Continuous selections and countable sets

by

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**Abstract.** Some known selection theorems for set-valued maps $\varphi\colon X \to 2^Y$ are strengthened by eliminating all hypotheses on $\varphi$, except lower semi-continuity, on an arbitrary countable subset of $X$.

**1. Introduction.** The purpose of this note is to strengthen some known selection theorems for set-valued maps $\varphi\colon X \to 2^Y$ by eliminating all hypotheses on $\varphi$, except lower semi-continuity, on an arbitrary countable subset of $X$. Analogous results for 0-dimensional subsets, which still required that $\varphi(x)$ be closed in $Y$ for every $x \in X$, were recently obtained in [12] and [13].

Let us establish some terminology. A map $\varphi\colon X \to 2^Y$ where $2^Y = \{S \subset Y\colon\ S \neq \emptyset\}$ is called *lower semi-continuous*, or *l.s.c.*, if $\{x \in X\colon\ \varphi(x) \cap V \neq \emptyset\}$ is open in $X$ for every open in $V$ in $Y$. A *selection* for a map $\varphi\colon X \to 2^Y$ is a continuous $f\colon X \to Y$ such that $f(x) \in \varphi(x)$ for every $x \in X$. If $A \subset X$ is closed, then $\varphi\colon X \to 2^Y$ has the *selection extension property*, or *SEP*, at $A$ if every selection for $\varphi|A$ extends to a selection for $\varphi$; if $\varphi$ has the SEP at every closed $A \subset X$, then we simply say that $\varphi$ has the SEP.

Our first result is the following theorem, which was announced without proof in [10].

THEOREM 1.1. *If $X$ is countable and regular, $Y$ first-countable, and $\varphi\colon X \to 2^Y$ l.s.c., then $\varphi$ has the SEP.*

It should be remarked that, if $Y$ is a complete metric space and each $\varphi(x)$ is closed in $Y$, then Theorem 1.1 follows from [7, Theorem 2], and if $X$ is a complete metric space it follows from a result of H. Reiter [15, Theorem 1] ([1]).

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([1]) A. V. Arhangel'skiĭ has kindly informed me that a result closely related to Theorem 1.1 was stated (without proof) by B. Pasynkov in [14, Theorem 2]. (There is a misprint in Pasynkov's hypothesis: He intended to assume that each fiber has a dense subset of points where the whole space — not just the fiber — is first-countable).

Examples 9.1 and 9.2 show that Theorem 1.1 becomes false if $X$ is not countable or if $Y$ is not first-countable.

All the remaining results in this introduction assume that $Y$ is metrizable, and for such $Y$ all these results except Theorem 1.7 generalize Theorem 1.1.

THEOREM 1.2. *Let $X$ be paracompact* ([2]), *$Y$ metrizable, $A\subset X$ closed with $X-A$ countable, and $\varphi\colon X\to 2^Y$ l.s.c. Then $\varphi$ has the SEP at $A$.*

Theorem 1.2 becomes false if $Y$ is only assumed first-countable (as in Theorem 1.1), even if $\varphi(x)=Y$ for all $x\in X$; see Example 9.3.

From now on, all our results assume that $Y$ is a complete metric space. All these results follow from established selection or extension theorems when the countable set $C$ is empty, and from more recent selection theorems in [12] and [13] when $\varphi(x)$ closed in $Y$ for all $x\in X$ (see Section 4).

Our next result, which generalizes Theorem 1.2, reduces to [6, Theorem 1.2] when $C=\emptyset$. Following [6], we say that $\dim_X S\leqslant n$, where $S\subset X$, if $\dim E\leqslant n$ for every set $E\subset S$ which is closed in $X$ (where $\dim E$ is the covering dimension of $E$).

THEOREM 1.3. *Let $X$ be paracompact, $Y$ complete metric, $A\subset X$ closed with $\dim_X(X-A)\leqslant 0$, $C\subset X$ countable, and $\varphi\colon X\to 2^Y$ l.s.c. with $\varphi(x)$ closed in $Y$ for all $x\notin C$. Then $\varphi$ has the SEP at $A$.*

The following result reduces to [5, Theorem 3.2″ and Proposition 1.4] when $C=\emptyset$.

