordinal, $t(\alpha) = 0$ implies $s(\alpha) = 0$ for all $\alpha < \gamma$, and the set $\{\alpha < \gamma \mid t(\alpha) = 0\}$ is a closed unbounded subset of γ. We order T by componentwise inclusion. Then T is a tree of height κ that is closed under increasing sequences of length less than κ and every node in T has κ-many direct successors, because every limit ordinal of countable cofinality in the interval $(\mathrm{lh}(s), \kappa)$ gives rise to a distinct direct successor of a node (s,t) in T. This shows that T is isomorphic to the tree ${}^{<\kappa}\kappa$. If we equip the set*

$$[T] \ = \ \{(x,y) \in {}^\kappa\kappa \times {}^\kappa 2 \mid \forall \alpha < \kappa \ \exists \alpha < \beta < \kappa \ (x \upharpoonright \beta, y \upharpoonright \beta) \in T\}$$

with the topology whose basic open sets consist of all extensions of elements of T, then we obtain a topological space homeomorphic to ${}^\kappa\kappa$. Since the projection $p : [T] \longrightarrow {}^\kappa\kappa$ onto the first coordinate is continuous and $\mathrm{ran}(p) = \mathrm{Club}_\kappa$, we can conclude that the set Club_κ is equal to a continuous image of ${}^\kappa\kappa$.

Given an infinite regular cardinal κ and cardinals $\mu_0, \ldots, \mu_n$, we call a subset T of the product ${}^{<\kappa}\mu_0 \times \ldots \times {}^{<\kappa}\mu_n$ a *subtree* if $\mathrm{lh}(t_0) = \ldots = \mathrm{lh}(t_n)$ and the tuple $(t_0 \upharpoonright \alpha, \ldots, t_n \upharpoonright \alpha)$ is an element of T whenever $(t_0, \ldots, t_n) \in T$ and $\alpha < \mathrm{lh}(t_0)$. We use $\leq$ to denote the natural tree-ordering on such a subtree T, i.e. if $s = (s_0, \ldots, s_n)$ and $t = (t_0, \ldots, t_n)$ are nodes in T, then we write $s \leq t$ to denote that $s_i \subseteq t_i$ holds for all $i \leq n$.

Given a subtree T of ${}^{<\kappa}\mu_0 \times \ldots \times {}^{<\kappa}\mu_n$, we say that an element $(x_0, \ldots, x_n)$ of ${}^\kappa\mu_0 \times \ldots \times {}^\kappa\mu_n$ is a *cofinal branch through* T if $(x_0 \upharpoonright \alpha, \ldots, x_n \upharpoonright \alpha) \in T$ for every $\alpha < \kappa$. It is easy to see that a subset A of ${}^\kappa\mu_0 \times \ldots \times {}^\kappa\mu_n$ is closed with respect to the topology introduced above if and only if it is equal to the set $[T]$ of all cofinal branches though some subtree T of ${}^{<\kappa}\mu_0 \times \ldots \times {}^{<\kappa}\mu_n$.

Example 1.2. *Suppose that $f : {}^{<\kappa}\kappa \longrightarrow \kappa$ is a bijection. Let T_κ denote the set of all $x \in {}^\kappa 2$ such that $x \circ f$ is the characteristic function of a subtree T of ${}^{<\kappa}\kappa$ with $[T] \neq \emptyset$. The results of* [MV93, Section 2] *show that T_κ is a $\mathbf{\Sigma}^1_1$-complete subset of*

${}^\kappa\kappa$, i.e. if A is a nonempty $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$, then there is a continuous function $f : {}^\kappa\kappa \longrightarrow {}^\kappa\kappa$ with $A = f^{-1}[T_\kappa]$.

The set T_κ is also equal to a continuous image of ${}^\kappa\kappa$. To see this, let S denote the set of all pairs (t,u) in ${}^\gamma 2 \times {}^\gamma\kappa$ with $\gamma < \kappa$ such that the set

$$T(t) \ = \ \{s \in {}^{<\kappa}\kappa \mid f(s) < \gamma, \ (t \circ f)(s) = 1\}$$

is a subtree of ${}^{<\kappa}\kappa$ with $u \restriction \bar{\gamma} \in T(t)$ for every $\bar{\gamma} < \gamma$. If we order the set S by componentwise inclusion, then the resulting tree is isomorphic to ${}^{<\kappa}\kappa$. As above, we equip the set

$$[S] \ = \ \{(x,y) \in {}^\kappa 2 \times {}^\kappa\kappa \mid \forall \alpha < \kappa \ \exists \alpha < \beta < \kappa \ (x \restriction \beta, y \restriction \beta) \in S\}$$

with the topology induced by S and obtain a space homeomorphic to ${}^\kappa\kappa$. Since T_κ is equal to the projection of $[S]$, this set is equal to a continuous image of ${}^\kappa\kappa$.

In this paper, we study the provable and relative consistent statements about the relationships between different classes of such continuous images in the case where κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. We will consider the following classes of subsets.

- The class C^κ of continuous images of ${}^\kappa\kappa$.
- The class $\mathbf{\Sigma}^1_1(\kappa)$ of continuous images of closed subsets of ${}^\kappa\kappa$.
- The class I^κ of continuous injective images of ${}^\kappa\kappa$.
- The class I^κ_{cl} of continuous injective images of closed subsets of ${}^\kappa\kappa$.

It will turn out that the following class is also important for this analysis.

- Let M be an inner model of set theory and $n < \omega$. We define $S^{M,\kappa}_n$ to be the class of all subsets A of ${}^\kappa\kappa$ such that
  $$A \ = \ \{x \in {}^\kappa\kappa \mid M[x,y] \models \varphi(x,y)\}$$
  for some $y \in {}^\kappa\kappa$ and a Σ_n-formula $\varphi(u,v)$, where $M[x,y] = \bigcup_{z \in M} \mathrm{L}[x,y,z]$ is the smallest transitive model of set theory containing $M \cup \{x,y\}$.

To motivate our work, we start by briefly reviewing the relations between these classes in the classical setting "$\kappa = \omega$".

- The classes $B(\omega)$ and I^ω_{cl} coincide (see [Kec95, 15.3]).
- Since every set in I^ω has no isolated points, the class I^ω is a proper subclass of $B(x)$ that contains every non-empty open set. Moreover, every Borel subset of ${}^\omega\omega$ is equal to the union of an element of I^ω and a countable set (this follows from [Kec95, 6.4] and [Kec95, 13.1]).
- A nonempty subset of ${}^\kappa\kappa$ is in $\mathbf{\Sigma}^1_1(\omega)$ if and only if it is in C^ω (see [Kec95, 2.8]).
- Given an inner model M of ZFC, the class $S^{M,\omega}_1$ coincides with the class of all $\mathbf{\Sigma}^1_2$-subsets of ${}^\omega\omega$, i.e. with the class of projections of complements of $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa \times {}^\kappa\kappa$ (this follows from [Jec03, Lemma 25.20] and [Jec03, Lemma 25.25]).

$$\mathbf{\Sigma}^0_1(\omega) \longrightarrow I^\omega \longrightarrow B(\omega) = I^\omega_{cl} \longrightarrow \mathbf{\Sigma}^1_1(\omega) = C^\omega \longrightarrow S^{M,\omega}_1$$

FIGURE 1. The provable and consistent relations between the considered classes in the case "$\kappa = \omega$".

We summarize the above relationships in Figure 1 with the help of a *complete diagram of provable and consistent relations*. Given two classes C and D of subsets, we use a solid arrow $A \longrightarrow B$ from A to B to indicate that it is provable from the axioms of ZFC that every non-empty subset in A is an element of B and a dashed arrow $A \dashrightarrow B$ to indicate that this inclusion is relatively consistent with these axioms. Such a diagram is *complete* if the transitive hull of the displayed implications contains all possible implications.

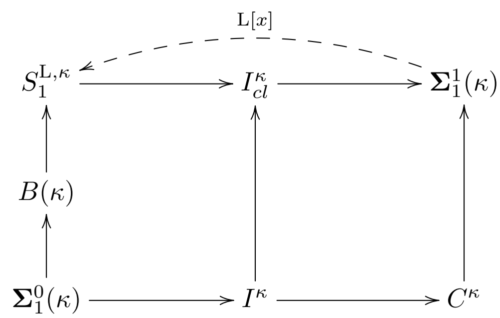

FIGURE 2. The provable and consistent relations in the case where κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$.

The results of this paper will show that the above classes relate in a fundamentally different way if κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. These results are summarized by Figure 2 and they will show that this diagram is also complete.

An important result in the above analysis is the observation that the class of continuous images of ${}^\kappa\kappa$ does not contain every nonempty closed subset of ${}^\kappa\kappa$. We will show that the closed subset constructed in the proof of this result is equal to a continuous image of the space ${}^\kappa(\kappa^+)$. Motivated by this fact, we also investigate the following classes in the case where μ is a cardinal with $\kappa < \mu < 2^\kappa$.

- The class $C^{\kappa,\mu}$ of continuous images of ${}^\kappa\mu$.
- The class $C^{\kappa,\mu}_{cl}$ of continuous images of closed subsets of ${}^\kappa\mu$.

As above, we first discuss known results about the relationships between these classes in the countable case.

- A nonempty subset of ${}^\kappa\kappa$ is in $C^{\omega,\mu}$ if and only if it is in $C^{\omega,\mu}_{cl}$ (see [Kec95, 2.8]).
- Every $\mathbf{\Sigma}^1_2$-subset of ${}^\omega\omega$ is equal to the projection of a subtree of ${}^\omega\omega \times {}^\omega\omega_1$ (see [Kec95, 38.9]). In particular, $\mathbf{\Sigma}^1_1(\omega)$ is a proper subclass of C^{ω,ω_1}.
- If every uncountable $\mathbf{\Sigma}^1_2$-subset of ${}^\omega\omega$ contains a perfect subset, then there is a set in C^{ω,ω_1} that is not a $\mathbf{\Sigma}^1_2$-subset.
- If every subset of ${}^\omega\omega$ of cardinality ω_1 is a $\mathbf{\Sigma}^1_2$-subset, then every set in the class C^{ω,ω_1} is a $\mathbf{\Sigma}^1_2$-subset (see the proof of Proposition 1.20 for a similar argument). By using *almost disjoint coding* (see [JS70] and [Har77, Section 1]), the above assumption can be seen to hold in every model of $\mathrm{MA}_{\omega_1} + \neg(\mathrm{CH}) +$ "$\omega_1 = \omega_1^{\mathrm{L}}$" (see [MS70, Section 3.2]).

The complete diagram shown in Figure 3 summarizes these results.

Analogous to the above results, these classes behave quite differently in the case where κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and μ is a cardinal with

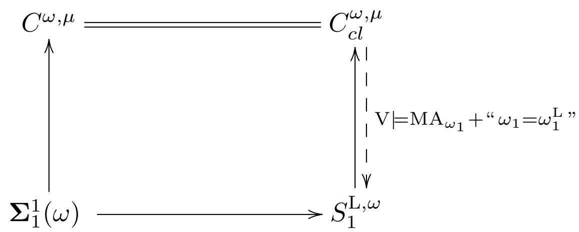

FIGURE 3. The provable and consistent relations between the considered classes in the case where “$\kappa = \omega$” and μ is a cardinal with $\omega < \mu < 2^\omega$.

$\kappa < \mu < 2^\kappa$. The results of this paper will show that the diagram shown in Figure 4 is complete.

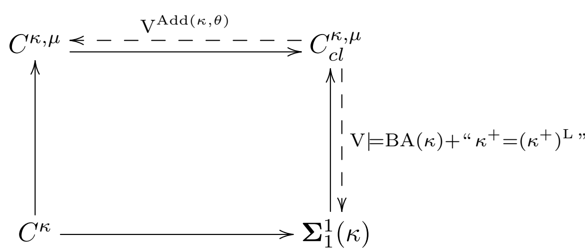

FIGURE 4. The provable and consistent relations between the considered classes in the case where κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and μ is a cardinal with $\kappa < \mu < 2^\kappa$.

In the remainder of this section, we present the results that imply that the above diagrams contain all provable and consistent implications.

1.1. **The class C^κ of continuous images of ${}^\kappa\kappa$.** An important topological property of spaces of the form ${}^\omega\mu$ is the fact that non-empty closed subsets are retracts of the whole space (see [Kec95, 2.8]), i.e. given a closed subset A of ${}^\omega\mu$ there is a continuous (even Lipschitz) surjection $f : {}^\omega\mu \longrightarrow A$ such that $f \restriction A = \mathrm{id}_A$. The proof of this statement generalizes to higher cardinalities for a certain class of closed subsets.

Given an infinite regular cardinal κ and cardinals $\mu_0, \ldots, \mu_n, \nu$, we say that a subtree T of ${}^{<\kappa}\mu_0 \times \ldots \times {}^{<\kappa}\mu_n$ is *$<\nu$-closed* if for every $\lambda < \nu$ and every $\leq$-increasing sequence $\langle (t_0^\alpha, \ldots, t_n^\alpha) \mid \alpha < \lambda \rangle$ in T, the tuple $(\bigcup_{\alpha<\lambda} t_0^\alpha, \ldots, \bigcup_{\alpha<\lambda} t_n^\alpha)$ is an element of T. We call a node t of such a tree T an *end node* if it is maximal in T with respect to the componentwise ordering.

Proposition 1.3. *Let κ be an infinite regular cardinal, $\mu_0, \ldots, \mu_n$ be cardinals and T be a subtree of ${}^{<\kappa}\mu_0 \times \ldots \times {}^{<\kappa}\mu_n$. If T is a $<\kappa$-closed tree without end nodes, then the closed set $[T]$ is a retract of ${}^\kappa\mu_0 \times \ldots \times {}^\kappa\mu_n$.*

Proof. Note that our assumptions imply that there is a cofinal branch through every node of T. We define ∂T to be the set

$$\{(s_0,\dots,s_n)\in({}^{\gamma}\mu_0\times\cdots\times{}^{\gamma}\mu_n)\setminus T \mid \gamma<\kappa, \forall\alpha<\gamma(s_0\restriction\alpha,\dots,s_n\restriction\alpha)\in T\}. \quad (1)$$

Then our assumptions imply that every element s of ∂T is the direct successor of an element t_s of T and there is a cofinal branch x_s through T with $t_s\subseteq x_s$. If $y=(y_0,\dots,y_n)\in({}^{\kappa}\mu_0\times\cdots\times{}^{\kappa}\mu_n)\setminus[T]$, then there is a unique $\alpha<\kappa$ such that $(y_0\restriction\alpha,\dots,y_n\restriction\alpha)\in\partial T$ and we let s_y denote this tuple. This allows us to define a surjection $f:{}^{\kappa}\mu_0\times\cdots\times{}^{\kappa}\mu_n\longrightarrow[T]$ by setting $f(x)=x$ if $x\in[T]$ and $f(y)=x_{s_y}$ otherwise. It is easy to see that f is continuous. □

The following observation shows that the above topological property does not hold for all closed subsets of ${}^{\kappa}\mu$ if κ is regular and uncountable.

Proposition 1.4. *Suppose that κ is an uncountable regular cardinal and $\mu>1$ is a cardinal. Let A denote the set of all x in ${}^{\kappa}\mu$ such that $x(\alpha)=0$ for only finitely many $\alpha<\kappa$. Then A is a closed subset of ${}^{\kappa}\mu$ that is not a retract of ${}^{\kappa}\mu$.*

Proof. Let T denote the subtree of ${}^{<\kappa}\mu$ consisting of all $t\in{}^{<\kappa}\mu$ such that $t(\alpha)=0$ for only finitely many $\alpha\in dom(t)$. Since $\mathrm{cof}(\kappa)>\omega$, we have $A=[T]$ and this shows that A is a closed subset of ${}^{\kappa}\mu$.

