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Continuous Nowhere Differentiable Functions

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Abstract

In the early nineteenth century, most mathematicians believed that a continuous function has derivative at a significant set of points. A. M. Ampère even tried to give a theoretical justification for this (within the limitations of the definitions of his time) in his paper from 1806. In a presentation before the Berlin Academy on July 18, 1872 Karl Weierstrass shocked the mathematical community by proving this conjecture to be false. He presented a function which was continuous everywhere but differentiable nowhere. The function in question was defined by

$$W(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x),$$

where a is a real number with $0 < a < 1$, b is an odd integer and $ab > 1+3\pi/2$. This example was first published by du Bois-Reymond in 1875. Weierstrass also mentioned Riemann, who apparently had used a similar construction (which was unpublished) in his own lectures as early as 1861. However, neither Weierstrass' nor Riemann's function was the first such construction. The earliest known example is due to Czech mathematician Bernard Bolzano, who in the years around 1830 (published in 1922 after being discovered a few years earlier) exhibited a continuous function which was nowhere differentiable. Around 1860, the Swiss mathematician Charles Cellérier also discovered (independently) an example which unfortunately wasn't published until 1890 (posthumously).

After the publication of the Weierstrass function, many other mathematicians made their own contributions. We take a closer look at many of these functions by giving a short historical perspective and proving some of their properties. We also consider the set of all continuous nowhere differentiable functions seen as a subset of the space of all real-valued continuous functions. Surprisingly enough, this set is even "large" (of the second category in the sense of Baire).

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On another *note*, I would like to extend my gratitude to Dissection, Chris Poland and Spawn of Possession for having provided some quality music that made the long nights of work less grating. Thanks to Jan Lindblom for helping me with some French texts as well.

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Introduction

I turn away with fear and horror from the lamentable plague of continuous functions which do not have derivatives...

– Hermite, letter to Stieltjes dated 20 May 1893[1].

Judging by the quote above, some mathematicians didn't like the possibility of continuous functions which are nowhere differentiable. Why was these functions so poorly received?
Observing the situation today, many students still find it strange that there exists a continuous function which is nowhere differentiable. When I first heard of it myself I was a bit perplexed, at least by the sheer magnitude of the number of such functions that actually exist. Usually beginning students of mathematics get the impression that continuous functions normally are differentiable, except maybe at a few especially "nasty" points. The standard example of $f(x) = |x|$, which only lacks derivative at $x = 0$, is one such function. This was also the situation for most mathematicians in the late 18th and early 19th century. They were not interested in the existence of the derivative of some hypothetical function but rather just calculating the derivative as some explicit expression. This was usually successful, except at a few points in the domain where the differentiation failed. These actions led to the belief that continuous functions have derivatives everywhere, except at some particular points. Ampère even tried to give a theoretical justification for this statement in 1806 (cf. Ampère [1]), although it is not exactly clear

[1] Quote borrowed from Pinkus [57].

if he attempted to prove this for all continuous functions or for some smaller subset (for further discussion see Medvedev [48], pages 214-219).

Therefore, with all this in mind, the reaction of a 19th century mathematician to the news of these functions doesn't seem that strange anymore. These functions caused a reluctant reconsideration of the concept of a continuous function and motivated increased rigor in mathematical analysis. Nowadays the existence of these functions is fundamental for "new" areas of research and applications like, for example, fractals, chaos and wavelets.

In this report we present a chronological review of some of the continuous nowhere differentiable functions constructed during the last 170 years. Properties of these functions are discussed as well as traits of more general collections of nowhere differentiable functions.

The contents of the thesis is as follows. We start in Chapter 2 with sequences and series of functions defined on some interval $I \subset \mathbb{R}$ and convergence of those. This is important for the further development of the subject since many constructions are based on infinite series. In Chapter 3 we take a stroll through the last couple of centuries and present some of the functions constructed. We do this in a concise manner, starting with a short historical background before giving the construction of the function and showing that it has the desired properties. Some proofs has been left out for various reasons, but in those cases a clear reference to a proof is given instead. Chapter 4 continues with an examination of the set of all continuous nowhere differentiable functions. It turns out that the "average" continuous function normally is nowhere differentiable and not the other way around. We do this both by a topological argument based on category and also by a measure theoretic result using prevalence (considered by Hunt, Sauer and York).

Table 1.1 gives a short timeline for development in the field of continuous nowhere differentiable functions.

Discoverer	Year	Page	What
B. Bolzano	≈1830	11	First known example
M. Ch. Cellérier	≈1830	17	Early example
B. Riemann	≈1861	18	"Nondifferentiable" function
K. Weierstrass	1872	20	First published example
H. Hankel	1870	29	"Condensation of singularities"
H. A. Schwarz	1873	28	Not differentiable on a dense subset
M. G. Darboux	1873-5	28	Example ('73) and generalization ('75)
U. Dini	1877	25	Large class including Weierstrass
K. Hertz	1879	27	Generalization of Weierstrass function
G. Peano	1890	32	Space-filling curve (nowhere differentiable)
D. Hilbert	1891	33	Space-filling curve (nowhere differentiable)
T. Takagi	1903	36	Easier (than Weierstrass) example
H. von Koch	1904	39	Continuous curve with tangent nowhere
G. Faber	1907-8	41	"Investigation of continuous functions"
W. Sierpiński	1912	44	Space-filling curve (nowhere differentiable)
G. H. Hardy	1916	27	Generalization of Weierstrass conditions
K. Knopp	1918	45	Generalization of Takagi-type functions
M. B. Porter	1919	27	Generalization of Weierstrass function
K. Petr	1922	47	Algebraic/arithmetic example
A. S. Besicovitch	1924	78	No finite *or* infinite one-sided derivative
B. van der Waerden	1930	36	Takagi-like construction
S. Mazurkiewicz	1931	74	$\mathcal{ND}[0,1]$ is of the second category
S. Banach	1931	74	$\mathcal{ND}[0,1]$ is of the second category
S. Saks	1932	78	The set of Besicovitch-functions is Ist category
I. J. Schoenberg	1938	48	Space-filling curve (nowhere differentiable)
W. Orlicz	1947	52	Intermediate result
J. McCarthy	1953	55	Example with very simple proof
G. de Rham	1957	36	Takagi generalization
H. Katsuura	1991	57	Example based on metric-spaces
M. Lynch	1992	62	Example based on topology
B. R. Hunt	1994	78	$\mathcal{ND}[0,1]$ is a prevalent set
L. Wen	2002	64	Example based on infinite products

Table 1.1: Timelime (partial) of the development in the field of continuous nowhere differentiable functions.

Chapter 2

Series and Convergence

Many constructions of nowhere differentiable continuous functions are based on infinite series of functions. Therefore a few general theorems about series and sequences of functions will be of great aid when we continue investigating the subject at hand. First we need a clear definition of convergence in this context.

Definition 2.1. *A sequence S_n of functions on the interval I is said to converge pointwise to a function S on I if for every $x \in I$*

$$\lim_{n\to\infty} S_n(x) = S(x),$$

that is

$$\forall x \in I \, \forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \geq N \, |S_n(x) - S(x)| < \epsilon.$$

The convergence is said to be uniform on I if

$$\lim_{n\to\infty} \sup_{x\in I} |S_n(x) - S(x)| = 0,$$

that is

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \geq N \, \sup_{x\in I} |S_n(x) - S(x)| < \epsilon.$$

Uniform convergence plays an important role to whether properties of the elements in a sequence are transfered onto the limit of the sequence. The following two theorems can be of assistance when establishing if the convergence of a sequence of functions is uniform.

Theorem 2.1. *The sequence S_n converges uniformly on I if and only if it is a uniformly Cauchy sequence on I, that is*

$$\lim_{m,n\to\infty} \sup_{x\in I} |S_n(x) - S_m(x)| = 0$$

or

$$\forall\epsilon > 0\, \exists N \in \mathbb{N}\, \forall m, n \geq N \sup_{x\in I} |S_n(x) - S_m(x)| < \epsilon.$$

Proof. First, assume that S_n converges uniformly to S on I, that is

$$\forall\epsilon > 0\, \exists N \in \mathbb{N}\, \forall n \geq N \sup_{x\in I} |S_n(x) - S(x)| < \frac{\epsilon}{2}.$$

For such $\epsilon > 0$ and for $m, n \in \mathbb{N}$ with $m, n \geq N$ we have

$$\begin{aligned} \sup_{x\in I} |S_n(x) - S_m(x)| &\leq \sup_{x\in I} (|S_n(x) - S(x)| + |S(x) - S_m(x)|) \\ &\leq \sup_{x\in I} |S_n(x) - S(x)| + \sup_{x\in I} |S(x) - S_m(x)| < 2\frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Conversely, assume that $\{S_n\}$ is a uniformly Cauchy sequence, i.e.

$$\forall\epsilon > 0\, \exists N \in \mathbb{N}\, \forall m, n \geq N \sup_{x\in I} |S_n(x) - S_m(x)| < \frac{\epsilon}{2}.$$

For any fixed $x \in I$, the sequence $\{S_n(x)\}$ is clearly a Cauchy sequence of real numbers. Hence the sequence converges to a real number, say $S(x)$. From the assumption and the pointwise convergence just established we have

$$\forall\epsilon > 0\, \exists N \in \mathbb{N}\, \forall m, n \geq N \sup_{x\in I} |S_n(x) - S_m(x)| < \frac{\epsilon}{2}$$

and

$$\forall\epsilon > 0\, \forall x \in I\, \exists m_x > N\, |S_{m_x}(x) - S(x)| < \frac{\epsilon}{2}.$$

If $\epsilon > 0$ is arbitrary and $n > N$, then

$$\sup_{x\in I} |S_n(x) - S(x)| \leq \sup_{x\in I} (|S_n(x) - S_{m_x}(x)| + |S_{m_x}(x) - S(x)|) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence the convergence of S_n to S is uniform on I. $\square$

Theorem 2.2 (Weierstrass M-test). *Let $f_k : I \to \mathbb{R}$ be a sequence of functions such that $\sup_{x\in I} |f_k(x)| \leq M_k$ for every $k \in \mathbb{N}$. If $\sum_{k=1}^{\infty} M_k < \infty$, then the series $\sum_{k=1}^{\infty} f_k(x)$ is uniformly convergent on I.*

Proof. Let $m, n \in \mathbb{N}$ with $n > m$. Then

$$\begin{aligned} \sup_{x\in I} |S_n(x) - S_m(x)| &= \sup_{x\in I} \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x) \right| \\ &= \sup_{x\in I} \left| \sum_{k=m+1}^{n} f_k(x) \right| \leq \sum_{k=m+1}^{n} \sup_{x\in I} |f_k(x)| \\ &\leq \sum_{k=m+1}^{n} M_k = \sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k. \end{aligned}$$

Since $M = \sum_{k=1}^{\infty} M_k < \infty$ it follows that

$$\sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k \to M - M = 0 \quad \text{as } m, n \to \infty$$

which gives that $\{S_n\}$ is a uniformly Cauchy sequence on I. Using Theorem 2.1 we obtain that the series $\sum_{k=1}^{\infty} f_k(x)$ is uniformly convergent on I. □

We are often interested in establishing the continuity of a limit of a sequence of continuous functions. To accomplish this, the following theorem and its corollary can be helpful.

Theorem 2.3. *If $\{S_n\}$ is a sequence of continuous functions on I and S_n converges uniformly to S on I, then S is a continuous function on I.*

Proof. Let $x_0 \in I$ be arbitrary. By assumption we have

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \geq N \, \sup_{x\in I} |S_n(x) - S(x)| < \frac{\epsilon}{3}$$

and

$$\forall \epsilon > 0 \, \exists \delta > 0 \text{ such that } |x - x_0| < \delta \Rightarrow |S_n(x) - S_n(x_0)| < \frac{\epsilon}{3}.$$

Let $\epsilon > 0$ be given, $x \in I$, $n \in \mathbb{N}$ with $n > N$ and $|x - x_0| < \delta$. Then

$$|S(x)-S(x_0)| \leq |S(x)-S_n(x)|+|S_n(x)-S_n(x_0)|+|S_n(x_0)-S(x_0)| < 3\frac{\epsilon}{3} = \epsilon$$

and therefore S is continuous at x_0. Since $x_0 \in I$ was arbitrary, S is continuous on I. $\square$

Corollary 2.4. *If $f_k : I \to \mathbb{R}$ is a continuous function for every $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly to $S(x)$ on I, then S is a continuous function on I.*

Chapter 3

Functions Through the Ages

3.1 Bolzano function (≈1830; published in 1922)

Probably the first example of a continuous nowhere differentiable function on an interval is due to Czech mathematician Bernard Bolzano. The history behind this example is filled with unfortunate circumstances. Due to these circumstances, Bolzano's manuscript with the name *"Functionenlehre"*, which was written around 1830 and contained the function, wasn't published until a century later in 1930. The publication came to since in 1920, after the first World War, another Czech mathematician Martin Jašek discovered a manuscript in the National Library of Vienna belonging to Bernard Bolzano (a photocopy is also in the archives of the Czech Academy of Sciences). It was named *"Functionenlehre"* and it was dated 1830. Originally it was supposed to be a part of Bolzano's more extensive work *"Grössenlehre"*. The manuscript *"Functionenlehre"* was published in Prague in 1930 (in the *"Schriften I"*), having 183 pages and containing an introduction and two parts. Bolzano proved in it that the set of points where the function is non-differentiable is dense in the interval where it is defined. The continuity was also deduced, however not completely correct. The full story on *"Functionenlehre"* can be found in Hyksǒvá [33] who also has written the following:

> *The first lecture of M. Jašek reporting on* **Functionenlehre** *was given on December 3, 1921. Already on February 3, 1922 Karel Rychlík presented to KČSN [Royal Czech Science Society] his treatise [61] where the correct proof of the continuity of*

Bolzano's function was given as well as the proof of the assertion that this function does not have a derivative at any point of the interval (a,b) (finite or infinite). The same assertion was proved by Vojtěch Jarník (1897 - 1970) at the same time but in a different way in his paper [34]. Both Jarník and Rychlík knew about the work of the other. Giving reference to Rychlík's paper, Jarník did not prove the continuity of Bolzano's function; on the other hand, Rychlík cited the work of Jarník (an idea of another way to the same partial result).

Unlike many other constructions of nowhere differentiable functions, Bolzano's function is based on a geometrical construction instead of a series approach. The Bolzano function, B, is constructed as the limit of a sequence $\{B_k\}$ of continuous functions. We can choose the domain of B_1 (which will be the domain of B as well) and the range of B_1. Let the interval $[a, b]$ be the desired domain and $[A, B]$ the desired range. Each piecewise linear and continuous function in the sequence is defined as follows.

(i) $B_1(x) = A + \frac{B-A}{b-a}(x - a)$;

(ii) $B_2(x)$ is defined on the intervals

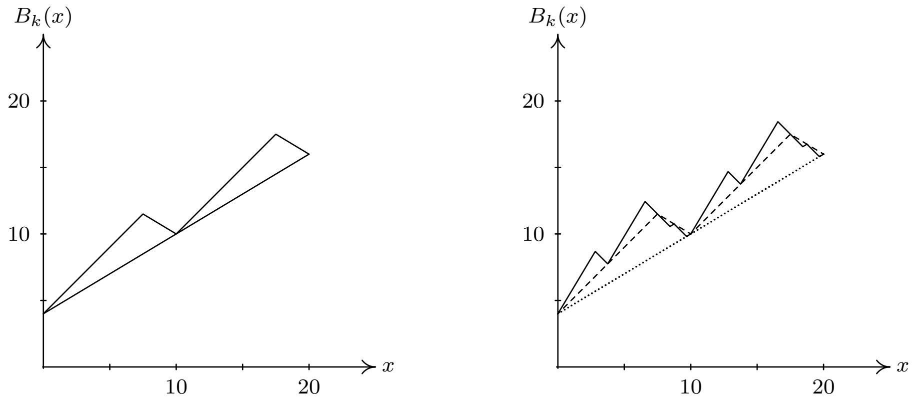
$$I_1 = \left[a, a + \frac{3}{8}(b-a)\right], \qquad I_2 = \left[a + \frac{3}{8}(b-a), \frac{1}{2}(a+b)\right],$$
$$I_3 = \left[\frac{1}{2}(a+b), a + \frac{7}{8}(b-a)\right], \quad I_4 = \left[a + \frac{7}{8}(b-a), b\right]$$

as the piecewise linear function having the values

$$B_2(a) = A, \qquad B_2\left(a + \frac{3}{8}(b-a)\right) = A + \frac{5}{8}(B-A),$$
$$B_2\left(\frac{1}{2}(a+b)\right) = A + \frac{1}{2}(B-A),$$
$$B_2\left(a + \frac{7}{8}(b-a)\right) = B + \frac{1}{8}(B-A), \qquad B_2(b) = B$$

at the endpoints;

(iii) $B_3(x)$ is constructed by the same procedure as in (ii) on each of the four subintervals I_i (with the corresponding values for a, b, A and B). This continues for $k = 4, 5, 6, \ldots$ and the limit of $B_k(x)$ as $k \to \infty$ is the Bolzano function $B(x)$.



(a) B_1 and B_2.

(b) B_1 (dotted), B_2 (dashed) and B_3 (whole).

Figure 3.1: The three first elements in the "Bolzano" sequence $\{B_k(x)\}$ with $[a, b] = [0, 20]$ and $[A, B] = [4, 16]$.

A fitting closing remark, before the proof of continuity and nowhere differentiability, can be found in Hyksŏvá [33]:

> *"Already the fact that it occurred to Bolzano at all that such a function might exist, deserves our respect. The fact that he actually succeeded in its construction, is even more admirable".*

Theorem 3.1. *The Bolzano function B is continuous and nowhere differentiable on the interval $[a, b]$.*

Proof. First we want to show that the function B is continuous. For fixed $k \in \mathbb{N}$ consider the function B_k. Let us find the slopes $M_k = \{M_{k,m}\}$ of each of the linear functions on the subintervals. Not to have too many indices we will just write M_k instead of $M_{k,m}$. For $k = 1$ it is immediate from the definition that $M_1 = \frac{B-A}{b-a}$ for all of $[a, b]$. Let $k \geq 2$. For each linear part $[a_k, b_k]$ of B_k we have the following

1. For $I = [t_1, t_2] = \left[a_k, a_k + \frac{3}{8}(b_k - a_k)\right]$,

$$M_{k+1}^{(1)} = \frac{B_k(t_2) - B_k(t_1)}{t_2 - t_1} = \frac{\frac{5}{8}(B_k - A_k)}{\frac{3}{8}(b_k - a_k)} = \frac{5}{3}\frac{B_k - A_k}{b_k - a_k} = \frac{5}{3}M_k;$$

2. for $I = [t_2, t_3] = \left[a_k + \frac{3}{8}(b_k - a_k), \frac{1}{2}(a_k + b_k)\right]$,

$$M_{k+1}^{(2)} = \frac{B_k(t_3) - B_k(t_2)}{t_3 - t_2} = \frac{\left(\frac{1}{2} - \frac{5}{8}\right)(B_k - A_k)}{\left(\frac{1}{2} - \frac{3}{8}\right)(b_k - a_k)} = \frac{-\frac{1}{8}}{\frac{1}{8}}\frac{B_k - A_k}{b_k - a_k} = -M_k;$$

3. for $I = [t_3, t_4] = \left[\frac{1}{2}(a_k + b_k), a_k + \frac{7}{8}(b_k - a_k)\right]$,

$$M_{k+1}^{(3)} = \frac{B_k(t_4) - B_k(t_3)}{t_4 - t_3} = \frac{\left(1 + \frac{1}{8} - \frac{1}{2}\right)(B_k - A_k)}{\left(\frac{7}{8} - \frac{1}{2}\right)(b_k - a_k)} = \frac{\frac{5}{8}}{\frac{3}{8}}\frac{B_k - A_k}{b_k - a_k} = \frac{5}{3}M_k;$$

4. for $I = [t_4, t_5] = \left[a_k + \frac{7}{8}(b_k - a_k), b_k\right]$,

$$M_{k+1}^{(4)} = \frac{B_k(t_5) - B_k(t_4)}{t_5 - t_4} = \frac{-\frac{1}{8}(B_k - A_k)}{\left(1 - \frac{7}{8}\right)(b_k - a_k)} = \frac{-\frac{1}{8}}{\frac{1}{8}}\frac{B_k - A_k}{b_k - a_k} = -M_k.$$

Let $\{I_{n,k}\} = \{[I_n(s_k), I_n(t_k)]\}$ be the collection of subintervals of $[a, b]$ where B_n is linear and define

$$L_n = \sup_{I \in \{I_{n+1,k}\}} (I(t_k) - I(s_k)) \quad \text{and} \quad M_n = \sup_{\substack{I \in \{I_{n+1,k}\} \\ i=1,2,3,4}} |M_n^{(i)}(I)|.$$

That is, L_n is the maximal length of an interval where B_{n+1} is linear and M_n is the maximum slope (to the absolute value) of B_{n+1}. Clearly

$$L_n \le \left(\frac{3}{8}\right)^{n+1} |b - a| \quad \text{and} \quad M_n \le \left(\frac{5}{3}\right)^{n+1} \left|\frac{B - A}{b - a}\right|$$

which gives that the maximum increase/decrease of the function from step n to $n + 1$ is bounded by $M_n L_n \le \left(\frac{5}{8}\right)^{n+1} |B - A|$. Hence, for $k \in \mathbb{N}$,

$$\sup_{x \in [a,b]} |B_{k+1}(x) - B_k(x)| \le \left(\frac{5}{8}\right)^{k+1} |B - A|.$$

Let $m, n \in \mathbb{N}$ with $m > n$. We have

$$\begin{aligned}
\sup_{x\in[a,b]} |B_m(x) - B_n(x)| &\le \sup_{x\in[a,b]} \left(\sum_{k=n+1}^{m} |B_k(x) - B_{k-1}(x)| \right) \\
&\le \sum_{k=n+1}^{m} \sup_{x\in[a,b]} |B_k(x) - B_{k-1}(x)| \\
&\le \sum_{k=n+1}^{m} \left(\frac{5}{8}\right)^k |B - A| \\
&= |B - A| \left(\sum_{k=1}^{m} \left(\frac{5}{8}\right)^k - \sum_{k=1}^{n} \left(\frac{5}{8}\right)^k \right) \\
&\to |B - A| \left(\frac{5}{3} - \frac{5}{3} \right) = 0 \quad \text{as } m, n \to \infty.
\end{aligned}$$

Thus $\{B_k\}$ is a uniformly Cauchy sequence on the interval $[a, b]$ and since each B_k is continuous it follows from Theorems 2.1 and 2.3 that Bolzano's function is continuous on $[a, b]$.
Secondly, we show that B is not differentiable at any $x \in [a, b]$. Again, let $\{I_{n,k}\} = \{[I_n(s_k), I_n(t_k)]\}$ be the collection of subintervals of $[a, b]$ where B_n is linear and define M as the set of all endpoints in $\{I_{n,k}\}$, i.e.

$$M = \{s, t \mid [s, t] \in \{I_{n,k}\}\} .$$

We show that M is dense in $[a, b]$. That is, for any $x_0 \in [a, b]$, $\exists x_n \in M$ such that $x_n \to x_0$. Let $x_0 \in [a, b]$ be arbitrary but fixed. If $x_0 = b$ we are done since $b \in M$. Assume that $x_0 \neq b$, we proceed as follows.

