Continuous Lattices and Domains

G. GIERZ
K. H. HOFMANN
K. KEIMEL
J. D. LAWSON
M. MISLOVE
D. S. SCOTT

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A Primer on Ordered Sets and Lattices

This introductory chapter serves as a convenient source of reference for certain basic aspects of complete lattices needed in what follows. The experienced reader may wish to skip directly to Chapter I and the beginning of the discussion of the main topic of this book: continuous lattices and domains.

Section O-1 fixes notation, while Section O-2 defines complete lattices, complete semilattices and directed complete partially ordered sets (**dcpo**s), and lists a number of examples which we shall often encounter. The formalism of Galois connections is presented in Section 3. This not only is a very useful general tool, but also allows convenient access to the concept of a Heyting algebra. In Section O-4 we briefly discuss meet continuous lattices, of which both continuous lattices and complete Heyting algebras (frames) are (overlapping) subclasses. Of course, the more interesting topological aspects of these notions are postponed to later chapters. In Section O-5 we bring together for ease of reference many of the basic topological ideas that are scattered throughout the text and indicate how ordered structures arise out of topological ones. To aid the student, a few exercises have been included. Brief historical notes and references have been appended, but we have not tried to be exhaustive.

O-1 Generalities and Notation

Partially ordered sets occur everywhere in mathematics, but it is usually assumed that the partial order is *antisymmetric*. In the discussion of nets and directed limits, however, it is not always so convenient to assume this property. We begin, therefore, with somewhat more general definitions.

Definition O-1.1. Consider a set L equipped with a reflexive and transitive relation $\leq$. Such a relation will be called a *preorder* and L a *preordered set*. We say that a is a *lower bound* of a set $X \subseteq L$, and b is an *upper bound*, provided that

$$\begin{aligned} a \le x \text{ for all } x \in X, &\quad \text{and} \\ x \le b \text{ for all } x \in X, &\quad \text{respectively.} \end{aligned}$$

A subset D of L is *directed* provided it is nonempty and every finite subset of D has an upper bound in D. (Aside from nonemptiness, it is sufficient to assume that every *pair* of elements in L has an upper bound in L.) Dually, we call a nonempty subset F of L *filtered* if every finite subset of F has a lower bound in F.

If the set of upper bounds of X has a unique smallest element (that is, the set of upper bounds contains exactly one of its lower bounds), we call this element the *least upper bound* and write it as $\bigvee X$ or $\sup X$ (for *supremum*). Similarly the *greatest lower bound* is written as $\bigwedge X$ or $\inf X$ (for *infimum*); we will not be dogmatic in our choice of notation. The notation $x = \bigvee^{\uparrow} X$ is a convenient device to express that, firstly, the set X is directed and, secondly, x is its least upper bound. In the case of pairs of elements it is customary to write

$$\begin{aligned} x \wedge y &= \inf\{x, y\}, \\ x \vee y &= \sup\{x, y\}. \end{aligned}$$

These operations are also often called *meet* and *join*, and in the case of meet the multiplicative notation xy is common and often used in this book. □

Definition O-1.2. A *net* in a set L is a function $j \mapsto x_j : J \to L$ whose domain J is a directed set. (Nets will also be denoted by $(x_j)_{j\in J}$, by (x_j), or even by x_j, if the context is clear.)

If the set L also carries a preorder, then the net x_j is called *monotone* (resp., *antitone*), if $i \le j$ always implies $x_i \le x_j$ (resp., $x_j \le x_i$).

If $P(x)$ is a property of the elements $x \in L$, we say that $P(x_j)$ holds *eventually* in the net if there is a $j_0 \in J$ such that $P(x_k)$ is true whenever $j_0 \le k$.

The next concept is slightly delicate: if L carries a preorder, then the net x_j is a *directed net* provided that for each fixed $i \in J$ one eventually has $x_i \le x_j$. A *filtered net* is defined dually. □

Every monotone net is directed, but the converse may fail. Exercise O-1.12 illustrates pitfalls to avoid in defining directed nets. The next definition gives us some convenient notation connected with upper and lower bounds. Some important special classes of sets are also singled out.

Definition O-1.3. Let L be a set with a preorder $\le$. For $X \subseteq L$ and $x \in L$ we write:

(i) $\downarrow X = \{y \in L : y \le x \text{ for some } x \in X\}$;
(ii) $\uparrow X = \{y \in L : x \le y \text{ for some } x \in X\}$;

(iii) $\downarrow x = \downarrow\{x\}$;
(iv) $\uparrow x = \uparrow\{x\}$.

We also say:

(v) X is a *lower set* iff $X = \downarrow X$;
(vi) X is an *upper set* iff $X = \uparrow X$;
(vii) X is an *ideal* iff it is a directed lower set;
(viii) X is a *filter* iff it is a filtered upper set;
(ix) an ideal is *principal* iff it has a maximum element;
(x) a filter is *principal* iff it has a minimum element;
(xi) $\mathrm{Id}\, L$ (resp., $\mathrm{Filt}\, L$) is the set of all ideals (resp. filters) of L;
(xii) $\mathrm{Id}_0\, L = \mathrm{Id}\, L \cup \{\emptyset\}$;
(xiii) $\mathrm{Filt}_0\, L = \mathrm{Filt}\, L \cup \{\emptyset\}$. □

Note that the principal ideals are just the sets $\downarrow x$ for $x \in L$. The set of lower bounds of a subset $X \subseteq L$ is equal to the set $\bigcap\{\downarrow x \colon x \in X\}$, and this is the same as the set $\downarrow \inf X$ in case $\inf X$ exists. Note, too, that

$$X \subseteq \downarrow X = \downarrow(\downarrow X),$$

and similarly for $\uparrow X$.

Remark O-1.4. *For a subset X of a preordered set L the following are equivalent:*

(1) *X is directed;*
(2) *$\downarrow X$ is directed;*
(3) *$\downarrow X$ is an ideal.*

Proof: (2) iff (3): By Definition O-1.3.

(1) implies (2): If A is a finite subset of $\downarrow X$, then there is a finite subset B of X such that for each $a \in A$ there is a $b \in B$ with $a \leq b$ by O-1.3(i). By (1) there is in X an upper bound of B, and this same element must also be an upper bound of A.