Suppose that $f:{}^{\kappa}\mu\longrightarrow A$ is a retraction onto A. We inductively construct sequences $\langle x_n\in A\mid n<\omega\rangle$ and $\langle\gamma_n<\kappa\mid n<\omega\rangle$ such that $x_n(\gamma_n)=0$, $\gamma_n<\gamma_{n+1}$ and $x_n\restriction\gamma_{n+1}=x_{n+1}\restriction\gamma_{n+1}$ for all $n<\omega$. Let $\gamma_0=0$ and x_0 be an arbitrary element of A with $x_0(0)=0$. Now assume that x_n and γ_n are already constructed. Then we find $\gamma_n<\gamma_{n+1}<\kappa$ with $f[N_{x_n\restriction\gamma_{n+1}}]\subseteq N_{x_n\restriction(\gamma_n+1)}$. Pick $x_{n+1}\in A$ with $x_{n+1}\restriction\gamma_{n+1}=x_n\restriction\gamma_{n+1}$ and $x_{n+1}(\gamma_{n+1})=0$.

Now pick $x_\omega\in{}^{\kappa}\mu$ with $x_n\restriction\gamma_{n+1}\subseteq x_\omega$ for all $n<\omega$ and set $\gamma_\omega=\sup_{n<\omega}\gamma_n$. Given $n<\omega$, we have $f(x_\omega)\in N_{x_n\restriction\gamma_n+1}$ and therefore $f(x_\omega)(\gamma_n)=x_n(\gamma_n)=0$. Hence $f(x_\omega)\notin A$, contradicting our assumption on f. □

Since every non-empty closed subset of the Baire space ${}^{\omega}\omega$ is a retract of ${}^{\omega}\omega$, it follows that every closed subset of ${}^{\omega}\omega$ is a continuous image of ${}^{\omega}\omega$ and this implies that the class C^{ω} is equal to the class of all $\mathbf{\Sigma}^1_1$-subsets of ${}^{\omega}\omega$. It is natural to ask whether these classes are also identical if κ is an uncountable regular cardinal. In Section 2, we will answer this question in the negative by proving the following result.

Theorem 1.5. *Let κ be an uncountable cardinal with $\kappa=\kappa^{<\kappa}$. Then there is a closed non-empty subset of ${}^{\kappa}\kappa$ that is not the continuous image of ${}^{\kappa}\kappa$.*

This result has the following corollary that shows that the class C^{κ} lacks important closure properties.

Corollary 1.6. *The class C^{κ} is not closed under finite intersections.*

Proof. Let $A=[T]$ be the closed subset provided by Theorem 1.5. Define ∂T as in (1) in the proof of Proposition 1.3. Given $i<2$, define T_i to be T together with the set of all extensions of elements of ∂T by strings of i's of length less than κ, i.e. we define

$$T_i \;=\; T\;\cup\;\{t\in{}^{<\kappa}\kappa\mid\exists\alpha\leq\mathrm{lh}(t)\;[t\restriction\alpha\in\partial T\wedge\forall\alpha\leq\beta<\mathrm{lh}(t)\;t(\beta)=i]\}.$$

Then T_i is a $<\kappa$-closed subtree of ${}^\kappa\kappa$ without end nodes. By Proposition 1.3, the subset $[T_i]$ is an element of C^κ. Our construction ensures that the intersection $[T_0] \cap [T_1]$ is equal to the set A and therefore is not an element of C^κ. $\square$

1.2. **The class I^κ of continuous injective images of ${}^\kappa\kappa$.** We discuss results separating the class I^κ from the other classes introduced above. Since every image of ${}^\kappa\kappa$ under a continuous injective map has no isolated points, it follows that the classes C^κ and I^κ_{cl} are both not contained in I^κ.

Proposition 1.7. *If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, then every non-empty open subset of ${}^\kappa\kappa$ is equal to an injective continuous image of ${}^\kappa\kappa$.*

Proof. We can write every open subset A as a disjoint union $A = \bigcup_{\alpha<\kappa} N_{s_\alpha}$ of basic open sets, where $s_\alpha \in {}^{<\kappa}\kappa$ for $\alpha < \kappa$. Next, we choose homeomorphisms $f_\alpha : N_{s_\alpha} \longrightarrow {}^\kappa\kappa$ and combine them to a homeomorphism $f : {}^\kappa\kappa \longrightarrow A$. $\square$

To separate I^κ from the class of all κ-Borel subsets of ${}^\kappa\kappa$, we need to introduce an important regularity property of subsets of ${}^\kappa\kappa$. We say that a subset A of ${}^\kappa\kappa$ *has the κ-Baire property* if there is an open subset U of ${}^\kappa\kappa$ and a sequence $\langle N_\alpha \mid \alpha < \kappa\rangle$ of nowhere dense subsets of ${}^\kappa\kappa$ such that the symmetric difference $A_\Delta U$ is a subset of $\bigcup_{\alpha<\kappa} N_\alpha$. Every κ-Borel subset of ${}^\kappa\kappa$ has the κ-Baire property (see [HS01]). It is consistent that every $\mathbf{\Delta}^1_1$-subset has this property and it is also consistent that there is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ which does not have the κ-Baire property (see [FHK, Theorem 49]). We will prove the following result in Section 3.

Theorem 1.8. *If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, then there is a sequence $\langle A_\gamma \subseteq {}^\kappa\kappa \mid \gamma < 2^\kappa\rangle$ of pairwise disjoint injective continuous images of ${}^\kappa\kappa$ such that A_γ and A_δ cannot be separated by sets with the κ-Baire property for all $\gamma < \delta < 2^\kappa$.*

Corollary 1.9. *If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, then there is a continuous injective image of ${}^\kappa\kappa$ that is not a κ-Borel subset of ${}^\kappa\kappa$.* $\square$

In Section 3, we will present a variation of the proof of the above theorem that will allow us to prove the following surprising implication.

Theorem 1.10. *Let ν be an infinite cardinal with $\nu = \nu^{<\nu}$ and $\kappa = \nu^+ = 2^\nu$. Assume that every κ-Aronszajn tree T that does not contain a κ-Souslin subtree is special.*[1] *Then there is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ that does not have the κ-Baire property.*

The assumptions of the above theorem are known to be consistent in the case "$\nu = \omega$" (see [AS85, Theorem 4.1]). If ν is uncountable, then this assumption implies that $\square(\kappa)$ fails and κ is weakly compact in L (see [Tod07, Corollary 6.3.3] and [Tod89, Theorem 3]).

[1]Let κ be an uncountable regular cardinal. A *κ-Aronszajn tree* is a tree of height κ without κ-branches and the property that every level has cardinality less than κ. A κ-Aronszajn tree is a *κ-Souslin tree* if every antichain in this tree has cardinality less than κ. If ν is an infinite cardinal, then a tree T of height ν^+ is *special* if there is a function $s : T \longrightarrow \nu$ that is injective on chains in T.

1.3. **The class I^κ_{cl} of continuous injective images of closed subsets of ${}^\kappa\kappa$.** By the above results, the class I^κ_{cl} is neither contained in the class $B(\kappa)$ nor in the class of nonempty sets in C^κ. We present results that imply the remaining implications shown in the diagram of Figure 2.

Lemma 1.11. *Every κ-Borel subset of ${}^\kappa\kappa$ is a continuous injective image of a closed subset of ${}^\kappa\kappa$.*

Proof. We can code a Borel set $B \subseteq {}^\kappa\kappa$ by a well-founded tree $T \subseteq {}^{<\omega}\kappa$ which has basic open sets N_{s_t} attached to the end nodes t, and labels $l_t \in \{c, u\}$ (for *complement* and *union*) at each non-end node t such that every node with label c has a unique successor in T.

Fix some well-ordering $\prec$ of T. We call a pair (y, z) *correct* if the following statements hold.

(i) $y : T \longrightarrow 2$ is a function with $y(\emptyset) = 1$, $l_{t_0} = c$ implies $y(t_0) = 1 - y(t_1)$ whenever t_1 is the unique successor of t_0 in T, and $l_{t_0} = u$ implies that $y(t_0) = 1$ holds if and only if there is a direct successor t_1 of t_0 in T with $y(t_1) = 1$.

(ii) $z : T \longrightarrow T$ is a function such that $z(t)$ is the $\prec$-minimal successor t' of t with $y(t') = 1$ whenever $l_t = u$ with $y(t) = 1$ and $z(t) = t$ otherwise.

Let End_T denote the set of end nodes of T. Define C to be the set

$$\{(x, y, z) \in {}^\kappa\kappa \times {}^T2 \times {}^TT \mid (y, z) \textit{ is correct } \wedge \ \forall t \in \mathrm{End}_T \ [y(t) = 1 \Longleftrightarrow x \in N_{s_t}]\}.$$

If we equip the set C with the initial segment topology, then the resulting topological space is homeomorphic to a closed subset of the space ${}^\kappa\kappa$ and the projection $p : C \longrightarrow {}^\kappa\kappa$ onto the first coordinate is continuous. Moreover, $B = p[C]$ and this projection is injective, because the pair (y, z) is uniquely determined by x for every triple $(x, y, z) \in C$. □

With the help of the above lemma, we will prove the following result that will allow us to show that it is consistent with the axioms of ZFC that every $\mathbf{\Sigma}^1_1$-set is equal to a continuous injective image of a closed subset of ${}^\kappa\kappa$.

Theorem 1.12. *Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. Then every subset in the class $S^{\mathrm{L},\kappa}_1$ is equal to an injective continuous image of a closed subset of ${}^\kappa\kappa$.*

The above statement now follows from the following observation.

Proposition 1.13. *Assume $V = \mathrm{L}[x]$ with $x \subseteq \kappa$ for some uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$. Then the classes $S^{\mathrm{L},\kappa}_1$ and $\mathbf{\Sigma}^1_1(\kappa)$ coincide.*

Proof. By the above remarks, a subset of ${}^\kappa\kappa$ is $\mathbf{\Sigma}^1_1$-definable if and only if it is definable over the structure $(\mathrm{H}(\kappa^+), \in)$ by a Σ_1-formula with parameters. Since $\mathrm{H}(\kappa^+) = \mathrm{L}_{\kappa^+}[x]$, these formulas are absolute between V and $\mathrm{H}(\kappa^+)$. □

Corollary 1.14. *Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. Assume that* $\mathrm{V} = \mathrm{L}[x]$ *for some $x \subseteq \kappa$. Then every $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ is equal to an injective continuous image of a closed subset of ${}^\kappa\kappa$.* □

In the other direction, we will show that it is also consistent that C^κ is not a subclass of I^κ_{cl}.

Theorem 1.15. *Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and let G be either* $\mathrm{Add}(\kappa, \kappa^+)$*-generic over* V *or* $\mathrm{Col}(\kappa, {<}\lambda)$*-generic over* V *for some inaccessible cardinal $\lambda > \kappa$. In* $\mathrm{V}[G]$*, the* club filter Club_κ *is not equal to an injective continuous image of a closed subset of ${}^\kappa\kappa$.*

We will also show that the classes I^κ_{cl} and $S^{\mathrm{L},\kappa}_1$ do not coincide in any $Col(\kappa, {<}\lambda)$-generic extension, where $\lambda > \kappa$ is inaccessible.

Theorem 1.16. *Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, G be* $\mathrm{Col}(\kappa, {<}\lambda)$*-generic over* V *for some inaccessible cardinal $\lambda > \kappa$ and M be an inner model of* $\mathrm{V}[G]$ *with $M \subseteq \mathrm{V}$. In* $\mathrm{V}[G]$*, every set contained in the class $S^{M,\kappa}_n$ for some $n < \omega$ has the κ-Baire property.*

The following corollary is a direct consequence of the combination of the above result and Theorem 1.8.

Corollary 1.17. *Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and let G be* $\mathrm{Col}(\kappa, {<}\lambda)$*-generic over* V *for some inaccessible cardinal $\lambda > \kappa$. In* $\mathrm{V}[G]$*, there is an element of the class I^κ that is not contained in the class $S^{\mathrm{L},\kappa}_n$ for any $n < \omega$.* □

1.4. **The class $C^{\kappa,\mu}$ of continuous images of ${}^\kappa\mu$.** We will consider images of continuous functions $f : {}^\kappa\mu \longrightarrow {}^\kappa\kappa$ for an arbitrary cardinal μ.

Let us first consider the case where μ is a cardinal with $1 < \mu < \kappa$. If κ is not weakly compact, then the spaces ${}^\kappa\mu$ and ${}^\kappa\kappa$ are homeomorphic by the results of [HN73]. Hence we may assume that κ is weakly compact. Then the class $C^{\kappa,\mu}_{cl}$ of images of closed subsets of ${}^\kappa\mu$ under continuous functions $f : {}^\kappa\mu \longrightarrow {}^\kappa\kappa$ consists of closed subsets and its elements are exactly the sets of the form $[T]$, where T is a subtree of ${}^{<\kappa}\kappa$ with the property that the α-th level $T(\alpha)$ has cardinality less than κ for every $\alpha < \kappa$. Theorem 1.5 shows that the class $C^{\kappa,\mu}$ of images of ${}^\kappa\mu$ under continuous functions $f : {}^\kappa\mu \longrightarrow {}^\kappa\kappa$ is a proper subclass of $C^{\kappa,\mu}_{cl}$ in this case. The images of ${}^\kappa\mu$ under injective continuous functions $f : {}^\kappa\mu \longrightarrow {}^\kappa\kappa$ are exactly the sets of the form $[T]$, where T is a perfect[2] subtree of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ with the property that the level $T(\alpha)$ has size less than κ for every $\alpha < \kappa$.

Next, we consider cardinals $\mu > \kappa$. We will later show (see Lemma 6.1) that the closed set constructed in the proof of Theorem 1.5 is equal to a continuous image of ${}^\kappa(\kappa^+)$. Since every subset of ${}^\kappa\kappa$ is obviously equal to a continuous image of ${}^\kappa(2^\kappa)$, this implies that

$$c(\kappa) \;=\; \min\{\mu \in \mathrm{On} \mid \textit{Every nonempty closed subset of } {}^\kappa\kappa \textit{ is an element of } C^{\kappa,\mu}\}$$

is a well-defined cardinal characteristic of every uncountable cardinal κ satisfying $\kappa = \kappa^{<\kappa}$. By Theorem 1.5 and the above remark, we have $\kappa < c(\kappa) \leq 2^\kappa$. In Section 6 and 7 we will prove the following result that shows that we can manipulate the value of this cardinal characteristic by forcing.

Theorem 1.18. *Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, $\mu \geq 2^\kappa$ be a cardinal with $\mu = \mu^\kappa$ and $\theta \geq \mu$ be a cardinal with $\theta = \theta^\kappa$. Then the following statements hold in a cofinality preserving forcing extension* $\mathrm{V}[G]$ *of the ground model* V.

(i) $2^\kappa = \theta$.
(ii) *Every closed subset of ${}^\kappa\mu$ is equal to a continuous image of ${}^\kappa\mu$.*

[2] A subtree T of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ is *perfect* if it is ${<}\kappa$-closed and its splitting nodes are cofinal.

(iii) *There is a closed subset A of ${}^{\kappa}\kappa$ that is not equal to a continuous image of ${}^{\kappa}\bar{\mu}$ for any $\bar{\mu} < \mu$ with $\bar{\mu}^{<\kappa} < \mu$.*

In particular, this result shows that the class of nonempty sets in $C^{\kappa,\mu}$ and the class $C^{\kappa,\mu}_{cl}$ can coincide for some $\kappa < \mu < 2^{\kappa}$ (see Lemma 6.2), but this statement does not follow from the axioms of ZFC. The following result will be one of the key ingredients in the proof of Theorem 1.18.

Theorem 1.19. *Let c be $\mathrm{Add}(\omega,1)$-generic over V. In $\mathrm{V}[c]$, if κ is an uncountable regular cardinal, then there is a closed subset A of ${}^{\kappa}\kappa$ such that A is not a continuous image of ${}^{\kappa}\mu$ for every cardinal μ with $\mu^{<\kappa} < 2^{\kappa}$.*

Given a cardinal μ, we next consider the class $C^{\kappa,\mu}_{cl}$ consisting of all continuous images of closed subsets of ${}^{\kappa}\mu$. This class contains all $\mathbf{\Sigma}^1_1$-subsets. We will discuss results showing that these classes can consistently be equal. Let $\prec\alpha,\beta\succ$ denote the Gödel pair of $\alpha,\beta \in \mathrm{On}$. Given $x \in {}^{\kappa}\mu$ and $\alpha < \kappa$, we let $(x)_\alpha$ denote the element of ${}^{\kappa}\mu$ defined by $(x)_\alpha(\beta) = x(\prec\alpha,\beta\succ)$ for all $\beta < \kappa$.