(i) Step 1: let $L = b - a$ and define

$$\begin{aligned}
J_0^{(0)} &= \left[a, a + \frac{3}{8}L\right), & J_0^{(1)} &= \left[a + \frac{3}{8}L, a + \frac{1}{2}L\right), \\
J_0^{(2)} &= \left[a + \frac{1}{2}L, a + \frac{7}{8}L\right) \text{ and } & J_0^{(3)} &= \left[a + \frac{7}{8}L, b\right).
\end{aligned}$$

Clearly there exists $i_0 \in \{0, 1, 2, 3\}$ such that $x_0 \in J_0^{(i_0)}$. We take $J_0 = J_0^{(i_0)}$.

(ii) Step n: we have $x_0 \in I_{n-1} = [a_n, b_n]$. Let $L_n = b_n - a_n$ and define

$$J_n^{(0)} = \left[a_n, a_n + \frac{3}{8}L_n\right), \qquad J_n^{(1)} = \left[a_n + \frac{3}{8}L_n, a_n + \frac{1}{2}L_n\right),$$
$$J_n^{(2)} = \left[a_n + \frac{1}{2}L_n, a_n + \frac{7}{8}L_n\right) \text{ and } \quad J_n^{(3)} = \left[a_n + \frac{7}{8}L_n, b_n\right).$$

As before, there exists $i_n \in \{0, 1, 2, 3\}$ such that $x_0 \in J_n^{(i_n)}$. We take $J_n = J_n^{(i_n)}$.

Hence M is dense in $[a, b]$ since

$$|x_0 - a_{n+1}| \leq \left(\frac{3}{8}\right)^{n+1} |b - a| \to 0 \text{ as } n \to \infty^1.$$

Now we show that B is non-differentiable for every $x_0 \in M$. Let $x_0 \in M$ be arbitrary but fixed, we consider two cases that exhaust all possibilities.
For $x_0 = a$: Let $x_n = a + \left(\frac{3}{8}\right)^n |b - a|$. Then $x_n \to a$ as $n \to \infty$ and $x_n \in M$ for every $n \in \mathbb{N}$. By the construction of the function B it is clear that $B(x_n) = B_{n+1}(x_n)$ for every $n \in \mathbb{N}$. Also, $B(a) = A$ and $B_{n+1}(x_n) = A + \left(\frac{5}{3}\right)^n \left(\frac{3}{8}\right)^n |b - a|$. Hence

$$\frac{B(x_n) - B(a)}{x_n - a} = \frac{A + \left(\frac{5}{3}\right)^n \left(\frac{3}{8}\right)^n |b - a| - A}{\left(\frac{3}{8}\right)^n |b - a|} = \left(\frac{5}{3}\right)^n \to \infty \text{ as } n \to \infty$$

and therefore $B'(x_0)$ does not exist.
For $x_0 \in M \setminus \{a\}$: let $x_n = x_0 - \left(\frac{1}{8}\right)^{n+q} |b - a|$, $q \in \mathbb{N}$. Since $x_0 \in M$, there exists $r \in \mathbb{N}$ such that $B(x_0) = B_p(x_0)$ for all $p \geq r$. We can choose $q > r$ so that $x_n \in (a, b]$ for every $n \in \mathbb{N}$. From the construction of B we see that $B(x_n) = B_{n+1}(x_n) = B_n(x_0) + (-1)^n K \left(\frac{1}{8}\right)^{n+q}$ where $K \in \mathbb{R}$ with $K \geq |b - a|/|B - A| \neq 0$. Moreover, since $q > r$, $B(x_0) = B_n(x_0)$ for every $n \in \mathbb{N}$. This implies that

$$(B(x_0) - B_n(x_0))\, 8^{n+q} = (B_n(x_0) - B_n(x_0))\, 8^{n+q} = 0.$$

So for $n \in \mathbb{N}$,

$$\begin{aligned}\frac{B(x_0) - B(x_n)}{x_0 - x_n} &= 8^{n+q} \left(B(x_0) - B_n(x_0) - (-1)^n K \left(\frac{1}{8}\right)^{n+q}\right) \\ &= (B(x_0) - B_n(x_0))\, 8^{n+q} - (-1)^n K = (-1)^{n+1} K.\end{aligned}$$

[1] And also $|x_0 - b_{n+1}| \to 0$ as $n \to \infty$.

But $(-1)^{n+1}K$ does not converge as $n \to \infty$ and thus $B'(x_0)$ does not exist. In no way is it clear from this that B is nowhere differentiable, only that it is non-differentiable on a dense subset of $[a, b]$ (which theoretically means that it might still be possible that B is differentiable almost everywhere). We will not complete the proof here but merely give a reference: the complete proof can be found in Jarník [34].

$\square$

3.2 Cellérier function (≈1860; published in 1890)

Charles Cellérier had proposed the function C defined as

$$C(x) = \sum_{k=1}^{\infty} \frac{1}{a^k} \sin(a^k x), \quad a > 1000$$

earlier than 1860 but the function wasn't published until 1890 (posthumously) in Cellérier [10]. When the manuscripts were opened after his death they were found to be containing sensational material. In an undated folder (according to the historians it is from around 1860) with heading

> *"Very important and I think new. Correct. Can be published as it is written."*

there was a proof of the fact that the function C is continuous and nowhere differentiable if a is a sufficiently large even number. The publication of Cellériers example in 1890 came as only a curiosity since it was already generally known from Weierstrass (see Section 3.4). Cellérier's function is strikingly similar to Weierstrass' function and its nowhere differentiability follows from Hardy's generalization of that function (see the remark to Theorem 3.4).
In Cellérier's paper (which, roughly translated, has the title "Notes on the fundamental principles of analysis") there is a section called "Example de fonctions faisant exception aux règles usuelles" – "Example of functions making departures from the usual rules". In this section Cellérier proposed the function C defined above and states that this function will provide an example of a function that is continuous, differentiable nowhere and never has any periods of growth or decay.

Cellérier's original condition on a was $a > 1000$ where a is an even integer (for nowhere differentiability) or $a > 1000$ where a is an odd integer (for no periods of growth or decay). According to Hardy [27], for the case of nowhere differentiability, the condition can be weakened to $a > 1$ (not necessarily an integer).

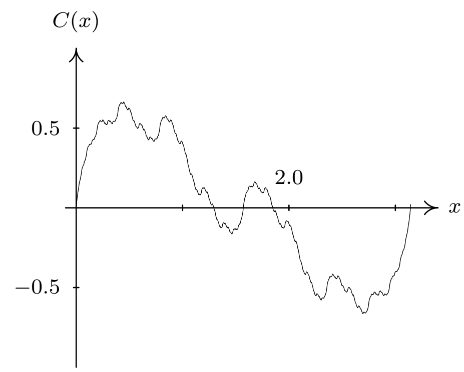


Figure 3.2: Cellerier's function $C(x)$ with $a = 2$ on $[0, \pi]$.

Theorem 3.2. *The Cellérier function*

$$C(x) = \sum_{k=1}^{\infty} \frac{1}{a^k} \sin(a^k x), \quad a > 1$$

is continuous and nowhere differentiable on $\mathbb{R}$.

Proof. The continuity of C follows exactly like in the proof for Weierstrass' function (Theorem 3.4). That C is nowhere differentiable follows from Hardy [27] (see the remark to Theorem 3.4) since $a \cdot a^{-1} \geq 1$ and that if $g(x)$ is nowhere differentiable than so is $g(x/\pi)$. $\square$

3.3 Riemann function (≈1861)

In a thesis from 1854 (*Habilitationsschrift*), Riemann [59] attempted to find necessary and sufficient conditions for representation of a function by Fourier series. In this paper he also generalized the definite integral and gave an example of a function that between any two points is discontinuous infinitely often but still is integrable (with respect to the Riemann-integral). The function he defined was

$$f(x) = \sum_{k=1}^{\infty} \frac{(nx)}{n^2}, \qquad \text{where } (x) = \begin{cases} 0, & \text{if } x = \frac{p}{2},\ p \in \mathbb{Z} \\ x - [x], & \text{elsewhere,} \end{cases}$$

and $[x]$ is the integer part of x. This function is interesting in this context for another reason. Consider, for $x \in [a, b]$, the function $F : [a, b] \to \mathbb{R}$ defined by the indefinite integral of f,

$$F(x) = \int_a^x f(\tau)\, d\tau.$$

It can quite easily be seen that this function is continuous and it is also clear that it is not differentiable on a dense subset of $[a, b]$. This, however, is not the function we will be concerned with here. What we will refer to as Riemann's function in this framework is the function $R : \mathbb{R} \to \mathbb{R}$ defined by

$$R(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sin(k^2 x).$$

Interesting to note is that there seems to be no other known sources for the claim that this was Riemann's construction than those that can be traced back to Weierstrass (cf. Butzer and Stark [7], Ullrich [74] and Section 3.4). Riemann's function isn't actually a nowhere differentiable function. It has been shown that R possess a finite derivative ($R'(x_0) = -\frac{1}{2}$) at points of the form

$$x_0 = \pi \frac{2p+1}{2q+1}, \quad p, q \in \mathbb{Z}.$$

These points however, are the only points where R has a finite derivative (cf. Gerver [24], [25] and Hardy [27] or for a more concise proof based on number theory see Smith [71]).
According to Weierstrass, Riemann used this function as an example of a "nondifferentiable" function in his lectures as early as 1861. It is unclear whether he meant that the function was nowhere differentiable or something else. Riemann claimed to have a proof, obtained from the theory of elliptic functions, but it was never presented nor was it found anywhere in his notes after his death (cf. Neuenschwander [49] and Segal [68]).

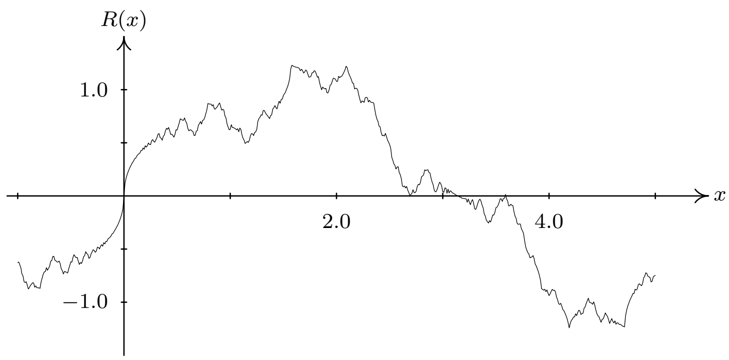


Figure 3.3: Riemann's function R on $[-1, 5]$.

Theorem 3.3. *The Riemann function*

$$R(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sin(k^2 x)$$

is continuous on all of $\mathbb{R}$ *and only has a derivative at points of the form*

$$x_0 = \pi \frac{2p+1}{2q+1}, \quad p, q \in \mathbb{Z}.$$

Proof. We start with showing that the function R is continuous. Since $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < \infty$ and $\sup_{x \in \mathbb{R}} |\frac{1}{k^2} \sin(k^2 x)| = \frac{1}{k^2}$, the Weierstrass M-test (Theorem 2.2) proves that the convergence is uniform and the Corollary 2.4 gives the continuity of R on $\mathbb{R}$. Secondly, the only points where R has a finite derivative (cf. Gerver [24], [25] and Hardy [27] or Smith [71]) is points of the form

$$x_0 = \pi \frac{2p+1}{2q+1}, \quad p, q \in \mathbb{Z}.$$

□

3.4 Weierstrass function (1872; published in 1875 by du Bois-Reymond)

On July 18, 1872 Karl Weierstrass presented in a lecture at the Royal Academy of Science in Berlin an example of a continuous nowhere differentiable function,

$$W(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x),$$

for $0 < a < 1$, $ab > 1 + 3\pi/2$ and $b > 1$ an odd integer. On the lecture Weierstrass said

> *As I know from some pupils of Riemann, he as the first one (around 1861 or earlier) suggested as a counterexample to Ampère's Theorem [which perhaps could be interpreted[2] as: every continuous function is differentiable except at a few isolated points]; for example, the function R does not satisfy this theorem. Unfortunately, Riemann's proof was unpublished and, as I think, it is neither in his notes nor in oral transfers. In my opinion Riemann considered continuous functions without derivatives at any point, the proof of this fact seems to be difficult...*

Weierstrass' function was the first continuous nowhere differentiable function to be published, which happened in 1875 by Paul du Bois-Reymond [19]. At this time, du Bois-Reymond was a professor at Heidelberg University in Germany and in 1873 he sent a paper to Borchardt's Journal ["Journal für die reine und angewandte Mathematik"]. This paper dealt with the function Weierstrass had discussed earlier (among several other topics). Borchardt gave the paper to Weierstrass to read through. Weierstrass wrote in a letter to du Bois-Reymond (dated 23 of November, 1873; cf. Weierstrass [77]) that he had made no new progress, except for some remarks about Riemann's function. In the letter, du Bois-Reymond had Weierstrass' function presented in the form

$$f(x) = \sum_{k=0}^{\infty} \frac{\sin(a^n x)}{b^n}, \quad \frac{a}{b} > 1,$$

which apparently was changed before the paper was published. Du Bois-Reymond accepted Weierstrass' remarks and put them in his paper together with some more historical notes about the subject and in 1875 the paper was published in Borchardt's Journal.

Since this was the first published continuous nowhere differentiable function it has been regarded by many as the first such function exhibited. This regardless of the fact that Weierstrass' function was not the earliest such

[2] See Medvedev [48], pages 214-219.

construction. Several others[3] had done it earlier, although non of those are believed to have been published before the publication of the Weierstrass function.

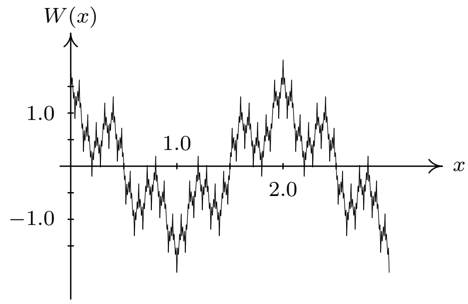


Figure 3.4: Weierstrass' function W with $a = \frac{1}{2}$ and $b = 5$ on $[0, 3]$.

In 1916, Hardy [27] proved that the function W defined above is continuous and nowhere differentiable if $0 < a < 1$, $ab \geq 1$ and $b > 1$ (not necessarily an odd integer).

Theorem 3.4. *The Weierstrass function,*

$$W(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x),$$

for $0 < a < 1$, $ab \geq 1$ *and* $b > 1$, *is continuous and nowhere differentiable on* $\mathbb{R}$.

Proof. Starting with establishing the continuity, observe that $0 < a < 1$ implies $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} < \infty$. This together with $\sup_{x\in\mathbb{R}} |a^n \cos(b^n \pi x)| \leq a^n$ gives, using the Weierstrass M-test (Theorem 2.2), that $\sum_{k=0}^{\infty} a^n \cos(b^n \pi x)$ converges uniformly to $W(x)$ on $\mathbb{R}$. The continuity of W now follows from the uniform convergence of the series just established and from the Corollary 2.4.
During the rest of this proof we assume that Weierstrass original assumptions hold, i.e. $ab > 1 + \frac{3}{2}\pi$ and $b > 1$ an odd integer. For a general proof with $ab \geq 1$ and $b > 1$ we refer to Hardy [27]. The rest of the proof follows,

[3] For example, Cellérier's and Bolzano's functions both described in earlier sections were constructed much earlier than Weierstrass' function.

quite closely, from the original proof of Weierstrass (as it is presented in du Bois-Reymond [19]).
Let $x_0 \in \mathbb{R}$ be arbitrary but fixed and let $m \in \mathbb{N}$ be arbitrary. Choose $\alpha_m \in \mathbb{Z}$ such that $b^m x_0 - \alpha_m \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ and define $x_{m+1} = b^m x_0 - \alpha_m$. Put

$$y_m = \frac{\alpha_m - 1}{b^m} \quad \text{and} \quad z_m = \frac{\alpha_m + 1}{b^m}.$$

This gives the inequality

$$y_m - x_0 = -\frac{1 + x_{m+1}}{b^m} < 0 < \frac{1 - x_{m+1}}{b^m} = z_m - x_0$$

and therefore $y_m < x_0 < z_m$. As $m \to \infty$, $y_m \to x_0$ from the left and $z_m \to x_0$ from the right.
First consider the left-hand difference quotient,

$$\begin{aligned}\frac{W(y_m) - W(x_0)}{y_m - x_0} &= \sum_{n=0}^{\infty} \left(a^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{y_m - x_0} \right) \\ &= \sum_{n=0}^{m-1} \left((ab)^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{b^n (y_m - x_0)} \right) \\ &\quad + \sum_{n=0}^{\infty} \left(a^{m+n} \frac{\cos(b^{m+n} \pi y_m) - \cos(b^{m+n} \pi x_0)}{y_m - x_0} \right) = S_1 + S_2.\end{aligned}$$

We treat these sums separately, starting with S_1. Since $\left|\frac{\sin(x)}{x}\right| \leq 1$ we can, using a trigonometric identity, bound the sum by

$$\begin{aligned}|S_1| &= \left| \sum_{n=0}^{m-1} (ab)^n (-\pi) \sin\left(\frac{b^n \pi (y_m + x_0)}{2} \right) \frac{\sin\left(\frac{b^n \pi (y_m - x_0)}{2} \right)}{b^n \pi \frac{y_m - x_0}{2}} \right| \\ &\leq \sum_{n=0}^{m-1} \pi (ab)^n = \frac{\pi((ab)^m - 1)}{ab - 1} \leq \frac{\pi (ab)^m}{ab - 1}.\end{aligned} \tag{3.1}$$

Considering the sum S_2 we can use (since $b > 1$ is an odd integer and $\alpha_m \in \mathbb{Z}$)

$$\begin{aligned}\cos(b^{m+n} \pi y_m) &= \cos\left(b^{m+n} \pi \frac{\alpha_m - 1}{b^m} \right) = \cos(b^n \pi (\alpha_m - 1)) \\ &= \left[(-1)^{b^n} \right]^{\alpha_m - 1} = -(-1)^{\alpha_m}\end{aligned}$$

and

$$\begin{aligned}\cos(b^{m+n}\pi x_0) &= \cos\left(b^{m+n}\pi\frac{\alpha_m + x_{m+1}}{b^m}\right)\\ &= \cos(b^n\pi\alpha_m)\cos(b^n\pi x_{m+1}) - \sin(b^n\pi\alpha_m)\sin(b^n\pi x_{m+1})\\ &= \left[(-1)^{b^n}\right]^{\alpha_m}\cos(b^n\pi x_{m+1}) - 0 = (-1)^{\alpha_m}\cos(b^n\pi x_{m+1})\end{aligned}$$

to express the sum as

$$\begin{aligned}S_2 &= \sum_{n=0}^{\infty} a^{m+n}\frac{-(-1)^{\alpha_m} - (-1)^{\alpha_m}\cos(b^n\pi x_{m+1})}{-\frac{1+x_{m+1}}{b^m}}\\ &= (ab)^m(-1)^{\alpha_m}\sum_{n=0}^{\infty} a^n\frac{1+\cos(b^n\pi x_{m+1})}{1+x_{m+1}}.\end{aligned}$$

Each term in the series above is non-negative and $x_{m+1} \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ so we can find a lower bound by

$$\sum_{n=0}^{\infty} a^n\frac{1+\cos(b^n\pi x_{m+1})}{1+x_{m+1}} \geq \frac{1+\cos(\pi x_{m+1})}{1+x_{m+1}} \geq \frac{1}{1+\frac{1}{2}} = \frac{2}{3}. \tag{3.2}$$

The inequalities (3.1) and (3.2) ensures the existence of an $\epsilon_1 \in [-1,1]$ and an $\eta_1 > 1$ such that

$$\frac{W(y_m) - W(x_0)}{y_m - x_0} = (-1)^{\alpha_m}(ab)^m\eta_1\left(\frac{2}{3} + \epsilon_1\frac{\pi}{ab-1}\right).$$

As with the left-hand difference quotient, for the right-hand quotient we do pretty much the same, starting by expressing the said fraction as

$$\frac{W(z_m) - W(x_0)}{z_m - x_0} = S_1' + S_2'.$$

As before, it can be deduced that

$$|S_1'| \leq \frac{\pi(ab)^m}{ab-1}. \tag{3.3}$$

The cosine-term containing z_m can be simplified as (again since b is odd and $\alpha_m \in \mathbb{Z}$)

$$\begin{aligned}\cos(b^{m+n}\pi z_m) &= \cos\left(b^{m+n}\pi\frac{\alpha_m+1}{b^m}\right) = \cos(b^n\pi(\alpha_m+1))\\ &= \left[(-1)^{b^n}\right]^{\alpha_m+1} = -(-1)^{\alpha_m},\end{aligned}$$

which gives

$$\begin{aligned} S_2' &= \sum_{n=0}^{\infty} a^{m+n} \frac{-(-1)^{\alpha_m} - (-1)^{\alpha_m} \cos(b^n \pi x_{m+1})}{\frac{1-x_{m+1}}{b^m}} \\ &= -(ab)^m (-1)^{\alpha_m} \sum_{n=0}^{\infty} a^n \frac{1+\cos(b^n \pi x_{m+1})}{1-x_{m+1}}. \end{aligned}$$

As before, we can find a lower bound for the series by

$$\sum_{n=0}^{\infty} a^n \frac{1+\cos(b^n \pi x_{m+1})}{1-x_{m+1}} \geq \frac{1+\cos(\pi x_{m+1})}{1-x_{m+1}} \geq \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}. \tag{3.4}$$

By the same argument as for the left-hand difference quotient (but by using the inequalities (3.3) and (3.4) instead), there exists an $\epsilon_2 \in [-1,1]$ and an $\eta_2 > 1$ such that

$$\frac{W(z_m) - W(x_0)}{z_m - x_0} = -(-1)^{\alpha_m} (ab)^m \eta_2 \left(\frac{2}{3} + \epsilon_2 \frac{\pi}{ab-1} \right).$$

By the assumption $ab > 1 + \frac{3}{2}\pi$, which is equivalent to $\frac{\pi}{ab-1} < \frac{2}{3}$, the left- and right-hand difference quotients have different signs. Since also $(ab)^m \to \infty$ as $m \to \infty$ it is clear that W has no derivative at x_0. The choice of $x_0 \in \mathbb{R}$ was arbitrary so it follows that $W(x)$ is nowhere differentiable on $\mathbb{R}$. □

Remark 1 (Dini). In a series of publications (cf. Dini [15], [16], [17] and [18]) in the years 1877-78, Italian mathematician Ulisse Dini proposed a more general class of continuous nowhere differentiable functions (under which Weierstrass function happen to fall). Our presentation here is largely based on Knopp's summary (cf. Knopp [38], pp. 23-26). Let $\{f_n\}$ be a sequence of differentiable functions $f_n : [0,1] \to \mathbb{R}$ that have bounded derivative on $[0,1]$ and such that

$$W_D(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges uniformly on $[0,1]$. We also require that

(i) each function f_n has a finite number of extrema and if δ_n is the maximum distance between two successive extrema then $\delta_n \to 0$ as $n \to \infty$;

(ii) if γ_n is the (to the absolute value) greatest difference between two successive extreme values then

$$\lim_{n\to\infty} \frac{\delta_n}{\gamma_n} = 0;$$

(iii) if $h_{n,x}$ denotes the two increments (one which is positive and one which is negative) for which $x + h_{n,x}$ gives the first right (respectively left) extremum for which

$$|f_n(x + h_{n,x}) - f_n(x)| \geq \frac{1}{2}\gamma_n,$$

then we can define a sequence $\{r_n\}$ of positive numbers such that

$$\sup_{x\in[0,1]} |R_n(x + h_{n,x}) - R_n(x)| \leq 2r_n$$

where $R_n(x)$ is the remainder of the series defining the function W_D;

(iv) if $\{c_n\}$ is a sequence of positive numbers such that $\sup_{x\in[0,1]} |f_n'(x)| \leq c_n$ then from some index on

$$\frac{4\delta_n}{\gamma_n} \sum_{k=1}^{n} c_k + \frac{4r_n}{\gamma_n} \leq \theta, \quad \theta \in [0, 1);$$

(v) the sign of $f_n(x + h_{n,x}) - f_n(x)$ is independent of $h_{n,x}$ from some n_0 onward for all $x \in [0, 1]$.