THEOREM 1.4. *Let $X$ be a paracompact, $Y$ a Banach space, $C\subset X$ countable, and $\varphi\colon X\to 2^Y$ l.s.c. with $\varphi(x)$ closed and convex for $x\notin C$. Then $\varphi$ has the SEP.*

In the following two results, we say that a map $\varphi\colon X\to 2^Y$ has the *selection neighborhood extension property*, or *SNEP*, at a closed set $A\subset X$ if every selection for $\varphi|A$ extends to a selection for $\varphi|U$ for some neighborhood $U$ of $A$ in $X$; if $\varphi$ has the SNEP at every closed $A\subset X$, we simply say that $\varphi$ has the SNEP.

THEOREM 1.5. *Let $X$ be paracompact, $Y$ a complete metric ANR* ([3]), *$C\subset X$ countable, and $\varphi\colon X\to 2^Y$ l.s.c. with $\varphi(x)=Y$ for $x\notin C$. Then $\varphi$ has the SNEP. If $Y$ is actually an AR, then $\varphi$ has the SEP.*

For $C=\emptyset$, Theorem 1.5 reduces to the known result that every complete metric AR (resp. ANR) has the extension property (resp. neighborhood extension property) with respect to paracompact spaces; see, for example, [2].

Our next result, a finite-dimensional version of Theorem 1.5, reduces to [6, Theorem 1.2] when $C=\emptyset$. When $n=-1$, Theorem 1.6 reduces to our Theorem 1.3, which was stated separately because of its greater simplicity. We assume that $n\geqslant -1$.

THEOREM 1.6. *Let $X$ be paracompact, $Y$ complete metric, $A\subset X$ closed with $\dim_X(X-A)\leqslant n+1$, $C\subset X$ countable, and $\varphi\colon X\to 2^Y$ l.s.c. with $\{\varphi(x)\colon x\notin C\}$*

([2]) It can be shown that it suffices for $X$ to be collectionwise normal.

([3]) A metric space $Y$ is an *AR* (resp. *ANR*) if it is a retract (resp. neighborhood retract) of any metric space containing it as a closed subset.



*a collection of closed sets which is equi-$LC^n$ in $Y$* ([4]). *Then $\varphi$ has the SNEP at $A$. If, moreover, $\varphi(x)$ is $C^n$ for all $x\notin C$, then $\varphi$ has the SEP at $A$.*

Our last result in this introduction is related to Theorems 1.5 and 1.6, but now local assumptions on $\varphi$ yield a global conclusion. Observe that, unlike other selection theorems, we need not assume $\varphi$ to be l.s.c., since that is automatic by Lemma 2.6. Observe also that, unlike Theorems 1.1–1.6, Theorem 1.7 demands something of $\varphi$ — beyond being l.s.c. — on the countable set $C$.

THEOREM 1.7. *Let $X$ be paracompact* (see Footnote ([2])), *$Y$ a complete metric space, $C\subset X$ countable, and $\varphi\colon X\to 2^Y$ such that $\varphi(x)=Y$ for $x\notin C$ and $(\varphi(x))^-=Y$ for all $x\in X$. Suppose also that either $Y$ is an ANR or that $Y$ is an $LC^n$-space* ([5]) *and $\dim X\leqslant n+1$. Then $\varphi$ has a selection* ([6]).

When $C=\emptyset$, Theorem 1.7 is of course trivial for *any* space $Y$, but in general Examples 9.4 and 9.5 show that the completeness and connectivity assumptions on $Y$ cannot be omitted. What can happen if $C$ is not assumed countable in indicated by Example 9.1.

It should perhaps be remarked that, although Theorem 1.7 is stated with considerable generality, its conclusion may be of some interest even when $Y$ is the real line ([7]).

As previously observed, our results are known in case $\varphi(x)$ is closed in $Y$ for all $x\in X$. We have two different methods of deriving our theorems about maps $\varphi\colon X\to 2^Y$ from the corresponding known theorems: The first method, based Proposition 2.4, construct a "smaller" map $\psi\colon X\to 2^Y$ (i.e. $\psi(x)\subset\varphi(x)$ for all $x\in X$) to which the corresponding known theorem applies. The second method is based on Proposition 5.1, which permits us to derive properties of $\varphi$ from known properties of the "larger" map $\bar{\varphi}\colon X\to 2^Y$ defined by $\bar{\varphi}(x)=(\varphi(x))^-$. While either method could be used to prove all the theorems in this introduction, we will apply each where it seems easiest, — Proposition 2.4 to prove Theorems 1.1–1.6, and Proposition 5.1 to prove Theorem 1.7

The paper is arranged as follows. Section 2 is devoted to some results on how to construct new l.s.c. maps from old ones; one of these is the basic Proposition 2.4 mentioned in the previous paragraph. Theorem 1.1 is then proved in Section 3, and Theorems 1.2–1.6 in Section 4. Section 5 introduces the selection approximation property, a useful property of set-valued maps which has already appeared implicitly in other papers, and proves the basic Proposition 5.1 about it. This proposition is

([4]) For the definitions of this concept and of $C^n$ ($=n$-connected), see [12].