(2) implies (1): If A is a finite subset of X, it is also contained in $\downarrow X$; therefore, by (2), there is an upper bound $y \in \downarrow X$ of A. By definition $y \leq x \in X$ for some x, and this x is an upper bound of A. □

Remark O-1.5. *The following conditions are equivalent for L and X as in* O-1.4*:*

(1) $\sup X$ *exists;*
(2) $\sup \downarrow X$ *exists.*

And if these conditions are satisfied, then $\sup X = \sup \downarrow X$. *Moreover, if every finite subset of* X *has a sup and if* F *denotes the set of all those finite sups, then* F *is directed, and* (1) *and* (2) *are equivalent to*

(3) $\sup F$ *exists.*

Under these circumstances, $\sup X = \sup F$. *If* X *is nonempty, we need not assume the empty sup belongs to* F.

Proof: Since, by transitivity and reflexivity, the sets X and $\downarrow X$ have the same set of upper bounds, the equivalence of (1) and (2) and the equality of the sups are clear. Now suppose that $\sup A$ exists for every finite $A \subseteq X$ and that F is the set of all these sups. Since $A \subseteq B$ implies $\sup A \leq \sup B$, we know that F is directed. But $X \subseteq F$, and any upper bound of X is an upper bound of $A \subseteq X$; thus, the sets X and F have the same set of upper bounds. The equivalence of (1) and (3) and the equality of the sups is again clear, also in the nonempty case. □

The – rather obvious – theme behind the above remark is that statements about arbitrary sups can often be reduced to statements about finite sups and sups of directed sets. Of course, both O-1.4 and O-1.5 have straightforward duals.

Definition O-1.6. A partial order is a transitive, reflexive, and antisymmetric relation $\leq$. (This last means $x \leq y$ and $y \leq x$ always imply $x = y$.) A *partially ordered set*, or *poset* for short, is a nonempty set L equipped with a partial order $\leq$. We say that L is *totally ordered*, or a *chain*, if all elements of L are comparable under $\leq$ (that is, $x \leq y$ or $y \leq x$ for all elements $x, y \in L$). An *antichain* is a partially ordered set in which any two different elements are incomparable, that is, in which $x \leq y$ iff $x = y$. □

We have remarked informally on duality several times already, and the next definition makes duality more precise.

Definition O-1.7. For $R \subseteq L \times L$ any binary relation on a set L, we define the *opposite relation* R^{op} (sometimes: the *converse relation*) by the condition that, for all $x, y \in L$, we have $x R^{\mathrm{op}} y$ iff $y R x$.

If in $(L, \leq)$, a set equipped with a transitive, reflexive relation, the relation is understood, then we write L^{op} as short for $(L, \leq^{\mathrm{op}})$. □

The reader should note that if L is a poset or a chain, then so is L^{op}. One should also be aware how the passage from L to L^{op} affects upper and lower bounds. Similar questions of duality are also relevant to the next (standard) definition.

Definition O-1.8. An *inf semilattice* is a poset S in which any two elements a, b have an inf, denoted by $a \wedge b$ or simply by ab. Equivalently, a semilattice is a poset in which every nonempty finite subset has an inf. A *sup semilattice* is a poset S in which any two elements a, b have a sup $a \vee b$ or, equivalently, in which every nonempty finite subset has a sup. A poset which is both an inf semilattice and a sup semilattice is called a *lattice*.

As we will deal with inf semilattices very frequently, we adopt the shorter expression "semilattice" instead of "inf semilattice".

If a poset L has a greatest element, it is called the *unit* or *top* element of L and is written as 1 (or, rarely, as $\top$). The top element is the inf of the empty set (which, if it exists, is the same as $\sup L$). A semilattice with a unit is called *unital*. If L has a smallest element, it is called the *zero* or *bottom* element of L and is written 0 (or $\perp$). The bottom element is the sup of the empty set (which, if it exists, is the same as $\inf L$). □

Note that in a semilattice an upper set is a filter iff it is a subsemilattice. A dual remark holds for lower sets and ideals in sup semilattices. We turn now to the discussion of maps between posets.

Definition O-1.9. A function $f : L \to M$ between two posets is called *order preserving* or *monotone* iff $x \leq y$ always implies $f(x) \leq f(y)$. A one-to-one function $f : L \to M$ where both f and f^{-1} are monotone is called an *isomorphism*. We denote by *POSET* the category of all posets with order preserving maps as morphisms.

We say that f preserves

(i) *finite sups*, or (ii) (*arbitrary*) *sups*, or (iii) *nonempty sups*, or (iv) *directed sups*

if, whenever $X \subseteq L$ is

(i) finite, or (ii) arbitrary, or (iii) nonempty, or (iv) directed,

and $\sup X$ exists in L, then $\sup f(X)$ exists in M and equals $f(\sup X)$. A parallel terminology is applied to the preservation of infs. □

In the case of (iv) above, the choice of expression may not be quite satisfactory linguistically, but the correct phrase "preserves least upper bounds of directed sets" is too long. The preservation of directed sups can be expressed in the form

$$f\left(\bigvee^{\uparrow} X\right) = \bigvee^{\uparrow} f(X).$$

For semilattices a map preserving nonempty finite infs might be called a *homomorphism* of semilattices. The reader should notice that a function preserving *all* finite infs preserves the inf of the empty set; that is, it maps the unit to the unit – provided that unit exists. In order to characterize maps f preserving only the nonempty finite infs (if this is the condition desired), we can employ the usual equation:

$$f(x \wedge y) = f(x) \wedge f(y),$$

for $x, y \in L$. Note that such functions are monotone, and the dual remark also holds for homomorphisms of sup semilattices.