Proposition 1.20. *Let κ be an uncountable regular cardinal and μ be a cardinal with $\mu = \mu^{<\kappa} < 2^{\kappa}$. Assume that every subset of ${}^{\kappa}\kappa$ of cardinality μ is a $\mathbf{\Sigma}^1_1$-subset. Then every set in $C^{\kappa,\mu}_{cl}$ is a $\mathbf{\Sigma}^1_1$-subset.*

Proof. Let A be a closed subset of ${}^{\kappa}\mu$ and $f : A \longrightarrow {}^{\kappa}\kappa$ be a continuous function. Then the graph of f is a closed subset of ${}^{\kappa}\mu \times {}^{\kappa}\kappa$ and $\mathrm{ran}(f) = p[T]$ for some subtree T of ${}^{<\kappa}\kappa \times {}^{<\kappa}\mu$. Fix a family $\langle y_t \mid t \in {}^{<\kappa}\mu\rangle$ of pairwise distinct elements of ${}^{\kappa}\kappa$. Our assumption implies that the sets

$$B \;=\; \{(x,y_t) \in {}^{\kappa}\kappa \times {}^{\kappa}\kappa \mid t \in {}^{<\kappa}\mu,\ (x \restriction \mathrm{lh}(t), t) \in T,\ \mathrm{supp}(x) \subseteq \mathrm{lh}(t)\}$$

and $C = \{(y_s,y_t) \mid s,t \in {}^{<\kappa}\mu,\ s \subsetneq t\}$ are $\mathbf{\Sigma}^1_1$-subsets of ${}^{\kappa}\kappa$. Then

$$\begin{aligned} x \in \mathrm{ran}(f) \iff \exists y\ [&\forall \alpha < \beta < \kappa\ ((y)_\alpha,(y)_\beta) \in C \\ &\wedge\ \forall \alpha < \kappa\ \exists (\bar{x},\bar{y}) \in B\ [x \restriction \alpha = \bar{x} \restriction \alpha \wedge (y)_\alpha = \bar{y}]] \end{aligned}$$

and this equivalence shows that $\mathrm{ran}(f)$ is definable in $(\mathrm{H}(\kappa^+),\in)$ by a Σ_1-formula with parameters. By the above remarks, this shows that $\mathrm{ran}(f)$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^{\kappa}\kappa$. □

We show that the above assumption holds in the canonical forcing extension of L that is a model of *Baumgartner's axiom* $\mathrm{BA}(\kappa)$, i.e. that is a model of the statement that for every κ-linked, $<\kappa$-closed, well-met partial order $\mathbb{P}$ and every collection $\mathcal{D}$ of κ^+-many dense subsets of $\mathbb{P}$, there is a $\mathcal{D}$-generic filter for $\mathbb{P}$ (see [Bau83, Section 4] and [Tal94]). Note that the canonical partial order that forces $\mathrm{BA}(\kappa)$ for some uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$ is $<\kappa$-closed and satisfies the κ^+-chain condition (see [Bau83, Theorem 4.2] and [Tal94, Theorem 0.3]). The following proof is a direct generalization of the arguments of [MS70, Section 3.2] to higher cardinalities.

Lemma 1.21. *Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and $\kappa^+ = (\kappa^+)^{\mathrm{L}}$. Assume that $\mathrm{BA}(\kappa)$ holds. Then every subset of ${}^{\kappa}\kappa$ of cardinality κ^+ is contained in the class $S^{\mathrm{L},\kappa}_1$. In particular, every such set is a $\mathbf{\Sigma}^1_1$-subset.*

Proof. Fix a subset $A = \{y_\gamma \mid \gamma < \kappa^+\}$ of ${}^{\kappa}\kappa$ of cardinality κ^+. Then there is a set $W = \{x_\gamma \mid \gamma < \kappa^+\} \in \mathrm{L}$ such that

$$(x_\gamma)_\alpha = (x_\delta)_\beta \iff \alpha = \beta\ \wedge\ \gamma = \delta$$

holds for all $\alpha, \beta < \kappa$ and $\gamma, \delta < \kappa^+$. Define

$$B = \{(x_\gamma)_{\prec\alpha, y_\gamma(\alpha)\succ} \mid \alpha < \kappa,\ \gamma < \kappa^+\} \subseteq \mathrm{L}.$$

Let $\mathbb{Q}(B)$ denote the corresponding generalization of the *almost disjoint coding forcing* to cardinality κ (see, for example, [Lüc12, Section 4]). Since $\mathbb{Q}(B)$ is κ-linked, $<\kappa$-closed and well-met, the axiom $\mathrm{BA}(\kappa)$ implies that there is an enumeration $\langle s_\alpha \mid \alpha < \kappa\rangle$ and a function $c \in {}^\kappa 2$ such that the equation

$$z \in B \iff \exists \alpha < \kappa\ \forall \alpha \leq \beta < \kappa\ [s_\beta \subseteq z \longrightarrow c(\beta) = 1]$$

holds for every $z \in ({}^\kappa\kappa)^{\mathrm{L}}$. Since

$$y \in A \iff \exists x \in B\ \forall \alpha, \beta < \kappa\ [y(\alpha) = \beta \longleftrightarrow (x)_{\prec\alpha,\beta\succ} \in B]$$

holds for every $y \in {}^\kappa\kappa$ and B is a subset of $({}^\kappa\kappa)^{\mathrm{L}}$, the parameter c witnesses that the set A is contained in the class $S_1^{\mathrm{L},\kappa}$. □

Corollary 1.22. *Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and $\kappa^+ = (\kappa^+)^{\mathrm{L}}$. If $\mathrm{BA}(\kappa)$ holds, then the classes C_{cl}^{κ,κ^+} and $\mathbf{\Sigma}_1^1(\kappa)$ are equal.* □

Conversely, it is also consistent that the GCH fails at κ and C_{cl}^{κ,κ^+} is not a subclass of $\mathbf{\Sigma}_1^1(\kappa)$.

Proposition 1.23. *Let κ be an uncountable regular cardinal, $\theta > \kappa$ be an inaccessible cardinal and $G * H$ be $(\mathrm{Col}(\kappa, <\theta) * \mathrm{Add}(\check{\kappa}, \check{\kappa}^{++}))$-generic over V. In $\mathrm{V}[G, H]$, there is a continuous image of ${}^\kappa(\kappa^+)$ that is not a $\mathbf{\Sigma}_1^1$-subset of ${}^\kappa\kappa$.*

Proof. By [Lüc12, Proposition 9.9], every $\mathbf{\Sigma}_1^1$-subset of ${}^\kappa\kappa$ in $\mathrm{V}[G, H]$ of cardinality greater than κ has cardinality at least κ^{++}. Since every subset of ${}^\kappa\kappa$ of cardinality κ^+ is a continuous image of ${}^\kappa(\kappa^+)$, the above statement follows directly. □

2. Thin sets

From now on, unless otherwise noted, we let κ denote an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. The goal of this section is to show that there is a closed subset of ${}^\kappa\kappa$ that is not contained in the class C^κ.

Definition 2.1. Let $\mu \leq \kappa$ be a cardinal. A set $A \subseteq {}^\kappa\kappa$ is *μ-thin* if $A \neq p[T]$ for every $<\mu$-closed subtree T of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ without end nodes.

We will construct a κ-thin closed subset of ${}^\kappa\kappa$. In the following, we call a subset D of ${}^{<\kappa}\mu_0 \times {}^{<\kappa}\mu_1$ a *$<\kappa$-closed subset of the tree ${}^{<\kappa}\mu_0 \times {}^{<\kappa}\mu_1$* if D consists of pairs of functions of equal length, and D is closed under increasing unions of length γ for all $\gamma < \kappa$. Given such a subset D, we let $[D]$ denote the set of all $(x, y) \in {}^\kappa\mu_0 \times {}^\kappa\mu_1$ such that the set $\{\alpha < \kappa \mid (x \restriction \alpha,\ y \restriction \alpha) \in D\}$ is unbounded in κ and define $p[D]$ to be the projection of $[D]$ onto the first coordinate.

Lemma 2.2. *Let μ and λ be cardinals with $\mu = \mu^{<\kappa}$ and $\lambda = \lambda^{<\kappa}$. Suppose that $f : {}^\kappa\lambda \longrightarrow {}^\kappa\mu$ is continuous. Then there is a $<\kappa$-closed subset D of the tree ${}^{<\kappa}\mu \times {}^{<\kappa}\lambda$ without end nodes such that $p[D] = \mathrm{ran}(f)$.*

Proof. We define

$$D = \{(s, t) \in {}^\gamma\lambda \times {}^\gamma\mu \mid \gamma < \kappa,\ f[N_s] \subseteq N_t\}.$$

Note that D is $<\kappa$-closed.

Claim. *For all $x \in {}^{\kappa}\lambda$, $s_0 \subsetneq x$, and $t_0 \subsetneq f(x)$, there is a pair (s,t) in D with $s_0 \subseteq s \subseteq x$ and $t_0 \subseteq t \subseteq f(x)$.*

Proof of the Claim. We construct strictly increasing sequences $\langle s_n \in {}^{<\kappa}\lambda \mid n < \omega\rangle$ and $\langle t_n \in {}^{<\kappa}\mu \mid n < \omega\rangle$ with $s_n \subseteq x$, $t_n \subseteq f(x)$, $f[N_{s_n}] \subseteq N_{t_n}$ and

$$\mathrm{lh}(t_{n+1}) \;\geq\; \mathrm{lh}(s_{n+1}) \;\geq\; \mathrm{lh}(t_n)$$

for all $n < \omega$, using the continuity of f. Let $(s,t) = (\bigcup_{n\in\omega} s_n, \bigcup_{n\in\omega} t_n) \in D$. □

It follows that D has no end nodes and that $(x, f(x)) \in [D]$ for all $x \in {}^{\kappa}\mu$.

Claim. $[D] = \{(x, f(x)) \mid x \in {}^{\kappa}\lambda\}$.

Proof of the Claim. Suppose that $(x,y) \in [D]$ and that $f(x) \neq y$. Let $t \in {}^{<\kappa}\lambda$ with $y \in N_t$ and $f(x) \notin N_t$. Find $(u,v) \in D$ with $u \subseteq x$, $v \subseteq y$, and $\mathrm{lh}(v) \geq \mathrm{lh}(t)$. Then $t \subseteq v$ and $f(x) \in f[N_u] \subseteq N_v \subseteq N_t$, contradicting the assumption that $f(x) \notin N_t$. □

This shows $p[D] = \mathrm{ran}(f)$, completing the proof of the lemma. □

Given a set D as above, we want to find a $<\kappa$-closed tree T without end nodes such that $p[T] = p[D]$. Note that the downwards-closure of a $<\kappa$-closed subset D does not necessarily have the same projection as the subset itself. For example, if we consider the set

$$D \;=\; \{(s,t) \in {}^{\gamma}2 \times {}^{\gamma}2 \mid \gamma < \kappa,\ \exists \alpha < \gamma\ s(\alpha) = 1\},$$

then the function with domain κ and constant value 0 is not an element of $p[D]$, but it is contained in the projection of the downwards-closure of D.

Lemma 2.3. *Suppose that $\lambda \leq \mu$ are cardinals with $\mu = \mu^{<\kappa}$. Suppose that D is a $<\kappa$-closed subset of the tree ${}^{<\kappa}\lambda \times {}^{<\kappa}\mu$ without end nodes. Then there is a $<\kappa$-closed subtree T of ${}^{<\kappa}\lambda \times {}^{<\kappa}\mu$ without end nodes such that $p[D] = p[T]$.*

Proof. Let $\langle d_\alpha \mid \alpha < \mu\rangle$ be an (not necessarily injective) enumeration of D. Let $\bar{T}$ denote the subtree of ${}^{<\kappa}\lambda \times {}^{<\kappa}\mu \times {}^{<\kappa}\mu$ consisting of all $(s,t,u) \in {}^{\gamma}\lambda \times {}^{\gamma}\mu \times {}^{\gamma}\mu$ such that $\gamma < \kappa$ and the following statements hold.

- $\langle d_{u(\alpha)} \mid \alpha < \gamma\rangle$ is a weakly increasing sequence in D.
- $(s \restriction \alpha,\ t \restriction \alpha)$ is a node equal to or below $u(\alpha)$ in ${}^{<\kappa}\lambda \times {}^{<\kappa}\mu$ for all $\alpha < \gamma$.

Claim. *$\bar{T}$ is $<\kappa$-closed.* □

Claim. *$\bar{T}$ has no end nodes.*

Proof of the Claim. Suppose that $(s,t,u) \in \bar{T}$ has length $\gamma < \kappa$. Since D is $<\kappa$-closed and has no end nodes, there is a $\beta < \mu$ such that d_β has length greater than γ and $d_{u(\alpha)}$ is below d_β in ${}^{<\kappa}\lambda \times {}^{<\kappa}\mu$ for every $\alpha < \gamma$. Let $d_\beta = (s_*, t_*)$ and define $u_* : \gamma + 1 \longrightarrow \kappa$ by $u_* \restriction \gamma = u \restriction \gamma$ and $u_*(\gamma) = \beta$. Then the tuple

$$(s_* \restriction (\gamma+1),\ t_* \restriction (\gamma+1),\ u_*)$$

is a direct successor of (s,t,u) in $\bar{T}$. □

Since $p[D] = p(p[\bar{T}])$, there is a $<\kappa$-closed subtree T of ${}^{<\kappa}\lambda \times {}^{<\kappa}\mu$ without end nodes such that $p[T] = p(p[\bar{T}]) = p[D]$. □

In the following, we construct a κ-thin closed subset of ${}^\kappa\kappa$. Given $\gamma \leq \kappa$ closed under Gödel pairing and $s : \gamma \longrightarrow 2$, we define

$$R_s \ = \ \{(\alpha,\beta) \in \gamma \times \gamma \mid s(\prec\alpha,\beta\succ) = 1\}.$$

Set

$$W \ = \ \{x \in {}^\kappa 2 \mid (\kappa, R_x) \ \textit{is a well-order}\} \tag{2}$$

Then W is a closed subset of ${}^\kappa\kappa$ with $W = [S]$, where $S = \{x \upharpoonright \alpha \mid x \in W, \ \alpha < \kappa\}$. Given $x \in W$ and $\alpha < \kappa$, let $\mathrm{rnk}_x(\alpha)$ denote the rank of α in (κ, R_x). In the following, we write $\alpha <_x \beta$ instead of $(\alpha,\beta) \in R_x$.

Lemma 2.4. *There is no ω-closed tree subtree T of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ with cofinal branches through all its nodes and $W = p[T]$.*

Proof. Assume, towards a contradiction, that T is such a tree. Given $(s,t) \in T$ and $\alpha < \kappa^+$, we define

$$T_{s,t} \ = \ \{(u,v) \in T \mid (s \subseteq u \vee u \subseteq s) \wedge (t \subseteq v \vee v \subseteq t)\}$$

to be subtree of T induced by the node (s,t) and

$$r(s,t,\alpha) \ = \ \sup\{\mathrm{rnk}_x(\alpha) \mid x \in p[T_{s,t}]\} \ \leq \ \kappa^+$$

to be the supremum of the ranks of α in (κ, R_x) with $x \in p[T_{s,t}]$. Then we have $r(\emptyset,\emptyset,\alpha) = \kappa^+$ for all $\alpha < \kappa$.