([5]) I.e., for every $y\in Y$, every neighborhood $V$ of $y$ in $Y$ contains a neighborhood $W$ of $y$ such that every continuous image of an $i$-sphere ($i\leqslant n$) in $W$ is contractible over $V$.

([6]) By Theorems 1.5 and 1.6, $\varphi$ also has the SNEP. It need not, however, have the SEP, even when $C=\emptyset$ and $Y$ is a two-point space.

([7]) In this special case, which is valid for any normal space $X$, one can outline a simple, direct proof: Write $C=(x_1,x_2,\ldots)$, and construct a uniformly Cauchy sequence of continuous functions $f_n\colon X\to Y$ such that $f_{n+1}(x_i)=f_n(x_i)\in\varphi(x_i)$ for $i\leqslant n$; then $\lim_n f_n$ is a selection for $\varphi$.

then applied in Section 6 to prove Theorem 1.7, and in Section 7 to prove a result which combines Theorems 1.3 and 1.4. Section 8 shows how "countable" may be weakened to "$\sigma$-discrete" in our results, and Section 9 is devoted to examples.

**2. Some results on l.s.c. maps.** If $\varphi\colon X\to 2^Y$, then we say that $\psi\subset\varphi$ if $\psi\colon X\to 2^Y$ and $\psi(x)\subset\varphi(x)$ for all $x\in X$. Now suppose that $\varphi\colon X\to 2^Y$ is l.s.c. and $\psi\subset\varphi$; under what conditions will $\psi$ also be l.s.c.? A very simple condition, which follows immediately from the definitions (or from Lemma 2.6 below), is that $\psi(x)$ is dense in $\varphi(x)$ for all $x\in X$. A considerably weaker — although more complicated — condition is given by the following lemma.

LEMMA 2.1. *Let $\varphi\colon X\to 2^Y$ be l.s.c. Suppose that $\psi\subset\varphi$ and that, for each $y\in Y$, every neighborhood $V$ of $y$ in $Y$ contains a neighborhood $W$ of $y$ in $Y$ such that*

$$A=\{x\in X\colon\ \varphi(x)\cap W\neq\emptyset,\ \psi(x)\cap V=\emptyset\}$$

*is closed in $X$. Then $\psi$ is also l.s.c.*

Proof. We need only show that, if $x_0\in X$, $y_0\in\psi(x_0)$, and if $V$ is a neighborhood of $y_0$ in $Y$, then $G=\{x\in X\colon\ \psi(x)\cap V\neq\emptyset\}$ is a neighborhood of $x_0$ in $X$. Pick a neighborhood $W\subset V$ of $y_0$ as in our hypothesis, and let

$$U=\{x\in X\colon\ \varphi(x)\cap W\neq\emptyset\}.$$

Then $U$ is open in $X$ because $\varphi$ is l.s.c. By assumption, the set

$$A=\{x\in U\colon\ \psi(x)\cap V\neq\emptyset\}$$

is closed in $X$, so $U-A$ is a neighborhood of $x_0$ contained in $G$. That completes the proof.

COROLLARY 2.2. *Suppose that $X$ is a $T_1$-space, that $\psi\colon X\to 2^Y$, and that each non-empty open $V\subset Y$ intersects $\psi(x)$ for all but finitely many $x\in X$. Then $\psi$ is l.s.c.*

Proof. Let $\varphi(x)=Y$ for all $x\in X$. The $\varphi$ is trivially l.s.c., so $\psi$ is also l.s.c. by Lemma 2.1 (with $W=V$ and $A$ finite).