Remark. It should be stressed that our definition of "preservation of sups" is quite strong, as we require that, whenever a set X in the domain has a sup, then its image has a sup in the range. As a consequence, if a function $f\colon L \to M$ preserves (directed) sups, it also preserves the order. Indeed, if $a \le b$ in L, then $\{a, b\}$ is a (directed) set that has a sup; as f preserves (directed) sups, then $f(a) \vee f(b)$ exists and $f(b) = f(a \vee b) = f(a) \vee f(b)$, whence $f(a) \le f(b)$.
Often in the literature a weaker definition is adopted: f "preserves sups" if whenever $\sup X$ and $\sup f(X)$ both exist, then $f(\sup X) = \sup f(X)$. In this weak sense, a one-to-one map from the two element chain to two incomparable elements preserves sups. Thus a function that preserves (directed) sups in this weak sense need not be order preserving. In order to avoid ambiguities one should keep in mind that if a map preserves (directed) sups in our sense, then it is automatically order preserving. This implies in particular that the image of a directed set is also directed.

Remark O-1.10. *Let $f\colon L \to M$ be a function between posets. The following are equivalent:*

(1) *f preserves directed sups;*
(2) *f preserves sups of ideals.*

Moreover, if L is a sup semilattice and f preserves finite sups, then (1) *and* (2) *are also equivalent to*

(3) *f preserves arbitrary sups.*

A dual statement also holds for filtered infs, infs of filters, semilattices and arbitrary infs.

Proof: Both conditions (1) and (2) imply the monotonicity of f. Then the equivalence of (1) and (2) is clear from O-1.4 and O-1.5. Now suppose L is a sup semilattice and f preserves finite sups. Let $X \subseteq L$ have a sup in L. By the method of O-1.5(3), we can replace X by a directed set F having the same sup. Hence, if (1) holds, then $f(\sup X) = \sup f(F)$. But since f preserves finite sups, it is clear that $f(F)$ is constructed from $f(X)$ in the same way as F was obtained from X. Thus, by another application of O-1.5(3), we conclude that $f(\sup X) = \sup f(X)$. That (3) implies (1) is obvious. □

Exercises

Exercise O-1.11. Let $f: L \to M$ be monotone on posets L and M, and let $X \subseteq L$. Show that $\downarrow f(X) = \downarrow f(\downarrow X)$. □

Exercise O-1.12. Construct a net $(x_j)_{j \in J}$ with values in a poset such that for all pairs $i, j \in J$ there is a $k \in J$ with $x_i \leq x_k$ and $x_j \leq x_k$ but such that $(x_j)_{j \in J}$ is *not* directed.

Hint. Consider the lattice $2 = \{0, 1\}$, let $J = \{0, 1, 2, \ldots\}$, and let the net be defined so that $x_i = 0$ iff i is even. □

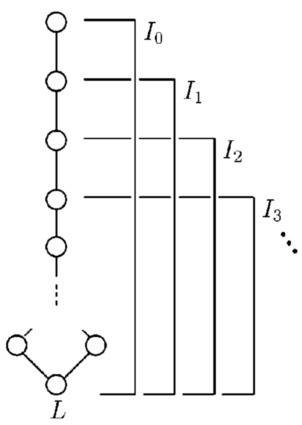
Exercise O-1.13. Modify O-1.10 so that for (3) we have only to assume that f preserves *nonempty* finite sups. □

Exercise O-1.14. Is the category of preordered sets and monotone maps *equivalent* to the category of posets and monotone maps? In these categories what sort of functor is $^{\mathrm{op}}$? □

Exercise O-1.15. Let L be a poset, and let the I_j for $j \in J$ be ideals of L. Prove the following.

(i) $\bigcap_j I_j$ is an ideal of L iff $\bigcap_j I_j \neq \emptyset$, for L a sup semilattice.
(ii) In general, $\bigcap_j I_j$ is not necessarily an ideal of L, even if $\bigcap_j I_j \neq \emptyset$.

Hint. Consider the semilattice and ideals in the following figure.



(iii) The intersection $I_1 \cap I_2$ of two ideals I_1, I_2 is an ideal, for L a semilattice.
(iv) If L is directed, $\bigcup_j I_j$ is contained in some ideal of L (however, even if this is the case, there need not be a smallest ideal containing $I_1 \cup I_2$) and the converse holds if this is true for any two ideals I_1, I_2.
(v) $\operatorname{Id} L$ is a sup semilattice iff L is a sup semilattice.

Hint. If L is a sup semilattice, then $I = \downarrow\{a \vee b : a \in I_1, b \in I_2\}$ is the sup of the ideals I_1 and I_2 of L. Conversely, if $\operatorname{Id} L$ is a sup semilattice, then we claim there is a unique element $c \in \downarrow a \vee \downarrow b$ with $a, b \leq c$. Indeed, there is at least one since $\downarrow a \vee \downarrow b$ is directed; moreover, if c and c_1 were two such elements, then $\downarrow c$ and $\downarrow c_1$ would be two ideals of L both containing a and b and both contained in $\downarrow a \vee \downarrow b$. Hence $\downarrow c = \downarrow c_1 = \downarrow a \vee \downarrow b$.

(vi) Dual statements hold for $\operatorname{Filt} L$, where one assumes L is a semilattice in part (v). □

Exercise O-1.16. Let L be a preordered set, and let $\mathcal{L}$ denote the family of all nonempty lower sets of L. Prove the following.

(i) $\operatorname{Id} L \subseteq \mathcal{L}$ and $\mathcal{L}$ is a sup semilattice.
(ii) If L is a poset, then the map $x \mapsto \downarrow x : L \to \mathcal{L}$ is an isomorphism of L onto the family of principal lower sets of L.
(iii) If L is a filtered poset, then $\mathcal{L}$ is a lattice with respect to intersection and union.
(iv) Let L and M be semilattices, $f : L \to M$ be a function, and $\mathcal{L}$ and $\mathcal{M}$ be the lattices of nonempty lower sets. Let $f_* = (A \mapsto \downarrow f(A)) : \mathcal{L} \to \mathcal{M}$. Then f is a semilattice morphism iff f_* is a lattice morphism. □

Old notes

The notion of a directed set goes back to the work of [Moore and Smith, 1922], where they use directed sets and nets to determine topologies. A convenient survey of this theory is provided in Chapter 2 of [Kelley, B1955]; we shall utilize this approach in our treatment of topologies on lattices, especially in Chapters II and III of this work. The material in this section is basic and elementary; a guide to additional reading – if more background is needed – is provided in the notes for Section O-2.