Claim. *Let $(s,t) \in T$ and $\alpha < \kappa$ with $r(s,t,\alpha) = \kappa^+$. If $\gamma < \kappa^+$, then there is $(u,v) \in T$ extending (s,t) and $\beta < \kappa$ such that* $\mathrm{dom}(u)$ *is closed under Gödel pairing, $\alpha < \beta < \mathrm{lh}(u)$, $\beta <_u \alpha$, and $r(u,v,\beta) \geq \gamma$.*

Proof of the Claim. The assumption $r(s,t,\alpha) = \kappa^+$ allows us to find $(x,y) \in [T]$ with $s \subseteq x$, $t \subseteq y$ and $\mathrm{rnk}_x(\alpha) \geq \gamma{+}\kappa$. This implies that there is a β with $\alpha < \beta < \kappa$ and $\gamma \leq \mathrm{rnk}_x(\beta) < \gamma{+}\kappa$. Pick $\delta > \max\{\alpha, \beta, \mathrm{lh}(s)\}$ closed under Gödel pairing and define (u,v) to be the node $(x \upharpoonright \delta, y \upharpoonright \delta)$ extending (s,t). Since R_u is a well-ordering of $\mathrm{lh}(u)$, we have $\beta <_u \alpha$. Finally, (x,y) witnesses that $r(u,v,\beta) \geq \gamma$. □

Claim. *If $(s,t) \in T$ and $\alpha < \kappa$ with $r(s,t,\alpha) = \kappa^+$, then there is a node (u,v) in T extending (s,t) and $\alpha < \beta < \mathrm{lh}(u)$ such that* $\mathrm{lh}(u)$ *is closed under Gödel pairing, $\beta <_u \alpha$ and $r(u,v,\beta) = \kappa^+$.*

Proof of the Claim. Given $\gamma < \kappa^+$, let $(u_\gamma, v_\gamma) \in T$ and $\beta_\gamma < \kappa$ denote the objects obtained by an application of the previous claim. Then we can find $(u,v) \in T$, $\beta < \kappa$ and $X \in [\kappa^+]^{\kappa^+}$ such that $(u,v) = (u_\gamma, v_\gamma)$ and $\beta = \beta_\gamma$ for every $\gamma \in X$. This implies that $r(u,v,\beta) = \kappa^+$. □

These claims imply that there are strictly increasing sequences $\langle (s_n,t_n) \mid n < \omega\rangle$ of nodes in T and $\langle \beta_n \mid n < \omega\rangle$ of elements of κ such that $\mathrm{lh}(s_n)$ is closed under Gödel pairing and $\beta_{n+1} <_{s_{n+1}} \beta_n$ for all $n < \omega$. Let $s = \bigcup_{n<\omega} s_n$ and $t = \bigcup_{n<\omega} t_n$. Then $\mathrm{lh}(s)$ is closed under Gödel pairing and R_s is ill-founded, hence $s \notin S$. But $(s,t) \in T$, since T is ω-closed. By our assumptions on T, there is a cofinal branch (x,y) in $[T]$ through (s,t) and this implies that $s = x \upharpoonright \mathrm{lh}(s) \in S$, a contradiction. □

In particular, the closed subset W is not κ-thin.

Proof of Theorem 1.5. Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and let W be the closed subset of ${}^{\kappa}\kappa$ defined in (2). Assume, towards a contradiction, that there is a continuous function $f : {}^{\kappa}\kappa \longrightarrow {}^{\kappa}\kappa$ with $W = \operatorname{ran}(f)$. By Lemma 2.2 and 2.3, there is a $<\kappa$-closed subtree T of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ without end nodes such that $W = p[T]$. This contradicts Lemma 2.4. □

We cannot omit the assumption on the existence of cofinal branches through T in Lemma 2.4 by the following remark.

Proposition 2.5. *Suppose that $\mu < \kappa$. Then every closed subset of ${}^{\kappa}\kappa$ is the projection of a $<\mu$-closed tree without end nodes.*

Proof. Suppose that $S \subseteq {}^{<\kappa}\kappa$ is a tree. We add a fully branching tree of height μ at each node t in the boundary ∂S of S (see (1)), i.e. we consider the tree

$$T \ = \ S \ \cup \ \{t \in {}^{<\kappa}\kappa \mid \exists \alpha \leq \operatorname{lh}(t) \ [t \restriction \alpha \in \partial S \ \wedge \ \operatorname{lh}(t) < \alpha + \mu]\}.$$

Then T is a $<\mu$-closed tree without end nodes and $[S] = [T]$. □

3. Injective images of ${}^{\kappa}\kappa$

We now turn to the class I^{κ} of images of ${}^{\kappa}\kappa$ under continuous injective maps. Note that there are κ-Borel subset of ${}^{\kappa}\kappa$ that are not contained in I^{κ}, because every set in I^{κ} has no isolated points.

Before we prove the more general results about this class mentioned in the Introduction (Theorem 1.8 and Corollary 1.9), we motivate their proofs by showing that, under certain cardinal arithmetic assumptions on κ, well-known combinatorial objects provide examples of sets in I^{κ} that do not have the κ-Baire property and therefore are not κ-Borel subsets of ${}^{\kappa}\kappa$. This set will consists of the *trivial ν-coherent sequences C-sequences* in the case where $\kappa = \nu^{+} = 2^{\nu}$ with ν regular.

Definition 3.1. Let ν be an infinite regular cardinal.

(i) Given $\gamma \in \mathrm{On}$, a sequence $\langle C_\alpha \mid \alpha < \gamma\rangle$ is a *C-sequence* if the following statements hold for all $\alpha < \gamma$.
   - If α is a limit ordinal, then C_α is a closed unbounded subset of α.
   - If $\alpha = \bar{\alpha} + 1$, then $C_\alpha = \{\bar{\alpha}\}$.

(ii) A C-sequence $\langle C_\alpha \mid \alpha < \gamma\rangle$ with $\gamma \in \mathrm{Lim}$ is *trivial* if there is a closed unbounded subset C of γ such that for every $\alpha < \gamma$ there is a $\alpha \leq \beta < \gamma$ with $C \cap \alpha \subseteq C_\beta$.

(iii) A C-sequence $\langle C_\alpha \mid \alpha < \gamma\rangle$ is *ν-coherent* if $C_\alpha = C_\beta \cap \alpha$ for all $\beta < \gamma$ and $\alpha \in \mathrm{Lim}(C_\beta)$ with $\operatorname{cof}(\alpha) = \nu$.

(iv) Given a ν-coherent C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \gamma\rangle$ with $\gamma \in \mathrm{Lim}$, we say that a closed unbounded subset C_γ of γ is a *ν-thread through $\vec{C}$* if the sequence $\langle C_\alpha \mid \alpha \leq \gamma\rangle$ is also a ν-coherent C-sequence.

It is easy to see that ν-coherent C-sequences are also coherent with respect to every regular cardinal greater than ν. This shows that ω-coherent C-sequences are *square sequences* (in the sense of [Tod07, Definition 7.1.1]).

Proposition 3.2. *Let ν is an infinite regular cardinal, λ be a limit ordinal with $\operatorname{cof}(\lambda) > \nu$ and $\vec{C} = \langle C_\alpha \mid \alpha < \lambda\rangle$ be a ν-coherent C-sequence. Then $\vec{C}$ is trivial if and only if there is a ν-thread through $\vec{C}$.*

Proof. Assume that $C \subseteq \gamma$ witnesses that $\vec{C}$ is trivial. Let D denote the set of all $\alpha \in \mathrm{Lim}(C)$ with $\mathrm{cof}(\alpha) = \nu$. Fix $\alpha, \bar{\alpha} \in D$ with $\bar{\alpha} < \alpha$. Then there is a $\alpha \leq \beta < \gamma$ such that $C \cap \alpha \subseteq C_\beta$. This shows that α is a limit point of C_β and ν-coherency implies $C \cap \alpha \subseteq C_\beta \cap \alpha = C_\alpha$. We can conclude that $\bar{\alpha} \in \mathrm{Lim}(C_\alpha)$ and $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$. Define $C_\lambda = \bigcup \{C_\alpha \mid \alpha \in D\}$. By the above computations, we have $C_\lambda \cap \alpha = C_\alpha$ for all $\alpha \in D$ and $\mathrm{cof}(\lambda) > \nu$ implies that C_λ is a ν-thread through $\vec{C}$.

Now, let C_λ be a ν-thread through $\vec{C}$. Given $\alpha < \lambda$, the assumption $\mathrm{cof}(\lambda) > \nu$ shows that there is a $\beta \in \mathrm{Lim}(C_\lambda) \setminus \alpha$ with $\mathrm{cof}(\beta) = \nu$ and in this situation, ν-coherency implies $C_\lambda \cap \alpha = C_\beta \cap \alpha \subseteq C_\beta$. This shows that C_λ witnesses that $\vec{C}$ is trivial. □

In the following, we assume that $\kappa = \nu^+ = 2^\nu$ for some regular cardinal ν. Let $\mathcal{Coh}(\kappa, \nu)$ denote the set of all ν-coherent C-sequences of length κ. We equip $\mathcal{Coh}(\kappa, \nu)$ with the topology whose basic open sets consist of all extensions of ν-coherent C-sequences of length less than κ.

Proposition 3.3. *The space $\mathcal{Coh}(\kappa, \nu)$ is homeomorphic to ${}^\kappa\kappa$.*

Proof. We will represent $\mathcal{Coh}(\kappa, \nu)$ as the set of branches of a tree T isomorphic to ${}^{<\kappa}\kappa$. Define T to be the tree consisting of all ν-coherent C-sequences $\langle C_\alpha \mid \alpha < \gamma \rangle$ such that either $\gamma < \kappa$ is a limit of ordinals of cofinality ν or $\gamma = \bar{\gamma} + 1 < \kappa$ and $\bar{\gamma}$ is a limit ordinal of cofinality ν that is not a limit of ordinals of cofinality ν. We first show that T is κ-branching.

Claim. *Suppose that $\vec{C}$ is a ν-coherent C-sequence of length $\gamma < \kappa$ and that δ is the least ordinal of cofinality ν greater than γ. Then there are κ-many sequences in T of length $\delta + 1$ that extend $\vec{C}$.*

Proof of the Claim. Since there are κ-many closed unbounded subsets of δ of order-type ν, there are also κ-many sequences $\langle C_\alpha \mid \alpha \in \mathrm{Lim} \cap [\gamma, \delta] \rangle$ with the property that C_α is a closed unbounded subset of α of order-type at most ν for every limit ordinal in the interval $[\gamma, \delta]$. Since every such sequence determines a unique direct successor of $\vec{C}$ in T, this completes the proof of the proposition. □

Since the tree T is closed under increasing unions of size $<\kappa$, the above claim implies that the trees T and ${}^{<\kappa}\kappa$ are isomorphic. This yields the statement of the proposition, because the collection of all subsets of $\mathcal{Coh}(\kappa, \nu)$ consisting of sets of all extensions of a given element of T forms a basis of the topology on this space. □

In the following, we will always identify the space $\mathcal{Coh}(\kappa, \nu)$ with ${}^\kappa\kappa$.

Theorem 3.4. *Suppose that ν is an infinite regular cardinal and $\kappa = \nu^+ = 2^\nu$. Let $\mathcal{Triv}(\kappa, \nu)$ denote the subset of $\mathcal{Coh}(\kappa, \nu)$ consisting of all trivial ν-coherent C-sequences.*

(i) *$\mathcal{Triv}(\kappa, \nu)$ is a continuous injective image of ${}^\kappa\kappa$.*
(ii) *$\mathcal{Triv}(\kappa, \nu)$ does not have the κ-Baire property.*

Proof. Define $\mathcal{Thr}(\kappa, \nu)$ to be the set of all pairs $(\vec{C}, C)$ such that $\vec{C}$ is an element of $\mathcal{Coh}(\kappa, \nu)$ and C is a ν-thread through $\vec{C}$. Then $\mathcal{Triv}(\kappa, \nu) = p[\mathcal{Thr}(\kappa, \nu)]$, where $p : \mathcal{Thr}(\kappa, \nu) \longrightarrow \mathcal{Coh}(\kappa, \nu)$ is the projection onto the first coordinate.

Let $\mathcal{S}$ be the tree consisting of all pairs $(\vec{D}, D)$ such that $\vec{D}$ is a ν-coherent C-sequence of length $\gamma \in \mathrm{Lim} \cap \kappa$ and D is a thread through $\vec{D}$. Given $(\vec{D}, D)$ in $\mathcal{S}$, the set D is unbounded in the length of $\vec{D}$ and this allows us to identify $\mathcal{T}\!hr(\kappa,\nu)$ with the set of all cofinal branches through $\mathcal{S}$ and equip $\mathcal{T}\!hr(\kappa,\nu)$ with the topology whose basic open sets consist of all extensions of elements of $\mathcal{S}$.

Claim. *The space $\mathcal{T}\!hr(\kappa,\nu)$ is homeomorphic to ${}^\kappa\kappa$.*

Proof of the Claim. By the above remarks, it again suffices to show that the tree $\mathcal{S}$ is isomorphic to ${}^{<\kappa}\kappa$. Since $\mathcal{S}$ is also $<\kappa$-closed, we have to show that every node in $\mathcal{S}$ has κ-many direct successors. Fix such a pair $(\langle D_\alpha \mid \alpha<\gamma\rangle, D_\gamma)\in\mathcal{S}$. Then the sequence $\vec{D}_* = \langle D_\alpha \mid \alpha\leq\gamma\rangle$ is also a ν-coherent C-sequence. Given an ordinal $\gamma<\delta<\kappa$ of countable cofinality, there is a ν-coherent C-sequence $\vec{D}_\delta = \langle D^\delta_\alpha \mid \alpha<\delta\rangle$ extending $\vec{D}_*$ such that $\operatorname{otp}\left(D^\delta_\alpha\right)\leq\nu$ for every $\gamma<\alpha<\delta$. If D^δ is a closed unbounded subset of δ with $\gamma\in D^\delta\subseteq\delta\setminus\gamma$ and $\operatorname{otp}\left(D^\delta\right)=\omega$, then the pair $(\vec{D}_\delta, D_\gamma\cup D^\delta)$ is a direct successor of $\vec{D}$ in $\mathcal{S}$. Since there are κ-many ordinals δ with these properties, this completes the proof of the claim. □

We now show that $\mathcal{T}\!riv(\kappa,\nu)$ is an element of I^κ.

Claim. *The projection $p:\mathcal{T}\!hr(\kappa,\nu)\longrightarrow\mathcal{C}\!oh(\kappa,\nu)$ is injective.*

Proof of the Claim. Suppose there is a sequence $\vec{C}$ in $\mathcal{C}\!oh(\kappa,\nu)$ and closed unbounded subsets C and D of κ such that $(\vec{C},C),(\vec{C},D)\in\mathcal{T}\!hr(\kappa,\nu)$. Then there are arbitrarily large $\alpha<\kappa$ with $\operatorname{cof}(\alpha)=\nu$ and $C\cap\alpha=C_\alpha=D\cap\alpha$. This implies $C=D$. □

Claim. *The projection $p:\mathcal{T}\!hr(\kappa,\nu)\longrightarrow\mathcal{C}\!oh(\kappa,\nu)$ is continuous.*

Proof of the Claim. Let $(\vec{C},C)\in\mathcal{T}\!hr(\kappa,\nu)$ and $\alpha<\kappa$. Let γ be a limit point of C above α. Then the pair $(\vec{C}\restriction\gamma, C\cap\gamma)$ is an element of $\mathcal{S}$ and the projection of every extension of this pair in $\mathcal{T}\!hr(\kappa,\nu)$ is an extension of $\vec{C}\restriction\alpha$ in $\mathcal{C}\!oh(\kappa,\nu)$. □

It remains to show that the set $\mathcal{T}\!riv(\kappa,\nu)$ does not have the κ-Baire property.

Claim. *The set $\mathcal{T}\!riv(\kappa,\nu)$ does not have the κ-Baire property in $\mathcal{C}\!oh(\kappa,\nu)$.*

Proof. We first argue that $\mathcal{T}\!riv(\kappa,\nu)$ is not comeager in any non-empty open set, i.e. if U is an nonempty open subset of $\mathcal{C}\!oh(\kappa,\nu)$ and $\langle U_\alpha \mid \alpha<\kappa\rangle$ is a sequence of dense open subsets of U, then there is a $\vec{C}\in\bigcap_{\alpha<\kappa}U_\alpha$ that is not an element of $\mathcal{T}\!riv(\kappa,\nu)$. Let $\vec{C}_0$ be a ν-coherent C-sequence of length $\gamma_0<\kappa$ and $\langle U_\alpha \mid \alpha<\kappa\rangle$ be a sequence of dense open subsets of the set $N_{\vec{C}_0}$ of all extensions of $\vec{C}_0$. In this situation, we can construct a sequence $\vec{C}=\langle C_\alpha \mid \alpha<\kappa\rangle$ and a strictly increasing continuous sequence $\langle\gamma_\alpha \mid \alpha<\kappa\rangle$ of ordinals less than κ such that the following statements hold for every $\alpha<\kappa$.