COROLLARY 2.3. *Let $X$ be a $T_1$-space, $Y$ a metric space and $\varphi\colon X\to 2^Y$ l.s.c. Suppose that $\psi\subset\varphi$ and that, for all $\varepsilon>0$, $\{x\in X\colon\ \varphi(x)\not\subset B_\varepsilon(\psi(x))\}$ ([8]) is finite. Then $\psi$ is also l.s.c.*

Proof. This follows from Lemma 2.1 (with $A$ finite), for if $y$ and $V$ are as in that lemma, one need only choose $r>0$ so that $B_r(y)\subset V$ and then let $W=B_{\frac{1}{2}r}(y)$. That completes the proof.

Using Corollary 2.3, we now prove the principal result of this section.

PROPOSITION 2.4. *Let $X$ be a $T_1$-space, $Y$ metrizable, $\varphi\colon X\to 2^Y$ l.s.c., and $C\subset X$ countable. Then there exists a l.s.c. map $\psi\subset\varphi$ such that $\psi(x)=\varphi(x)$ for $x\notin C$ and $\psi(x)$ is closed in $Y$ for $x\in C$.*

Proof. Write $C=\{x_1,x_2,\ldots\}$. Fix a metric $d$ on $Y$, and for each $n$ pick $S_n\subset\varphi(x_n)$ such that $S_n$ is closed in $Y$ and $\varphi(x_n)\subset B_{1/n}(S_n)$. (One can take $S_n$ to be any maximal subset of $\varphi(x_n)$ such that $d(y,y')\geqslant 1/n$ for $y,y'\in S_n$ and $y\neq y'$.) Let $\psi(x)=\varphi(x)$ for $x\notin C$ and $\psi(x_n)=S_n$ for all $n$. Then $\psi$ is l.s.c. by Corollary 2.3, and that completes the proof.

([8]) $B_\varepsilon(S)$ denotes the open $\varepsilon$-neighborhood of $S$.



Remark. Proposition 2.4 becomes false if $C$ is not countable; see Footnote ([11]) following Example 9.1.

We conclude this section by recording two known results which are needed in the sequel.

LEMMA 2.5 [5, Example 1.3]. *Let $\varphi\colon X\to 2^Y$ be l.s.c., $A\subset X$ closed, and $g$ a selection for $\varphi|A$. Define $\varphi_g\colon X\to 2^Y$ by $\varphi_g(x)=\varphi(x)$ for $x\notin A$ and $\varphi_g(x)=\{g(x)\}$ if $x\in A$. Then $\varphi_g$ is also l.s.c.*

We now introduce the following notation: If $\varphi\colon X\to 2^Y$, then $\bar\varphi\colon X\to 2^Y$ is defined by $\bar\varphi(x)=(\varphi(x))^-$.

LEMMA 2.6 [5, Proposition 2.3]. *If $\varphi\colon X\to 2^Y$, then $\varphi$ is l.s.c. if and only if $\bar\varphi$ is l.s.c.*

**3. Proof of Theorem 1.1.** We prove this result in three steps.

First, let us show that $\varphi$ has a selection if $Y$ is a metric space: Let $Y^*$ be the completion of $Y$. By Proposition 2.4, there is a $\psi\subset\varphi$ such that $\psi(x)$ is closed in $Y^*$ for all $x\in X$. Since a countable regular space is paracompact and 0-dimensional, $\psi$ has a selection $f$ by [7, Theorem 2], and this $f$ is also selection for $\varphi$.

Next, let us show that $\varphi$ has a selection if $Y$ is first-countable. By [9, Theorem 4.3], there exists an open, continuous $u\colon M\to Y$ from a metric space $M$ onto $Y$. Define $\theta\colon X\to 2^M$ by $\theta(x)=u^{-1}(\varphi(x))$. Since $u$ is open, $\theta$ is l.s.c. Hence $\theta$ has a selection $g$ by the previous paragraph, and now $f=u\circ g$ is a selection for $\varphi$.

Finally, let us prove our theorem. Let $A\subset X$ be closed, and $g$ a selection for $\varphi|A$. Then $\varphi_g$ is l.s.c. by Lemma 2.5, so $\varphi_g$ has a selection $f$ by the previous paragraph, and this $f$ is a selection for $\varphi$ which extends $g$.

**4. Proof of Theorems 1.2–1.6.** Observe first that Theorem 1.2 follows from Theorem 1.3: Under the hypotheses of Theorem 1.2, where we may assume that $Y$ is complete, let $g$ be a selection for $\varphi|A$. Then $\varphi_g$ is l.s.c. by Lemma 2.5, so Theorem 1.3 implies that $\varphi_g$ has a selection $f$, and this $f$ is a selection for $\varphi$ which extends $g$.