Then the function W_D is continuous and nowhere differentiable on $[0, 1]$.
As two concrete examples of functions in Dini's classification, consider for $|a| > 1 + 3\pi/2$

$$W_{D_1}(x) = \sum_{k=1}^{\infty} \frac{a^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cos(1 \cdot 3 \cdot 5 \cdots (2n-1)\pi x)$$

and for $a > 1 + 3\pi/2$

$$W_{D_2}(x) = \sum_{k=1}^{\infty} \frac{a^n}{1 \cdot 5 \cdot 9 \cdots (4n+1)} \sin(1 \cdot 5 \cdot 9 \cdots (4n+1)\pi x).$$

Remark 2 (Hertz). Polish mathematician Karol Hertz gave in his paper [28] from 1879 a generalization of Weierstrass function, namely

$$W_H(x) = \sum_{k=1}^{\infty} a^k \cos^p(b^k \pi x),$$

where $a > 1$, $p \in \mathbb{N}$ is odd, b an odd integer and $ab > 1 + \frac{2}{3}p\pi$.
Remark 3 (Hardy). Hardy proved (in Hardy [27]) that if $0 < a < 1$, $b > 1$ and $ab \geq 1$ then both

$$W_1(x) = \sum_{k=0}^{\infty} a^k \sin(b^k \pi x) \qquad \text{and} \qquad W_2(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$$

are continuous and nowhere differentiable on all of $\mathbb{R}$.
Remark 4 (Porter). M. B. Porter generalized Weierstrass function in an article (Porter [58]) published in 1919. He proposed two classes of functions $W_i : [a, b] \to \mathbb{R}$ defined by

$$W_1(x) = \sum_{k=0}^{\infty} u_k(x) \sin(b_n \pi x) \qquad \text{and} \qquad W_2(x) = \sum_{k=0}^{\infty} u_k(x) \cos(b_n \pi x)$$

where $\{b_n\}$ is a sequence of integers and $\{u_k\}$ is a sequence of differentiable functions. We have the following requirements:

(i) W_i converges uniformly on $[a, b]$ for $i = 1, 2$;

(ii) b_n divides b_{n+1} and for an unlimited number of n's, b_{n+1}/b_n must be divisible by four or increase to infinity with n;

(iii) $\sum_{k=0}^{\infty} u_n'(x)$ converge uniformly on $[a, b]$ by the Weierstrass M-test;

(iv) $(3\pi/2) \sum_{k=0}^{N-1} |b_n u_n(x)| < |b_N u_N(x)|$ for all $x \in [a, b]$.

If this holds then both W_1 and W_2 are continuous and nowhere differentiable. The following concrete functions are examples that falls under Porter's generalization.

$$\text{(a)} \sum_{k=0}^{\infty} \frac{a^n}{n!} \sin(n!\,\pi x) \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{a^n}{n!} \cos(n!\,\pi x), \quad \text{where } |a| > 1 + \frac{3}{2}\pi;$$

$$\text{(b)} \sum_{k=0}^{\infty} \frac{1}{a^n} \sin(n!\,a^n \pi x) \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{1}{a^n} \cos(n!\,a^n \pi x), \quad \text{where } |a| \in \mathbb{N} \setminus \{1\};$$

$$\text{(c)} \sum_{k=1}^{\infty} \frac{a_k}{10^k} \sin(10^{3k}\pi x) \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{a_k}{10^k} \cos(10^{3k}\pi x), \quad \text{where } a_k \text{ is chosen}$$

such that $\sum_{k=1}^{\infty} \frac{a_k}{10^k}$ is a non-terminating decimal. Both Dini functions in the remark above falls under this generalization as well.

3.5 Darboux function (1873; published in 1875)

Darboux's function, discovered independently of Weierstrass, was presented on 19 March 1873 (two years earlier than the first publication of Weierstrass' function) and was published two years later in Darboux [11]. In this publication (whose title translates to "paper on the discontinuous functions") Darboux spends much of the discussion on the subject of Riemann-integration of discontinuous functions but he also investigated when a continuous function possess a finite derivative. Contained in this document is his description of a continuous function which is nowhere differentiable and this function is defined as the infinite series

$$D(x) = \sum_{k=1}^{\infty} \frac{1}{k!} \sin\left((k+1)!\,x\right).$$

Darboux constructed this function after having analyzed and generalized results from Schwarz and Hankel, who in the years before had studied and made suggestions about the subject. One of Schwarz ideas, proposed in 1873 in Schwarz [67], was a function $S : (0, \infty) \to \mathbb{R}$ defined by

$$S(x) = \sum_{k=0}^{\infty} \frac{\varphi(2^k x)}{4^k}, \qquad \text{where } \varphi(x) = [x] + \sqrt{x - [x]}$$

and $[x]$ means the integer part of x. The function S is continuous and monotonically increasing, but there is no derivative at infinitely many points in any interval so S is not differentiable on a dense subset of $(0, M)$ (which we will prove). Interesting to note is that Schwarz (and many others) seem to have considered these types of functions "without derivative", but today, with measure theoretic background, we call many of them differentiable almost everywhere.
Hankel had introduced the concept of "Condensation of singularities" some years before (cf. Hankel [26]). This is a process where by letting each term in an absolutely convergent series have a singularity, a function with singularities at all rational points[4] is created. An example of this procedure could be the function g defined by

$$g(x) = \sum_{n=1}^{\infty} \frac{\psi(\sin(n\pi x))}{n^s}, \qquad \text{where } \psi(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and $s > 1$. Hankel's treatment of the subject, however, wasn't entirely accurate as other mathematicians pointed out after the publication. Darboux writes in his paper that he thought it was a shame that Hankel had died before he had a chance to correct some of his ideas himself.

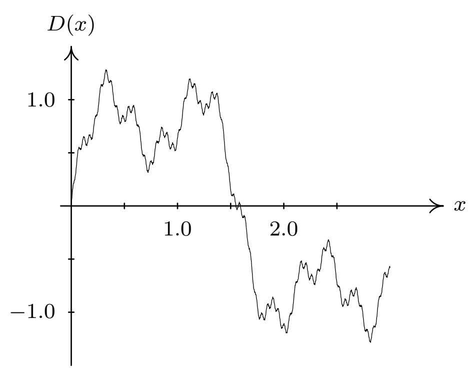


Figure 3.5: Darboux's function $D(x)$ on $[0, 3]$.

In a subsequent paper (Darboux [12]), Darboux generalized his example. He

[4]If x is a rational number, say $x = p/q$, then $\sin(n\pi x) = \sin(n\pi p/q) = \sin(\pm p\pi) = 0$ for $n = q$. Hence x is a singular point of $\psi(\sin(n\pi x))$ (for a special n) and this behavior can be shown to transfer onto the sum of the series as well.

considered the series

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{f(a_n b_n x)}{a_n}$$

where a_n and b_n are sequences of real numbers and $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function with a bounded second derivative. By adding some restrictions to the two sequences $\{a_n\}$ and $\{b_n\}$,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$$

and for some fixed $k \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n-k} a_m b_m^2}{a_n} = 0,$$

Darboux states that φ is a continuous function. Moreover, it is possible to make some additional restrictions on the parameters to ensure that φ is nowhere differentiable as well as continuous. For example, with $b_n = 1$ and $k = 1$ it is enough to have

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n-1} a_m}{a_n} = 0$$

for φ to be nowhere differentiable for an infinite number of functions f. For example with $a_n = n!$ and $f(x) = \cos(x)$. Another example would be $b_n = n + 1$, $a_n = n!$, $k = 3$ and $f(x) = \sin(x)$ which is the function D introduced by Darboux in his earlier paper (Darboux [11]) and which was defined at the beginning of this section.

Theorem 3.5. *The Darboux function*

$$D(x) = \sum_{k=1}^{\infty} \frac{1}{k!} \sin\left((k+1)!\, x\right)$$

is continuous and nowhere differentiable on $\mathbb{R}$.

Proof. Since $\sum_{k=0}^{\infty} \frac{1}{k!} = e$ it is clear that $\sum_{k=1}^{\infty} \frac{1}{k!} < \infty$. This and the fact that $\sup_{x\in\mathbb{R}} \left|\frac{1}{k!} \sin((k+1)!\, x)\right| \leq \frac{1}{k!}$ implies, by the Weierstrass M-test (Theorem 2.2), that the convergence is uniform. The Corollary 2.4 gives the continuity of D. A proof of the fact that D is nowhere differentiable can be found in Darboux [12]. $\square$

Theorem 3.6. *The Schwarz function $S : (0, M) \to \mathbb{R}$ defined by*

$$S(x) = \sum_{k=0}^{\infty} \frac{\varphi(2^k x)}{4^k}, \qquad \textit{where } \varphi(x) = [x] + \sqrt{x - [x]},$$

is continuous and non-differentiable on a dense subset of $(0, M)$. Here $M > 0$ is any real number.

Proof. We start by proving that S is continuous. The only possible discontinuities of the function φ is for $x \in \mathbb{N}$. Let $p \in \mathbb{N}$, we show that φ is both left and right continuous at p. From the right we have

$$\lim_{x \to p^+} \varphi(x) = \lim_{x \to p^+} \left([x] + \sqrt{x - [x]} \right) = p + \sqrt{p - p} = p$$

and from the left

$$\lim_{x \to p^-} \varphi(x) = \lim_{x \to p^-} \left([x] + \sqrt{x - [x]} \right) = p - 1 + \sqrt{p - (p-1)} = p.$$

Hence φ is continuous on $(0, M)$ (and $\varphi(p) = p$ for $p \in \mathbb{N}$). Now we show that the series converge uniformly so that also S is continuous on $(0, M)$. Let $h \in (0, 1)$ and $p \in \mathbb{N} \cup \{0\}$. Then

$$\varphi(p + h) = [p + h] + \sqrt{p + h - [p + h]} = p + \sqrt{h}.$$

Define $q(h) = \varphi(p + h) - (p + h)$, then $q(h) \le p + h + 1/4$ since

$$q'(h) = \frac{1}{2\sqrt{h}} - 1 = 0 \quad \Rightarrow \quad h = \frac{1}{4}$$

and $q''(1/4) < 0$ so the maximum is attained at $h = 1/4$ ($q(0) = q(1) = 0$). From this we get the inequality

$$\varphi(x) \le x + \frac{1}{4}.$$

Now it follows that

$$\sup_{x \in (0,M)} \left| \frac{1}{4^n} \varphi(2^n x) \right| \le \sup_{x \in (0,M)} \left| \frac{2^n x + 1/4}{4^n} \right| \le \frac{M}{2^n} + \frac{1}{4^{n+1}}$$

and since

$$\sum_{n=0}^{\infty}\left(\frac{M}{2^n}+\frac{1}{4^{n+1}}\right)<\infty,$$

the Weierstrass' M-test (Theorem 2.2) and the Corollary 2.4 gives that S is continuous on $(0, M)$.
We turn to the non-differentiable part. Let $x_0, x_1 \in (0, M)$ with $x_0 < x_1$ be arbitrary. We show that between any two such points there exists a point where S is without derivative (which implies that S is non-differentiable on a dense subset of $(0, M)$).
Let x be a dyadic rational such that $x_0 < x < x_1$. Then $x = i2^{-m}$ for some $i, m \in \mathbb{N}$. Let $0 < h < 2^{-m}$, then, since each term in the series is non-negative,

$$\frac{S(x+h)-S(x)}{h}=\sum_{k=0}^{\infty}\frac{\varphi(2^n(x+h))-\varphi(2^n x)}{4^n h}\geq\frac{\varphi(2^m(x+h))-\varphi(2^m x)}{4^m h}.$$

Since $2^m h < 1$ and $2^m x = i \in \mathbb{N}$ we see that

$$\begin{aligned}\varphi(2^m(x+h))-\varphi(2^m x) &= [2^m x+2^m h]+\sqrt{2^m x+2^m h-[2^m x+2^m h]}\\ &\quad -[2^m x]-\sqrt{2^m x-[2^m x]}\\ &= i+\sqrt{i+2^m h-i}-i-\sqrt{i-i}=\sqrt{2^m h}.\end{aligned}$$

Hence

$$\frac{S(x+h)-S(x)}{h}\geq\frac{\sqrt{2^m h}}{4^m h}=\frac{1}{2^m\sqrt{2^m}}\cdot\frac{1}{\sqrt{h}}\to\infty \text{ as } h\to 0$$

and therefore $S'(x)$ does not exist. □

3.6 Peano function (1890)

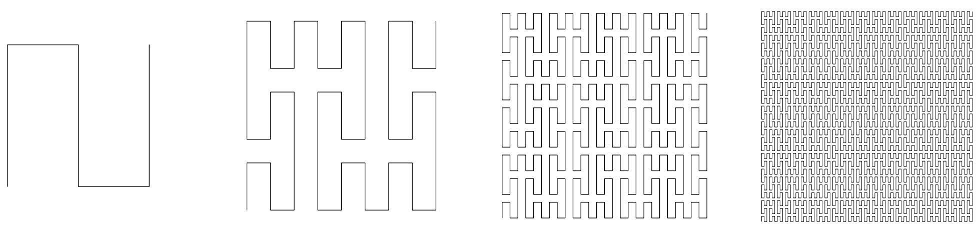
Let $t = (t_1 t_2 t_3 \cdots)_3$ be a ternary representation of $t \in [0, 1]$ (that is, $t = \sum_{k=1}^{\infty} t_k 3^{-k}$ with $t_k \in \{0, 1, 2\}$). Then Peano's function P is expressed as

$$\begin{cases} \quad P:[0,1]\to[0,1]\times[0,1], \\ (t_1t_2t_3\cdots)_3 \mapsto \begin{pmatrix} (t_1(k^{t_2}t_3)(k^{t_2+t_4}t_5)(k^{t_2+t_4+t_6}t_7)\cdots)_3 \\ ((k^{t_1}t_2)(k^{t_1+t_3}t_4)(k^{t_1+t_3+t_5}t_6)\cdots)_3 \end{pmatrix}, \end{cases}$$

where the operator k is defined as

$$kt_j = 2 - t_j, \qquad t_j = 0, 1, 2$$

and $k^l t_j$ is the l'th element in the sequence $\{kt_j, k(kt_j), k(k(kt_j)), \ldots\}$ (and we adhere to the convention that $k^0 t_j = t_j$).
It can be shown (cf. Sagan [64], pp. 32-33) that P is independent of which[5] ternary representation of t is chosen and that P is surjective (i.e. a space-filling curve, that is a "1-dimensional" curve that fills two-dimensional space[6]).



(a) $n = 1$. (b) $n = 1$. (c) $n = 2$. (d) $n = 3$.

Figure 3.6: First four steps in the geometric generation of Peano's curve.

Peano's curve was the first space-filling curve discovered and it was published in 1890 (in Peano [54]). After his publication several other mathematicians proposed new examples and among those were Hilbert's function (published in 1891, see Sagan [64]) and Schoenberg's curve (proposed in 1938). Both of those happen to be nowhere differentiable (and in Section 3.13 we take a closer look on Schoenberg's curve). It is not, however, the case that all space-filling curves are nowhere differentiable (although Peano's turns out to be). For example, Lebesgue's space filling curve[7]is differentiable almost everywhere (it is differentiable everywhere except on the Cantor set, which incidentally has Lebesgue measure zero).

[5]The representation is not unique, e.g. $(1)_3 = 1/3$ and also $(022\cdots)_3 = 1/3$.

[6]Or more generally, a curve that passes through every point of some subset of n-dimensional Euclidean space (or even more general as is stated in the Hahn-Mazurkiewicz theorem).

[7]Henri Lebesgue constructed his curve in 1904 as a continuous extension of a known mapping. The original mapping had the Cantor set as domain and mapped it onto $[0,1] \times [0,1]$. The extension is done by linear interpolation, see Sagan [64].

Let ϕ_p and ψ_p be the component functions of P. Peano stated in his presentation that both components were continuous and nowhere differentiable but left the proof of nowhere differentiability out of his paper. The proof presented here is due to Sagan [64], pp. 33-34.

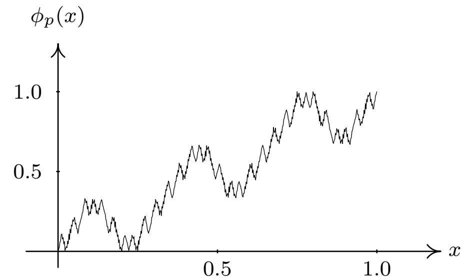


Figure 3.7: The component ϕ_p of Peano's curve.

Theorem 3.7. *The components ϕ_p and ψ_p of the Peano function P are continuous and nowhere differentiable on the interval* $[0, 1]$.

Proof. First we establish the continuity of ϕ_p. We do this in two steps, first we show that ϕ_p is continuous from the right.
For $t_0 \in [0, 1)$, let $t_0 = (t_1 t_2 t_3 \cdots t_{2n} t_{2n+1} \cdots)_3$ be the ternary representation of t_0 that doesn't end in infinitely many 2's. Choose

$$\delta = 3^{-2n} - (00 \cdots t_{2n+1} t_{2n+2} \cdots)_3.$$

Clearly $\delta \to 0$ as $n \to \infty$. The definition of δ gives

$$\begin{aligned} t_0 + \delta &= (t_1 t_2 t_3 \cdots t_{2n} t_{2n+1} \cdots)_3 + 3^{-2n} - (00 \cdots t_{2n+1} t_{2n+2} \cdots)_3 \\ &= (t_1 t_2 t_3 \cdots t_{2n} 00 \cdots)_3 + 3^{-2n} = (t_1 t_2 t_3 \cdots t_{2n} 22 \cdots)_3. \end{aligned}$$

So for any $t \in [t_0, t_0 + \delta)$, the first $2n$ digits in the ternary expansion are equal, i.e. $t = (t_1 t_2 t_3 \cdots t_{2n} \tau_{2n+1} \tau_{2n+2} \cdots)_3$. Let $\epsilon_n = \sum_{i=1}^{n} t_{2i}$. We have

$$\begin{aligned} |\phi(t) - \phi(t_0)| &= |(t_1 (k^{t_2} t_3) \cdots (k^{\epsilon_n} \tau_{2n+1}) \cdots)_3 - (t_1 (k^{t_2} t_3) \cdots (k^{\epsilon_n} t_{2n+1}) \cdots)_3| \\ &\leq \sum_{i=n}^{\infty} \frac{1}{3^{i+1}} |k^{\epsilon_i} \tau_{2i+1} - k^{\epsilon_i} t_{2i+1}| \leq \sum_{i=n}^{\infty} \frac{2}{3^{i+1}} \\ &= \frac{2}{3^{n+1}} \sum_{i=0}^{\infty} \frac{1}{3^i} = \frac{1}{3^n} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence ϕ_p is continuous from the right. Now we show that ϕ_p is also continuous from the left. The argument follows similarly as above.
For $t_0 \in (0,1]$, let $t_0 = (t_1 t_2 t_3 \cdots t_{2n} t_{2n+1} \cdots)_3$ be the ternary representation with infinitely many non-zero terms. Pick

$$\delta = (00 \cdots 0 t_{2n+1} t_{2n+1} \cdots)_3.$$

Then

$$t_0 - \delta = (t_1 t_2 t_3 \cdots t_{2n} 00 \cdots)_3.$$

Hence, for $t \in (t_0 - \delta, t_0]$, t's ternary representation has the same first $2n$ digits as t_0. Thus

$$\begin{aligned} |\phi(t) - \phi(t_0)| &= |(t_1(k^{t_2} t_3) \cdots (k^{\epsilon_n} \tau_{2n+1}) \cdots)_3 \\ &\qquad - (t_1(k^{t_2} t_3) \cdots (k^{\epsilon_n} t_{2n+1}) 00 \cdots)_3| \\ &\leq \frac{2}{3^{n+1}} \sum_{i=0}^{\infty} \frac{1}{3^i} = \frac{1}{3^n} \to 0 \text{ as } n \to \infty. \end{aligned}$$

So ϕ_p is continuous from the left on $(0,1]$. Since we established that ϕ_p also is continuous from the right on $[0,1)$ it is clear that ϕ_p is continuous on $[0,1]$.
Next we show that ϕ_p is nowhere differentiable on $[0,1]$. For arbitrary $t \in [0,1]$, let $t = (t_1 t_2 t_3 \cdots t_{2n} t_{2n+1} \cdots)_3$ be a ternary representation of t. Define the sequence $\{t_n\}$ by $t_n = (t_1 t_2 t_3 \cdots t_{2n} \tau_{2n+1} t_{2n+2} \cdots)_3$, where τ_{2n+1} is chosen as $\tau_{2n+1} = (t_{2n+1} + 1) \bmod 2$. This implies that

$$|t - t_n| = \frac{1}{3^{2n+1}}.$$

From the definition of P and t_n, $\phi_p(t)$ and $\phi_p(t_n)$ only differs at position $n+1$ in the ternary representation. Therefore we have

$$|\phi_p(t) - \phi_p(t_n)| = \frac{1}{3^{n+1}} |k^{\epsilon_n} t_{2n+1} - k^{\epsilon_n} \tau_{2n+1}| = \frac{1}{3^{n+1}}.$$

Analyzing the differential quotient we see that

$$\left| \frac{\phi_p(t) - \phi_p(t_n)}{t - t_n} \right| = \frac{1}{3^{n+1}} \frac{3^{2n+1}}{1} = 3^n \to \infty \text{ as } n \to \infty.$$

Hence ϕ_p is not differentiable at t. Since $t \in [0,1]$ was arbitrary it follows that ϕ_p is nowhere differentiable on $[0,1]$.
Moreover, since $\psi_p(t) = 3\phi_p(t/3)$, the fact that ψ_p is continuous and nowhere differentiable on $[0,1]$ follows from what we just established for ϕ_p. $\square$

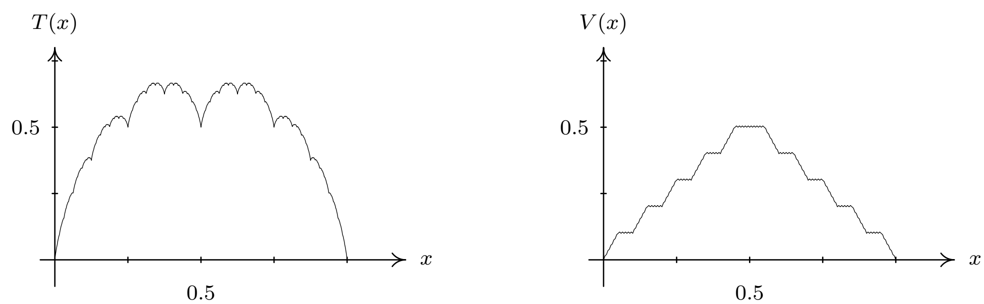
3.7 Takagi (1903) and van der Waerden (1930) functions

Takagi's and van der Waerden's functions are very similar in their construction. Takagi presented his example in 1903 (cf. Takagi [73]) as an example of a "simpler" continuous nowhere differentiable function than Weierstrass. Van der Waerden published his function in 1930 (van der Waerden [75]), apparently unaware of Takagi's very similar idea.
The definition of Takagi's function is expressed as the infinite series

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \operatorname{dist}\left(2^k x, \mathbb{Z}\right) = \sum_{k=0}^{\infty} \frac{1}{2^k} \inf_{m \in \mathbb{Z}} \left|2^k x - m\right|$$

and Van der Waerden's function is defined as

$$V(x) = \sum_{k=0}^{\infty} \frac{1}{10^k} \operatorname{dist}\left(10^k x, \mathbb{Z}\right) = \sum_{k=0}^{\infty} \frac{1}{10^k} \inf_{m \in \mathbb{Z}} \left|10^k x - m\right| .$$



(a) Takagi's function. (b) Van der Waerden's function.