(i) $\langle C_\beta \mid \beta<\gamma_\alpha\rangle$ is a ν-coherent C-sequence extending $\vec{C}_0$.
(ii) $N_{\langle C_\beta \mid \beta<\gamma_{\alpha+1}\rangle}$ is a subset of U_α.
(iii) $\operatorname{otp}\left(C_{\gamma_\alpha}\right)\leq\nu$.

Then $\vec{C}$ is a ν-coherent C-sequence that is contained in $\bigcap_{\alpha<\kappa}U_\alpha$. Moreover, our construction ensures that $\operatorname{otp}\left(C_\gamma\right)\leq\nu$ holds for every element γ of the closed unbounded subset $\{\gamma_\alpha \mid \alpha<\kappa\}$ of κ. Assume, towards a contradiction, that $\vec{C}$ is trivial and let D be a thread through $\vec{C}$ witnessing this. Then there are limit points $\gamma<\delta$ of $C\cap D$ of cofinality ν. By Proposition 3.2, we have

$$\nu \;=\; \operatorname{otp}\left(C_\gamma\right) \;=\; \operatorname{otp}\left(D\cap\gamma\right) \;<\; \operatorname{otp}\left(D\cap\delta\right) \;=\; \operatorname{otp}\left(C_\delta\right) \;=\; \nu,$$

a contradiction. This shows that $\mathcal{T}riv(\kappa,\nu)$ is not comeager in any open set.

To see that the complement of $\mathcal{T}riv(\kappa,\nu)$ is not comeager in any non-empty open set, pick $\vec{C}_0$ and $\langle U_\alpha \mid \alpha<\kappa\rangle$ as above. Again, we construct a sequence $\vec{C}=\langle C_\alpha \mid \alpha<\kappa\rangle$ and a strictly increasing continuous sequence $\langle \gamma_\alpha \mid \alpha<\kappa\rangle$ of ordinals less than κ that satisfy the first two of the above statements and the following statement for every $\alpha \in \mathrm{Lim}\cap\kappa$.

(iii)$'$ $C_{\gamma_\alpha} \ = \ \{\gamma_{\bar{\alpha}} \mid \bar{\alpha}<\alpha\}$.

If we define $C=\{\gamma_\alpha \mid \alpha<\kappa\}$, then the pair $(\vec{C},C)$ is an element of $\mathcal{T}hr(\kappa,\nu)$ with $\vec{C}\in\bigcap_{\alpha<\kappa}U_\alpha$. Hence the complement of $\mathcal{T}riv(\kappa,\nu)$ is not comeager in any open set.

Now assume, towards a contradiction, that there is an open subset U of $\mathcal{C}oh(\kappa,\nu)$ and a sequence $\langle N_\alpha \mid \alpha<\kappa\rangle$ of closed nowhere dense subsets of $\mathcal{C}oh(\kappa,\nu)$ such that $\mathcal{T}riv(\kappa,\nu)_\Delta U \subseteq \bigcup_{\alpha<\kappa}N_\alpha$. Then U contains a basic open set $N_{\vec{C}_0}$, because the complement of $\mathcal{T}riv(\kappa,\nu)$ is not comeager in $\mathcal{C}oh(\kappa,\nu)$. Since $\mathcal{T}riv(\kappa,\nu)$ is not comeager in $N_{\vec{C}_0}$, there is a $\vec{C}\in N_{\vec{C}_0}\setminus\mathcal{T}riv(\kappa,\nu)$ such that such that $\vec{C}\notin N_\alpha$ for all $\alpha<\kappa$, a contradiction. □

This completes the proof of the theorem. □

Corollary 3.5. *Let G be $\mathrm{Add}(\kappa,\kappa^+)$-generic over* V*. In $V[G]$, the set $\mathcal{T}riv(\kappa,\nu)$ of trivial ν-coherent C-sequences is not a $\boldsymbol{\Delta}^1_1$-subset of $\mathcal{C}oh(\kappa,\nu)$.*

Proof. This follows from the previous theorem and the fact that $\boldsymbol{\Delta}^1_1$-subsets of ${}^\kappa\kappa$ have the κ-Baire property in every $\mathrm{Add}(\kappa,\kappa^+)$-generic extension of the ground model (see [FHK, Theorem 49]). □

Proof of Theorem 1.10. Let ν be an infinite regular cardinal with $\nu=\nu^{<\nu}$ and $\kappa=\nu^+=2^\nu$. Assume that every κ-Aronszajn tree that does not contain a κ-Souslin subtree is special. By Theorem 3.4, it suffices to show the set $\mathcal{T}riv(\kappa,\nu)$ is a $\boldsymbol{\Delta}^1_1$-subset of $\mathcal{C}oh(\kappa,\nu)$ to prove the existence of a $\boldsymbol{\Delta}^1_1$-subset of ${}^\kappa\kappa$ that does not have the κ-Baire property.

We associate to every C-sequence $\vec{C}=\langle C_\alpha \mid \alpha<\kappa\rangle$ a tree $(\kappa,<^{\vec{C}}_\nu)$ defined by

$$\alpha <^{\vec{C}}_\nu \beta \iff \alpha\in\mathrm{Lim}(C_\beta)\ \wedge\ \mathrm{cof}(\alpha)=\nu.$$

Claim. *Let $\vec{C}\in\mathcal{C}oh(\kappa,\nu)$. Then $\vec{C}$ is trivial if and only if there is a κ-branch through the tree $(\kappa,<^{\vec{C}}_\nu)$.*

Proof of the Claim. Let $B\subseteq\kappa$ be a κ-branch through the tree $(\kappa,<^{\vec{C}}_\nu)$ and set $C=\bigcup\{C_\alpha \mid \alpha\in B\}$. Then $C\cap\alpha=C_\alpha$ for every $\alpha\in B$ and this shows that C is a closed unbounded subset of κ. Since B is a branch, we can conclude that B is equal to the set of all limit points of C of cofinality ν. This shows that C is a ν-thread through $\vec{C}$.

In the other direction, let C be a ν-thread through $\vec{C}$ and define B to be the set of all limit points of C of cofinality ν. It follows directly that B is a κ-branch through the tree $(\kappa,<^{\vec{C}}_\nu)$. □

In the following, we say that a C-sequence $\vec{C}$ is ν-special if the corresponding tree $(\kappa,<^{\vec{C}}_\nu)$ is special. Since special trees have no cofinal branches, the following statement is a direct consequence of the above claim.

Claim. *Let $\vec{C} \in \mathcal{Coh}(\kappa, \nu)$. If $\vec{C}$ is ν-special, then $\vec{C}$ is not trivial.* □

We now want to use the above assumption to conclude that the converse of the above implication also holds. Since trees of the form $(\kappa, <_\nu^{\vec{C}})$ might have levels of cardinality κ, we have to associate a κ-Aronszajn tree to a given C-sequence and apply our assumption to these trees. We will use Todorčević's method of *minimal walks* to obtain those trees.

Fix a non-trivial ν-coherent C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$. Let $T(\rho_0)$ denote the tree obtained from the *full codes of walks through* $\vec{C}$ (see [Tod87, Section 1]). Then $T(\rho_0)$ is a tree of height κ. By [Tod87, 1.7], the tree $T(\rho_0)$ has a κ-branch if and only if there is a closed unbounded subset C of κ and $\xi < \kappa$ such that for every $\xi \leq \alpha < \kappa$ there is a $\alpha \leq \beta < \kappa$ with $C \cap \alpha = C_\beta \cap [\xi, \alpha)$. By considering limit points of such a club of cofinality ν, we can conclude that the existence of such a subset would imply that $\vec{C}$ is trivial. Hence there are no κ-branches through $T(\rho_0)$. Next, [Tod87, 1.3] shows that the α-th level of $T(\rho_0)$ has cardinality at most $|\{C_\beta \cap \alpha \mid \alpha \leq \beta < \kappa\}| + \aleph_0$ for every $\alpha < \kappa$. The ν-coherency of $\vec{C}$ and the assumption $\nu = \nu^{<\nu}$ allow us to inductively show that this cardinality is at most ν for all $\alpha < \kappa$. This shows that $T(\rho_0)$ is a κ-Aronszajn tree. Since [Tod07, Corollary 6.3.3] says that trees of the form $T(\rho_0)$ do not contain κ-Souslin subtrees, our assumption implies that the tree $T(\rho_0)$ is special. The obvious adaptation of the argument of the proof of [Tod07, Lemma 7.1.7] shows that there is a strictly increasing mapping of $(\kappa, <_\nu^{\vec{C}})$ into $T(\rho_0)$ and we can conclude that $\vec{C}$ is ν-special. We summarize the above computations in the following claim.

Claim. *Let $\vec{C} \in \mathcal{Coh}(\kappa, \nu)$. Then $\vec{C}$ is ν-special if and only if $\vec{C}$ is not trivial.* □

By considering the tree consisting of all pairs $(\vec{D}, s)$ such that $\vec{D} = \langle D_\alpha \mid \alpha < \gamma \rangle$ is a ν-coherent C-sequences of length $\gamma < \kappa$ and $s : \gamma \longrightarrow \nu$ is a function with $s(\alpha) \neq s(\beta)$ for all $\beta < \gamma$ and $\alpha \in \mathrm{Lim}(C_\beta)$ with $\mathrm{cof}(\alpha) = \nu$, it is easy to see that the set of all ν-special ν-coherent C-sequences is a $\mathbf{\Sigma}^1_1$-subset of $\mathcal{Coh}(\kappa, \nu)$. By the above claim, we can conclude that $\mathcal{Triv}(\kappa, \nu)$ is a $\mathbf{\Delta}^1_1$-subset of $\mathcal{Coh}(\kappa, \nu)$. □

In the remainder of this section, we let κ denote an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. In order to prove the more general Theorem 1.8, we will construct subsets of ${}^\kappa\kappa$ with the following property that was already used in the proof of Theorem 3.4.

Definition 3.6. We call a subset A of ${}^\kappa\kappa$ *super-dense* if $A \cap (\bigcap_{\alpha<\kappa} U_\alpha) \neq \emptyset$ whenever $\langle U_\alpha \mid \alpha < \kappa \rangle$ is a sequence of dense open subsets of some non-empty open subset of ${}^\kappa\kappa$.

The following statement was essentially shown in the proof of Theorem 3.4.

Proposition 3.7. *Assume that A and B are disjoint super-dense subsets of ${}^\kappa\kappa$. If $A \subseteq X \subseteq {}^\kappa\kappa \setminus B$, then X does not have the κ-Baire property.* □

Proof of Theorem 1.8. Suppose that $f : {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa \longrightarrow \{2 \cdot \alpha \mid \alpha < \kappa\}$ is an injection. Fix a stationary subset S of κ and let $\chi_S : \kappa \longrightarrow 2$ denote its characteristic function. Let $\mathcal{A}_S$ denote the set of all pairs $(x, y) \in {}^\kappa\kappa \times {}^\kappa\kappa$ such that the set $C = \{\alpha < \kappa \mid x(\alpha) = y(\alpha)\}$ is a club in κ and

$$x(\alpha) \ = \ f(x \restriction \alpha, \ y \restriction \alpha) \ + \ \chi_S(\alpha)$$

for all $\alpha \in C$. Define $\mathcal{T}_S$ be the tree consisting of all pairs $(s,t) \in {}^{\gamma}\kappa \times {}^{\gamma}\kappa$ such that $\gamma \in \mathrm{Lim} \cap \kappa$, $D = \{\alpha < \gamma \mid s(\alpha) = t(\alpha)\}$ is a club in γ and

$$s(\alpha) \ = \ f(s \restriction \alpha, \ t \restriction \alpha) \ + \ \chi_S(\alpha)$$

for all $\alpha \in D$. We equip $\mathcal{A}_S$ with the topology whose basic open sets are of the form $\{(x,y) \in \mathcal{A} \mid s \subseteq x, \ t \subseteq y\}$ with $(s,t) \in \mathcal{T}_S$.

Claim. *The space $\mathcal{A}_S$ is homeomorphic to ${}^{\kappa}\kappa$.*

Proof of the Claim. Since we can identify $\mathcal{A}_S$ with the set of all cofinal branches through the tree $\mathcal{T}_S$, it suffices to show that the trees $\mathcal{T}_S$ and ${}^{<\kappa}\kappa$ are isomorphic. Given $(s,t) \in \mathcal{T}_S$ and $\gamma \geq \mathrm{lh}(s)$, we can inductively construct an element (s_*, t_*) in $\mathcal{T}_S$ extending (s,t) such that $\mathrm{lh}(s_*) \geq \gamma$ and $s_*(\alpha) \neq t_*(\alpha)$ for all $\alpha \in [\mathrm{lh}(s), \gamma)$. This shows that every node in $\mathcal{T}_S$ has κ-many direct successors. Since $\mathcal{T}_S$ is $<\kappa$-closed, it follows that the trees $\mathcal{T}_S$ and ${}^{<\kappa}\kappa$ are isomorphic. □

Let $p : \mathcal{A}_S \longrightarrow {}^{\kappa}\kappa$ be the projection onto the first coordinate. Define $A_S = p[\mathcal{A}_S]$.

Claim. *The projection $p : \mathcal{A}_S \longrightarrow {}^{\kappa}\kappa$ is continuous.* □

Claim. *The projection $p : \mathcal{A}_S \longrightarrow {}^{\kappa}\kappa$ is injective.*

Proof of the Claim. Suppose that $(x,y), (x,y') \in \mathcal{A}_S$. Let C and C' be the corresponding club subsets of κ. Fix $\alpha \in C \cap C'$. Then

$$f(x \restriction \alpha, \ y \restriction \alpha) \ + \ \chi_S(\alpha) \ = \ x(\alpha) \ = \ f(x \restriction \alpha, \ y' \restriction \alpha) \ + \ \chi_S(\alpha).$$

Since f is injective, we have $y \restriction \alpha = y' \restriction \alpha$ for arbitrarily large $\alpha < \kappa$. □

The above claims show that all sets of the form A_S are contained in the class I^{κ}. The next claim implies that disjoint sets of the form A_S and $A_{S'}$ cannot be separated by a set that has the κ-Baire property.

Claim. *The set A_S is super-dense.*

Proof of the Claim. Let $\langle U_\alpha \mid \alpha < \kappa \rangle$ be a sequence of dense open subsets of some basic open subset N_{s_0} of ${}^{\kappa}\kappa$. Let $\gamma_0 = \mathrm{lh}(s_0)$ and define $t_0 \in {}^{\gamma_0}\kappa$ by setting $t_0(\alpha) = s_0(\alpha) + 1$ for all $\alpha < \gamma_0$. Now we can construct a strictly increasing continuous sequence $\langle \gamma_\alpha \mid \alpha < \kappa \rangle$ and a sequence $\langle (s_\alpha, t_\alpha) \in {}^{\gamma_\alpha}\kappa \times {}^{\gamma_\alpha}\kappa \mid \alpha < \kappa \rangle$ such that the following statements hold for all $\alpha < \kappa$.

(i) If $\bar{\alpha} < \alpha$, then $s_{\bar{\alpha}} = s_\alpha \restriction \gamma_{\bar{\alpha}}$ and $t_{\bar{\alpha}} = t_\alpha \restriction \gamma_{\bar{\alpha}}$.
(ii) $s_{\alpha+1}(\gamma_\alpha) \ = \ t_{\alpha+1}(\gamma_\alpha) \ = \ f(s_\alpha, \ t_\alpha) \ + \ \chi_S(\alpha)$.
(iii) $N_{s_{\alpha+1}} \subseteq U_\alpha$.
(iv) $t_{\alpha+1}(\beta) \ = \ s_{\alpha+1}(\beta) + 1$ for all β with $\gamma_\alpha < \beta < \gamma_{\alpha+1}$.

If $x = \bigcup_{\alpha<\kappa} s_\alpha$, then $x \in A_S \cap (\bigcap_{\alpha<\kappa} U_\alpha)$. □

It remains to construct 2^κ-many pairwise disjoint sets of the form A_S. Let $\langle S_\alpha \mid \alpha < \kappa \rangle$ be a partition of κ into κ-many stationary subsets. Given $\emptyset \neq X \subseteq \kappa$, define $S_X = \bigcup_{\alpha \in X} S_\alpha$. Then $S_X {}_\Delta S_Y$ is stationary for all $\emptyset \neq X, Y \subseteq \kappa$ with $X \neq Y$ and the above statement follows from the following claim.