It remains to prove Theorems 1.3–1.6. Now, as observed in the introduction, these results are known in case $\varphi(x)$ is closed in $Y$ for all $x\in X$, for in that case, Theorem 1.3 reduces to [6, Theorem 1.2], Theorem 1.4 follows from [13, Theorem 1.1], Theorem 1.5 follows from [12, Theorem 1.3], and Theorem 1.6 follows from [12, Theorem 1.4]. To prove Theorems 1.3–1.6, we will use Proposition 2.4 to reduce them to these known cases. Specifically, we will do that for Theorem 1.3; the process for Theorems 1.4–1.6 is essentially the same.

Let $g$ be a selection for $\varphi|A$. Then $\varphi_g$ is l.s.c. by Lemma 2.5. Let $C'=C-A$. Applying Proposition 2.4 to $\varphi_g$ and $C'$, we obtain a l.s.c. map $\psi\subset\varphi_g$ such that $\psi(x)$ is closed in $Y$ when $x\in C'$ and $\psi(x)=\varphi_g(x)$ when $x\notin C'$. Now $\psi$ also satisfies the hypotheses of Theorem 1.3, and $\psi(x)$ is closed in $Y$ for *every* $x\in X$. By the previous paragraph, $\psi$ therefore has a selection $f$, and this $f$ is a selection for $\varphi$ which extends $g$. That completes the proof.

**5. The selection approximation property.** Let us say that a map $\varphi\colon X\to 2^Y$, with $Y$ a metric space, has the *selection approximation property*, or *SAP*, if to every $\varepsilon>0$ corresponds a $\delta=\delta_\varphi(\varepsilon)>0$ satisfying the following condition: If $h\colon X\to Y$ is continuous with $d(h,\varphi)<\delta$ (9), and if $A\subset X$ is closed, then every selection $g$ for $\varphi|A$ with $d(g,h|A)<\delta$ extends to a selection $f$ for $\varphi$ with $d(f,h)<\varepsilon$.

All set-valued maps which appear in general selection theorems known to the author seem to have the SAP, and a closely related property (see [12, [3.2)]) is actually used — implicitly or explicitly — in the proofs of almost all of these theorems. Explicit appearances of the SAP (although without that name) can be found, for example, in [6, Corollary 4.2 and Theorem 9.1], in [12, Footnotes 8 and 10], and especially in [12, Section 3].

The purpose of this section is to prove the following result, which is somewhat more detailed and general than is needed for our applications in Sections 6 and 7.

PROPOSITION 5.1. *Suppose that $\varphi\colon X\to 2^Y$, with $Y$ complete metric, that $\bar\varphi$ has the SAP, and that $\varphi(x)=\bar\varphi(x)$ for all but countably many $x\in X$. Then*:

(a) *$\varphi$ has the SAP (and one can take $\delta_\varphi(\varepsilon)=\delta_{\bar\varphi}(\frac12\varepsilon)$).*

(b) *If $\bar\varphi$ has a selection, so does $\varphi$.*

(c) *If $\bar\varphi$ has the SEP, so does $\varphi$.*

(d) *If $\bar\varphi$ has the SNEP, if $X$ is normal, and if $\bar\varphi|E$ has the SAP for every closed $E\subset X$, then $\varphi$ has the SNEP.*

Proof. We shall first prove (a), and then show that (a) implies (b), (c) and (d).

(a) Suppose that $A\subset X$ is closed, that $g$ is a selection for $\varphi|A$, and that $h\colon X\to Y$ is continuous with $d(h,\varphi)<\delta_{\bar\varphi}(\frac12\varepsilon)$ and $d(h,g|A)\leqslant\delta_{\bar\varphi}(\frac12\varepsilon)$. We must extend $g$ to a selection $f$ for $\varphi$ such that $d(f,h)<\varepsilon$.