Figure 3.8: Takagi's and van der Waerden's functions on $[0, 1]$.

The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = \operatorname{dist}(x, \mathbb{Z}) = \inf_{m \in \mathbb{Z}} |x - m|$, which both series above are superpositions of, can be seen graphically in figure 3.13. More variations have been developed and in 1918 Knopp [38] did a generalization, which we consider in Section 3.11. Also de Rham treated these kinds of functions in his article de Rham [14]. He gives a proof that the function referred to as Takagi's function here is continuous and nowhere differentiable.

De Rham also considers the function

$$f(x) = \sum_{k=0}^{\infty} a^{-k}\phi(a^k x)$$

where a is an even positive integer. He claims that his proof can be adapted to show that f is continuous and nowhere differentiable as well. Moreover, he points out that the function f is a solution to the functional equation

$$f(x) - \frac{1}{a}f(ax) = \phi(x)$$

and that it is the only solution that is bounded. He proceeds to generalize this equation to

$$F(x) - bF(ax) = g(x)$$

where g is a given function and a and b are constants. De Rham claims that the only bounded solution for $b \in (0, 1)$ is

$$F(x) = \sum_{k=0}^{\infty} b^k g(a^k x).$$

Interesting to note is that for $g(x) = \cos(x)$ and a an odd integer with $ab > 1 + 3\pi/2$ we have the Weierstrass function (see Section 3.4).
We will use the following lemma when proving that Takagi's function (and several others) is nowhere differentiable.

Lemma 3.8. *Let $a < a_n < x < b_n < b$ for all $n \in \mathbb{N}$ and let $a_n \to x$ and $b_n \to x$. If $f : [a, b] \to \mathbb{R}$ is a continuous function and $f'(x)$ exists then*

$$\lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n} = f'(x).$$

Proof. Since

$$\left|\frac{b_n - x}{b_n - a_n}\right| \leq \frac{b_n - a_n}{b_n - a_n} = 1 \qquad \text{and} \qquad \left|\frac{x - a_n}{b_n - a_n}\right| \leq \frac{b_n - a_n}{b_n - a_n} = 1$$

we can estimate by

$$\begin{aligned}\left|\frac{f(b_n)-f(a_n)}{b_n-a_n}-f'(x)\right| &= \left|\frac{b_n-x}{b_n-a_n}\left(\frac{f(b_n)-f(x)}{b_n-x}-f'(x)\right)\right.\\ &\qquad \left.+\frac{x-a_n}{b_n-a_n}\left(\frac{f(a_n)-f(x)}{a_n-x}-f'(x)\right)\right|\\ &\le \left|\frac{f(b_n)-f(x)}{b_n-x}-f'(x)\right|+\left|\frac{f(a_n)-f(x)}{a_n-x}-f'(x)\right|\\ &\to 0 \text{ as } n\to\infty.\end{aligned}$$

Hence

$$\lim_{n\to\infty}\frac{f(b_n)-f(a_n)}{b_n-a_n}=f'(x).$$

□

Theorem 3.9. *Both the Takagi function and the van der Waerden function are continuous and nowhere differentiable on* $\mathbb{R}$*.*

Proof. That both T and V are continuous follows from the proof of the continuity of the Knopp function in Section 3.11.
We show that T is nowhere differentiable. The proof is based on an argument by Billingsley [5] and in a similar way it can be shown that also $V(x)$ is nowhere differentiable (cf. van der Waerden [75]).
Let $x\in\mathbb{R}$ be arbitrary and assume that $T'(x)$ exists. By Lemma 3.8, if $u_n\le x\le v_n$ (with $u_n<v_n$) and $v_n-u_n\to 0$, then

$$\frac{T(v_n)-T(u_n)}{v_n-u_n}\to T'(x).$$

We will define two sequences that contradicts this. Let $\phi(x)=\inf_{m\in\mathbb{Z}}|x-m|$. Then

$$T(x)=\sum_{k=0}^{\infty}\frac{1}{2^k}\inf_{m\in\mathbb{Z}}\left|2^kx-m\right|=\sum_{k=0}^{\infty}\frac{1}{2^k}\phi\left(2^kx\right).$$

Let $\mathbb{D}=\{i2^{-n}\,|\,i,n\in\mathbb{Z}\}$ be the dyadic rationals. If $u\in\mathbb{D}$ is of order n then, for every integer $k\ge n$, $2^ku\in\mathbb{Z}$. Hence, since $\phi(p)=0$ for $p\in\mathbb{Z}$, we have

$$T(u)=\sum_{k=0}^{n-1}\frac{1}{2^k}\phi\left(2^ku\right).$$

Let $u_n, v_n \in \mathbb{D}$ be successive numbers of order n for which $u_n \leq x < v_n$. Then $v_n - u_n = i2^{-n} - (i-1)2^{-n} = 2^{-n}$ and

$$\frac{T(v_n) - T(u_n)}{v_n - u_n} = \sum_{k=0}^{n-1} \frac{1}{2^k} \frac{\phi(2^k v_n) - \phi(2^k u_n)}{v_n - u_n}.$$

Obviously $\phi(x)$ is linear for $x \in \left[2^k u_n, 2^k v_n\right]$ since $\left[2^k u_n, 2^k v_n\right] = \left[\frac{i-1}{2^l}, \frac{i}{2^l}\right]$ where $l = n - k \in \mathbb{N}$. Hence, for $0 \leq k < n$,

$$\frac{1}{2^k} \frac{\phi(2^k v_n) - \phi(2^k u_n)}{v_n - u_n} = \frac{\pm 2^{-l}}{2^{-l}} = \pm 1$$

which gives

$$\frac{T(v_n) - T(u_n)}{v_n - u_n} = \sum_{k=0}^{n-1} \pm 1.$$

As $n \to \infty$, the series on the right does not converge. This contradicts the assumption that $T'(x)$ exists. Since $x \in \mathbb{R}$ was arbitrary, the function T is nowhere differentiable.
Note that Cater [8] has shown that T has no one-sided derivative at any point (whereas what we just proved above is that T has no two-sided derivative at any point). $\square$

A further property of Takagi's function is deduced in Shidfar and Sabetfakhri [69], it turns out that it's also continuous in the Hölder sense for $0 < \alpha < 1$ (or Lipschitz class of order α). That is, for every $\alpha \in (0, 1)$ there exists $M_\alpha > 0$ such that for every $x, y \in \mathbb{R}$

$$|T(x) - T(y)| \leq M_\alpha |x - y|^\alpha.$$

3.8 Koch "snowflake" curve (1904)

In 1904, Swedish mathematician Helge von Koch published (in Koch [39]) an article about a curve of infinite length with tangent nowhere. It was republished two years later with some added pages in Koch [40]. Koch writes[8] about the previous misconception that all continuous curves has a well determined tangent except at some isolated points:

[8]Translation from Edgar [20].

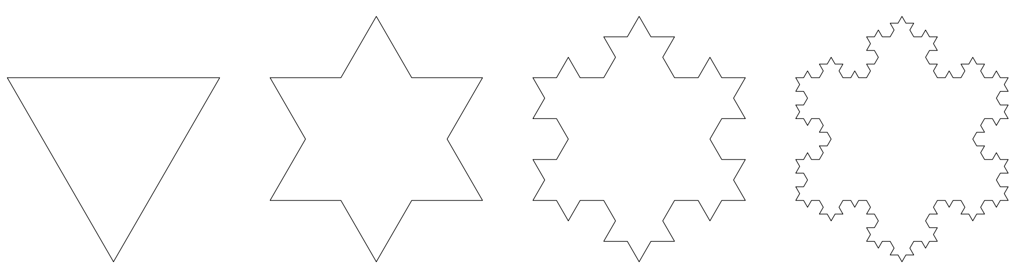
"Even though the example of Weierstrass [Section 3.4] has corrected this misconception once and for all, it seems to me that his example is not satisfactory from the geometrical point of view since the function is defined by an analytic expression that hides the geometrical nature of the corresponding curve and so from this point of view one does not see why the curve has no tangent."

Koch's "snowflake" curve (named after its shape) is constructed as follows: Take an equilateral triangle and split each line in three equal parts. Replace the middle segments by two sides of a new equilateral triangle that is constructed with the removed segment as its base. Repeat this procedure on each of the four new lines (for each of the original three sides). Repeat indefinitely. The limit of the process gives rise to a curve that is continuous and has a tangent nowhere (which is shown in Koch's paper). In figure 3.9 the first few iterations are shown graphically.

Koch also shows that there exists a parameterization

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

of the curve for $t \in [0, 1]$ and that both functions f and g are continuous and nowhere differentiable (on $[0, 1]$).



(a) $n = 0$. (b) $n = 1$. (c) $n = 2$. (d) $n = 3$.

Figure 3.9: First four steps in the construction of Koch's "snowflake".

3.9 Faber functions (1907, 1908)

In 1907, German mathematician Georg Faber [21] presented an example of a continuous nowhere differentiable function defined by

$$F_1(x) = \sum_{k=1}^{\infty} \frac{1}{10^k} \inf_{m \in Z} \left| 2^{k!}x - m \right| = \sum_{k=1}^{\infty} \frac{1}{10^k} \operatorname{dist}\left(2^{k!}x, \mathbb{Z}\right).$$

In 1908, Faber proceeded to publish a second article [22] named "Über stetige Funktionen" and two years later a longer article [23] about a similar subject. In the article from 1908, Faber presented another continuous nowhere differentiable function:

$$F_2(x) = \sum_{k=1}^{\infty} \frac{1}{k!} \inf_{m \in Z} \left| 2^{k!}x - m \right| = \sum_{k=1}^{\infty} \frac{1}{k!} \operatorname{dist}\left(2^{k!}x, \mathbb{Z}\right).$$

Faber set out to do an investigation of what he referred to as the "deep gap" between the differentiable and the merely continuous functions. By doing this, Faber intended to give a better insight on the infinitesimal structure of continuous functions in general.

His investigation makes it possible to construct, by superposition of piecewise linear functions, examples of continuous functions with special properties like, for example, nowhere differentiability. The construction is to a large degree geometrical and the fact that Faber's function has the desired properties is proven as it is constructed. This is in contrast with most other proofs of this kind, which are done on a fixed analytical expression that is given. Unfortunately we won't go through all of Faber's construction but merely give a brief description leading to a fixed expression and thereby neglecting some of the beauty in his architecture.

To make the understanding easier, Faber wished to characterize his functions by a countable (and dense) subset of the interval $[0, 1]$, namely the set

$$M = \left\{ \frac{k}{2^n} \,\middle|\, k, n \in \mathbb{N},\ k \leq 2^n \right\}.$$

He then proceeded to define the real numbers $\delta_{\frac{k}{2^n}}$ such that

$$\delta_{\frac{1}{2}} = F_2\left(\frac{1}{2}\right) - \frac{F_2(0) + F_2(1)}{2}$$

and for $m, n \in \mathbb{N}$ with $2m+1 < 2^n$

$$\delta_{\frac{2m+1}{2^n}} = F_2\left(\frac{2m+1}{2^n}\right) - \frac{F_2\left(\frac{m}{2^{n-1}}\right) + F_2\left(\frac{m+1}{2^{n-1}}\right)}{2}$$

where F_2 is the function we will show is continuous and nowhere differentiable. We take $F_2(0) = F_2(1) = 0$. From the relations above it is clear that we can recursively express $F_2(x_i)$ in terms of "δ's" for any $x_i \in M$.
To give a more geometrical view, we consider a sequence of piecewise linear continuous functions $\{f_k\}$, where, for any $k \in \mathbb{N}$, $f_k : [0,1] \to \mathbb{R}$ is the function that binds together the points

$$(0,0),\ \left(\frac{1}{2^n}, \delta_{\frac{1}{2^n}}\right),\ \left(\frac{2}{2^n}, 0\right),\ \left(\frac{3}{2^n}, \delta_{\frac{3}{2^n}}\right),\ \ldots,\ \left(\frac{2^n-1}{2^n}, \delta_{\frac{2^n-1}{2^n}}\right),\ (1,0)$$

in a continuous and piecewise linear manner. Figure 3.11(a) shows two functions graphically.

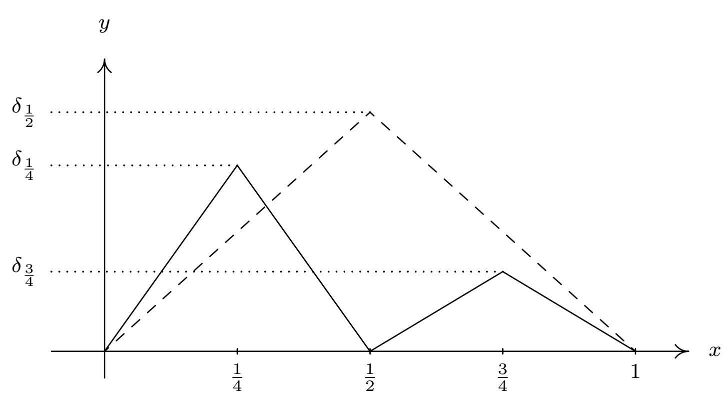


Figure 3.10: The functions f_1 (dashed) and f_2 (whole).

Now, for any $x_i \in M$ we can deduce that

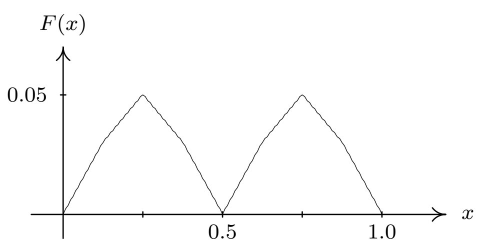
$$F_2(x_i) = \sum_{k=1}^{\infty} f_k(x_i).$$

We choose a subsequence $\{n_k\}$ and for each n_k we choose all "δ's" equal to $\frac{1}{k!}$ (for simplicity; Faber made the same choice in his article). Naturally, for arbitrary $x \in [0,1]$ we define

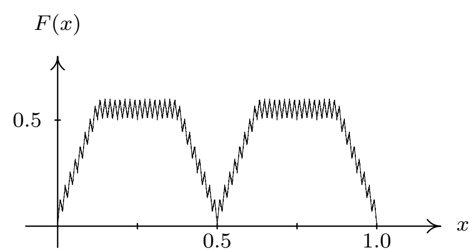
$$F_2(x) = \sum_{k=1}^{\infty} f_{n_k}(x).$$

This gives rise to the function which we refer to as Faber's function:

$$F_2(x) = \sum_{k=1}^{\infty} \frac{1}{k!} \inf_{m \in Z} \left|2^{k!}x - m\right| = \sum_{k=1}^{\infty} \frac{1}{k!} \operatorname{dist}\left(2^{k!}x, \mathbb{Z}\right).$$



(a) Faber's first function F_1.



(b) Faber's second function $F_2(x)$.

Figure 3.11: Faber's functions $F_i(x)$ on $[0,1]$.

Theorem 3.10. *The Faber functions*

$$F_1(x) = \sum_{k=1}^{\infty} \frac{1}{10^k} \inf_{m \in Z} \left|2^{k!}x - m\right| \quad \textit{and} \quad F_2(x) = \sum_{k=1}^{\infty} \frac{1}{k!} \inf_{m \in Z} \left|2^{k!}x - m\right|$$

are continuous and nowhere differentiable on $\mathbb{R}$.

We will not repeat Faber's proof here but instead opt for a direct proof based on the analytic expression given above. The proof is based on the same argument that was used in the proof of Theorem 3.9 (for Takagi's function).

Proof. We only prove this for F_2. The proof for F_1 follows similarly.
First, that F_2 is continuous follows from the Weierstrass M-test (Theorem 2.2) and the Corollary 2.4 since

$$\sup_{x \in [0,1]} \frac{1}{k!} \inf_{m \in Z} \left|2^{k!}x - m\right| \leq \frac{1}{2k!}$$

and $\sum_{k=1}^{\infty} 1/(2k!) < \infty$.
We show that F_2 also is nowhere differentiable. Let $x \in \mathbb{R}$ be arbitrary. As before we construct sequences u_n and v_n of successive dyadic rationals (of the same order) such that $u_n \leq x \leq v_n$ (with $u_n < v_n$) and $v_n - u_n = 2^{-n}$. Then we show that

$$\frac{F_2(v_n) - F_2(u_n)}{v_n - u_n}$$

does not converge as $n \to \infty$ which implies that $F_2'(x)$ does not exist (by Lemma 3.8).
Let $\phi(x) = \inf_{m \in \mathbb{Z}} |x - m|$, then

$$F_2(x) = \sum_{k=1}^{\infty} \frac{1}{k!} \phi\left(2^{k!} x\right).$$

If $u \in \mathbb{D}$ is a dyadic rational of order n, then

$$F_2(u) = \sum_{k! < n} \frac{1}{k!} \phi\left(2^{k!} u\right).$$

Now,

$$\frac{F_2(v_n) - F_2(u_n)}{v_n - u_n} = \sum_{k! < n} \frac{1}{k!} \frac{\phi(2^{k!} v_n) - \phi(2^{k!} u_n)}{v_n - u_n}.$$

As before, $\phi(x)$ is linear for $x \in \left[2^{k!} u_n, 2^{k!} v_n\right]$. Hence, for $0 \leq k! < n$,

$$\frac{1}{k!} \frac{\phi(2^{k!} v_n) - \phi(2^{k!} u_n)}{v_n - u_n} = \pm \frac{2^{k!}}{k!} \to \pm\infty \text{ as } n \to \infty$$

which gives

$$\frac{F_2(v_n) - F_2(u_n)}{v_n - u_n} = \sum_{k! < n} \pm \frac{2^{k!}}{k!}.$$

This series does not converge as $n \to \infty$ so $F_2'(x)$ does not exist. Since $x \in \mathbb{R}$ was arbitrary, F_2 is nowhere differentiable. $\square$

3.10 Sierpiński curve (1912)

Wacław Sierpiński published another example of a space-filling curve in 1912 in his paper Sierpiński [70]. He found a bounded, continuous and even function S_W such that, for $t \in [0,1]$, the mapping

$$\begin{cases} x = S_W(t) \\ y = S_W(t - 1/4) \end{cases}$$

is surjective onto $[-1,1]$. Sierpiński deduced the following expression for S_W:

$$S_W(t) = \frac{\Theta(t)}{2}\left(1 + \sum_{k=1}^{\infty}(-1)^k \frac{\prod_{l=1}^{k}\Theta(\tau_l(t))}{2^k}\right)$$

where both Θ and τ are periodic functions with period 1 defined by

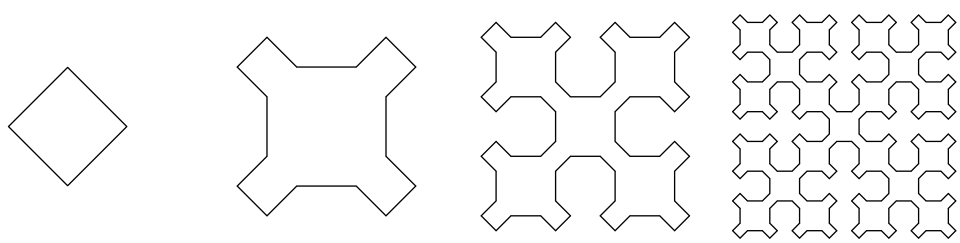
$$\Theta(t) = \begin{cases} -1 & \text{if } t \in [1/4, 3/4), \\ 1 & \text{if } t \in [0, 1/4) \cup [3/4, 1) \end{cases}$$

and

$$\tau_l(t) = \begin{cases} 1/8 + 4t & \text{if } t \in [0, 1/4) \cup [1/2, 3/4), \\ 1/8 - 4t & \text{if } t \in [1/4, 1/2) \cup [3/4, 1), \end{cases}$$

$$\tau_{l+1}(t) = \tau_l(\tau_1(t)), \text{ for every } l \in \mathbb{N}.$$

Moreover, he demonstrated that S_W is the limit of a sequence of polygonal curves, of which the first four can be seen graphically in figure 3.12.