Claim. *Let S and S' be stationary subsets of κ. If $S {}_\Delta S'$ is stationary in κ, then the subsets A_S and $A_{S'}$ are disjoint.*

Proof of the Claim. Suppose that $(x,y) \in \mathcal{A}_S$ and $(x,y') \in \mathcal{A}_{S'}$. Let C and C' be the corresponding clubs. Pick $\alpha \in (C \cap C' \cap (S_\Delta S')$. Then

$$f(x \upharpoonright \alpha,\ y \upharpoonright \alpha) \ + \ \chi_S(\alpha) \ = \ x(\alpha) \ = \ f(x \upharpoonright \alpha,\ y \upharpoonright \alpha) \ + \ \chi_{S'}(\alpha).$$

But this is a contradiction, because the ordinals $f(x \upharpoonright \alpha,\ y \upharpoonright \alpha)$ and $f(x \upharpoonright \alpha,\ y \upharpoonright \alpha)$ are both even and $\chi_S(\alpha) + \chi_{S'}(\alpha) = 1$. □

This completes the proof of the theorem. □

4. Continuous injective images of closed sets

In this section, we consider images of injective continuous functions $f : A \longrightarrow {}^\kappa\kappa$ for some closed subset A of ${}^\kappa\kappa$. We start by proving Theorem 1.12.

Proof of Theorem 1.12. Suppose that $\varphi(u,v)$ is a Σ_1-formula, $z \in {}^\kappa\kappa$ and A is the set of all $x \in {}^\kappa\kappa$ with $L[x,z] \models \varphi(x,z)$. In the following, we will code models of the form $(L_\gamma[x], \in)$ by some (not unique) $y \in {}^\kappa\kappa$ with the help of bijections $f : L_\gamma[x] \longrightarrow \kappa$ with $f[\kappa] = \{2 \cdot \alpha \mid \alpha < \kappa\}$. This allows us to easily calculate from any $v \subseteq \kappa$ and y whether v is an element of $L_\gamma[x]$.

Let C denote the set of triples $(x, y, \langle y_n \mid n \in \omega \rangle) \in {}^\kappa\kappa \times {}^\kappa\kappa \times {}^\omega({}^\kappa\kappa)$ such that each y_n codes a minimal well-founded model M_n of *Kripke-Platek set theory* KP and "$\mathrm{V} = \mathrm{L}[x,z]$" such that the following statements hold for all $n < \omega$.

- If $i < n$, then $x, y, y_i, z \in M_n$.
- $M_0 \models$ "*y is $<_{\mathrm{L}[x,z]}$-least element of ${}^\kappa\kappa$ with $\varphi(x,y)$*".
- If $n = k+1$, then $M_{k+1} \models$ "*y_k is the $<_{\mathrm{L}[x,z]}$-least code for M_k*".

The set C is κ-Borel, since the conjunction of $M_n \models \mathsf{KP} +$ "$\mathrm{V} = \mathrm{L}[x,z]$" and the remaining conditions is equivalent to an arithmetic sentence about the elements x, y, $\langle y_n \mid n < \omega \rangle$ and z. Then C is an element of the class I^κ_{cl} by the Lemma 1.11. We have $A = p_0[C]$, where

$$p_0 : {}^\kappa\kappa \times {}^\kappa\kappa \times {}^\omega({}^\kappa\kappa) \longrightarrow {}^\kappa\kappa;\ (x, y, \langle y_n \mid n < \omega \rangle) \longmapsto x.$$

Moreover $p_0 \upharpoonright C$ is injective, since every $x \in p_0[C]$ uniquely determines an element $(x, y, \langle y_n \mid n < \omega \rangle)$ of C. □

The next observation will allow us to simplify the following proofs.

Proposition 4.1. *Let A be an element of the class I^κ_{cl}. Then there is a subtree T of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ such that $A = p[T]$ and the projection $p : [T] \longrightarrow A$ is injective.*

Proof. Let T_0 be a subtree of ${}^{<\kappa}\kappa$ and $f : [T_0] \longrightarrow A$ be a continuous bijection. Fix an enumeration $\langle u_\alpha \mid \alpha < \kappa \rangle$ of ${}^{<\kappa}\kappa$. Define T to consist of all pairs (s,t) such that $s \in {}^{<\kappa}\kappa$, $\langle u_{t(\alpha)} \mid \alpha < \mathrm{lh}(s) \rangle$ is a strictly increasing sequence of nodes in T_0 and $f[T_{t(\alpha)}] \subseteq N_{s \upharpoonright (\alpha+1)}$ holds for all $\alpha < \mathrm{lh}(s)$.

Pick $y \in A$ and let x denote the unique element of $[T_0]$ with $f(x) = y$. We can construct a strictly increasing sequence $\langle \beta_\alpha \mid \alpha < \kappa \rangle$ of ordinals less than κ such that $f[T_{x \upharpoonright \beta_\alpha}] \subseteq N_{y \upharpoonright (\alpha+1)}$ holds for all $\alpha < \kappa$. Pick $z \in {}^\kappa\kappa$ with $x \upharpoonright \beta_\alpha = u_{z(\alpha)}$ for all $\alpha < \kappa$. Then the pair (y,z) is a κ-branch through T. In the other direction, if (y,z) is an element of $[T]$, then $x = \bigcup_{\alpha<\kappa} u_{z(\alpha)}$ is the unique element of $[T_0]$ with $f(x) = y$. □

Lemma 4.2. *The class I^κ_{cl} is closed under continuous preimages.*

Proof. Let T be a subtree of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ with $p \upharpoonright [T]$ injective and $f : {}^{\kappa}\kappa \longrightarrow {}^{\kappa}\kappa$ be continuous. Set $B = p[T]$ and $A = f^{-1}[B]$. Define

$$C \ = \ \{(x, y, z) \mid (x, y) \in \mathrm{Graph}(f),\ (y, z) \in [T]\}.$$

Then $A = p_0[C]$, C is closed and $p_0 \upharpoonright C$ is injective. □

Remark 4.3. *The class I^{κ} of injective continuous images of ${}^{\kappa}\kappa$ contains no singletons, and therefore not every nonempty continuous preimage of a set in I^{κ} is in I^{κ}.*

Next, we prove Theorem 1.15 that allows us to construct models that separate the class I^{κ}_{cl} from $\mathbf{\Sigma}^1_1(\kappa)$.

Proof of Theorem 1.15. Let $\lambda > \kappa$ be an inaccessible and G be $\mathrm{Col}(\kappa, {<}\lambda)$-generic over V. Assume, towards a contradiction, that there is a subtree $T \in \mathrm{V}[G]$ of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ such that $\mathrm{Club}_{\kappa} = p[T]$ and $p \upharpoonright [T]$ is injective in $\mathrm{V}[G]$. We may assume that T is an element of V by passing to a suitable intermediate model.

Work in V. Let $\dot{\mathbb{Q}}$ denote the canonical $\mathrm{Add}(\kappa, 1)$-name for the partial order that shoots a club through the $\mathrm{Add}(\kappa, 1)$-generic subset of κ by initial segments. It is easy to see that both $\mathrm{Add}(\kappa, 1) * \dot{\mathbb{Q}}$ and $\mathrm{Add}(\kappa, 1) * (\dot{\mathbb{Q}} \times \dot{\mathbb{Q}})$ contain ${<}\kappa$-closed dense subsets and hence these partial orders are forcing equivalent to $\mathrm{Add}(\kappa, 1)$.

Suppose that g is $\mathrm{Add}(\kappa, 1)$-generic over V and h is $\dot{\mathbb{Q}}^g$-generic over $\mathrm{V}[g]$ such that there is a $\mathrm{Col}(\kappa, {<}\lambda)$-generic filter H over $\mathrm{V}[g, h]$ with $\mathrm{V}[g, h, H] = \mathrm{V}[G]$. Let $\dot{x}$ be the canonical $\mathrm{Add}(\kappa, 1)$-name for the characteristic function of the subset of κ induced by the generic filter. By a classical theorem of Silver (see, for example, [Lüc12, Proposition 7.3]), we have $(\mathrm{H}(\kappa^+)^{\mathrm{V}[g,h]}, \in) \prec_{\Sigma_1} (\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in)$ and this shows that there is some $y \in ({}^{\kappa}\kappa)^{\mathrm{V}[g,h]}$ with $(\dot{x}^g, y) \in [T]^{\mathrm{V}[g,h]}$. Since the partial order $\mathrm{Add}(\kappa, 1)$ is homogeneous, we have

$$\mathbb{1}_{\mathrm{Add}(\kappa,1) * \dot{\mathbb{Q}}} \Vdash \exists y\ (\dot{x}, y) \in [\check{T}].$$

Now suppose that g is $\mathrm{Add}(\kappa, 1)$-generic over V and $h_0 \times h_1$ is $(\dot{\mathbb{Q}} \times \dot{\mathbb{Q}})^g$-generic over $\mathrm{V}[g]$ such that there is a $\mathrm{Col}(\kappa, {<}\lambda)$-generic filter H over $\mathrm{V}[g, h_0, h_1]$ with $\mathrm{V}[G] = \mathrm{V}[g, h_0, h_1, H]$. By the above computations, there are $y_0 \in \mathrm{V}[g, h_0]$ and $y_1 \in \mathrm{V}[g, h_1]$ such that $(x, y_i) \in [T]^{\mathrm{V}[g,h_i]} \subseteq [T]^{\mathrm{V}[G]}$. Then $y_0 = y_1$ and hence $y_0 \in \mathrm{V}[g]$ by mutual genericity. Since $\mathrm{Add}(\kappa, 1)$ is homogeneous, we have

$$\mathbb{1}_{\mathrm{Add}(\kappa,1)} \Vdash \exists y\ (\dot{x}, y) \in [\check{T}].$$

Finally, let g be $\mathrm{Add}(\kappa, 1)$-generic over V and H be $\mathrm{Col}(\kappa, {<}\lambda)$-generic over $\mathrm{V}[g]$ with $\mathrm{V}[G] = \mathrm{V}[g, H]$. In this situation, the above computations show that there is some $y \in \mathrm{V}[g]$ with $(x, y) \in [T]^{\mathrm{V}[g]} \subseteq [T]^{\mathrm{V}[G]}$ and this shows that $\dot{x}^g$ is an element of Club_{κ} in $\mathrm{V}[G]$. Since $(\mathrm{H}(\kappa^+)^{\mathrm{V}[g]}, \in) \prec_{\Sigma_1} (\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in)$ holds in this situation, this shows that the subset of κ induced by g contains a club subset in $\mathrm{V}[g]$, a contradiction.

The proof for $\mathrm{Add}(\kappa, \kappa^+)$ instead of $\mathrm{Col}(\kappa, {<}\lambda)$ is analogous. □

The next statement is a direct consequence of Theorem 1.15 and Lemma 4.2.

Corollary 4.4. *No $\mathbf{\Sigma}^1_1$-complete set (see Example 1.2) is contained in the class I^{κ}_{cl} after forcing with either $\mathrm{Add}(\kappa, \kappa^+)$ or $\mathrm{Col}(\kappa, {<}\lambda)$ with $\lambda > \kappa$ inaccessible. In* *particular, if ν is regular with $\kappa = \nu^+ = 2^\nu$, then the set $\mathcal{Triv}(\kappa,\nu)$ of trivial ν-coherent sequences is not a $\mathbf{\Sigma}^1_1$-complete subset of $\mathcal{Coh}(\kappa,\nu)$ in the above forcing extensions.* □

Remark 4.5. *The results of* [Lüc12] *show that an arbitrary subset A of ${}^\kappa\kappa$ is contained in the class I^κ_{cl} in a forcing extension of the ground model by a $<\kappa$-closed partial order $\mathbb{P}(A)$ that satisfies the κ^+-chain condition. Given some enumeration $\langle s_\alpha \mid \alpha < \kappa\rangle$ of ${}^{<\kappa}\kappa$ with $\mathrm{lh}(s_\alpha) \leq \alpha$ for all $\alpha < \kappa$, the forcing $\mathbb{P}(A)$ constructed in* [Lüc12, Section 3] *adds a subtree T of ${}^{<\kappa}2$ and a bijection $f : A \longrightarrow [T]$ such that for every $x \in A$, the branch $f(x)$ is the unique element y of $[T]$ with the property that there exists a $\beta < \kappa$ with*

$$s_\alpha \subseteq x \iff y(\prec\alpha,\beta\succ) = 1$$

for all $\alpha < \kappa$. In a $\mathbb{P}(A)$-generic extension of V, we define D to consist of all tuples (s,t,u) in ${}^\gamma\kappa \times {}^\gamma 2 \times {}^\gamma\kappa$ with γ closed under Gödel pairing, $t \in T$, u constant with value β and $t(\prec\alpha,\beta\succ) = 1$ if and only if $s_\alpha \subseteq s$ for all $\alpha < \mathrm{lh}(s)$. Let T_* be the subtree of ${}^{<\kappa}\kappa \times {}^{<\kappa}2 \times {}^{<\kappa}\kappa$ obtained by the downwards closure of D. By the above remarks, the projection onto the first coordinate is a continuous bijection between the subsets $[T_*]$ and A.

5. Classes defined from inner models

We start this section by proving Theorem 1.16. This result yields the remaining relations between the classes $S^{\mathrm{L},\kappa}_1$, I^κ, I^κ_{cl} and $\mathbf{\Sigma}^1_1(\kappa)$ shown in Figure 2.

Proof of Theorem 1.16. Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, G be $\mathrm{Col}(\kappa,<\lambda)$-generic over V for some inaccessible cardinal $\lambda > \kappa$ and M be an inner model of $\mathrm{V}[G]$ with $M \subseteq \mathrm{V}$. We work in $\mathrm{V}[G]$.

Let $\varphi(u,v)$ be a Σ_n-formula, $z \in {}^\kappa 2$ and

$$A \ = \ \{x \in {}^\kappa\kappa \mid M[x,z] \models \varphi(x,z)\}.$$

Suppose that $z_0 \in ({}^\kappa\kappa)^{\mathrm{V}}$ codes a bijection between κ and ${}^{<\kappa}\kappa$. Let C denote the set of elements of ${}^\kappa\kappa$ that are $\mathrm{Add}(\kappa,1)$-generic over $M[z,z_0]$ and let $\dot{x} \in M[z,z_0]$ be the canonical $\mathrm{Add}(\kappa,1)$-name for the generic element of ${}^\kappa\kappa$. Set

$$U \ = \ \bigcup\{N_s \mid s \Vdash^{M[z,z_0]}_{\mathrm{Add}(\kappa,1)} \varphi^{M[\dot{x},\check{z}]}(\dot{x},\check{z})\}.$$

Since we are working in a $\mathrm{Col}(\kappa,<\lambda)$-generic extension, the set of dense open subsets of $\mathrm{Add}(\kappa,1)$ contained in $M[z,z_0]$ has cardinality κ in $\mathrm{V}[G]$. Hence C is a comeager subset of ${}^\kappa\kappa$, i.e. it is equal to the intersection of κ-many dense open subsets of ${}^\kappa\kappa$. We claim that $A_\Delta U \subseteq {}^\kappa\kappa \setminus C$. It is easy to see that $C \cap U \subseteq A$. Suppose that $x \in A \cap C$. Then $M[x,z] \models \varphi(x,z)$ and this implies that

$$x \restriction \alpha \ \Vdash^{M[z,z_0]}_{\mathrm{Add}(\kappa,1)} \ \varphi^{M[\dot{x},\check{z}]}(\dot{x},\check{z})$$

for some $\alpha < \kappa$. We can conclude that x is an element of U in this case. □

We present an interesting consequence of Theorem 1.16 and Corollary 1.17. A different example of such a set was given in [ST13].