Let $C=\{x\in X\colon \varphi(x)\neq\bar\varphi(x)\}$, and write $C=\{x_1,x_2,\ldots\}$. Let $E_0=A$, and let $E_n=A\cup\{x_1,\ldots,x_n\}$ for $n\geqslant 1$. By induction, we shall construct selections $f_n$ $(n\geqslant 0)$ for $\bar\varphi$ which extend $g$ such that $f_n(x)\in\varphi(x)$ for $x\in E_n$, $f_{n+1}|E_n=f_n|E_n$, $d(f_0,g)<\frac12\varepsilon$ and $d(f_{n+1},f_n)<2^{-(n+1)}\varepsilon$. That will suffice, for then $f=\lim_n f_n$ will have all the required properties.

Let $f_0$ be any selection for $\bar\varphi$ which extends $g$ such that $d(f_0,h)<\frac12\varepsilon$. Now suppose that $f_0,\ldots,f_n$ have been chosen. Extend $f_n|E_n$ to a selection $k$ for $\varphi|E_{n+1}$ such that

$$d(k,f_n|E_{n+1})<\delta_{\bar\varphi}(2^{-(n+1)}\varepsilon)\,.$$

Then $k$ can be extended to a selection $f_{n+1}$ for $\bar\varphi$ such that $d(f_{n+1},f_n)<2^{-(n+1)}\varepsilon$. That completes the induction, and thus the proof of (a).

(a)→(b). Suppose $f$ is a selection for $\bar\varphi$. Then $d(f,\varphi)<\delta$ for all $\delta>0$, so $\varphi$ must have selection by (a).

(a)→(c). Proved similarly to (a)→(b).

(9) Here $d$ is the metric on $Y$, and $d(g,\varphi)<\varepsilon$ means that $d(g(x),\varphi(x))<\varepsilon$ for all $x\in X$.



(a)→(d). Suppose $A\subset X$ is closed, and $g$ is a selection for $\varphi|A$. Then $g$ is also a selection for $\bar\varphi|A$, so it can be extended to a selection $h$ for $\bar\varphi|U$ for some open $U\supset A$. Choose an open $V$ in $X$ such that $A\subset V\subset\bar V\subset U$. Then $d(h|\bar V,\varphi|\bar V)<\delta$ for all $\delta>0$, so we can apply (a) to $\varphi|\bar V$ to obtain a selection $f$ for $\varphi|\bar V$ which extends $g$.

That completes the proof.

**6. Proof of Theorem 1.7.** Since $\bar\varphi(x)=Y$ for all $x\in X$, the map $\bar\varphi$ clearly has a selection. By Proposition 5.1, it will therefore suffice to establish that $\bar\varphi$ has the SAP for some suitable metric on $Y$. In case $Y$ is an ANR, that follows [11, Theorem 7.2 and 1.1] (or the more general assertion in [12, Footnote 10]); in case $\dim X\leqslant n+1$ and $Y$ is $LC^n$, it follows from [6, Theorem 4.1].

**7. Another application of Proposition 5.1.** In this section, we use Proposition 5.1 to prove a theorem which combines Theorems 1.3 and 1.4.

THEOREM 7.1. *Let $X$ be paracompact, $Y$ a Banach space, $Z\subset X$ with $\dim_X Z\leqslant 0$, $C\subset X$ countable, and $\varphi\colon X\to 2^Y$ l.s.c. such that $\varphi(x)$ is closed in $Y$ when $x\notin C$ and $\bar\varphi(x)$ is convex when $x\notin Z$. Then $\varphi$ has the SEP.*

Proof. By Proposition 5.1, it will suffice to show that $\bar\varphi$ has the SEP and the SAP. Now $\bar\varphi$ satisfies the hypotheses of [13, Theorem 1.1], so that theorem implies that $\bar\varphi$ has the SEP. It remains to show that $\bar\varphi$ has the SAP.

Let $A\subset X$ be closed and $h\colon X\to Y$ continuous with $d(h,\varphi)<\frac12\varepsilon$. We will show that every selection $g$ for $\bar\varphi|A$ with $d(g,h|A)<\frac12\varepsilon$ extends to a selection $f$ for $\bar\varphi$ with $d(f,h)\leqslant\varepsilon$. Define $\psi\colon X\to 2^Y$ by

$$\psi(x)=\bar\varphi_g(x)\cap B_{\frac12\varepsilon}(h(x))\,.$$

Now $\bar\varphi_g$ is l.s.c. by Lemmas 2.6 and 2.5, and hence so is $\psi$ by [5, Proposition 2.5]. But then $\bar\psi$ is also l.s.c. by Lemma 2.6, so $\bar\psi$ satisfies the hypotheses of [13, Theorem 1.1] and therefore has a selection $f$. But then $f$ is a selection for $\bar\varphi$ which extends $g$, and $d(f,h)\leqslant\frac12\varepsilon<\varepsilon$. That completes the proof.