O-2 Completeness Conditions for Lattices and Posets

No excuse need be given for studying complete lattices, because they arise so frequently in practice. Perhaps the best infinite example (aside from the lattice of all subsets of a set) is the unit interval $\mathbb{I} = [0, 1]$. Many more examples will be found in this text – especially involving nontotally ordered lattices.

Definition O-2.1. (i) A poset is said to be *complete with respect to directed sets* (shorter: *directed complete* or also *up-complete*) if every directed subset has a sup. A **d***irected* **c***omplete* **po***set* is called a **dcpo** for short. A **dcpo** with a least element is called a *pointed* **dcpo**, or a **dcpo** *with zero* 0 or *with bottom* $\perp$.

(ii) A poset which is a semilattice and directed complete will be called a *directed complete semilattice*.

(iii) A *complete lattice* is a poset in which *every* subset has a sup and an inf. A totally ordered complete lattice is called a *complete chain*.

(iv) A poset is called a *complete semilattice* iff every nonempty (!) subset has an inf and every directed subset has a sup.

(v) A poset is called *bounded complete*, if every subset that is bounded above has a least upper bound. In particular, a bounded complete poset has a smallest element, the least upper bound of the empty set. □

We advise the reader to keep in mind that "up-complete poset" and "**dcpo**" are completely synonymous expressions; this advice is appropriate since the second terminology has become prevalent in the theoretical computer science community and since we use it in this book. We observe in the following that *a poset is a complete lattice iff it is both a* **dcpo** *and a sup semilattice with a smallest element.* In the exercises for this section we comment further on the relation of the concepts we have just introduced.

Proposition O-2.2. *Let L be a poset.*

(i) *For L to be a complete lattice it is sufficient to assume the existence of arbitrary sups (or the existence of arbitrary infs).*
(ii) *For L to be a complete lattice it is sufficient to assume the existence of sups of finite sets and of directed sets (or the existence of finite infs and filtered infs).*
(iii) *If L is a unital semilattice, then for completeness it is sufficient to assume the existence of filtered infs.*
(iv) *L is a complete semilattice iff L is a bounded complete* **dcpo**.

Proof: For (i) we observe that the existence of arbitrary sups implies the existence of arbitrary infs. Let $X \subseteq L$ and let

$$B = \bigcap\{\downarrow x \colon x \in X\}$$

be the set of lower bounds of X. (If X is empty, we take $B = L$.) We wish to show that

$$\sup B = \inf X.$$

If $x \in X$, then x is an upper bound of B; whence, $\sup B \leq x$. This proves that $\sup B \in B$; as it clearly is the maximal element of B, this also proves that X has a greatest lower bound. (There is obviously a dual argument assuming infs exist.)

For (ii) we first observe by Remark O-1.5 that the existence of finite sups and of directed sups implies the existence of arbitrary sups and then apply part (i).

For (iii), since the existence of finite infs is being assumed, the existence of all infs follows from (the dual of) (ii).

For a proof of (iv) if L is a complete semilattice and $A \subseteq L$ is bounded above, then the set of upper bounds has a greatest lower bound which will be the least upper bound of A. Conversely, for a bounded complete **dcpo** L and $\emptyset \neq A \subseteq L$ the 0 is contained in the set B of lower bounds of A. Any member of A is an upper bound of B and hence B has a least upper bound which is the greatest lower bound of A. □

Many subsets of complete lattices are again complete lattices (with respect to the restricted partial ordering). Obviously, if we assume that $M \subseteq L$ is *closed* under arbitrary sups and infs of the complete lattice L, then M is itself a complete lattice. But this is a very strong assumption on M. In view of O-2.2, if we assume only that M is closed under the sups of L, then M is a complete lattice (in itself as a poset). The well-worn example is with L equal to *all* subsets of a topological space X and with M the lattice of *open* subsets of X. This example is instructive because in general M is not closed under the infs of L (open sets are not closed under the formation of infinite intersections). Thus the infs of M (as a complete lattice) are *not* the infs of L. (**Exercise**: What is the simple topological definition of the infs of M?)

An even more general construction of subsets which form complete lattices is provided by the next theorem from [Tarski, 1955]. This theorem is of great interest in itself, as it implies that every monotone self-map on a complete lattice has a greatest fixed-point and a least fixed-point.

Theorem O-2.3. (The Tarski Fixed-Point Theorem) *Let $f : L \to L$ be a monotone self-map on a complete lattice L. Then the set* $\mathrm{fix}(f) = \{x \in L : x = f(x)\}$ *of fixed-points of f forms a complete lattice in itself. In particular, f has a least and a greatest fixed-point.* □

Proof: Let us consider first the set $M = \{x \in L : x \le f(x)\}$ of *pre-fixed-points* of f. We first show that the sup (formed in L) of every subset $X \subseteq M$ belongs to M again. Indeed, $x \le \sup X$ implies $x \le f(x) \le f(\sup X)$ by the monotonicity of f for all $x \in X$; hence $\sup X \le f(\sup X)$ which shows that $\sup X \in M$. By O-2.2(i) we conclude that M is a complete lattice in itself. Furthermore, f maps M into itself, as $x \le f(x)$ implies $f(x) \le f(f(x))$ by the monotonicity of f and $M \ne \emptyset$ since $0 \in M$. Thus, restricting f yields a monotone self-map on the complete lattice M. A dual argument to the above shows that the set $F = \{x \in M : f(x) \le x\}$ also is a complete lattice. But F is exactly the set of all fixed-points of f as the elements of F are exactly those elements of L that satisfy both inequalities $x \le f(x)$ and $f(x) \le x$. □

If we consider again the topological example with L the powerset lattice of the space X, the mapping assigning to a subset its interior is monotone; so the completeness of the lattice of open sets also follows from O-2.3. We shall see many other examples of monotone maps. In particular, a function preserving directed sups is monotone (see Remark preceding O-1.10).