Proposition 5.1. *Suppose that $\mathrm{V} = \mathrm{L}[G]$, where G is $\mathrm{Col}(\kappa,<\lambda)$-generic over L with $\lambda > \kappa$ inaccessible in L. Then the class $S^{\mathrm{L},\kappa}_1$ contains a proper $\mathbf{\Sigma}^1_1$-set which is not $\mathbf{\Sigma}^1_1$-complete.*

Proof. Let $A = \{(x,y) \in {}^\kappa\kappa \times {}^\kappa\kappa \mid y \in \mathrm{L}[x]\}$. To show that the complement of A is not a $\mathbf{\Sigma}^1_1$-subset in $\mathrm{L}[G]$, suppose that $x \in A$ if and only if $\varphi(x, x_0)$ holds in $\mathrm{H}(\kappa^+)$, where $\varphi(u,v)$ is a Σ_1-formula and $x_0 \in \mathrm{H}(\kappa^+)$. Then there is some $x \in {}^\kappa\kappa$ such that $x_0 \in \mathrm{L}[x]$ and $(\mathrm{H}(\kappa^+)^{\mathrm{L}[x]}, \in) \prec_{\Sigma_1} (\mathrm{H}(\kappa^+), \in)$. Since ${}^\kappa\kappa \nsubseteq \mathrm{L}[x]$, it follows that $\exists y \in {}^\kappa\kappa \ \varphi(x,y)$ holds in $\mathrm{H}(\kappa^+)$ and there is a $z \in ({}^\kappa\kappa)^{\mathrm{L}[x]}$ such that $\varphi(x,z)$ holds in $\mathrm{H}(\kappa^+)$. But this implies $z \notin \mathrm{L}[x]$ by the choice of $\varphi(u,v)$, a contradiction.

By coding continuous functions $f : {}^\kappa\kappa \longrightarrow {}^\kappa\kappa$ as elements of ${}^\kappa\kappa$, it is easy to see that the class $S^{\mathrm{L},\kappa}_1$ is closed under continuous preimages. This shows that the set A is not $\mathbf{\Sigma}^1_1$-complete, because otherwise every $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ would be contained in the class $S^{\mathrm{L},\kappa}_1$, contradicting Corollary 1.17. □

By Proposition 1.13, the classes $S^{\mathrm{L},\kappa}_1$ and $\mathbf{\Sigma}^1_1(\kappa)$ coincide in L. In contrast, it also consistent that for a fixed $n < \omega$, the class $S^{\mathrm{L},\kappa}_n$ consists of $\mathbf{\Delta}^1_1$-subsets.

Remark 5.2. *Given $0 < n < \omega$, it is possible to modify coding techniques developed in* [HL] *to construct a forcing extension of the ground model* V *in which the class $S^{\mathrm{L},\kappa}_n$ consists of $\mathbf{\Delta}^1_1(\kappa)$ subsets of ${}^\kappa\kappa$. In this construction, we fix a universal Σ_n-formula $\Phi(v_0, v_1)$ and recursively construct a notion of forcing $\mathbb{P}$ with the property that the set*

$$Sat^{\mathrm{L}}_n \ = \ \{(n,x) \in \omega \times ({}^\kappa\kappa)^{\mathrm{V}[G]} \mid \Phi(n,x)^{\mathrm{L}[x]}\}$$

is definable over the structure $(\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in)$ by a Σ_1-formula with parameters in every $\mathbb{P}$-generic extension $\mathrm{V}[G]$ of V*. This is possible, because the validity of the statement $\Phi(n,x)^{\mathrm{L}[x]}$ is absolute between $\mathrm{V}[G]$ and any intermediate extension given by some complete subforcing of $\mathbb{P}$. It is easy to see that the $\mathbf{\Sigma}^1_1(\kappa)$-definability of Sat^{L}_{n+1} implies that every subset in $S^{\mathrm{L},\kappa}_n$ is $\mathbf{\Delta}^1_1(\kappa)$-definable.*

Sets contained in the class $S^{\mathrm{L},\kappa}_1$ are always $\mathbf{\Sigma}^1_1$-definable. We close this section by showing that for arbitrary inner models M, sets in the class $S^{M,\kappa}_1$ are not necessarily definable from ordinals and subsets of κ.

Proposition 5.3. *Let $\mu > \kappa$ be a cardinal and $G \times H$ be $(\mathrm{Add}(\kappa,\mu) \times \mathrm{Add}(\kappa,\mu))$-generic over* V*. In $\mathrm{V}[G,H]$, there is a set A contained in the class $S^{\mathrm{V}[G],\kappa}_1$ which is not definable from ordinals and subsets of κ.*

Proof. Let $\mu = X_0 \sqcup X_1$ with $|X_0| = |X_1| = \mu$. We may assume that $G \times H$ is $(\mathbb{P} \times \mathbb{Q})$-generic over V, where $\mathbb{P} = \mathrm{Add}(\kappa, X_0)$ and $\mathbb{Q} = \mathrm{Add}(\kappa, X_1)$. Given $x \in {}^\kappa 2$, define $s(x) = \{\alpha < \kappa \mid s(\alpha) = 1\}$. Define A to be the set of all $x \in {}^\kappa 2$ such that $s(x)$ contains a club in $\mathrm{V}[G,x]$.

Assume, towards a contradiction, that there is a formula $\psi(v_0, v_1, v_2)$ that defines A using parameters $a \in {}^\kappa\kappa$ and $\gamma \in \mathrm{On}$. Suppose that $\dot{a} \in \mathrm{V}$ is a nice $(\mathbb{P} \times \mathbb{Q})$-name with $\dot{a}^{G \times H} = a$. Then there is a condition $p \in G \times H$ which forces over V that A is defined by $\psi(\cdot, \dot{a}, \check{\gamma})$. Let $S = \mathrm{supp}(p) \cup \mathrm{supp}(\dot{a})$. Since $|S| \leq \kappa$, we can find $\alpha \in X_0 \setminus S$, $\beta \in X_1 \setminus S$ and an automorphism π of $\mathbb{P} \times \mathbb{Q}$ which switches only the coordinates α and β.

Let $\dot{\mathbb{Q}}$ denote the canonical $\mathrm{Add}(\kappa,1)$-name for a partial order that shoots a club through the generic subset of κ by initial segments. Then $\mathrm{Add}(\kappa,1) * \dot{\mathbb{Q}}$ is forcing equivalent to $\mathrm{Add}(\kappa,1)$. Given $\delta < \mu$, $\dot{x}_\delta$ and $\dot{y}_\delta$ be canonical $(\mathbb{P} \times \mathbb{Q})$-names such that $\dot{x}_\delta^{\bar{G} \times \bar{H}}$ is the generic subset induced by g and $\dot{y}_\delta^{\bar{G} \times \bar{H}}$ is the corresponding generic club subset whenever $\bar{G} \times \bar{H}$ is $(\mathbb{P} \times \mathbb{Q})$-generic over V, $\bar{G}_\delta$ is the filter on $\mathrm{Add}(\kappa,1)$

induced by the δ-th component of $\bar{G}$ and $g * h$ is the filter in $\mathrm{Add}(\kappa, 1) * \dot{\mathbb{Q}}$ induced by $\bar{G}_\delta$.

Since $\alpha \in X_0$, we have $\dot{x}_\alpha^{G \times H}, \dot{y}_\alpha^{G \times H} \in \mathrm{V}[G]$ and hence $\dot{x}_\alpha^{G \times H} \in A$. By the homogeneity of $\mathbb{P} \times \mathbb{Q}$, we have $p \Vdash \psi(\dot{x}_\alpha, \dot{a}, \check{\gamma})$. By applying π, we get $p \Vdash \psi(\dot{x}_\beta, \dot{a}, \check{\gamma})$ and $\dot{x}_\beta^{G \times H}$ is an element of A. But this yields a contradiction, because $\dot{x}_\beta^{G \times H}$ is $\mathrm{Add}(\kappa, 1)$-generic over $\mathrm{V}[G]$ and hence $\dot{x}_\beta^{G \times H}$ does not contain a club subset in $\mathrm{V}[G, \dot{x}_\beta^{G \times H}]$. □

6. Trees of higher cardinalities

The following observation shows that the class of projections of $<\kappa$-closed trees of height κ without end nodes increases if we consider subtrees of ${}^{<\kappa}\kappa \times {}^{<\kappa}(\kappa^+)$, i.e. the class C^κ is always a proper subclass of the class C^{κ,κ^+}.

Lemma 6.1. *The closed subset W defined by equation (2) in Section 2 is the projection of a $<\kappa$-closed subtree of ${}^{<\kappa}\kappa \times {}^{<\kappa}(\kappa^+)$ without end nodes.*

Proof. Let D denote the set of all pairs (s, t) in ${}^\gamma 2 \times {}^\gamma(\kappa^+)$ such that $\gamma < \kappa$ is closed under Gödel pairing and $t : \gamma \longrightarrow \kappa^+$ is an injection such that for all $\alpha, \beta < \gamma$, $\alpha <_s \beta$ if and only if $t(\alpha) < t(\beta)$, where the relation $<_s := R_s$ is defined as in Section 2. Then D is a $<\kappa$-closed subset of ${}^{<\kappa}\kappa \times {}^{<\kappa}(\kappa^+)$.

Claim. *Every element of D is extended by an element of $[D]$. In particular, D has no end nodes.*

Proof of the Claim. Suppose that $(s, t) \in D$ with $\mathrm{lh}(s) = \gamma$. We extend s to $x \in {}^\kappa 2$ such that $x(\prec\alpha, \beta\succ) = 1$ is equivalent to $\alpha < \beta$ for all $\alpha, \beta < \kappa$ with $\beta \geq \gamma$ and extend t to $y \in {}^\kappa(\kappa^+)$ such that $\mathrm{ran}(t) \subseteq y(\gamma)$ and $y \restriction [\gamma, \kappa)$ is strictly increasing. Then (x, y) is an element of $[D]$. □

In this situation, Lemma 2.3 implies that there is a $<\kappa$-closed subtree T of ${}^{<\kappa}\kappa \times {}^{<\kappa}(\kappa^+)$ with $W = p[D] = p[T]$. □

The following result shows that it is consistent that the classes $C^{\kappa,\mu}$ and $C^{\kappa,\mu}_{cl}$ coincide for some $\mu < 2^\kappa$. In particular, there can be a $\mu < 2^\kappa$ such that every $\mathbf{\Sigma}^1_1$-subset is an element of $C^{\kappa,\mu}$.

Lemma 6.2. *Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, $\mu = 2^\kappa$ and G be $\mathrm{Add}(\kappa, \theta)$-generic over V for some cardinal θ. In $\mathrm{V}[G]$, every closed subset of ${}^\kappa\mu$ is equal to a continuous image of ${}^\kappa\mu$.*

Proof. We may assume that $\theta > \mu$, because otherwise $\mu = |{}^\kappa\mu|^{\mathrm{V}[G]}$ and the statement of the lemma holds trivially. Let $A = [T]^{\mathrm{V}[G]}$ be a closed subset of ${}^\kappa\mu$ in $\mathrm{V}[G]$. Since $\mathrm{Add}(\kappa, \theta)$ satisfies the κ^+-chain condition and we can identify T with a subset of μ, we may assume that T is an element of V. By Proposition 1.3, it suffices to show that A is equal to the projection of a $<\kappa$-closed subtree of ${}^{<\kappa}\mu \times {}^{<\kappa}\mu$ without end nodes in $\mathrm{V}[G]$.

Let σ be an $\mathrm{Add}(\kappa, \theta)$-nice name for an element of $[T]$ in V. Since $\mathrm{Add}(\kappa, \theta)$ satisfies the κ^+-chain condition, the set $X = \mathrm{supp}(\sigma)$ is a subset of θ of cardinality at most κ. Define T_σ to be the subtree of ${}^{<\kappa}\mu \times {}^{<\kappa}\mathrm{Add}(\kappa, X)$ that consists of all pairs $(t, \vec{p}) \in {}^\gamma\kappa \times {}^\gamma\mathrm{Add}(\kappa, X)$ such that $\gamma < \kappa$ and the following statements hold.

- $\langle \vec{p}(\alpha) \mid \alpha < \gamma \rangle$ is a descending sequence of conditions in $\mathrm{Add}(\kappa, X)$.
- If $\alpha < \gamma$, then $\vec{p}(\alpha) \Vdash^V_{\mathrm{Add}(\kappa, X)}$ "$\check{s} \restriction (\check{\alpha} + 1) \subseteq \sigma$".

Then $[T_\sigma]$ is a $<\kappa$-closed tree without end notes. We have $\sigma^G \in p[T_\sigma]^{\mathrm{V}[G]}$ and $p[T_\sigma]^{\mathrm{V}[G]} \subseteq A$, because A is closed.

We can find an automorphism π of $\mathrm{Add}(\kappa,\theta)$ in V with $\mathrm{supp}(\pi(\sigma)) \subseteq \kappa$. If $(t, \langle \vec{p}(\alpha) \mid \alpha < \gamma \rangle)$ is an element of T_σ, then it follows that $(t, \langle \pi(\vec{p}(\alpha)) \mid \alpha < \gamma \rangle)$ is an element of $T_{\pi(\sigma)}$. This shows that $p[T_\sigma]^{\mathrm{V}[G]} = p[T_{\pi(\sigma)}]^{\mathrm{V}[G]}$.

The above computations show that for every element x of A there is a $\mathrm{Add}(\kappa,\kappa)$-nice name σ for an element of $[T]$ in V such that $x \in p[T_\sigma]^{\mathrm{V}[G]} \subseteq A$. Since there are at most μ-many $\mathrm{Add}(\kappa,\kappa)$-nice names for elements of $[T]$ in V and $\mathrm{Add}(\kappa,\kappa)$ has cardinality κ in V, we can conclude that A is equal to the union of μ-many projections of $<\kappa$-closed subtrees of ${}^{<\kappa}\mu \times {}^{<\kappa}\kappa$ without end nodes. But this shows that A is equal to the projection of a $<\kappa$-closed subtree of ${}^{<\kappa}\mu \times {}^{<\kappa}\mu$ without end nodes. □

7. Perfect embeddings

In this section, we isolate a property of subtrees of ${}^{<\kappa}\kappa$ that implies that the corresponding closed subset of ${}^{\kappa}\kappa$ is not a continuous image of a space of the form ${}^{\kappa}\mu$. We will use this implication to prove several consistency results mentioned in Subsection 1.4.

Theorem 7.1. *Let κ be an uncountable regular, U be an unbounded subset of κ and T be a subtree of ${}^{<\kappa}\kappa$. If μ is a cardinal with $\mu^{<\kappa} < |[T]|$ and $c : {}^{\kappa}\mu \longrightarrow [T]$ is a continuous surjection, then there is a strictly increasing sequence $\langle \lambda_n \in U \mid n < \omega \rangle$ with least upper bound λ and an injection*

$$i : \prod_{n<\omega} \lambda_n \longrightarrow T(\lambda).$$

such that

$$x \restriction n = y \restriction n \iff i(x) \restriction \lambda_n = i(y) \restriction \lambda_n$$

holds for all $x, y \in \prod_{n<\omega} \lambda_n$ and $n < \omega$.

Proof. We start by proving two claims.

Claim. *If U is an open subset of ${}^{\kappa}\nu$ with $|c[U]| > \mu^{<\kappa}$, then there is an $x \in U$ with $|c[N_{x \restriction \alpha}]| > \mu^{<\kappa}$ for all $\alpha < \kappa$.*

Proof of the Claim. Assume, towards a contradiction, that for every $x \in U$ there is an $\alpha_x < \kappa$ with $|c[N_{x \restriction \alpha_x}]| \leq \mu^{<\kappa}$. Define $A = \{x \restriction \alpha_x \mid x \in U\} \subseteq {}^{<\kappa}\mu$. Then

$$|c[U]| \;\leq\; |\bigcup_{t \in A} c[N_t]| \;\leq\; \sum_{t \in A} |c[N_t]| \;\leq\; \mu^{<\kappa},$$

a contradiction. □

Claim. *Let $\bar{\gamma}, \lambda < \kappa$ and $s \in {}^{<\kappa}\mu$ with $|c[N_s]| > \mu^{<\kappa}$. Then there is an ordinal $\bar{\gamma} \leq \gamma < \kappa$ and a sequence $\langle y_\alpha \in N_s \mid \alpha < \lambda \rangle$ such that the following statements hold for all $\alpha, \bar{\alpha} < \lambda$.*

(i) *$|c[N_{y_\alpha \restriction \beta}]| > \mu^{<\kappa}$ for all $\beta < \kappa$.*
(ii) *$c(y_\alpha) \restriction \gamma = c(y_{\bar{\alpha}}) \restriction \gamma$ if and only if $\alpha = \bar{\alpha}$.*

Proof of the Claim. We recursively construct

- a strictly increasing sequence $\langle \gamma_\alpha \mid \alpha < \lambda \rangle$ of ordinals and
- a sequence $\langle y_\alpha \in N_s \mid \alpha < \lambda \rangle$

such that the following statements hold for all $\alpha < \lambda$.