Remark. Theorem 7.1 reduces to [13, Theorem 1.1] when $C=\varnothing$, and to Theorem 1.4 when $C=Z$. To see that Theorem 7.1 also implies Theorem 1.3, let $\varphi\colon X\to 2^Y$, $A\subset X$ and $C\subset X$ be as in Theorem 1.3; since every metric space can be embedded isometrically in a Banach space, we may assume that $Y$ is a Banach space. Now if $g$ is a selection for $\varphi|A$, then $\varphi_g$ is l.s.c. by Lemma 2.5, so $\varphi_g$ has a selection $f$ by Theorem 7.1 (with $Z=X-A$), and this $f$ is a selection for $\varphi$ which extends $g$.

Remark. Observe that, unless $C\subset Z$, Theorem 7.1 does not meet our goal of eliminating all hypotheses on $\varphi$ — except lower semi-continuity — on the countable set $C$. To meet that goal, the assumption in Theorem 7.1 that "$\bar\varphi(x)$ is convex for $x\notin Z$" would have to be weakened to "$\bar\varphi(x)$ is convex for $x\notin(Z\cup C)$". Unfortunately, that would make the theorem false: For example, the weakened hypothesis is satisfied if $X=Y=R$, $C=Q$ (rationals), $Z=R-Q$, $\varphi(0)=\{0\}$ $\varphi(1)=\{1\}$

and $\varphi(x) = \{0, 1\}$ for all other $x \in X$, but this $\varphi$ clearly has no selection. The trouble, essentially, is that even though $C$ and $Z$ both are 0-dimensional, their union $C \cup Z$ is not.

Remark. In the same spirit as Theorem 7.1, one can obtain other compound theorems by applying Proposition 5.1 to [12, Theorem 1.2 and Footnote 8] or to [12, Theorem 1.3 and Footnote 10].

**8. The case of $\sigma$-discrete subsets.** Recall that a subset $S$ of a space $X$ is *discrete* if it has no accumulation point in $X$, and that $S$ is *$\sigma$-discrete* if $S = \bigcup_{n=1}^{\infty} S_n$ with each $S_n$ discrete. Now it is easy to check that our basic Propositions 2.4 and 5.1 remain valid, with essentially the same proofs, if "countable" is weakened to "$\sigma$-discrete", and hence the same is true of the other results in this paper ([10]). Only in Theorem 1.1 must we exercise some care: If "countable" is weakened to "$\sigma$-discrete" in that result, then "regular" must be strengthened "paracompact", since — unlike a countable regular space — a $\sigma$-discrete regular space need not be paracompact or even normal, and Theorem 1.1 cannot be true for a non-normal space $X$.

Since every metric space has a dense, $\sigma$-discrete subset, the above remarks yield the following corollary to Theorem 1.1.

COROLLARY 8.1. *Let $X$ be metrizable, $Y$ first-countable, $\varphi$: $X \to 2^Y$ l.s.c. and $A \subset X$ closed. Then every selection for $\varphi|A$ extends to a selection for $\varphi|D$ for some $D \supset A$ which is dense in $X$.*

**9. Examples.** Our first example illustrates the importance of $X$ being countable in Theorem 1.1. We denote the closed interval by $I$.

EXAMPLE 9.1. If $X$ is any separable metric space of cardinality $c$, then there exists an l.s.c. map $\varphi$: $X \to 2^I$, with $\bar{\varphi}(x) = I$ for all $x \in X$, which admits no selection ([11]).

Proof. Let $S = X \times I$, and let $\pi$: $S \to X$ be the projection. Precisely as in [6, Example 4.1], we can choose points $s_x \in I$ for each $x \in X$ such that, if $E = S - \{(x, s_x): x \in X\}$, then there is no $A \subset E$ such that $A$ is closed in $S$ and $\pi(A) = X$. Let $\varphi(x) = I - \{s_x\}$ for all $x \in X$. Then $\bar{\varphi}(x) = I$ for all $x \in X$, so $\varphi$ is l.s.c. by Lemma 2.5. If $\varphi$ had a selection $g$, and if $f(x) = (x, g(x))$, then $f(X) \subset E$, $\pi(f(X)) = X$, and $f(X)$ is closed in $S$, which is impossible.