(a) $n = 1$. (b) $n = 2$. (c) $n = 3$. (d) $n = 4$.

Figure 3.12: Polygonal approximations (of order n) to Sierpiński's curve.

3.11 Knopp function (1918)

Define the function $K : \mathbb{R} \to \mathbb{R}$ as

$$K(x) = \sum_{k=0}^{\infty} a^k \phi(b^k x),$$

where

$$\phi(x) = \inf_{m \in \mathbb{Z}} |x - m| = \operatorname{dist}(x, \mathbb{Z})$$

and $a \in (0,1)$, $ab > 4$ and $b > 1$ an even integer. ϕ is a "saw-tooth" function and can be seen graphically in figure 3.13.

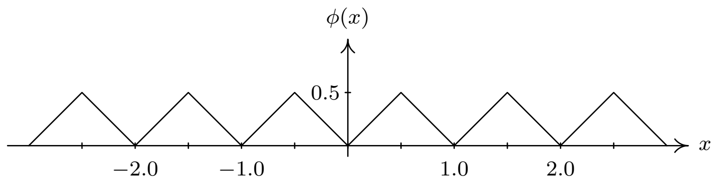


Figure 3.13: The "saw-tooth" function $\phi(x)$ on $[-3, 3]$.

K is the Knopp function which was introduced by Konrad Knopp in 1918 (cf. Knopp [38]). Both Takagi's and van der Waerden's functions are special cases of this function. Originally Knopp had the restrictions

$$0 < a < 1, \quad ab > 4 \quad \text{and} \quad b > 1 \text{ an even integer}$$

on the parameters but in an article published in 1994 Baouche and Dubuc [3] weakened the restrictions to

$$0 < a < 1, \qquad ab > 1$$

where b is not necessarily an integer. Further investigations were done on the case when $ab = 1$ and in another article published in 1994 by Cater [9], F.S. Cater proved[9] that

$$\sum_{k=0}^{\infty} b^{-n} \phi(b^n x)$$

is nowhere differentiable if $b \geq 10$.

Theorem 3.11. *The Knopp function*

$$K(x) = \sum_{k=0}^{\infty} a^k \phi(b^k x) = \sum_{k=0}^{\infty} a^k \operatorname{dist}\left(b^k x, \mathbb{Z}\right)$$

[9]Actually, both in Baouche and Dubuc [3] and in Cater [9] the results are proved in a more general case when an additional phase sequence $\{c_n\}$ is added to the argument, i.e. $\widehat{K}(x) = \sum_{k=0}^{\infty} a^n \phi(b^n x + c_n)$. Moreover, Cater actually proves that for non-zero sequences $\{a_n\}$ and $\{b_n\}$ with $b_n > 0$ and $|a_n b_n| = 1$, $\widehat{K}$ has no unilateral derivative if $b_{n+1} \geq 10 b_n$.

is continuous and nowhere differentiable on $\mathbb{R}$ for $a \in (0,1)$ and $ab > 1$.

Proof. We can establish the continuity of K similarly as for Weierstrass' function. In fact, if $0 < a < 1$ then $\sum_{k=0}^{\infty} a^k < \infty$ and since ϕ is a bounded function, $\sup_{x\in\mathbb{R}} |\phi(x)| \leq \frac{1}{2}$, it follows that $\sup_{x\in\mathbb{R}} |a^k \phi(b^k x)| \leq \frac{1}{2} a^k$. The Weierstrass M-test (Theorem 2.2) shows that K converges uniformly on $\mathbb{R}$. As before, the continuity of K now follows from the Corollary 2.4.
The proof of nowhere differentiability can be found in Baouche and Dubuc [3] for $ab > 1$ or in Knopp [38] for the original constraints $ab > 4$ and $b > 1$ an even integer. $\square$

3.12 Petr function (1920)

The Czech mathematician Karel Petr published, in 1920, a simple example of a continuous nowhere differentiable function. The Petr function $P_k : [0,1] \to \mathbb{R}$ in question is defined as follows. For any $x \in [0,1]$, let

$$x = \sum_{k=1}^{\infty} \frac{a_k}{10^k}, \qquad \text{where } a_k \in \{0,1,\dots,9\},$$

be a decimal expansion of x and define

$$P_K(x) = \sum_{k=1}^{\infty} \frac{c_k b_k}{2^k}$$

where $b_k = a_k \bmod 2$, $c_1 = 1$ and for $k \geq 2$

$$c_k = \begin{cases} -c_{k-1} & \text{if } a_{k-1} \in \{1,3,5,7\}, \\ c_{k-1} & \text{else.} \end{cases}$$

In the same year as Petr's function was published, another Czech mathematician, Karel Rychlík, gave a generalization (cf. Rychlík [60],[62]) where he carried over the definition from $\mathbb{R}$ to the ring[10] $\mathbb{Q}_p$ of p-adic numbers. For $x \in \mathbb{Q}_p$, that is,

$$x = \sum_{k=r}^{\infty} a_k p^k, \qquad \text{where } a_k \in \{0,1,\dots,p-1\},$$

[10] Or field when certain properties hold, like, for example, if p is a prime number.

we define the function f by

$$f(x) = \sum_{k=0}^{\infty} a_{r+2k} p^{r+2k}.$$

Rychlík proves in his papers that f is continuous and nowhere differentiable in $\mathbb{Q}_p$.

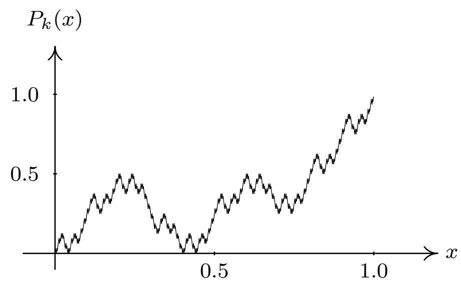


Figure 3.14: Petr's function in a 4-adic system.

Theorem 3.12. *The Petr function P_K is continuous and nowhere differentiable on $(0,1)$.*

Proof. Petr proved this result himself in Petr [56]. $\square$

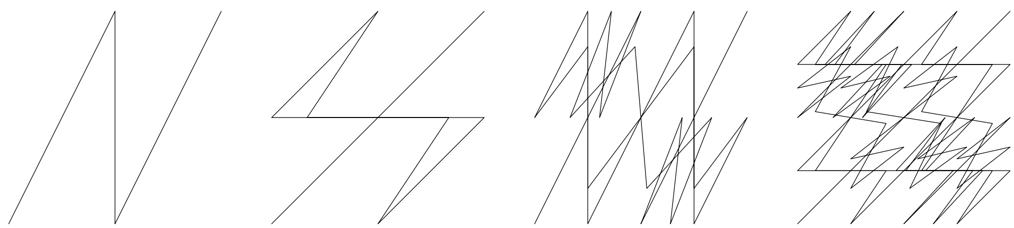
3.13 Schoenberg function (1938)

Schoenberg's two functions ϕ_s and ψ_s are defined as

$$\begin{cases} \phi_s(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} p(3^{2k}x), \\ \psi_s(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} p(3^{2k+1}x), \end{cases} \quad \text{where} \quad p(x) = \begin{cases} 0 & x \in [0, 1/3], \\ 3x - 1 & x \in [1/3, 2/3], \\ 1 & x \in [2/3, 4/3], \\ 5 - 3x & x \in [4/3, 5/3], \\ 0 & x \in [5/3, 2] \end{cases}$$

and $p(x+2) = p(x)$ for every $x \in \mathbb{R}$. Figure 3.16(a) gives a more intuitive description of the function p. Schoenberg's curve is actually another example of a space-filling curve (like Peano's curve, which was discussed earlier). The "space-filling" is accomplished by the parameterization (for $t \in [0, 1]$)

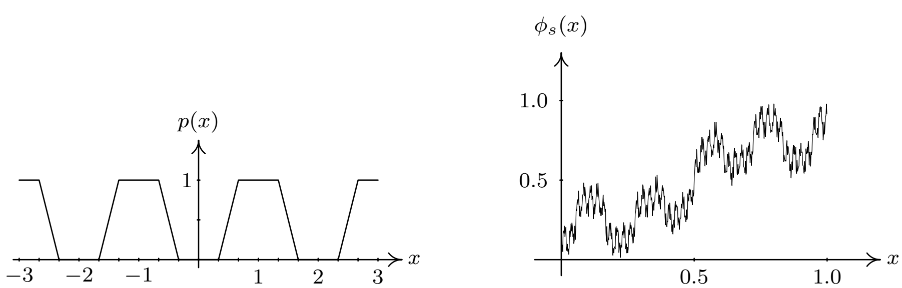
$$\begin{cases} x = \phi_s(t), \\ y = \psi_s(t). \end{cases}$$



(a) $n = 1$. (b) $n = 2$. (c) $n = 3$. (d) $n = 3$.

Figure 3.15: First four approximation polygons in the construction of Schoenberg's curve (sampled at $t_k = m/3^n$, $m = 0, 1, \ldots, 3^n$).

Schoenberg constructed the curve in 1938 as an extension of the same map that Henri Lebesgue had used in the construction of his space-filling function decades earlier. Schoenberg's curve resulted in a much easier proof of the continuity (that is, easier than for Lebesgue's case) and the curve also turned out to be nowhere differentiable (a fact proven later). For more discussion, see Sagan [64], pp, 119-130.



(a) p for $-3 \leq x \leq 3$. (b) Schoenberg's function ϕ_s.

Figure 3.16: Schoenberg's function ϕ_s and the auxiliary function p.

Theorem 3.13. *Both the Schoenberg functions ϕ_s and ψ_s are continuous and nowhere differentiable on the interval* $(0,1)$.

The proof of this theorem is based on Sagan's proof from [63].

Proof. The continuity of both ϕ_s and ψ_s follows immediately from the Weierstrass M-test (Theorem 2.2) and Corollary 2.4 since $\sup|(1/2^k)p(x)| \leq 1/2^k$. We turn to the nowhere differentiable part. Let $t \in (0,1)$ be arbitrary and assume that $\phi_s'(x)$ exists. By Lemma 3.8, if $0 < a_n < t < b_n < 1$, $a_n \to t$ and $b_n \to t$ then

$$\frac{\phi_s(b_n) - \phi_s(a_n)}{b_n - a_n} \to \phi_s'(x) \quad \text{as } n \to \infty.$$

We construct two sequences $\{a_n\}$ and $\{b_n\}$ that contradicts this.
Take $\hat{k}_n = [9^n t]$, where $[x]$ denotes the integer part of x, and put $\hat{a}_n = \hat{k}_n 9^{-n}$ and $\hat{b}_n = \hat{k}_n 9^{-n} + 9^{-n}$. Then $a_n \to t$ and $b_n \to t$ with $0 < a_n < t < b_n < 1$ for n large enough.
Now, infinitely many $\hat{k}_n$ are even, odd or both. We consider two cases.
(i) If there are infinitely many even $\hat{k}_n$ then take k_n as the corresponding subsequence of $\hat{k}_n$ (and the same subsequences a_n and b_n of $\hat{a}_n$ and $\hat{b}_n$ respectively). Then we have

$$\begin{aligned}
\phi_s(b_n) - \phi_s(a_n) &= \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(9^{k-n}k_n + 9^{k-n}) - \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(9^{k-n}k_n) \\
&= \frac{1}{2}\sum_{k=0}^{n-1} \frac{1}{2^k}\left(p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)\right) \\
&\quad + \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k}\left(p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n)\right) = M_1 + M_2.
\end{aligned}$$

For $0 \leq k < n$, $9^{k-n} \leq \frac{1}{9}$ and from the definition of $p(x)$ we can obtain the lower bound

$$p\left(9^{k-n}k_n + 9^{k-n}\right) - p\left(9^{k-n}k_n\right) \geq -3 \cdot 9^{k-n}.$$

This gives a lower bound for M_1 by

$$M_1 \geq -\frac{3}{2}\sum_{k=0}^{n-1} \frac{1}{2^k} 9^{k-n} = -\frac{3}{2 \cdot 9^n}\sum_{k=0}^{n-1}\left(\frac{9}{2}\right)^k = -\frac{3}{7 \cdot 9^n}\left(\left(\frac{9}{2}\right)^n - 1\right).$$

For $k \geq n$, $9^{k-n} \geq 1$ is odd which implies that $u_k = 9^{k-n}k_n$ is even and $v_k = 9^{k-n}k_n + 9^{k-n}$ is odd. Hence

$$M_2 = \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k}\left(p(v_k) - p(u_k)\right) = \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k}\left(1 - 0\right) = \frac{1}{2^n}.$$

Now,

$$\begin{aligned} \frac{\phi_s(b_n) - \phi_s(a_n)}{b_n - a_n} &= 9^n(M_1 + M_2) \geq 9^n\left(\frac{1}{2^n} - \frac{3}{7\cdot 9^n}\left(\left(\frac{9}{2}\right)^n - 1\right)\right) \\ &= \frac{4}{7}\left(\frac{9}{2}\right)^n + \frac{3}{7} \to \infty \text{ as } n \to \infty. \end{aligned}$$

(ii) If there are infinitely many odd $\hat{k}_n$ instead, we can define the corresponding subsequences k_n, a_n and b_n similarly as before but with the odd subsequence. Instead of a lower bound for M_1 we estimate by

$$M_1 \leq \frac{3}{2}\sum_{k=0}^{n-1} \frac{1}{2^k}9^{k-n} = \frac{3}{7\cdot 9^n}\left(\left(\frac{9}{2}\right)^n - 1\right)$$

and for $k \geq n$ we have $u_k = 9^{k-n}k_n$ odd and $v_k = 9^{k-n}k_n + 9^{k-n}$ even, which gives

$$M_2 = \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k}\left(p(v_k) - p(u_k)\right) = \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k}\left(0 - 1\right) = -\frac{1}{2^n}.$$

From this we get

$$\begin{aligned} \frac{\phi_s(b_n) - \phi_s(a_n)}{b_n - a_n} &\leq 9^n\left(\frac{3}{7\cdot 9^n}\left(\left(\frac{9}{2}\right)^n - 1\right) - \frac{1}{2^n}\right) \\ &= -\frac{4}{7}\left(\frac{9}{2}\right)^n - \frac{3}{7} \to -\infty \text{ as } n \to \infty. \end{aligned}$$

Hence $\phi_s'(t)$ does not exist. Since $t \in (0,1)$ was arbitrary, ϕ_s is nowhere differentiable on $(0,1)$. Moreover, since $\psi_s(t) = \phi_s(3t)$ it is clear that ψ_s is nowhere differentiable on $(0,1)$ as well. □

Remark. It can quite easily be shown, with similar technique, that both ϕ_s and ψ_s lacks derivative at both $t = 0$ and $t = 1$ as well (cf. Sagan [63]).

3.14 Orlicz functions (1947)

In 1947 Polish mathematician Władysław Orlicz [50] put forth a slightly different approach to continuous functions without derivative. Instead of going for a pure existence proof or a direct construction he presents a form of an intermediate result in terms of both a fairly general construction and Baire category. Orlicz did more research in this area and in two of his subsequent papers (cf. Orlicz [51],[52]) he dealt with a more general form of Lipschitz conditions from which several new constructions of continuous nowhere differentiable functions sprang to life.

Let $\{f_n\}$ be a sequence of functions $f_n : [a,b] \to \mathbb{R}$ for which $\sum_{n=1}^{\infty} |f_n(x)|$ converges uniformly on $[a,b]$ and let $\mathcal{N}$ be the metric space of all sequences $\eta = \{\eta_n\}$, $\eta_n \in \{0,1\}$ for every $n \in \mathbb{N}$, with the metric d defined by

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x_k - y_k|.$$

It can be seen that $(\mathcal{N}, d)$ is complete, and by Baire's category theorem (Theorem 4.2) it is therefore of the second category in itself. The metric space terminology used here can be reviewed in Section 4.1.

We define the first Orlicz function as:

$$O_1(x) = \sum_{n=1}^{\infty} \eta_n f_n(x).$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function with period $l = b - a$ and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive numbers such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\beta_1 < \beta_2 < \cdots < \beta_n \to \infty$.

We define the second Orlicz function as:

$$O_2(x) = \sum_{n=1}^{\infty} \eta_n \alpha_n \varphi(\beta_n x).$$

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous, periodic function satisfying a Lipschitz condition (on $\mathbb{R}$) and let $s > 1$ be a real number. Let also α and β be real numbers such that $\alpha \in (0,1)$ and $\alpha\beta > 1$. We define the third and fourth Orlicz functions as:

$$O_3(x) = \sum_{n=1}^{\infty} \frac{1}{2^{n^2}} \psi(2^{sn^2} x) \qquad \text{and} \qquad O_4(x) = \sum_{n=1}^{\infty} \alpha^n \psi(\beta^n x).$$

Theorem 3.14.

(i) If

(a) $f_n'^{+}(x)$ *exists for every* $x \in [a,b)$ *and is continuous except for possibly on a finite subset of* $[a,b)$*;*

(b) there exists $\lambda > 0$ *and a sequence* $\{\delta_n\}$ *of positive numbers with* $\delta_n \to 0$ *such that*

$$\left| \frac{f_n(x+h) - f_n(x)}{h} - f_n'^{+}(x) \right| > \lambda \qquad \text{for every } x \in [a',b'] \subset [a,b)$$

for some h *(possibly dependent on* x *and* n*) with* $0 < h < \delta_n$ *and* $h < b - x$*,*

then O_1 *has no right-hand derivative at any* $x \in [a',b']$ *for any* η *in a residual subset of* $\mathcal{N}$*.*

(ii) If

(a) f_n *satisfies a Lipschitz condition on* $[a,b]$*;*

(b) There exists a real sequence $\{k_n\}$ *with* $k_n \to \infty$ *such that*

$$\left| \frac{f_n(x+h) - f_n(x)}{h} \right| \geq k_n$$

is satisfied for some h *(possibly dependent on* x*) with* $0 < h < b - x$*,*

then for any η *in a residual subset of* $\mathcal{N}$ *we have*

$$\limsup_{h \to 0^+} \left| \frac{O_1(x+h) - O_1(x)}{h} \right| = \infty$$

which implies that there exist no right-hand derivative.

(iii) If φ *is a non-constant function with a continuous derivative everywhere and* $\alpha_n \beta_n > c > 0$ *for every* $n \in \mathbb{N}$*, then for each* η *in a residual subset of* $\mathcal{N}$*,* O_2 *has no right-hand derivative.*

(iv) The third Orlicz function O_3 *is continuous and nowhere differentiable.*

(v) *If K is the Lipschitz constant for ψ and*

$$0 < \alpha < \frac{1}{1 + \frac{4\sigma \max|\varphi(x)|}{cr\tau}}, \qquad \alpha\beta > 1 + \frac{2K\sigma}{(1-c)r},$$

where $c \in (0,1)$ is arbitrary and σ, τ and r are suitable[11] *real numbers, then the fourth Orlicz function O_4 is continuous and nowhere differentiable.*

Proof. The proof of (i) can be found in the proof of Theorem 7 in Orlicz [50], (ii) in the proof of Theorem 8 and (iii) in the proof of Theorem 9. (iv) and (v) are proven in Section 4 of Orlicz [51]. □

Remark 1. With $\psi(x) = \cos(x)$, $r = 2$, $K = \pi$, $\sigma = 1$, $c = 1/2$ and $\tau = 1/2$ in (v) above we get Weierstrass function W but with slightly different conditions on α and β.

Remark 2. Orlicz also considered series with sequences ϵ from a metric space $\mathcal{E}$ consisting of all sequences $\epsilon = \{\epsilon_n\}$, where $\epsilon_n \in \{-1, 1\}$, with the same metric as for $(\mathcal{N}, d)$.

Remark 3. In Orlicz [50], Orlicz also gave measure theoretic results on the differentiability of especially O_1 and O_2. The term "almost every" when applied to the metric spaces $\mathcal{E}$ and $\mathcal{N}$ has the following meaning: Let $\epsilon_n(t) = \text{sgn}[\sin(2^n\pi t)]$ (the Rademacher system) and $\eta_n(t) = \frac{1}{2}(1 - \epsilon_n(t))$. Both $\{\epsilon_n(t)\}$ and $\{\eta_n(t)\}$ are orthonormal sequences in $[0,1]$. By neglecting a countable set in $\mathcal{E}$ (or $\mathcal{N}$) and a countable set in the interval $[0,1]$ there is a bijective mapping between these two sets. One says "for almost every" sequence in $\mathcal{E}$ (or $\mathcal{N}$) if the set of numbers from $[0,1]$ for which the sequence $\{\epsilon_n(t)\}$ (or $\{\eta_n(t)\}$) does not have this property is of measure zero.
With this terminology, Orlicz stated and proved the following:

(i) If $f_n'^{+}(x)$ exists for almost every $x \in [a,b]$ and

$$\limsup_{h \to 0^+} \sum_{n=1}^{\infty} \left(\frac{f_n(x+h) - f_n(x)}{h} - f_n'^{+}(x) \right)^2 > 0$$

for almost every $x \in [a,b]$, then both

$$O_{1,\epsilon}(t,x) = \sum_{n=1}^{\infty} \epsilon_n(t) f_n(x) \quad \text{and} \quad O_{1,\eta}(t,x) = \sum_{n=1}^{\infty} \eta_n(t) f_n(x)$$

[11] See Orlicz [51], the existence of suitable constants are given by an existence proof.

have no right-hand derivative almost everywhere for almost every $t \in [0, 1]$.

(ii) If φ is also absolutely continuous,

$$0 < \int_{[a,b]} [\varphi'(x)]^2 \, dx < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n^2 \beta_n^2 = \infty,$$

then both

$$O_{2,\epsilon}(t, x) = \sum_{n=1}^{\infty} \epsilon_n(t) \alpha_n \varphi(\beta_n x) \quad \text{and} \quad O_{2,\eta}(t, x) = \sum_{n=1}^{\infty} \eta_n(t) \alpha_n \varphi(\beta_n x)$$

have no derivative almost everywhere for almost every $t \in [0, 1]$.

3.15 McCarthy function (1953)

McCarthy's function M is defined as the infinite series

$$M(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} g\left(2^{2^k} x\right),$$

where

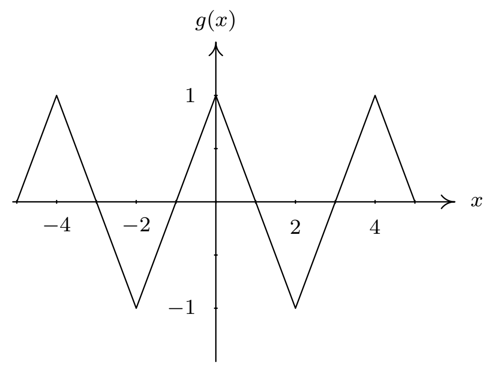
$$g(x) = \begin{cases} 1 + x, & x \in [-2, 0] \\ 1 - x, & x \in [0, 2] \end{cases}$$

and $g(x + 4) = g(x)$ for any $x \in \mathbb{R}$.
John McCarthy [47] writes that this function has the easiest proof of continuity and nowhere differentiabillity of any such function he has seen and I'm inclined to agree that the proof indeed is one of the shorter I have seen.