(a) $|c[N_{y_\alpha \restriction \beta}]| > \mu^{<\kappa}$ for all $\beta < \kappa$.
(b) If $\bar{\alpha} < \alpha$, then $c(y_\alpha) \restriction \gamma_\alpha \neq c(y_{\bar{\alpha}}) \restriction \gamma_\alpha$.

Assume that the above sequences are constructed up to $\alpha < \lambda$. We define

$$U \ = \ N_s \setminus c^{-1}[\{c(y_{\bar{\alpha}}) \mid \bar{\alpha} < \alpha\}].$$

Then U is open and $|c[U]| > \mu^{<\kappa}$. In this situation, the above claim implies that there is a $y_\alpha \in U$ such that the statement (a) holds. Since $c(y_\alpha) \neq c(y_{\bar{\alpha}})$ for all $\bar{\alpha} < \alpha$, there is a $\bar{\gamma} \leq \gamma < \kappa$ with $\gamma \geq \operatorname{lub}_{\bar{\alpha}<\alpha}\gamma_{\bar{\alpha}}$ and $c(y_{\bar{\alpha}}) \notin N_{c(y_\alpha)\restriction\gamma_\alpha}$ for all $\bar{\alpha} < \alpha$. This implies that the above statement (b) holds.

If we define $\gamma = \sup_{\alpha<\lambda} \gamma_\alpha < \kappa$, then this construction ensures that the statement of the lemma holds. □

Now we recursively construct

- a strictly increasing sequence $\langle \lambda_n < \kappa \mid n < \omega\rangle$ of ordinals,
- a sequence $\langle s_p \in {}^{<\kappa}\mu \mid n < \omega,\ p \in \prod_{m<n}(\lambda_m + 1)\rangle$ of functions, and
- a sequence $\langle t_p \in T(\lambda_n) \mid n < \omega,\ p \in \prod_{m<n}(\lambda_m + 1)\rangle$ of nodes in T

such that the following statements hold for all $n < \omega$ and $p, q \in \prod_{m<n}(\lambda_m + 1)$.

(i) $0 < \lambda_{n+1} \in U$.
(ii) $|c[N_{s_p}]| > \mu^{<\kappa}$ and $c[N_{s_p}] \subseteq N_{t_p}$.
(iii) $N_{t_p} \cap N_{t_q} \neq \emptyset$ if and only if $p = q$.
(iv) If $m < n$, then $s_{p\restriction m} \subsetneq s_p$ and $t_{p\restriction m} = t_p \restriction \lambda_m$.

Set $s_\emptyset = t_\emptyset = \emptyset$ and $\lambda_0 = 0$. Now assume that $n < \omega$ and the sequences $\langle \lambda_m \mid m \leq n\rangle$, $\langle s_p \mid p \in \prod_{m<n}(\lambda_m + 1)\rangle$ and $\langle t_p \mid p \in \prod_{m<n}(\lambda_m + 1)\rangle$ are already constructed. Pick $p \in \prod_{m<n}(\lambda_m + 1)$. Apply the second claim to λ_n, $\lambda_n + 1$ and s_p to obtain an ordinal $\lambda_n \leq \gamma_p < \kappa$ and a sequence $\langle y^p_\alpha \in N_{s_p} \mid \alpha < \lambda_n + 1\rangle$ satisfying the properties (i) and (ii) stated in the claim. Define

$$\lambda_{n+1} \ = \min(U \setminus \operatorname{lub}\{\gamma_p \mid p \in \prod_{m<n}(\lambda_m + 1)\}) \ < \ \kappa.$$

Given $p \in \prod_{m<n}(\lambda_m + 1)$ and $\alpha < \lambda_n + 1$, we define

$$t_{p^\frown\langle\alpha\rangle} \ = \ c(y^p_\alpha) \restriction \lambda_{n+1}$$

and

$$s_{p^\frown\langle\alpha\rangle} \ = \ y^p_\alpha \restriction \min\{\beta < \kappa \mid c[N_{y^p_\alpha\restriction\beta}] \subseteq N_{t_{p^\frown\langle\alpha\rangle}}\}.$$

Let $\lambda = \sup_{n<\omega} \lambda_n$. Fix a $z \in \prod_{n<\omega}(\lambda_n + 1)$. Then we have $s_{z\restriction n} \subseteq s_{z\restriction(n+1)}$ and $t_{z\restriction n} \subseteq t_{z\restriction(n+1)}$ for all $n < \omega$. Hence there is an $x_z \in \bigcap_{n<\omega} N_{s_{z\restriction n}}$ and our construction yields

$$c(x_z) \ \in \ [T] \ \cap \ \bigcap_{n<\omega} N_{t_{z\restriction n}}.$$

Pick $z_0, z_1 \in \prod_{n<\omega}(\lambda_n + 1)$ and $n < \omega$. Assume $z_0 \restriction (n+1) \neq z_1 \restriction (n+1)$. Our construction ensures $N_{t_{z_0\restriction(n+1)}} \cap N_{t_{z_1\restriction(n+1)}} = \emptyset$ and $c(x_{z_i}) \in N_{t_{z_i\restriction(n+1)}}$ for all $i < 2$. Hence $c(x_{z_0}) \restriction \lambda_{n+1} \neq c(x_{z_1}) \restriction \lambda_{n+1}$. In the other direction, if $z_0 \restriction n = z_1 \restriction n$, then $c(x_{z_0}) \in N_{t_{z_0\restriction n}} \cap N_{t_{z_1\restriction n}} \neq \emptyset$ and we get $z_0 \restriction n = z_1 \restriction n$. In particular, we can conclude that the induced map

$$i_0 : \prod_{n<\omega}(\lambda_n + 1) \longrightarrow T(\lambda);\ z \longmapsto c(x_z) \restriction \lambda$$

is an injection. By embedding the product $\prod_{0<n<\omega} \lambda_n$ into $\prod_{n<\omega}(\lambda_n + 1)$, the statement of the theorem follows. □

We use Theorem 7.1 to prove the following result that will imply Theorem 1.18 and Theorem 1.19.

Theorem 7.2. *Assume that there is an inner model M of* ZFC *such that $\mathbb{R} \nsubseteq M$ and M has the ω_1-cover property (see* [Ham03]*), i.e. every countable set of ordinals in* V *is covered by a set that is an element of M and countable in M. If κ is an uncountable regular cardinal, then there is a closed subset A of ${}^\kappa\kappa$ such that A is not a continuous image of ${}^\kappa\mu$ for every cardinal μ with $\mu^{<\kappa} < |(2^\kappa)^M|^{\mathrm{V}}$.*

The following result due to Veličković and Woodin will be the key ingredient of this proof. Remember that a tree T is *superperfect* if above every node of T there is a node with infinitely many direct successors.

Theorem 7.3 ([VW98, Theorem 2])**.** *Let T be a superperfect subtree of ${}^{<\omega}\omega$. If M is an inner model of* ZFC *with $[T]^{\mathrm{V}} \subseteq M$, then $\mathbb{R}^{\mathrm{V}} \subseteq M$.*

Proof of Theorem 7.2. Let M be an inner model of ZFC with $\mathbb{R}^{\mathrm{V}} \nsubseteq M$ and the property that every countable set of ordinals in V is covered by a set that is contained in M and countable in M. We work in V. Fix an uncountable regular cardinal κ and let T denote the subtree $({}^{<\kappa}2)^M$ of ${}^{<\kappa}\kappa$. Assume, towards a contradiction, that there is a continuous surjection $c : {}^\kappa\mu \longrightarrow [T]$ for some cardinal μ with $\mu^{<\kappa} < |(2^\kappa)^M|$. Since T has at least $|(2^\kappa)^M|$-many cofinal branches, we can use Theorem 7.1 to find a strictly increasing sequence $\langle \lambda_n < \kappa \mid n < \omega \rangle$ with limit λ and an injection $i : {}^\omega\omega \longrightarrow T(\lambda)$ with

$$x \restriction n = y \restriction n \iff i(x) \restriction \lambda_n = i(y) \restriction \lambda_n$$

for all $x, y \in {}^\omega\omega$ and $n < \omega$. By our assumptions, we have $\mathrm{cof}(\lambda)^M = \omega$ and there is a strictly increasing sequence $\langle \eta_n \mid n < \omega \rangle$ contained in M that is cofinal in M. Moreover, with the help of a well-ordering of T in M we can find $C \in M$ such that C is countable in M and the set $\{i(x) \restriction \eta_n \mid x \in {}^\omega\omega,\ n < \omega\}$ is contained in C. Define

$$B \ = \ \{t \in T(\lambda) \mid \forall n < \omega \ t \restriction \eta_n \in C\} \ \in \ M.$$

Then $\mathrm{ran}(i) \subseteq B$.

Claim. *If $t \in {}^\lambda 2$ such that for every $k < \omega$ there is an $x \in {}^\omega\omega$ satisfying $t \restriction \eta_k = i(x) \restriction \eta_k$, then $t \in \mathrm{ran}(i) \subseteq T(\lambda) \subseteq M$.*

Proof of the Claim. Fix $n < \omega$. Then there is a $k < \omega$ with $\eta_k \geq \lambda_n$ and our assumption gives us an $x_n \in {}^\omega\omega$ with $t \restriction \lambda_n = i(x_n) \restriction \lambda_n$. Define $s_n = x_n \restriction n$. Given $x \in {}^\omega\omega$, the properties of i imply that $i(x) \restriction \lambda_n = t \restriction \lambda_n$ if and only if $s_n \subseteq x$. This shows that $x_t = \bigcup_{n<\omega} s_n$ is an element of ${}^\omega\omega$ with $i(x_t) = t$. □

Fix an injective enumeration $\langle c_n \mid n < \omega \rangle$ of C that is contained in M and let $j : B \longrightarrow ({}^\omega\omega)^M$ denote the injection in M defined by

$$j(t)(n) = k \iff t \restriction \eta_n = c_k.$$

Claim. *The set $\mathrm{ran}(j \circ i)$ is closed in ${}^\omega\omega$.*

Proof of the Claim. Let z be a limit point of $\mathrm{ran}(j \circ i)$ in ${}^{\omega}\omega$. Fix $k \leq n < \omega$. Then there is an $x \in {}^{\omega}\omega$ with $(j \circ i)(x) \restriction (n+1) = z \restriction (n+1)$ and

$$c_{z(k)} \; = \; c_{(j \circ i)(x)(k)} \; = \; i(x) \restriction \eta_k \; \subseteq \; i(x) \restriction \eta_n \; = \; c_{(j \circ i)(x)(n)} \; = \; c_{z(n)} \; \in \; T(\eta_n).$$

This shows that $t = \bigcup_{n<\omega} c_{z(n)}$ is an element of ${}^{\lambda}2$ with the property that for every $n < \omega$ there is an $x_n \in {}^{\omega}\omega$ with $t \restriction \eta_n = i(x_n) \restriction \eta_n$. By the above claim, there is an $x \in {}^{\omega}\omega$ with $i(x) = t$. If $k < \omega$, then $i(x) \restriction \eta_k = t \restriction \eta_k = c_{z(k)}$. This shows that $z = (j \circ i)(x) \in \mathrm{ran}(j \circ i)$. □

Let $S = \{(j \circ i)(x) \restriction n \mid x \in {}^{\omega}\omega, \ n < \omega\}$. Then $[S] = \mathrm{ran}(j \circ i) \subseteq M$.

Claim. *S is a superperfect subtree of ${}^{<\omega}\omega$.*

Proof of the Theorem. Let $x \in {}^{\omega}\omega$ and $k < \omega$. Then there are $m, n < \omega$ such that $\eta_k \leq \lambda_m < \lambda_{m+1} \leq \eta_n$. Then there is a sequence $\langle x_p \in {}^{\omega}\omega \mid p < \omega \rangle$ such that $x_p \restriction m = x \restriction m$ and $x_p(m) = p$. Given $p < \omega$, we have $i(x) \restriction \eta_l = i(x_p) \restriction \eta_l$ for every $l < k$ and therefore $(j \circ i)(x) \restriction k \subseteq (j \circ i)(x_p) \restriction (n+1)$. Since

$$(j \circ i)(x_p) \restriction (n+1) \neq (j \circ i)(x_q) \restriction (n+1)$$

for all $p < q < \omega$, this shows that $(j \circ i)(x) \restriction k$ has infinitely many successor at level $n+1$. But this shows that there is a node above $(j \circ i)(x) \restriction k$ in S of length at most n that has infinitely many direct successors. □

The combination of the last claim and Theorem 7.3 shows that $\mathbb{R} \subseteq M$, a contradiction. □

We close this section with the proofs of the theorems mentioned above.

Proof of Theorem 1.19. Assume that V is an $\mathrm{Add}(\omega, 1)$-generic extension of some ground model M. Then M satisfies the assumptions of Theorem 7.2, because $\mathrm{Add}(\omega, 1)$ satisfies the countable chain condition and adds a new real. Let κ be an uncountable regular and A be the closed subset of ${}^{\kappa}\kappa$ given by Theorem 7.2. If μ is a cardinal with $\mu^{<\kappa} < 2^{\kappa}$, then $\mu^{<\kappa} < (2^{\kappa})^M$ and A is not equal to a continuous image of ${}^{\kappa}\mu$. □

Proof of Theorem 1.18. Fix an uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$, a cardinal $\mu \geq 2^{\kappa}$ with $\mu = \mu^{\kappa}$ and a cardinal $\theta \geq \mu$ with $\theta = \theta^{\kappa}$. Let G be $\mathrm{Add}(\kappa, \mu)$-generic over V, H be $\mathrm{Add}(\omega, 1)$-generic over $\mathrm{V}[G]$ and K be $\mathrm{Add}(\kappa, \theta)^{\mathrm{V}[G,H]}$-generic over $\mathrm{V}[G, H]$. Then $\mu = (2^{\kappa})^{\mathrm{V}[G]} = (2^{\kappa})^{\mathrm{V}[G,H]}$, $\kappa = (\kappa^{<\kappa})^{\mathrm{V}[G,H]}$, $\theta = (2^{\kappa})^{\mathrm{V}[G,H,K]}$ and all models have the same cardinals.

We work in $\mathrm{V}[G, H, K]$. Then the inner model $\mathrm{V}[G]$ satisfies the assumptions of Theorem 7.2 and there is a closed subset A of ${}^{\kappa}\kappa$ such that A is not equal to a continuous image of ${}^{\kappa}\bar{\mu}$ for any cardinal $\bar{\mu}$ with $\bar{\mu}^{<\kappa} < |(2^{\kappa})^{\mathrm{V}[G]}| = \mu$. Finally, Lemma 6.2 implies that every closed subset of ${}^{\kappa}\mu$ is equal to a continuous image of ${}^{\kappa}\mu$. □

8. Questions

We close this paper with questions raised by the above results.

Proposition 1.13 shows that the classes $\mathbf{\Sigma}^1_1(\kappa)$ and $S_1^{\mathrm{L},\kappa}$ coincide in models of the form $\mathrm{L}[x]$ with $x \subseteq \kappa$. This induces the question whether there are other models in which these classes are identical.

Question 8.1. *If the classes* $\mathbf{\Sigma}^1_1(\kappa)$ *and* $S^{\mathrm{L},\kappa}_1$ *coincide, is there an* $x \subseteq \kappa$ *with* ${}^\kappa\kappa \subseteq \mathrm{L}[x]$*?*

By Proposition 1.13 and Theorem 1.15, the statement "*the club filter is a continuous injective image of a closed subset of* ${}^\kappa\kappa$" is not absolute between different models of set theory. In particular, it is consistent that the club filter is not a continuous injective image of the whole space ${}^\kappa\kappa$. Therefore it is natural to ask whether the negation of this statement is also consistent.

Question 8.2. *Is it consistent that the club filter* Club_κ *on* κ *is an element of* I^κ*?*

The discussion presented in Subsection 1.4 shows that it is consistent that all subsets in C^{κ,κ^+} can consistently be $\mathbf{\Sigma}^1_1$-definable. Therefore we may ask if these continuous images can consistently be contained in the smaller classes considered in this paper.

Question 8.3. *Is it consistent that the class* C^{κ,κ^+} *is contained in the class* I^κ_{cl}*?*

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