Remark O-2.4. *Let $f : L \to M$ be a map between complete lattices preserving sups. Then $f(L)$ is closed under sups in M and is a complete lattice in itself.*

Proof: Let $Y \subseteq f(L)$ and let $X = f^{-1}(Y)$. Then $f(X) = Y$. Also

$$\sup Y = \sup f(X) = f(\sup X),$$

because f preserves sups. Hence, $\sup Y \in f(L)$. □

The above argument is not sufficient to show that if f preserves directed sups, then its image is closed under directed sups. We have to be satisfied with a special case: a self-map $p : L \to L$ on a poset L will be called a *projection operator* or a *projection*, for short, if it is monotone and idempotent, i.e., if $p = p \circ p$. Note that a self-map is idempotent if $p(x) = x$ for all x in the image. Projections will play a prominent role in the theory of domains.

Remark O-2.5. *For a projection p on a poset L, consider its image $p(L)$ in L with the induced ordering. Then the following properties hold.*

(i) *If X is a subset of $p(L)$ which has a sup in L, then X has a sup in $p(L)$ and*

$$\sup_{p(L)} X = p(\sup_L X).$$

The same holds for meets.

(ii) *If L is a semilattice, a lattice, a* **dcpo**, *a bounded complete* **dcpo**, *a complete lattice, respectively, the same holds for $p(L)$.*

(iii) *If, in addition, p preserves directed sups, then $p(L)$ is closed in L for directed sups, i.e., every directed subset $D \subseteq p(L)$ that has a sup in L also has a sup in $p(L)$ and*

$$\sup_{p(L)} D = \sup_L D.$$

Proof: (i) Let $X \subseteq p(L)$ have a sup in L. From $x \leq \sup_L X$ we deduce that $p(x) \leq p(\sup_L X)$ for every $x \in X$ by the monotonicity of p. By the idempotence of p, we obtain $x = p(x) \leq p(\sup_L X)$ and we we conclude that $p(\sup_L X)$ is an upper bound of X in $p(L)$. Let $a \in p(L)$ be another upper bound of X. Then $a \geq \sup_L X$, whence $a = p(a) \geq p(\sup_L X)$ again by monotonicity and idempotence of p. Thus $p(\sup_L X)$ is the least upper bound of X in $p(L)$.

Part (ii) is an immediate consequence of (i).

(iii) If $D \subseteq p(L)$ is directed and has a sup in L, then by (i), $\sup_{p(L)} D = p(\sup_L D)$. If $p\colon L \to L$ preserves directed sups, then $p(\sup_L D) = \sup_L p(D) = \sup_L D$, which finishes the proof. □

As a very simple example of the application of O-2.5, let V be a vector space (say, over the reals $\mathbb{R}$) and let L be the lattice of all *subsets* of V. For $x \in L$, define $f(x)$ to be the *convex closure* of the set x (no topology here, only convex linear combinations). The fact that an element of $f(x)$ depends on only *finitely* many elements of x is responsible for f preserving directed unions (sups) of subsets of V. Obviously we have $f(f(x)) = f(x)$. By O-2.5, the convex subsets of V form a complete lattice. Note, however, that $x \leq f(x)$ for all $x \in L$. This special property of the function f gives a special property to $f(L)$, as we shall see in Chapter I. In particular, with this property, the set of fixed-points of f is closed under infs – which is a simpler reason why $f(L)$ is a complete lattice. And, of course, this can all be verified directly for convex sets.

The next definition introduces some classical kinds of complete lattices that we shall often refer to in what follows; however, it should be noted that they only partly overlap with the class of continuous lattices.

Definition O-2.6. A *Boolean algebra* (sometimes also called *Boolean lattice*) is a lattice with 0 and 1 which is *distributive* in the sense that, for all elements x, y, z,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \tag{D}$$

and where every element x has a *complement* x' in the sense that

$$x \wedge x' = 0 \text{ and } x \vee x' = 1. \tag{C}$$

It is well known that (D) implies its dual, and that indeed every Boolean algebra is *isomorphic* to its opposite. Also well known is the fact that complements are *unique*.

A *complete Boolean algebra* (cBa for short) is a Boolean algebra that is complete as a lattice.

A *frame* (we also use the term *complete Heyting algebra* (cHa) as a synonym) is a complete lattice which satisfies the following infinite distributive law:

$$x \wedge \bigvee Y = \bigvee \{x \wedge y \colon y \in Y\}, \tag{ID}$$

for all elements x and all subsets Y. □

The proper definition of a Heyting algebra *without* completeness will emerge in the next section. From the above definition it is not immediately obvious that every cBa is a cHa, but this is the case. We return to these ideas in Exercise O-3.20.

We turn now to a list of complete lattices that, so to speak, "occur in nature". This list is far from exhaustive, and many more examples are contained in the remainder of this work. The reader may take these assertions as exercises.

Examples O-2.7. (1) We have already often referred to the *set of all subsets*, or *powerset*, of a set X. We employ the notation 2^X and of course regard this as a lattice under inclusion with union and intersection as sup and inf. It is a cBa but a rather special one. (It is atomic, for instance; and all atomic cBa's are of this form. Here, *atomic* means that every nonzero element contains a minimal nonzero element – an *atom*; for cBa's this is the same as saying that every element is the sup of atoms.)

(2) Generalizing (1), we can form the direct power L^X of any poset L; this is just the poset of *all* functions $f\colon X \to L$ under the pointwise ordering. Similarly, we can form direct products $\prod_{j \in J} L_j$ of any family of posets in the well-known way. If all the factors L_j are **dcpo**s, semilattices, lattices, complete lattices, etc., respectively, then the same holds for the direct product $\prod_{j \in J} L_j$.

(3) If X is a topological space, our notation for the *topology*, or *set of open subsets*, of X is $\mathcal{O}(X)$. It is a sublattice of 2^X closed under finite intersections and under arbitrary unions. It is clear then that $\mathcal{O}(X)$ is a frame since we know the truth of O-2.6 (ID) for the set theoretical operations. In general $\mathcal{O}(X)$ is *not* closed under arbitrary intersections, and its opposite is *not* a frame. (Consider the case of $X = \mathbb{R}$, the real line.)