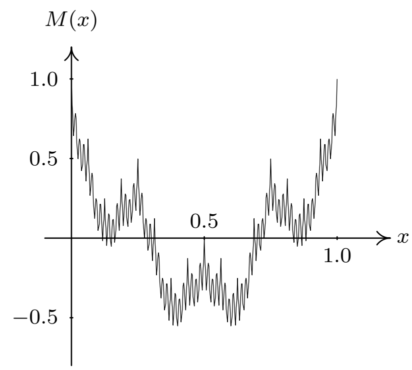
Theorem 3.15. *The McCarthy function*

$$M(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} g\left(2^{2^k} x\right), \qquad \textit{where } g(x) = \begin{cases} 1 + x, & x \in [-2, 0] \\ 1 - x, & x \in [0, 2] \end{cases}$$

and $g(x + 4) = g(x)$ for any $x \in \mathbb{R}$, is continuous and nowhere differentiable on $\mathbb{R}$.



(a) $g(x)$ for $-5 \leq x \leq 5$.



(b) McCarthy's function M.

Figure 3.17: McCarthy's function M and the auxiliary function $g(x)$.

Proof. First we show that M is continuous on $\mathbb{R}$. Obviously g is continuous and since $\sup_{x\in\mathbb{R}} |2^{-k}g(2^{2^k}x)| = 2^{-k}$ with $\sum_{k=1}^{\infty} 2^{-k} < \infty$ it follows from the Weierstrass M-test (Theorem 2.2) and the Corollary 2.4 that M is continuous. Secondly, we show that M is nowhere differentiable on $\mathbb{R}$. Let $x \in \mathbb{R}$ be arbitrary but fixed and let $n \in \mathbb{N}$ be arbitrary. Choose $h_n = \pm 2^{-2^n}$ where the sign is chosen such that x and $x + h_n$ are on the same linear segment of $g(2^{2^n}x)$.
Let $k \in \mathbb{N}$, for $k > n$ we have

$$g\left(2^{2^k}(x + h_n)\right) - g\left(2^{2^k}x\right) = g\left(2^{2^k}x\right) - g\left(2^{2^k}x\right) = 0$$

since g has period 4 and $2^{2^k}h_n = 4q$ for some $q \in \mathbb{Z}$.
For $k = n$ we obtain

$$\left|g\left(2^{2^n}(x + h_n)\right) - g\left(2^{2^n}x\right)\right| = \left|g\left(1 + 2^{2^n}x\right) - g\left(2^{2^n}x\right)\right| = 1.$$

For $k < n$ we can estimate by

$$\sup_{k=1,\dots,n-1}\left|g\left(2^{2^k}(x + h_n)\right) - g\left(2^{2^k}x\right)\right| \leq 2^{2^{n-1}}2^{-2^n} = 2^{-2^{n-1}}$$

and therefore

$$\left|\sum_{k=1}^{n-1} 2^{-k}\left(g\left(2^{2^k}(x + h_n)\right) - g\left(2^{2^k}x\right)\right)\right| \leq (n-1)2^{-2^{n-1}} < 2^n 2^{-2^{n-1}} \leq 1.$$

Hence,

$$\begin{aligned}\left|\frac{M(x+h_n)-M(x)}{h_n}\right| &= 2^{2^n}\left|\sum_{k=1}^{\infty} 2^{-k}\left(g\left(2^{2^k}(x+h_n)\right)-g\left(2^{2^k}x\right)\right)\right| \\ &= 2^{2^n}\left|\sum_{k=1}^{n} 2^{-k}\left(g\left(2^{2^k}(x+h_n)\right)-g\left(2^{2^k}x\right)\right)\right| \\ &\geq 2^{2^n}\left(1-\left|\sum_{k=1}^{n-1} 2^{-k}\left(g\left(2^{2^k}(x+h_n)\right)-g\left(2^{2^k}x\right)\right)\right|\right) \\ &\geq 2^{2^n}\left(1-2^n 2^{-2^{n-1}}\right) = 2^{2^{n-1}}\left(2^{2^{n-1}}-2^n\right) \\ &\to \infty \text{ as } n\to\infty.\end{aligned}$$

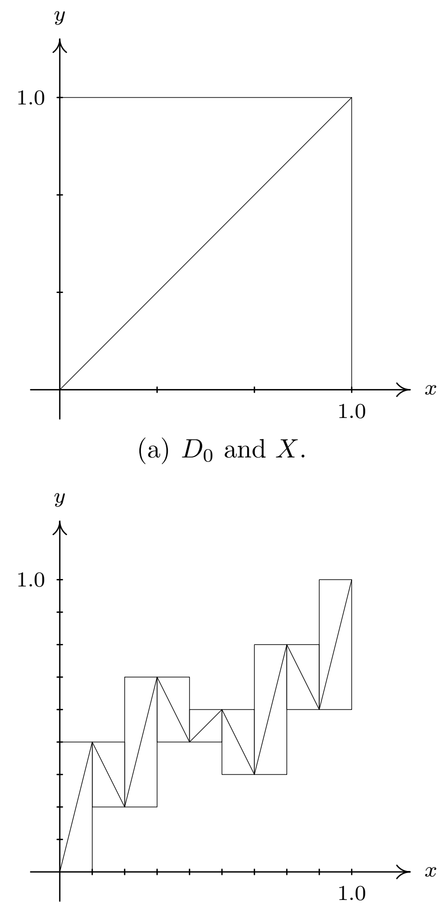
It is now clear that the function M cannot be differentiable at x and since $x \in \mathbb{R}$ was arbitrary it follows that M is nowhere differentiable. □

3.16 Katsuura function (1991)

Hidefumi Katsuura claims in his paper Katsuura [37], published in 1991, that he got the idea for this function when attending a master's thesis defense about attractors of contraction mappings. We construct the function as follows. Let $X = [0,1]\times[0,1]$ be the closed unit square and let $F(X)$ by the collection of all non-empty closed subsets of X. For $i = 1,2,3$, define the mappings $T_i : X \to X$ by

$$\begin{aligned} T_1(x,y) &= \left(\frac{x}{3}, \frac{2y}{3}\right), \\ T_2(x,y) &= \left(\frac{2-x}{3}, \frac{1+y}{3}\right) \text{ and} \\ T_3(x,y) &= \left(\frac{2+x}{3}, \frac{1+2y}{3}\right). \end{aligned}$$

We define the mapping $T : F(X) \to F(X)$ by $T(A) = T_1(A)\cup T_2(A)\cup T_3(A)$. Let $D_0 = \{(x,x) \in X\}$ (i.e. the diagonal) and for $n \in \mathbb{N}$ define $D_n = T(D_{n-1})$. Each D_n is the graph of a function $K_n : [0,1] \to [0,1]$ and Hidefumi Katsuura's function $K_H : [0,1] \to [0,1]$ is the function whose graph D is the limit of this process. Figure 3.18 shows a few steps of graphically.



(a) D_0 and X.

(b) D_1 and $T(X)$.

(c) D_2 and $T^2(X)$.

(d) D_3 and $T^3(X)$.

Figure 3.18: The graphs of the first four "iterations" of the Katsuura function and the corresponding mappings of X (the rectangles).

A few things about this construction are interesting to consider. It can be shown that the metric space $(F(X), d_H)$ is complete (where d_H is the Hausdorff[12] metric induced by the Euclidean metric) and also that T is a contraction[13] mapping on this space (again with respect to the Hausdorff metric). Since these properties are fulfilled, Banach's fixed point theorem implies that T has a unique fixed point in $F(X)$ and no matter what set $A \in F(X)$ we start with, the sequence $\{T^n(A)\}$ always converge to D in the Hausdorff metric.

Theorem 3.16. *The Katsuura function K_H is continuous and nowhere differentiable on the interval $(0, 1)$.*

The proof is based on Hidefumi's proof in Katsuura [37].

Proof. First we show that K_H is continuous. For $m \leq n$, $D_n \subset T^m(X)$ and $T^m(X)$ is the union of 3^m rectangles of height bounded by $(2/3)^m$. Hence

$$\sup_{x\in[0,1]} |K_m(x) - K_n(x)| \leq \left(\frac{2}{3}\right)^m \to 0 \text{ as } m, n \to \infty.$$

Thus the sequence $\{K_n\}$ of functions is uniformly Cauchy and therefore the convergence is uniform by Theorem 2.1. Since each function K_n obviously is continuous it follows by Theorem 2.3 that the limit function K_H also is continuous

We show that the function K_H is nowhere differentiable on $(0, 1)$ in two steps.
(i) For $x \in (0, 1)$ when x is a ternary rational which has a finite ternary representation. Then, for some $n \in \mathbb{N}$,

$$x = \sum_{k=1}^{n} \frac{x_k}{3^k}, \qquad \text{where } x_k \in \{0, 1, 2\} \text{ and } x_n \neq 0.$$

Let the sequence $\{y_k\}$ be defined by $y_k = x + 3^{-(n+k)}$ for $k \in \mathbb{N}$. Then $y_k \to x$ and $y_k - x = 3^{-(n+k)}$.

[12]The Hausdorff metric d_H can be defined by

$$d_H(A, B) = \max\left\{\sup_{x\in A}\left(\inf_{y\in B} d(x, y)\right), \sup_{y\in B}\left(\inf_{x\in A} d(x, y)\right)\right\},$$

where d is the Euclidean metric (in our case).

[13]Meaning that $\exists\alpha \in (0, 1)$ such that $\forall x, y \in F(X)$ $d_H(Tx, Ty) \leq \alpha d_H(x, y)$.

We claim that

$$\left|\frac{K_H(y_k)-K_H(x)}{y_k-x}\right| \geq 2^{k-1} \qquad \text{for all } k\in\mathbb{N} \tag{3.5}$$

which will imply that K_H has no derivative at x.
The proof of the claim is by induction on k. For $k=1$ we have, by the geometry of the construction,

$$\left|\frac{K_H(y_1)-K_H(x)}{y_1-x}\right| = 3^{n+1}|K_H(y_1)-K_H(x)| \geq 3^{n+1}\frac{1}{3^{n+1}} = 1.$$

Assume that equation (3.5) holds for $k=q$. The geometry of the construction implies that

$$\begin{cases} |K_H(y_{q+1})-K_H(x)| & = \dfrac{2}{3}|K_H(y_q)-K_H(x)|, \\ |y_{q+1}-x| & = \dfrac{1}{3}|y_q-x|, \end{cases} \tag{3.6}$$

which gives

$$\left|\frac{K_H(y_{q+1})-K_H(x)}{y_{q+1}-x}\right| = \frac{(2/3)|K_H(y_q)-K_H(x)|}{(1/3)|y_q-x|} \geq 2\cdot 2^{q-1} = 2^{(q+1)-1}$$

and this completes the proof of the claim.
(ii) Now, if $x\in(0,1)$ isn't a ternary rational with a finite ternary representation, then

$$x=\sum_{k=1}^{\infty}\frac{x_k}{3^k} \qquad \text{where infinitely many } x_k \text{ are non-zero.}$$

We choose two sequences $\{y_n\}$ and $\{z_n\}$ of ternary rationals with finite ternary representation such that $y_n < x < z_n$ and $z_n - y_n = 3^{-(n+q)}$ for some $q\in\mathbb{N}$. We take q as the smallest element in $\mathbb{N}$ for which there exists natural numbers $r_1\leq q$ and $r_2\leq q$ for which $x_{r_1}\neq 2$ and $x_{r_2}\neq 0$ (these elements exists since $0<t<1$ and are necessary to ensure that our sequences will satisfy $y_n>0$ and $z_n<1$). Define the sequences by

$$y_n=\sum_{k=1}^{n+q}\frac{x_k}{3^k} \qquad \text{and} \qquad z_n = y_n + \frac{1}{3^{n+q}}.$$

Then both y_n and z_n are ternary rationals with $n+q$ digits and clearly $0 < y_n < t < z_n < 1$. Moreover, that $z_n - y_n = 3^{-(n+q)}$ is immediate from the definition of the sequences.
We claim that

$$\left| \frac{K_H(z_n) - K_H(y_n)}{z_n - y_n} \right| \geq 1 \qquad \text{for all } n \in \mathbb{N} \tag{3.7}$$

and prove this by induction. For $n = 1$ we have

$$\left| \frac{K_H(z_1) - K_H(y_1)}{z_1 - y_1} \right| = 3^{q+1} |K_H(z_1) - K_H(y_1)| \geq 3^{q+1} \frac{1}{3^{q+1}} = 1$$

by the construction of K_H. Assume that equation (3.7) holds for $n = k$. We consider two cases. First, if $z_k = z_{k+1}$ or if $y_k = y_{k+1}$ it follows, similarly as for equation (3.6), that

$$\left| \frac{K_H(z_{k+1}) - K_H(y_{k+1})}{z_{k+1} - y_{k+1}} \right| = \frac{(2/3)|K_H(z_k) - K_H(y_k)|}{(1/3)|z_k - y_k|} \geq 1.$$

where the last inequality is the induction assumption. Secondly, if $z_k \neq z_{k+1}$ and $y_k \neq y_{k+1}$, then the geometry of the construction implies that

$$\begin{cases} K_H(z_{k+1}) - K_H(y_{k+1}) & = \dfrac{-1}{3}(K_H(z_k) - K_H(y_k)), \\ z_{k+1} - y_{k+1} & = \dfrac{1}{3}(z_k - y_k) \end{cases} \tag{3.8}$$

and thus

$$\left| \frac{K_H(z_{k+1}) - K_H(y_{k+1})}{z_{k+1} - y_{k+1}} \right| = \frac{(1/3)|K_H(z_k) - K_H(y_k)|}{(1/3)|z_k - y_k|} \geq 1$$

by the induction assumption.
Since x is not a ternary rational we must have $z_k \neq z_{k+1}$ and $y_k \neq y_{k+1}$ for infinitely many k. Taking this subsequence of $\{y_k\}$ and $\{z_k\}$ (with the same index to avoid sub-subscripts) it follows from equation (3.8) that

$$\frac{K_H(z_{k+1}) - K_H(y_{k+1})}{z_{k+1} - y_{k+1}} = -\frac{K_H(z_k) - K_H(y_k)}{z_k - y_k}.$$

Hence the only possible limit would be zero, but by equation (3.7) this is not possible. Thus the limit

$$\lim_{n\to\infty} \frac{K_H(z_n) - K_H(y_n)}{z_n - y_n}$$

does not exist and therefore K_H has no derivative at x since this would contradict Lemma 3.8. □

3.17 Lynch function (1992)

In an article from 1992 (Lynch [43]), Mark Lynch presented a function which is continuous and nowhere differentiable by using a topological argument. As a result, no theorems involving infinite series and uniform convergence were needed.
We define the mapping $T : \mathbb{R}^2 \to \mathbb{R}$ by $(x, y) \mapsto x$ (i.e. the projection on the first coordinate). For any $x \in \mathbb{R}$ and any $A \subset \mathbb{R}^2$ let $A[x] = \{y \mid (x, y) \in A\}$. We will define a sequence $\{C_n\}$ of compact sets with $C_{n+1} \subset C_n \subset \mathbb{R}^2$ for all $n \in \mathbb{N}$ such that

(i) $T(C_n) = [0, 1]$ for all $n \in \mathbb{N}$;

(ii) $\mathrm{diam}(C_n[x]) < 1/n$ for each $x \in [0, 1]$ and $n \in \mathbb{N}$;[14]

(iii) for each $x \in [0, 1]$ there exists $y \in [0, 1]$ with $0 < |x - y| < 1/n$ such that $p \in C_n[x]$ and $q \in C_n[y]$ implies that

$$\left| \frac{p - q}{x - y} \right| > n.$$

We choose the elements in the sequence $\{C_n\}$ as the closures of band neighborhoods of the graph of polygonal arcs defined on $[0, 1]$. It is quite easy to see that (i) and (ii) holds (the first is trivial and the second one can be obtained by choosing the thickness of the bands appropriately to compensate

[14] The diameter $\mathrm{diam}(A)$ of a set A is defined by

$$\mathrm{diam}(A) = \sup_{x,y\in A} d(x, y).$$

for the steepnes of each segment). To show that (iii) will hold, we start by considering a linear segment of a polygonal arc.
Let $f(x) = mx + b$ with $|m| > n$ and let both $\delta > 0$ and $x \in [0, 1]$ be arbitrary. If $m > n$ (the other case, when $m < -n$, is handled similarly) we choose $y = x+\delta$ and take a band neighborhood $N_\epsilon(f)$ of the graph of f. For $p \in \overline{N_\epsilon(f)[x]}$ and $q \in \overline{N_\epsilon(f)[y]}$ it is obvious from figure 3.19 that $|p-q|/|x-y|$

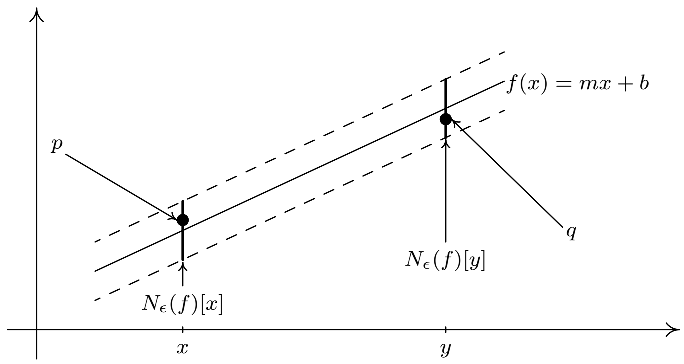


Figure 3.19: Line segment with band neighborhoods for Lynch's function.

is the absolute value of the slope of the line between the points (x, p) and (y, q). The minimum of this slope is attained when we choose $p = mx+b+\epsilon$ and $q = m(x+\delta)+b-\epsilon$ and for this case we can choose $\epsilon > 0$ small enough so that

$$\left|\frac{p-q}{x-y}\right| = \left|m - \frac{2\epsilon}{\delta}\right| > n$$

since $m > n$ by assumption.
So, assuming that C_{n-1} is constructed, we construct C_n in the following manner. First, take a polygonal arc P in the interior of C_{n-1} where each segment P_n has a slope whose absolute value exceeds n. For each $i \in \{0, 1, \ldots k\}$ choose δ_i such that

$$0 < \delta_i < \min\left\{\frac{|T(P_i)|}{2}, \frac{1}{n}\right\}$$

where $|T(P_i)|$ is the length of the interval $T(P_i)$. From our result above for linear segments we get an ϵ_i-neighborhood for each P_i (and since we have $\delta_i < |T(P_i)|/2$ we can always choose $y \in T(p_i)$). Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_k\}$, then $\overline{N_\epsilon(P)}$ is a closed neighborhood of P that clearly satisfies (iii). Furthermore, if we should happen to be unlucky enough so that $\overline{N_\epsilon(P)} \not\subset C_{n-1}$ we can

always choose a smaller $\epsilon > 0$ so that both $\overline{N_\epsilon(P)} \subset C_{n-1}$ and (i)-(iii) are satisfied. We take $C_n = \overline{N_\epsilon(P)}$.
In other words, we construct a sequence of bands that get steeper and narrower for each element in the sequence. This results in a "zig-zag" pattern which is transferred onto our function.

Theorem 3.17. *The sequence $\{C_n\}$ defines a continuous function $L : [0,1] \to \mathbb{R}$ that is nowhere differentiable on the interval $[0,1]$.*

Proof. Let $C = \bigcap_n C_n$. Since $\mathrm{diam}(C_n[x]) < 1/n$ for any $x \in [0,1]$ it follows that $\mathrm{diam}(C[x]) = 0$ for any $x \in [0,1]$ as well. Hence C is the graph of a well-defined function $L : [0,1] \to \mathbb{R}$. Since each C_n is compact (and non-empty) and C is a nested intersection of $\{C_n\}$, it is clear that also C must be compact (and non-empty). Thus the graph of L is compact and therefore L is continuous.
We prove that L is also nowhere differentiable. Let both $x \in [0,1]$ and $\delta > 0$ be arbitrary. We can choose $n \in \mathbb{N}$ so that $1/n < \delta$. By (iii) there exists $y \in [0,1]$ with $0 < |x-y| < 1/n$ such that $p \in C_n[x]$ and $q \in C_n[y]$ implies that

$$\left|\frac{p-q}{x-y}\right| > n.$$

Since $L(x) \in C_n[x]$ and $L(y) \in C_n[y]$ the difference quotient

$$\left|\frac{L(x)-L(y)}{x-y}\right|$$

is unbounded (as we let $\delta \to 0$). Hence L is not differentiable at x. □

3.18 Wen function (2002)

The Chinese mathematician Liu Wen has during the last few years proposed several continuous nowhere differentiable functions. One of these is an interesting function that is based on an infinite product instead of a series (Wen [80]). Let $W_L : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$W_L(x) = \prod_{n=1}^{\infty} (1 + a_n \sin(b_n \pi x)),$$

where the parameters a_n and b_n are chosen such that $0 < a_n < 1$ for all n,

$$\sum_{k=1}^{\infty} a_k < \infty \qquad \text{and} \qquad b_n = \prod_{k=1}^{n} p_k,$$

and p_k is an even integer for all $k \in \mathbb{N}$. Moreover, we require that

$$\lim_{n\to\infty} \frac{2^n}{a_n p_n} = 0.$$

We show later that W_L is both continuous and nowhere differentiable.

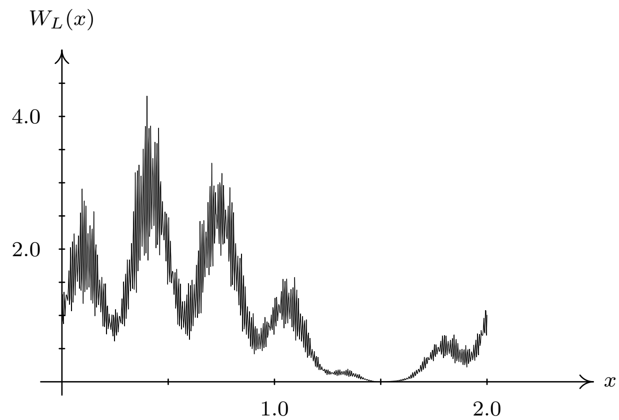


Figure 3.20: Wen's function W_L with $a_n = 2^{-n}$ and $p_n = 6^n$ for $x \in [0, 2]$.