The opposite of $\mathcal{O}(X)$ is a complete lattice and is obviously isomorphic to the lattice $\Gamma(X)$ of *closed* subsets of X. The isomorphism between $\mathcal{O}(X)^{\mathrm{op}}$ and $\Gamma(X)$ is by complements: $U \mapsto X \backslash U$.

Contained in $\mathcal{O}(X)$ is a very interesting complete lattice $\mathcal{O}_{\mathrm{reg}}(X)$ of *regular open sets*, that is, those sets equal to the interiors of their closures. The sup is *not* the union of the regular open sets but the *interior of the closure of the union*. The inf is the *interior of the intersection* (which is the same as the inf in $\mathcal{O}(X)$). Remarkably, $\mathcal{O}_{\mathrm{reg}}(X)$ is a cBa where the lattice complement of a $U \in \mathcal{O}_{\mathrm{reg}}(X)$ is the interior of $(X \backslash U)$. Actually this construction of a cBa can be done abstractly in any frame (cHa), and we return to it in the next section (see Exercise O-3.21).

For much more on Boolean algebras and the proof that *every* cBa is isomorphic to $\mathcal{O}_{\mathrm{reg}}(X)$ for some space X, the reader is referred to Halmos, B1963. (It is interesting to note that $\mathcal{O}_{\mathrm{reg}}(\mathbb{R})$ is an *atomless* cBa. That is to say, there are no minimal nonzero elements.)

(4) Let $\mathcal{A}$ be an abstract algebra with any number of operations. The poset (Cong $\mathcal{A}, \subseteq$) of all *congruence relations* under inclusion (of the graphs of the relations) forms a complete lattice, because congruence relations are closed under arbitrary intersections. This example includes numerous special cases:

(i) If $\mathcal{A}$ is a *group*, then Cong $\mathcal{A}$ can be identified with the lattice of all *normal* subgroups in the usual way, and if $\mathcal{A}$ is an *abelian group* (or a module or a vector space), with the lattice of *all* subgroups (submodules, vector subspaces). In general this lattice is *not* distributive.

(ii) If $\mathcal{A}$ is a *ring*, then Cong $\mathcal{A}$ is canonically isomorphic to the lattice of all two-sided ideals. If $\mathcal{A}$ is a *lattice ordered group* (lattice ordered ring), then Cong $\mathcal{A}$ can be identified with the lattice of all order convex normal subgroups (ideals) which are also sublattices. In general the ideals of a ring *do not* form a distributive lattice.

(iii) If $\mathcal{A}$ is a *lattice*, then Cong $\mathcal{A}$ cannot generally be identified with either the ideals or the filters of $\mathcal{A}$, but it *does* form a frame. (**Exercise**: Prove the distributivity.) If $\mathcal{A}$ is a Boolean algebra, then identification with the lattice of ideals is possible.

Note that in the case of algebras with finitary operations, Cong $\mathcal{A}$ is closed under directed unions. The significance of this remark will become clear in Section I-4.

(5) If $\mathcal{A}$ is an abstract algebra, then (Sub $\mathcal{A}, \subseteq$), the structure of all *subalgebras* of $\mathcal{A}$ under inclusion, also becomes a complete lattice. The reader can supply special cases easily. In the case of vector spaces, the lattice of

subspaces has complements but not unique ones owing to the failure of the distributive law.

(6) Let $\mathcal{A}$ be a compact Hausdorff topological algebra. Then the set $\mathrm{Cong}^- \mathcal{A}$ of *closed congruences* (congruences $R \subseteq \mathcal{A} \times \mathcal{A}$ closed in the product space) also forms a complete lattice. The relevance of this example is that these congruences correspond precisely to compact Hausdorff quotient algebras.

(7) Let $\mathcal{A}$ be a Hausdorff topological ring, then the set $\mathrm{Id}^- \mathcal{A}$ of *closed two-sided ideals* forms a complete lattice. Again the interest lies in the fact that the quotient rings are Hausdorff.

(8) Let $\mathcal{H}$ be a Hilbert space. Then $\mathrm{Sub}^- \mathcal{H}$, the *closed subspaces* of $\mathcal{H}$, forms a complete lattice. This generalizes to the lattice of projections in any von Neumann algebra.

(9) Every nonempty compact interval of real numbers in its natural order is a complete lattice, and all nonsingleton intervals are isomorphic to $\mathbb{I} = [0, 1]$ and to the infinite interval

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty].$$

As complete lattices are closed under direct products (see (2) above), we can form $\mathbb{I}^X$, where X is an arbitrary set. Such lattices are called *cubes*. In Exercise O-2.10, we note what can be said if X is a topological space and only *certain* functions are admitted; this connects with the ideas of semicontinuous functions and real-valued random variables, to which we return in I-1.22. An easy example of a restricted function space which is a complete lattice would be the subspace $M \subseteq \mathbb{I}^{\mathbb{I}}$ of all *monotone* functions from $\mathbb{I}$ into itself.

(10) Let $\mathcal{F}$ be the set of all *partial* functions from the set $\mathbb{N}$ of natural numbers into itself (this could be generalized to any other set besides the set $\mathbb{N}$). Thus, if the function $f \in \mathcal{F}$, then its *domain*, $\mathrm{dom}\, f$, is a subset of $\mathbb{N}$ and $f\colon \mathrm{dom}\, f \to \mathbb{N}$. The empty function $\varnothing\colon \varnothing \to \mathbb{N}$ is allowed. We define $f \leq g$ to mean that

$$\mathrm{dom}\, f \subseteq \mathrm{dom}\, g \text{ and } f = g|_{\mathrm{dom}\, f},$$

that is, whenever f is defined, then g is defined and they have the same value. This definition makes $\mathcal{F}$ into a poset with directed sups and arbitrary *nonempty* infs: $\mathcal{F}$ is a complete semilattice, it fails to be a lattice only in lacking a top.

In Exercise O-2.12 we show how to adjoin a top to such structures. Another repair would be to expand $\mathbb{N}$ to $\mathbb{N}^* = \mathbb{N} \cup \{\bot, \top\}$, which is a poset under the ordering where for $x, y \in \mathbb{N}^*$ we have

$$x \leq y \text{ iff } x = \bot \text{ or } x = y \text{ or } y = \top.$$

Then $\mathcal{F}$ can be regarded as a subset of $(\mathbb{N}^*)^{\mathbb{N}}$ under the pointwise ordering (we define $f(x) = \bot$ if $x \notin \operatorname{dom} f$). (Note that this ordering has nothing to do with natural ordering of $\mathbb{N}$.) Now $(\mathbb{N}^*)^{\mathbb{N}}$ is a complete lattice, but it is *much larger* than $\mathcal{F} \cup \{\top\}$, because for $f \in (\mathbb{N}^*)^{\mathbb{N}}$ the values taken in $\{\bot, \top\}$ and in $\mathbb{N}$ can be very mixed.

For applications to the theory of computation this proliferation of top elements is most inconvenient. If we read $f \leq g$ as an "information ordering" (roughly, f and g are consistent but g has possibly more information than f), then the only interpretation of $\top$ is to consider it as the *inconsistent* element. (The words "overdefined" for $\top$ and "underdefined" for $\bot$ have also been used.) As we generally try to keep our values "consistent" as much as possible, it seems natural to avoid $\top$. Because of the importance of the applications to computability, we should keep in mind the need to cover examples like this in our general theory. □

The following also deals with examples, but they play such a very prominent role in what follows that we separate them out.

Examples O-2.8. Let L be a poset.

(1) The family of all *lower sets* of L and the family of all *upper sets* are both complete lattices under $\subseteq$; indeed, both of these families are closed under arbitrary intersections and unions in 2^L.

(2) In any poset L, $\operatorname{Filt}_0 L$ and $\operatorname{Filt} L$ are closed under directed unions and hence **dcpo**s. If L is a semilattice, then $\operatorname{Filt}_0 L$ is a complete lattice; if L is also unital, then $\operatorname{Filt} L$ is complete. In the latter case both lattices of sets are closed under arbitrary intersections in 2^L. In a semilattice the ideals only form a semilattice, since in 2^L both $\operatorname{Id}_0 L$ and $\operatorname{Id} L$ are only closed under finite intersections. We note that the infinite intersection of ideals in a semilattice need not be an ideal (cf. O-1.15 and its figure).

(3) In a lattice, both $\operatorname{Filt}_0 L$ and $\operatorname{Id}_0 L$ are complete lattices; and if L has a top and bottom, then $\operatorname{Filt} L$ and $\operatorname{Id} L$ are complete lattices.

(4) The function $x \mapsto \downarrow x : L \to \operatorname{Id} L$ is an embedding preserving arbitrary infs and finite sups; it is called the *principal ideal embedding*. (There is a dual principal filter embedding.) The example $L = \mathbb{N} \cup \{\infty\}$ (with its natural ordering) shows that the principal ideal embedding need *not* preserve arbitrary (or even directed) sups.

(5) If L is a Boolean algebra, we can construe it as an algebra of "propositions" (0 is *false* and 1 is *true*, $\wedge$ and $\vee$ are *conjunction* and *disjunction*, complementation is *negation*). $\operatorname{Filt} L$ can be thought of as the lattice of *theories*. Any subset $A \subseteq L$ can be taken as a set of "axioms" generating the following "theory",

which is just a filter and corresponds to the propositions "implied" by the axioms:

$$\{x \in L : (\exists a_0, \ldots, a_{n-1} \in A) \quad a_0 \wedge \cdots \wedge a_{n-1} \leq x\}.$$

The "inconsistent" theory is L, that is, the top filter generated by $\{0\}$. If we eliminate L, then Filt $L \backslash \{L\}$ is closed under arbitrary nonempty intersections and directed unions. This is similar to the poset of O-2.7(10). As is well known, the lattice Filt$L$ is lattice isomorphic to the lattice of open subsets of the Stone space of the Boolean algebra L. □

Exercises

Exercise O-2.9. (Clopen sets) Let X be a topological space and let $\Gamma\mathcal{O}(X) = \mathcal{O}(X) \cap \Gamma(X)$ be the sublattice of 2^X of all *closed-and-open sets* (sometimes: *clopen sets*). Show that $\Gamma\mathcal{O}(X)$ is not complete in general, but it is always a Boolean algebra. For a compact totally disconnected space, show that $\Gamma\mathcal{O}(X)$ is complete iff the closure of every open set is open (such spaces are called *extremally disconnected*). (This complements Example O-2.7(3).) □

Exercise O-2.10. (Semicontinuous functions) Let X be a topological space, and let $C(X, \mathbb{R}^*)$ be the set of continuous extended real-valued functions. Verify the following assertions: under the pointwise ordering, $C(X, \mathbb{R}^*)$ is not complete, but it is a lattice with a top and bottom. For compact X, it is complete iff X is extremely disconnected.

Over an arbitrary space to have a complete lattice we must pass to a larger lattice. The *lower semicontinuous* functions $f \in \mathrm{LSC}(X, \mathbb{R}^*)$ are characterized by the condition that the set $\{x \in X : r < f(x)\}$ is open in X for every $r \in \mathbb{R}^*$. (For *upper semicontinuous* functions we reverse the inequality.) The lattice $\mathrm{LSC}(X, \mathbb{R}^*)$ is complete because it is closed under arbitrary pointwise sups. Notice that $\mathrm{LSC}(X, \mathbb{R}^*)$ is also closed under finite pointwise infs but not under arbitrary pointwise infs. The lattices $\mathrm{LSC}(X, \mathbb{R}^*)$ and $\mathrm{USC}(X, \mathbb{R}^*)$ are anti-isomorphic and

$$C(X, \mathbb{R}^*) = \mathrm{LSC}(X, \mathbb{R}^*) \cap \mathrm{USC}(X, \mathbb{R}^*).$$ □

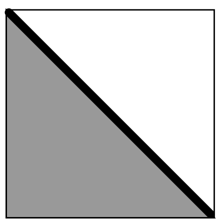
In the next exercises, and many times elsewhere in this text, we shall have occasion to discuss weaker forms of completeness as was already indicated in Definition O-2.1. In order to compare the definitions of a complete lattice and a complete semilattice we suggest that the reader recall that a complete lattice is a poset with all conceivable completeness properties which a lattice may have and which are *symmetric* (i.e., remain invariant under passage to the opposite poset); whereas a complete semilattice has, coarsely speaking, the maximal completeness properties which a semilattice may have, short of becoming a lattice. Every *finite* semilattice of course is a complete semilattice. Every complete semilattice which is, in addition, unital is clearly a complete lattice (O-2.2).