In two other articles, from 2000 (Wen [78]) and 2001 (Wen [79]), Liu Wen presented two other continuous nowhere differentiable functions. Both are based on an expansion of the real numbers in $[0, 1]$ in a different base than the usual base-10 representation.

In the first article the base-b representation is used (where $b \in \mathbb{N} \setminus \{1\}$). The construction of this function is done as follows. Let $b \geq 2$ be an integer and for $x \in [0, 1]$ let $(x_1 x_2 \cdots)_b$ be the base-b expansion of x, i.e.

$$x = \sum_{k=1}^{\infty} \frac{x_k}{b^k} \qquad \text{where } x_k \in \{0, 1, \ldots, b-1\}.$$

Let $\lambda > 1$ be a real number and define the sequence $\{u_n\}$ by $u_1 = 1$ and for $n > 1$ let

$$u_n = \begin{cases} u_{n-1} & \text{if } x_n = x_{n-1}, \\ \phi(u_{n-1}) & \text{if } x_n \neq x_{n-1} \end{cases}$$

where ϕ is a function that is chosen so that f_1 is continuous and nowhere differentiable when $f_1 : [0, 1] \to \mathbb{R}$ is defined by

$$f_1(x) = \sum_{k=1}^{\infty} \frac{u_k}{\lambda^k}.$$

In this article, Liu choose $\phi(u) = (1 - \lambda)(u - c)$ where c is any real constant. This article also presents a proof that f_1 is right-continuous but lacks a finite right-hand derivative (which can be extended similarly to the left-hand side). In figure 3.21(a) an example (with fixed parameters) of f_1 is shown graphically.
The second article uses the Cantor series representation to construct a function. Let $q_n \geq 2$ be an integer for all n and let $x \in [0, 1]$. Then the Cantor series expansion of x is defined as

$$x = \sum_{n=1}^{\infty} \frac{x_n}{q_1 q_2 \cdots q_n} \qquad \text{where } x_n \in \{0, 1, \ldots, q_n - 1\}.$$

The function $f_2 : [0, 1] \to \mathbb{R}$ is expressed as

$$f_2(x) = \sum_{n=1}^{\infty} \frac{u_n}{n(n+1)},$$

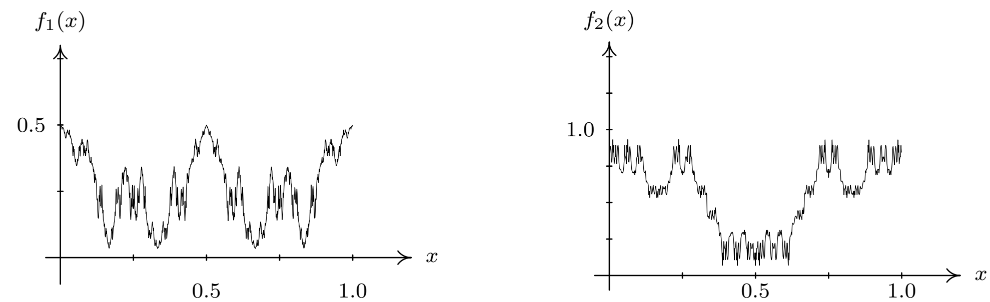
where the sequence $\{u_n\}$ is defined as $u_1 = 1$ and for $n \in \mathbb{N}$

$$u_{n+1} = \begin{cases} -\dfrac{u_n}{n}, & \begin{array}{l} \text{if } (x_{n+1} = 0 \text{ and } x_n \neq 0) \\ \text{or if } (x_{n+1} = q_{n+1} - 1 \text{ and } x_n \neq q_n - 1), \end{array} \\ u_n, & \text{else.} \end{cases}$$

In Wen [79] it is shown that f_2 is well-defined and is right continuous but lacks right-hand derivative on $[0, 1)$ (the argument can be done similarly for the left-hand side).
We turn to prove that the function that was based on an infinite product, W_L, is continuous and nowhere differentiable on $\mathbb{R}$.

Theorem 3.18. *The Wen function W_L is continuous and nowhere differentiable on $\mathbb{R}$.*

The proof follows Liu Wen's proof in Wen [80] but is a bit more explicit.



(a) f_1 with $b = 3$, $\lambda = 3$ and$c = 1/10$.

(b) f_2.

Figure 3.21: Two of Liu Wen's functions with $0 \le x \le 1$.

Proof. We start by establishing the continuity of W_L. The following well-known inequality is needed,

$$\frac{x}{1+x} \le \ln(1+x) \le x \quad \text{if } x > -1. \tag{3.9}$$

Let $a = \max_{n\ge1} a_n$. By the restrictions on a_n it is clear that $0 < a < 1$ so from the inequality (3.9) we get

$$\begin{aligned} |\ln(1 + a_n \sin(b_n\pi x))| &\le a_n|\sin(b_n\pi x)| \max\left\{\frac{1}{|1 + a_n \sin(b_n\pi x)|}, 1\right\} \\ &\le a_n \max\left\{\frac{1}{1-a}, 1\right\} \le \frac{a_n}{1-a}. \end{aligned}$$

Since $\sum_{k=1}^{\infty} a_k < \infty$ it follows from the Weierstrass' M-test (Theorem 2.2) and the Corollary 2.4 that

$$\sum_{k=1}^{\infty} \ln(1 + a_n \sin(b_n\pi x))$$

converges to a continuous function and therefore

$$W_L(x) = \prod_{n=1}^{\infty}(1 + a_n \sin(b_n\pi x)) = \exp\left(\sum_{k=1}^{\infty} \ln(1 + a_n \sin(b_n\pi x))\right)$$

is also continuous.

We turn to prove that W_L is nowhere differentiable. For every $x \in \mathbb{R}$ there exists a sequence $\{N_n\}$ with $N_n \in \mathbb{Z}$ such that

$$x \in \left[\frac{N_n}{b_n}, \frac{N_n+1}{b_n}\right) \qquad \text{for all } n \in \mathbb{N}.$$

Define the sequences $\{y_n\}$ and $\{z_n\}$ by

$$y_n = \frac{N_n+1}{b_n} \qquad \text{and} \qquad z_n = \frac{N_n+3/2}{b_n}.$$

Clearly $x < y_n < z_n$ and $0 < z_n - x < 3/(2b_n)$. Moreover, $z_n - y_n = 1/(2b_n)$ and also

$$z_n - x = \frac{N_n+3/2}{b_n} - x \leq \frac{N_n+3/2}{b_n} - \frac{N_n}{b_n} = \frac{3}{2b_n} = 3(z_n - y_n).$$

From the relations above we have the inequalities

$$z_n - y_n \geq \frac{1}{3}(z_n - x) > \frac{1}{3}(y_n - x). \tag{3.10}$$

We define a, b and $L_n : \mathbb{R} \to \mathbb{R}$ as

$$a = \prod_{k=1}^{\infty}(1-a_k), \quad b = \prod_{k=1}^{\infty}(1+a_k) \quad \text{and} \quad L_n(x) = \prod_{k=1}^{n}(1 + a_k \sin(b_k \pi x)).$$

We will consider the expression

$$\Delta_n = W_L(z_n) - W_L(y_n) = \prod_{k=1}^{\infty}(1 + a_k \sin(b_k \pi z_n)) - \prod_{k=1}^{\infty}(1 + a_k \sin(b_k \pi y_n)).$$

First, for $k > n$ it is obvious that b_k/b_n is an even integer. Thus, for $k > n$ and some $q_k \in \mathbb{Z}$, we have

$$\sin(b_k \pi y_n) = \sin\left(\frac{b_k}{b_n}\pi(N_n+1)\right) = \sin(2q_k(N_n+1)\pi) = 0$$

and

$$\sin(b_k \pi z_n) = \sin\left(\frac{b_k}{b_n}\pi(N_n+3/2)\right) = \sin(3q_k\pi) = 0.$$

Moreover, for $k = n$ we obtain

$$\sin(b_n \pi y_n) = \sin(\pi(N_n + 1)) = 0$$

and

$$\sin(b_n \pi z_n) = \sin(\pi(N_n + 3/2)) = -(-1)^{N_n}.$$

With these equalities in mind we can rewrite Δ_n as

$$\begin{aligned}\Delta_n &= L_{n-1}(z_n)(1 + a_n \sin(b_n \pi z_n)) - L_{n-1}(y_n)(1 + a_n \sin(b_n \pi y_n)) \\ &= L_{n-1}(z_n) - L_{n-1}(y_n) - (-1)^{N_n} a_n L_{n-1}(z_n).\end{aligned}$$

Now, for $k < n$ we have

$$\begin{aligned}|a_k \sin(b_k \pi z_n) - a_k \sin(b_k \pi y_n)| &= 2a_k \left| \sin\left(b_k \pi \frac{z_n - y_n}{2}\right) \cos\left(b_k \pi \frac{z_n + y_n}{2}\right) \right| \\ &\leq a_k |b_k \pi (z_n - y_n)| = \frac{a_k b_k \pi}{2b_n} < \frac{\pi}{2p_n}\end{aligned}$$

so there exists $\sigma_k \in \mathbb{R}$ with $|\sigma_k| < \pi/(2p_n) < 1$ such that

$$a_k \sin(b_k \pi z_n) = a_k \sin(b_k \pi y_n) + \sigma_k.$$

Now we can estimate Δ_n, but first we need the following bound

$$\begin{aligned}|L_{n-1}(z_n) - L_{n-1}(y_n)| &= \left| \prod_{k=1}^{n-1} [(1 + a_k \sin(b_k \pi z_n)) + \sigma_k] \right. \\ &\qquad \left. - \prod_{k=1}^{n-1} [1 + a_k \sin(b_k \pi y_n)] \right| \\ &= \left| \sum_{i=1}^{2^{(n-1)}-1} \sigma_{l_i} \left(\prod_{j \in I_i} \sigma_j \right) \left(\prod_{j \in J_i} (1 + a_j \sin(b_j \pi y_n)) \right) \right| \\ &\leq \sum_{i=1}^{2^{(n-1)}-1} |\sigma_{l_i}| \left(\prod_{j \in I_i} |\sigma_j| \right) \left(\prod_{j \in J_i} |1 + a_j \sin(b_j \pi y_n)| \right) \\ &\leq \frac{\pi}{2p_n} \sum_{i=1}^{2^{(n-1)}-1} \left(\prod_{j \in J_i} |1 + a_j| \right) \\ &\leq \frac{b\pi}{2p_n}(2^{n-1} - 1) \leq \frac{b\pi}{p_n} 2^{n-2}\end{aligned}$$

where, for $0 < i < n$, $I_i \subset \mathbb{N}$ and $J_i \subset \mathbb{N}$ are some index sets and l_i some index (we also adhere to the convention that $\prod_{j\in\emptyset} x_j = 1$). Thus we can find a lower bound for $|\Delta_n|$ by

$$\begin{aligned}|\Delta_n| &= |L_{n-1}(z_n) - L_{n-1}(y_n) - (-1)^{N_n} a_n L_{n-1}(z_n)| \\ &\geq a_n L_{n-1}(z_n) - |L_{n-1}(z_n) - L_{n-1}(y_n)| \\ &\geq a_n a - \frac{b\pi}{p_n} 2^{n-2} = a_n \left(a - \frac{2^{n-2}}{a_n p_n} b\pi \right)\end{aligned}$$

where the last inequality follows from the bound above and the fact that $a < L_n(x) < b$. Now, since $\lim_{n\to\infty} 2^n/(a_n p_n) = 0$ by assumption (which also implies that $a_n b_n \to \infty$ since $a_n b_n \geq a_n p_n$), we have

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{W_L(z_n) - W_L(y_n)}{z_n - y_n} \right| &= \lim_{n\to\infty} |2b_n \Delta_n| \\ &\geq \lim_{n\to\infty} \left| 2a_n b_n \left(a - \frac{2^{n-2}}{a_n p_n} b\pi \right) \right| = \infty \cdot a = \infty.\end{aligned}$$

By the triangle inequality and inequality (3.10) we can estimate

$$\begin{aligned}\left| \frac{W_L(z_n) - W_L(y_n)}{z_n - y_n} \right| &\leq \frac{|W_L(z_n) - W_L(x)|}{z_n - y_n} + \frac{|W_L(y_n) - W_L(x)|}{z_n - y_n} \\ &\leq \frac{3|W_L(z_n) - W_L(x)|}{z_n - x} + \frac{3|W_L(y_n) - W_L(x)|}{y_n - x}.\end{aligned}$$

If we let $n \to \infty$ it is clear that W_L is not differentiable at x. Since $x \in \mathbb{R}$ was arbitrary it follows that W_L is nowhere differentiable. $\square$

Chapter 4

How “Large” is the Set $\mathcal{ND}[a,b]$

From the previous chapter it is clear that there exists continuous nowhere differentiable functions but how many are there? A simple answer would be “infinitely many” but could we perhaps say something else about the size of the set of continuous nowhere differentiable functions? As a matter of fact we can. One such way is based on topology in metric spaces and for this we will need some definitions and theorems. But first, where does all continuous nowhere differentiable functions live? We refer to this place as $\mathcal{ND}[a,b]$, or more exactly as in the following definition.

Definition 4.1. *Let $\mathcal{ND}[a,b]$ $(a < b)$ be the set of all continuous nowhere differentiable functions $f : [a,b] \to \mathbb{R}$.*

4.1 Metric spaces and category

We collect a few ideas from the theory of metric spaces, starting with defining exactly what we mean by a metric space.

Definition 4.2. *A metric space is a pair (X,d) of a set X and a metric d defined on X. A metric $d : X \times X \to [0,\infty)$ is a mapping that, for any $x, y, z \in X$, satisfies*

(i) $d(x,y) \geq 0$ *is a real number;*

(ii) $d(x,y) = 0$ *if and only if* $x = y$*;*

(iii) $d(x,y) = d(y,x)$*;*

(iv) $d(x, y) \leq d(x, z) + d(z, y)$.

A metric space (X, d) *is said to be complete if every Cauchy sequence in* X *converges. That is, if* $\{x_n\}$ *is a Cauchy sequence in* (X, d)*, i.e.*

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall m, n \geq N \, d(x_m, x_n) < \epsilon,$$

then there exists $x \in X$ *such that*

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

Remark. We often write X instead of (X, d) when the metric is implicit.
A metric space is a very general construction, a bit too general for our application, so we will need to make some more restrictions. We introduce the concept of a normed vector space.

Definition 4.3. *A normed space is a pair* $(X, \|\cdot\|)$ *of a vector space* X *and a norm* $\|\cdot\|$ *defined on* X*. A norm* $\|\cdot\| : X \to [0, \infty)$ *is a mapping that for any* $x, y \in X$ *and any* $\alpha \in \mathbb{R}$ *(or* $\mathbb{C}$ *when* X *is a complex space) satisfies*

(i) $\|x\| \geq 0$ *is a real number;*

(ii) $\|x\| = 0$ *if and only if* $x = 0$*;*

(iii) $\|\alpha x\| = |\alpha| \|x\|$*;*

(iv) $\|x + y\| \leq \|x\| + \|y\|$.

A Banach space $(X, \|\cdot\|)$ *is a normed space which is complete seen as a metric space* (X, d) *with the metric* d *induced by the norm, that is for* $x, y \in X$

$$d(x, y) = \|x - y\|.$$

Remark. As with metric spaces, we often write X instead of $(X, \|\cdot\|)$ when the norm is implicit.
Our results in the next section will be presented in a specific normed space, namely the vector space of all continuous functions with supremum norm.

Definition 4.4. *Let* $C[a, b]$ $(a < b)$ *be the normed (real) vector space of all continuous functions* $f : [a, b] \to \mathbb{R}$ *with the supremum norm, i.e.*

$$\|f\| = \sup_{x \in [a,b]} |f(x)|.$$

Remark. Clearly $\mathcal{ND}[a,b] \subset C[a,b]$ properly.
As it turns out, this normed space is actually a Banach space.

Theorem 4.1. *The space $C[a,b]$ of all real-valued (or complex-valued) continuous functions on $[a,b]$ with the supremum norm,*

$$\|f\| = \sup_{x \in [a,b]} |f(x)|,$$

is a Banach space.

Proof. See Kreyszig [42], pages 36-37. $\square$

To get some understanding of the "size" of subsets of a metric space we start by giving a definition of some topological properties. We will later establish that the set $\mathcal{ND}[a,b]$ is of the second category (actually, what we will show is that it is residual).

Definition 4.5. *We say that a set M in a metric space X is*

(i) **nowhere dense** *if the closure $\overline{M}$ contains no non-empty open sets,*

(ii) of the **first category** *if*

$$M = \bigcup_{k=1}^{\infty} M_k,$$

where each M_k is nowhere dense (in X),

(iii) of the **second category** *if M is not of the first category.*

A set which is a complement (in X) of a set of the first category is called residual and a property that holds on a residual set is called a (topologically) generic property.

We will rely heavily on the following theorem when proving that the set $\mathcal{ND}[a,b]$ is of the second category. This will be possible since we know that $C[a,b]$ is complete.

Theorem 4.2 (Baire's Category Theorem). *If a metric space $X \neq \emptyset$ is complete it is of the second category in itself.*

Proof. See Kreyszig [42], pages 247-248. $\square$

Remark. The formulation of Theorem 4.2 is equivalent with the following: If $X \neq \emptyset$ is a complete metric space and

$$X = \bigcup_{k=1}^{\infty} M_k$$

where each M_k is closed, then at least one M_{k_0} contains a nonempty open subset.
The equivalence is obvious since (i), if no M_k ($= \overline{M_k}$ since M_k is closed) contains a non-empty open subset then X would be of the first category in itself and (ii), if X is of the second category in itself we cannot write X as a countable union of nowhere dense sets (hence there is a non-empty open subset in some M_k).

4.2 Banach-Mazurkiewicz theorem

We can get some topological results on the size of $\mathcal{ND}[0,1]$ from a theorem that was originally done in 1931 by Banach (cf. Banach [2]) and Mazurkiewicz (cf. Mazurkiewicz [45]). In 1929, H. Steinhaus posed the question "of what category is the set of all continuous nowhere differentiable functions in the space of all continuous functions" in his paper Steinhaus [72], pp. 81. This as a reaction to his statement (in the same paper, pp. 63) that the set of all 2π-periodic continuous nowhere differentiable functions is of the second category seen as a subset of all 2π-periodic continuous functions (with supremum norm). The papers of Banach and Mazurkiewicz gives an answer to Steinhaus question.
The Banach-Mazurkiewicz theorem is based on Baire's category theorem (Theorem 4.2) which states that a complete metric space is of the second category in itself. The proof presented here is largely due to Oxtoby [53] and we need the following lemma.

Lemma 4.3. *The set $\mathcal{P}[a,b]$ of all piecewise linear continuous functions defined on the interval $[a,b]$ is dense in $C[a,b]$.*

Proof. Let $g \in C[a,b]$ be arbitrary but fixed. Put h_n as the piecewise linear function on the partition $P_n : a = t_0 < t_1 < \cdots < t_n = b$ defined by

$$h_n(x) = g(t_i)\frac{t_{i+1} - x}{t_{i+1} - t_i} + g(t_{i+1})\frac{x - t_i}{t_{i+1} - t_i}, \quad x \in [t_i, t_{i+1}].$$

Clearly $h_n \in \mathcal{P}[a,b]$ for every partition P_n. Let $\epsilon > 0$ be given, we show that $\|g - h_n\| < \epsilon$ for some partition P_n.
The function g is continuous on $[a,b]$, i.e.

$$\forall \epsilon > 0\, \exists \delta > 0 \text{ such that } |x - x_0| < \delta \Rightarrow \; |S_n(x) - S_n(x_0)| < \frac{\epsilon}{4}.$$

Choose the partition P_n so that

$$\max_{i=0\ldots n-1} |t_{i+1} - t_i| < \delta.$$

For $x \in [t_i, t_{i+1}]$ we have $|t_{i+1} - t_i| < \delta$ and

$$\begin{aligned}
|g(x) - h_n(x)| &= \left| g(x) - \frac{1}{t_{i+1} - t_i} \left[t_{i+1} g(t_i) - t_i g(t_{i+1}) + x \left(g(t_{i+1}) - g(t_i) \right) \right] \right| \\
&= \left| g(x) - \frac{t_{i+1} g(t_i) - t_i g(t_{i+1})}{t_{i+1} - t_i} - x \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i} \right| \\
&= \left| g(x) - g(t_{i+1}) - \frac{t_{i+1} - x}{t_{i+1} - t_i} \left(g(t_i) - g(t_{i+1}) \right) \right| \\
&\leq |g(x) - g(t_{i+1})| + \left| \frac{t_{i+1} - x}{t_{i+1} - t_i} \right| |g(t_{i+1}) - g(t_i)| \\
&\leq \frac{\epsilon}{4} + 1 \cdot \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned}$$

Now,

$$\|g - h_n\| \leq \max_{i=0,\ldots,n-1} \left(\sup_{x \in [t_i, t_{i+1}]} |g(x) - h_n(x)| \right) \leq \frac{\epsilon}{2} < \epsilon$$

and we are done. □

Remark From the lemma above and the construction in Section 3.17 it is clear that $\mathcal{ND}[a,b]$ is dense in $C[a,b]$. Consider the following (for the interval $[0,1]$): Let $g \in C[0,1]$ and $\epsilon > 0$ be arbitrary but fixed. Let P be a polygonal arc (a piecewise linear function) within $\epsilon/2$ of g. This is no problem because of what we just established in Lemma 4.3. We can, as in Section 3.17, construct $C_n \subset N_{\epsilon/2}(P)$ that satisfies conditions (i)-(iii) in said section. In the same manner as in Theorem 3.17, $\bigcap_n C_n$ defines a well-defined, continuous and nowhere differentiable function on $[0,1]$ that clearly is within $\epsilon/2$ of P and henceforth within ϵ of g. Thus $\mathcal{ND}[0,1]$ is dense in $C[0,1]$.

The main theorem of this section states that in the normed (real) vector space of all real-valued continuous functions on $[a, b]$ with the supremum norm, nowhere differentiability is a topologically generic property. This implies that the set $\mathcal{ND}[a, b]$ is of the second category in $C[a, b]$.

Theorem 4.4 (Banach-Mazurkiewicz Theorem). *The set* $\mathcal{ND}[a, b]$ *of all nowhere differentiable continuous functions on* $[a, b]$ *is of the second category in* $C[a, b]$.

Proof. It is enough to prove the theorem for $[a, b] = [0, 1]$. Let

$$E_n = \left\{ f \,\middle|\, \exists\, x \in \left[0, 1 - \frac{1}{n}\right] \text{ s.t. } \forall\, h \in (0, 1 - x)\, |f(x + h) - f(x)| \le nh \right\}$$

where $f \in C[0, 1]$.
We show that the sets E_n are closed for all $n \in \mathbb{N}$.
Take $f \in \overline{E_n}$, then $\exists\, f_k \in E_n$ such that $f_k \to f$ uniformly on $[0, 1]$. Since $f_k \in E_n$, $\exists x_k \in [0, 1 - \frac{1}{n}]$ for every $k \in \mathbb{N}$ by the definition of E_n. The sequence $\{x_k\}$ is clearly bounded so by the Bolzano-Weierstrass theorem it has a convergent subsequence, say $\{x_{k_l}\}$, that converges to some element $x \in [0, 1 - \frac{1}{n}]$. Let $\{f_{k_l}\}$ be the corresponding subsequence of $\{f_k\}$. By the construction, $|f_{k_l}(x_{k_l} + h) - f_{k_l}(x_{k_l})| \le nh$ for all $0 < h < 1 - x_{k_l}$. Since $x_{k_l} \to x$ and $0 < h < 1 - x$ we can always choose some $l_0 \in \mathbb{N}$ large enough so that $0 < h < 1 - x_{k_l}$ for $l > l_0$. Then (for l large enough)

$$\begin{aligned}
|f(x + h) - f(x)| &\le |f(x + h) - f(x_{k_l} + h)| + |f(x_{k_l} + h) - f_{k_l}(x_{k_l} + h)| \\
&\quad + |f_{k_l}(x_{k_l} + h) - f_{k_l}(x_{k_l})| + |f_{k_l}(x_{k_l}) - f(x_{k_l})| \\
&\quad + |f(x_{k_l}) - f(x)| \\
&\le |f(x + h) - f(x_{k_l} + h)| + \|f - f_{k_l}\| + nh + \|f_{k_l} - f\| \\
&\quad + |f(x_{k_l}) - f(x)|.
\end{aligned}$$

If we let $l \to \infty$ then the continuity of f at x and $x + h$ and the convergence of f_{k_l} (in the norm) gives the inequality $|f(x + h) - f(x)| \le nh$ for every $0 < h < 1 - x$ and thus $f \in E_n$. Hence E_n is closed.
Now consider the set $\mathcal{P}[0, 1]$ of all piecewise linear continuous functions on the interval $[0, 1]$. This set is dense in $C[0, 1]$ by Lemma 4.3.
The sets E_n are nowhere dense if we show that for any $g \in \mathcal{P}[0, 1]$ and any $\epsilon > 0$ there exists $h \in C[0, 1] \setminus E_n$ such that $\|g - h\| < \epsilon$.

Let $\epsilon > 0$ be given and let M be the maximum slope of any "piece" of g. Choose $m \in \mathbb{N}$ such that $m\epsilon > n+M$. Let $\phi(x) = \inf_{k\in\mathbb{Z}} |x-k|$ ("saw-tooth" function, see figure 3.13 as well) and take $h(x) = g(x) + \epsilon\phi(mx)$. Clearly $h \in C[0,1]$.
Then, for all $x \in [0,1)$, $h(x)$ has a right-hand side derivative, $h'^{+}(x)$, such that

$$|h'^{+}(x)| = |g'^{+}(x) + \epsilon m \phi'^{+}(mx)| > n$$

since we have chosen $m\epsilon > n + M$. Hence $h \in C[0,1] \setminus E_n$.
We also have

$$\|g-h\| = \sup_{x\in[0,1]} |g(x) - (g(x) + \epsilon\phi(mx))| = \epsilon \sup_{x\in[0,1]} |\phi(mx)| = \frac{\epsilon}{2} < \epsilon$$

and thus E_n is clearly nowhere dense in $C[0,1]$.
Since E_n is nowhere dense, we see that $E = \bigcup_{k=1}^{\infty} E_k$ is of the first category in $C[0,1]$. This is the set of all elements in $C[0,1]$ with bounded right hand difference quotients at some point $x \in [0,1]$ (i.e. the complement to E in $C[0,1]$ does not possess a finite right-hand derivative anywhere in $[0,1)$). Since $C[0,1]$ is complete and thus by Baire's theorem (Theorem 4.2) of the second category it is clear that the set of functions in $C[0,1]$ which are nowhere differentiable constitutes a set of the second category. □

Remark 1. Banach and Mazurkiewicz did not prove exactly the same thing in their respective articles, however their results coincide when formulated as in the theorem above. Mazurkiewicz shows that the set of continuous functions which have a bounded one-sided derivative at some point is of the first category while Banach proved that the set of functions which have a bounded Dini-derivative[1] at some point is of the first category. This makes the theorem of Banach stronger than Mazurkiewicz's similar result.

Remark 2. What was shown in the proof of the theorem above is that the set of continuous functions that have a finite right-hand derivative at some point $x \in [0,1]$ is of the first category. It can similarly be shown that the subset of $C[0,1]$ having a finite left-hand derivative at some point $x \in [0,1]$ is of the first category. Thus the subset of $C[0,1]$ consisting of functions with a finite one-sided derivative at some point is also of the first category.

[1]The four Dini-derivatives are defined as one-sided derivatives with limes superior and limes inferior instead of only limes.

From the second remark above, the following question arise: what about the set of continuous functions without finite *or* infinite one-sided derivative everywhere? Saks solved the question in a paper published in 1932 (Saks [65]). He proved that the set of continuous functions which have a finite or infinite right-hand derivative at some point is of the second category. This is the complement of the set above so that set is of the first category. The first example of such a function wasn't constructed until 1922 when Besicovitch managed the feat (and published the function in 1924, cf. Besicovitch [4][2]). These types of functions are usually referred to as functions of the Besicovitch type.

4.3 Prevalence of $\mathcal{ND}[0,1]$

Prevalence[3] is a concept that can be used when one is interested in a measure theoretic result of how "large" a set in an infinite dimensional vector space is. It enables us to use terms such as "almost every" and "measure zero" on these spaces (without a specific measure like, for example, the Wiener measure). Its development was partially motivated by wanting to keep some of the properties that the Lebesgue measure on finite dimensional spaces possess, one of which is the translation invariance. Prevalence is a more useful property than topological properties like category and denseness when a probabilistic result on the likelihood of a given property is desired. This in part due to the fact that it actually turns out that a property that is topologically generic in $\mathbb{R}^n$ can have very low probability[4] (and also that a first category set can contain almost every [Lebesgue] point in the space).
In Hunt, Sauer and Yorke [31] there are a few examples of this phenomenon as well as detailed development of the concept of prevalence. We borrow a few definitions from this paper and show that $\mathcal{ND}[0,1]$ constitutes a prevalent set in $C[0,1]$. The main part of this section is gathered from Hunt [30].

Definition 4.6. *Let X be a complete metric vector space. A measure μ is said to be* **transverse** *to a Borel set $S \subset X$ if the following conditions hold.*

[2]In 1928, E. Pepper published an article (Pepper [55]) about functions of the Besicovitch type where he produced the same function as Besicovitch but with simpler reasoning.

[3]After the publication of Hunt, Sauer and Yorke it became clear that this was closely related to another concept, more specifically, that so called shy sets are very closely related to the notion of a Haar zero set for Abelian polish groups (cf. Hunt, Sauer and Yorke [32])

[4]Actually, in some cases, probability equal to zero.

(i) There exists a compact set $K \subset X$ for which $0 < \mu(K) < \infty$.

(ii) $\mu(\{x + s | s \in S\}) = 0$ for every $x \in X$.

Definition 4.7. *A Borel set $S \subset X$ is called* **shy** *if there exists a measure transverse to S. If a set W is contained in a shy Borel set then W is also said to be shy. The complement of a shy set is called a* **prevalent** *set.*

Definition 4.8. *We call a finite-dimensional subspace $P \subset C[0,1]$ a* **probe** *for a set $S \subset C[0,1]$ if Lebesgue measure supported on P is transverse to a Borel set which contains $S^c = C[0,1] \setminus S$.*

The following inequality is central for our main result.

Lemma 4.5. *Let $g(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \cos(2^k \pi x)$ and $h(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sin(2^k \pi x)$. Then there exists $c > 0$ such that for every $\alpha, \beta \in \mathbb{R}$ and any closed interval $I \subset [0,1]$ with length $\epsilon \leq \frac{1}{2}$,*

$$\sup_{x \in I}(\alpha g(x) + \beta h(x)) - \inf_{x \in I}(\alpha g(x) + \beta h(x)) \geq \frac{c\sqrt{\alpha^2 + \beta^2}}{(\log \epsilon)^2}.$$

Proof. Let $f = \alpha g + \beta h$. Then, for some $\theta \in [0, 2\pi]$,

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \left(\alpha \cos(2^k \pi x) + \beta \sin(2^k \pi x)\right) = \sqrt{\alpha^2 + \beta^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \cos(2^k \pi x + \theta).$$

We may assume that $\alpha^2 + \beta^2 = 1$ without loss of generality. Let I be some closed interval in $[0,1]$ with length 2^{-m}, where $m \in \mathbb{N}$. We claim that for any continuous function f,

$$\sup_{x \in I} f(x) - \inf_{x \in I} f(x) \geq \sup_{j \in \mathbb{N}} 2^m \pi \int_I f(x) \cos(2^{m+j} \pi x + \theta)\, dx. \tag{4.1}$$

We may assume that $\sup_{x \in I} f(x) = -\inf_{x \in I} f(x) = K$ for some $K \geq 0$ since adding a constant to both sides of (4.1) does not change the inequality (since $\int_I \cos(2^{m+j} \pi x)\, dx = 0$). Then $|f| \leq 1$ on I and hence

$$\begin{aligned} 2^m \pi \int_I f(x) \cos(2^{m+j} \pi x + \theta)\, dx &\leq 2^m \pi \int_I K |\cos(2^{m+j} \pi x + \theta)|\, dx \\ &= 2^m \pi K \frac{2}{\pi} 2^{-m} = 2K, \end{aligned}$$

which is equivalent to (4.1). We then have, for f defined as above and for any $j \in \mathbb{N}$, that

$$\begin{aligned}\sup_{x\in I} f(x) - \inf_{x\in I} f(x) &\geq 2^m\pi \int_I \sum_{k=1}^{\infty} \frac{1}{k^2}\cos(2^k\pi x+\theta)\cos(2^{m+j}\pi x+\theta)\,dx \\ &= \sum_{k=1}^{\infty}\frac{2^m\pi}{k^2}\int_I \frac{\cos((2^{m+j}-2^k)\pi x)+\cos((2^{m+j}+2^k)\pi x+2\theta)}{2}\,dx.\end{aligned} \tag{4.2}$$

Since I has length 2^{-m}, $\int_I \cos((2^{m+j}\pm 2^k)\pi x+\varphi)\,dx = 0$ whenever $k > m$, except when $k = m+j$ (with the "-" sign). For $k \leq m$, let $\omega = \pm 2^k$ and let y be the left endpoint of I. Then

$$\begin{aligned}&\int_I \cos((2^{m+j}+\omega)\pi x+\varphi)\,dx \\ &\quad= \frac{\sin((2^{m+j}+\omega)\pi(y+2^{-m})+\varphi)-\sin((2^{m+j}+\omega)\pi y+\varphi)}{(2^{m+j}+\omega)\pi} \\ &\quad= \frac{\sin((2^{m+j}+\omega)\pi y+\varphi+2^{-m}\pi\omega)-\sin((2^{m+j}+\omega)\pi y+\varphi)}{(2^{m+j}+\omega)\pi} \\ &\quad\geq -\frac{|2^{-m}\pi\omega|}{(2^{m+j}+\omega)\pi} = -\frac{|\omega|}{2^m(2^{m+j}+\omega)}.\end{aligned}$$

It then follows from (4.2) that

$$\begin{aligned}\sup_{x\in I} f(x) - \inf_{x\in I} f(x) &\geq \frac{\pi}{2(m+j)^2} - \sum_{k=1}^{m}\frac{\pi}{2k^2}\left(\frac{2^k}{2^{m+j}-2^k}+\frac{2^k}{2^{m+j}+2^k}\right) \\ &\geq \frac{\pi}{2(m+j)^2} - \frac{\pi}{2^m(2^j-1)}\sum_{k=1}^{m}\frac{2^k}{k^2}.\end{aligned} \tag{4.3}$$

Now we claim that

$$\sum_{k=1}^{m}\frac{2^k}{k^2} \leq 5\frac{2^m}{m^2} \tag{4.4}$$

for all $m \in \mathbb{N}$. For $m = 1, 2, 3, 4$ it can quite easily be seen to hold and for $m \geq 4$ we prove by induction. Assume that equation (4.4) holds for $m = n$.

Let $m = n+1$,

$$\sum_{k=1}^{n+1} \frac{2^k}{k^2} \leq 5\frac{2^n}{n^2} + \frac{2^{n+1}}{(n+1)^2} = \left(\frac{5(n+1)^2}{2n^2} + 1\right) \frac{2^{n+1}}{(n+1)^2}$$
$$\leq \left(\frac{125}{32} + 1\right) \frac{2^{n+1}}{(n+1)^2} \leq 5\frac{2^{n+1}}{(n+1)^2}$$

and the claim follows by induction on m. This gives a new estimate for (4.3),

$$\sup_{x \in I} f(x) - \inf_{x \in I} f(x) \geq \frac{\pi}{2(m+j)^2} - \frac{5\pi}{(2^j - 1)m^2}.$$

We let $j = 10$ and assume that $m \geq 2$. Then

$$\begin{aligned}\sup_{x \in I} f(x) - \inf_{x \in I} f(x) &\geq \frac{\pi}{2(m+j)^2} - \frac{5\pi}{(2^j - 1)m^2} \\ &\geq \frac{\pi}{2(6m)^2} - \frac{\pi}{200m^2} = \frac{2\pi}{225m^2}.\end{aligned}$$

Finally, if $I \subset [0,1]$ has arbitrary length $\epsilon \leq 1/2$, choose an $m \geq 2$ such that $2^{1-m} \geq \epsilon > 2^{-m}$. Then, for any closed subinterval $J \subset I$ with length 2^{-m}, we have

$$\begin{aligned}\sup_{x \in I} f(x) - \inf_{x \in I} f(x) &\geq \sup_{x \in J} f(x) - \inf_{x \in J} f(x) \\ &\geq \frac{2\pi}{225m^2} \geq \frac{\pi}{450(m-1)^2} \geq \frac{(\log 2)^2 \pi}{450(\log \epsilon)^2},\end{aligned}$$

which proves the lemma. $\square$

It turns out that we can't work directly with the set $\mathcal{ND}[0,1]$ (since it's not a Borel set, which was proved by Mazurkiewicz [46] in 1936, see Mauldin [44]) so we will instead consider the set of nowhere Lipschitz functions. As we shall see, this set is actually a subset of the set of continuous nowhere differentiable functions and the results we prove in this section will therefore hold for a class of functions that is actually smaller than $\mathcal{ND}[0,1]$.

Definition 4.9. *A function $f \in C[a,b]$ is said to be M-Lipschitz at $x \in [a,b]$ if*

$$\exists M > 0 \text{ such that } \forall y \in [a,b]\, |f(x) - f(y)| \leq M|x-y|.$$

We define $\mathcal{NL}_M[a,b]$ as the set of nowhere M-Lipschitz functions on $[a,b]$, i.e.

$$\mathcal{NL}_M[a,b] = \{f \in C[a,b] \,|\, \forall x \in [a,b]\, \forall y \in [a,b]\, |f(x) - f(y)| > M|x-y|\}\,.$$

We collect some properties of these sets of nowhere Lipschitz functions.

Lemma 4.6. *Let*

$$\mathcal{NL}[a,b] = \bigcap_{M\in\mathbb{N}} \mathcal{NL}_M[a,b],$$

i.e. the set of all nowhere Lipschitz functions. Then the following properties hold.

(i) $\mathcal{NL}_M[a,b]$ *is an open set for every* $M \in \mathbb{N}$.

(ii) $\mathcal{NL}[a,b]$ *is a Borel set.*

(iii) $\mathcal{NL}[a,b] \subset \mathcal{ND}[a,b]$.

Proof. It is enough to prove the theorem for $[a,b] = [0,1]$.
(i) Let $M \in \mathbb{N}$ be arbitrary. Take $f \in \overline{C[0,1] \setminus \mathcal{NL}[0,1]_M}$, then $\exists f_n \in C[0,1] \setminus \mathcal{NL}_M[0,1]$ such that $f_n \to f$ uniformly on $[0,1]$. For every $n \in \mathbb{N}$, there exists $x_n \in [0,1]$ such that f_n is M-Lipschitz at x_n. That is,

$$\forall y \in [0,1] \quad |f_n(x_n) - f_n(y)| \le M|x_n - y|.$$

The sequence $\{x_n\}$ is bounded so by the Bolzano-Weierstrass theorem there exists a convergent subsequence $\{x_{n_k}\}$, say $x_{n_k} \to x \in [0,1]$. Let $y \in [0,1]$ be arbitrary,

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f(x_{n_k})| + |f(x_{n_k}) - f_{n_k}(x_{n_k})| \\ &\quad + |f_{n_k}(x_{n_k}) - f_{n_k}(y)| + |f_{n_k}(y) - f(y)| \\ &\le |f(x) - f(x_{n_k})| + \|f - f_{n_k}\| + M|x_{n_k} - y| + \|f_{n_k} - f\| \\ &\to M|x-y| \quad \text{as } k \to \infty. \end{aligned}$$

Hence f is M-Lipschitz at x and henceforth $f \in C[0,1] \setminus \mathcal{NL}_M[0,1]$. Thus $C[0,1] \setminus \mathcal{NL}_M[0,1]$ is closed and therefore $\mathcal{NL}_M[0,1]$ is open.
(ii) This is obvious from the definition of a Borel set and (i).
(iii) Take $f \in \mathcal{NL}[0,1]$. Then for every $M \in \mathbb{N}$ and for every $x, y \in [0,1]$

$$|f(x) - f(y)| > M|x-y| \quad \Rightarrow \quad \frac{|f(x) - f(y)|}{|x-y|} > M$$

and so the difference quotient is unbounded for all $x, y \in [0,1]$. Hence f has no derivative and thus $f \in \mathcal{ND}[0,1]$. $\square$

Now follows the two main theorems of this section. The first ensures the existence of a probe and the second proves the desired prevalence of $\mathcal{ND}[0,1]$.

Theorem 4.7. *There exists $g, h \in C[0,1]$ such that for all $f \in C[0,1]$*

$$m\left(\mathbb{R}^2 \setminus \left\{(\lambda,\nu) \,\middle|\, (\lambda,\nu) \in \mathbb{R}^2,\ (f+\lambda g+\nu h) \in \bigcap_{m\in\mathbb{N}} \mathcal{NL}_m[0,1]\right\}\right) = 0,$$

where m is the Lebesgue measure on $\mathbb{R}^2$.

Proof. Take g and h as in Lemma 4.5. That both functions are continuous is shown similarly as for Weierstrass' function in Section 3.4. Let $f \in C[0,1]$ be arbitrary and put

$$S = \left\{(\alpha,\beta) \in \mathbb{R}^2 \mid f+\alpha g+\beta h \text{ is Lipschitz at some } x \in [0,1]\right\}.$$

We want to show that S has Lebesgue measure zero. Let

$$S_M = \left\{(\alpha,\beta) \in \mathbb{R}^2 \mid f+\alpha g+\beta h \text{ is M-Lipschitz at some } x \in [0,1]\right\}.$$

From this definition it is clear that $S = \bigcup_{M\in\mathbb{N}} S_M$. So if m is the Lebesgue measure on $\mathbb{R}^2$ and we show that $m(S_M) = 0$ for each $M \in \mathbb{N}$ it is clear that $m(S) = 0$ (by the countable sub-additivity of m).
Let $N \in \mathbb{N} \setminus \{1\}$ and cover $[0,1]$ by N closed intervals of length $\epsilon = \frac{1}{N}$. Let I be anyone of those intervals and put

$$J_I = \{(\alpha,\beta) \in S_M \mid f+\alpha g+\beta h \text{ is M-Lipschitz at some } x \in I\}.$$

Let $(\alpha_1,\beta_1), (\alpha_2,\beta_2) \in J_I$ be arbitrary. Let $f_i = f+\alpha_i g+\beta_i h$ and $x_i \in I$ be an M-Lipschitz point for f_i where $i = 1, 2$. Then

$$\sup_{x\in I} |f_i(x) - f_i(x_i)| \leq \sup_{x\in I} M|x - x_i| \leq M\epsilon$$

which gives

$$\begin{aligned}\sup_{x\in I} |f_1(x) - f_2(x) - [f_1(x_1) - f_2(x_2)]\,| &\leq \sup_{x\in I} |f_1(x) - f_1(x_1)| \\ &\quad + \sup_{x\in I} |f_2(x) - f_2(x_2)| \leq 2M\epsilon.\end{aligned}$$

This gives

$$\sup_{x\in I}(f_1(x) - f_2(x)) - \inf_{x\in I}(f_1(x) - f_2(x)) \leq 4M\epsilon$$

and since $f_1 - f_2 = (\alpha_1 - \alpha_2)g + (\beta_1 - \beta_2)h$, Lemma 4.5 gives the bound

$$\sqrt{(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2} \leq \frac{4M\epsilon\log(\epsilon)^2}{c_I}.$$

Let $c = \min_I\{c_I\}$. Since the points were arbitrary, it follows that J_I is enclosed by a disk of radius $\frac{4M}{c}\epsilon\log(\epsilon)^2$ (for all the intervals I). It now follows that S_M can be covered by $N = \frac{1}{\epsilon}$ such discs, giving that the total area of the covering is bounded by

$$A = \pi\left(\frac{4M}{c}\epsilon(\log\epsilon)^2\right)^2\frac{1}{\epsilon} = \frac{\pi 16M^2}{c^2}\epsilon(\log\epsilon)^4.$$

As $\epsilon \to 0$, $A \to 0$. Hence S_M has measure zero and thus S has measure zero. $\square$

Theorem 4.8. *Almost every function in $C[0,1]$ is nowhere differentiable; that is, $\mathcal{ND}[0,1]$ is a prevalent subset of $C[0,1]$.*

Proof. Let $\mathcal{NL}[0,1]$ be the set of all nowhere Lipschitz functions. By Theorem 4.7 there exists a probe P (spanned by g and h) for $\mathcal{NL}[0,1]$. This is clear since, by the conclusion of said theorem, Lebesgue measure supported on P is transverse to $\mathcal{NL}[0,1]^c = C[0,1] \setminus \mathcal{NL}[0,1]$ (which is a Borel set by Lemma 4.6(ii) since $\mathcal{NL}[0,1]$ is a Borel set and obviously there exists a compact set $K \subset C[0,1]$ such that $0 < m(K) < \infty$ (where m is the Lebesgue measure)). Hence $\mathcal{NL}[0,1]^c$ is shy and therefore $\mathcal{NL}[0,1]$ is a prevalent set. By Lemma 4.6(iii) it is also clear that $\mathcal{ND}[0,1]$ must be a prevalent set in $C[0,1]$. $\square$

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