Exercise O-2.11. Let S be a poset in which every nonempty subset has an inf. Show that every $X \subseteq S$ with an upper bound has a sup. □

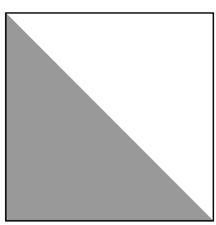
Exercise O-2.12. Let again S be a poset in which every nonempty subset has an inf. Adjoin an identity by forming $S^1 = S \cup \{1\}$ with an element $1 \notin S$ and $x \leq 1$ for all $x \in S$. Show that S^1 is a complete lattice. □

As a consequence, the adjunction of an identity to a complete semilattice will produce a complete lattice.

Exercise O-2.13. Let S be the closed lower left triangle $\{(x, y)\colon x + y \leq 1\}$ in the square $[0, 1]^2$.



Verify the following assertions: S is a complete semilattice but *not* a complete lattice. (Actually, the subsemilattice T of S consisting of the three corner points serves to illustrate this.) The interior of the triangle, $\{(x, y)\colon x + y < 1\}$, is a semilattice in which every nonempty subset has an inf, but it is *not* a complete semilattice.



The half open interval $]0, 1]$ is a directed complete lattice, but it is *not* a complete semilattice. □

Exercise O-2.14. Prove the following.

(i) A poset is directed complete iff all ideals have sups.
(ii) A semilattice is a complete semilattice iff all filters have infs and all ideals have sups. □

Exercise O-2.15. Prove the following.

(i) Every poset may be embedded into a complete lattice with the preservation of all existing infs.
(ii) Every lattice may be embedded into a complete lattice with the preservation of all finite lattice operations and all existing infs.
(iii) Every lattice may be embedded into a complete lattice with the preservation of all existing sups and infs.

Hint. Parts (i) and (ii) are easily accomplished with the means available in Section 2. For (i) use the complete lattice of all lower sets and the embedding $x \mapsto \downarrow x$. For (ii) use the complete lattice $\operatorname{Id} L$ and the principal ideal embedding. Finally, (iii) is the so-called *MacNeille completion*, which is likewise constructed by using suitable ideals; we refer to the existing literature for details, e.g., [Balbes and Dwinger, B1974], p. 235. □

Exercise O-2.16. Prove the following.

(i) For every semilattice S, the poset $\operatorname{Id} S$ is a directed complete semilattice.
(ii) If S is a semilattice in which every nonempty subset has an inf, then $\operatorname{Id} S$ is a complete semilattice. □

Exercise O-2.17. In a Boolean algebra, is the lattice of finitely axiomatizable "theories" complete? directed complete? □

Exercise O-2.18. Let G be a group and let H be any subgroup. Let L be the lattice of all subsets of G, that is, $L = 2^G$. Let M be the collection of *double cosets* of H; that is, let

$$M = \{X \subseteq G : X = XH = HX\}.$$

Prove that M is a cBa, and discuss the closure properties of M within L with respect to sups and infs.

Hint. Consider the map $X \mapsto HXH$. □

Exercise O-2.19. Let $\mathcal{F}$ be as in O-2.7(10). Define $\mathcal{G} \subseteq \mathcal{F}$ to be the collection of all *one-to-one* partial functions. Is $\mathcal{G}$ a complete semilattice? □

Exercise O-2.20. (Least Fixed-Point Theorem for dcpos) Let L be a **dcpo** with a bottom element $\perp$. Show that every monotone self-map $f: L \to L$ has a least fixed-point.

The preceding result generalizes the fixed-point theorem O-2.3 for complete lattices. We will indicate two proofs for this fact. The first proof uses transfinite induction:

Hint. We define $a_0 = \perp$ and, by transfinite induction, $a_{\alpha+1} = f(a_\alpha)$ for every ordinal α and $a_\alpha = \sup_{\beta<\alpha} a_\beta$ for limit ordinals. As the cardinality of L is bounded, there is an ordinal γ such that $a_{\gamma+1} = a_\gamma$. This a_γ is a fixed-point of f, and it is the least one, as one can verify readily. □

A second proof avoiding transfinite or equivalent reasonings due to D. Pataraia (unpublished) is included in the following exercise.

Exercise O-2.21. Let L be a **dcpo** with a bottom element $\perp$. We denote by $\mathcal{L}$ the set of all monotone self-maps $g: L \to L$ that are *inflationary*, i.e., $x \leq g(x)$ for all $x \in L$. We equip $\mathcal{L}$ with the pointwise ordering of functions. Let f be an arbitrary monotone self-map of L. Prove the following.

(i) $\mathcal{L}$ is a **dcpo** with a greatest element T.

Hint. First, let us remark that $\mathcal{L}$ is nonempty, as it contains the identity map as least element. As $g \leq g \circ h$ and $h \leq g \circ h$ for inflationary maps g and h, we conclude that $\mathcal{L}$ is directed. It is readily verified that $\mathcal{L}$ is complete with respect to directed pointwise suprema. Hence, $\mathcal{L}$ has a greatest element that we denote by T.

(ii) For every $x \in L$, $T(x)$ is a common fixed-point of all $g \in \mathcal{L}$.

Hint. Clearly, $g \circ T \in \mathcal{L}$ for every $g \in \mathcal{L}$. Hence, $g \circ T \leq T$ as T is the top element of $\mathcal{L}$. On the other hand, $g \circ T \geq T$ for inflationary g. Consequently, $g \circ T = T$ which implies the claim.

(iii) Let $M = \{x \in L : x \leq f(x)\}$ be the set of pre-fixed-points of f. Show that (a) $\perp \in M$, (b) M is closed for directed sups, and (c) M is mapped into itself by f.

Hint. Compare the proof of O-2.3.

(iv) Every monotone self-map $f: L \to L$ has a least fixed-point.