Our next example, due to E. van Douwen and R. Pol [3], shows that Theorem 1.1 becomes false (even with $\varphi(x) = Y$ for all $x \in X$) if the assumption that $Y$ is first-countable is omitted.

EXAMPLE 9.2 [3]. There exists a closed subset $A$ of a countable, regular space $X$ which is not a retract of $X$.

([10]) I am grateful to R. Telgarski for calling this to my attention.

([11]) It can be shown that there does not even exist a l.s.c. map $\psi$: $X \to 2^Y$ with $\psi(x) \subset \varphi(x)$ and $\psi(x)$ closed in $Y$ for all $x \in X$.



The following example shows that Theorem 1.2 becomes false (even with $\varphi(x) = Y$ for all $x \in X$) if $Y$ is only assumed first-countable (as in Theorem 1.1).

EXAMPLE 9.3. There exists a paracompact space $X$, and a first-countable, compact $A \subset X$ with $X - A$ countable, such that $A$ is not a retract of $X$.

Proof. Let $Z$ be any first-countable, compact Hausdorff space of cardinality $c$ which is not separable. (For example, $Z$ can be the unit square topologized by dictionary ordering). Since card $Z = c$, $Z$ is homeomorphic to subset $A$ of $I^c$. By the Pondiczery–Hewitt–Marczewski theorem (see [4, p. 77, Theorem 7]), $I^c$ has a countable dense subset $D$. Let $X = A \cup D$. Then $X$ is separable while $A$ is not, so $A$ is not a retract of $X$. Since $X$ is regular and $\sigma$-compact, it is paracompact. That completes the proof.

Our next example shows why the completeness assumptions on the range $Y$ cannot be dropped from many of our results.

EXAMPLE 9.4. Suppose $Y$ is any non-empty first-category space ([12]), and $X$ any non-empty compact metric space without isolated points. Then there exists a map $\varphi$: $X \to 2^Y$, with $\bar{\varphi}(x) = Y$ for all $x \in X$ and $\varphi(x) = Y$ for all but countably many $x \in X$, which has no selection ([13]).

Proof. By assumption, $Y = \bigcup_n Y_n$ with each $Y_n$ a closed set with empty interior in $Y$. Since $X$ has no isolated points, it has a disjoint sequence $(C_n)$ of countable, dense subsets. Let $\varphi(x) = Y - Y_n$ if $x \in C_n$, and let $\varphi(x) = Y$ if $x \in X - \bigcup_n C_n$.

Suppose $\varphi$ had a selection $f$. Let $X_n = f^{-1}(Y_n)$. Then $X = \bigcup_n X_n$, and the $X_n$ are closed in $X$, so $\operatorname{Int} X_m \neq \emptyset$ for some $m$. Pick $x \in X_m \cap C_m$. Then $f(x) \in f(X_m) = Y_m$, and $f(x) \in f(C_m) \subset Y - Y_m$, a contradiction.

Our last example shows why the ANR and $LC^n$ assumptions on $Y$ are needed in Theorem 1.7.

EXAMPLE 9.5. A map $\varphi$: $X \to 2^Y$, with $X$ and $Y$ compact metric spaces, $\bar{\varphi}(x) = Y$ for all $x \in X$ and $\varphi(x) = Y$ for all but two $x \in X$, which admits no selection.

Proof. Let $X = I$, let $Y$ by the Cantor set, and let $Y_0$, $Y_1$ be disjoint, dense subsets of $Y$. Let $\varphi(0) = Y_0$, $\varphi(1) = Y_1$, and $\varphi(x) = Y$ for all other $x \in I$. Since $Y$ is totally disconnected, every continuous $f$: $X \to Y$ must be constant. However, there are clearly no constant selections for $\varphi$.

([12]) I.e. $Y = \bigcup_n Y_n$, with each $Y_n$ nowhere dense in $Y$. Recall that a non-empty complete metric space is never of the first category.

([13]) Our map $\varphi$ actually has the following stronger property: If $K \subset X \times Y$ is compact, and if $K \subset \bigcup \{\{x\} \times \varphi(x): x \in X\}$, then $\pi_1(K)$ has empty interior (where $\pi_1$: $X \times Y \to X$ is the projection).

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# Ultraparacompactness in certain Pixley–Roy hyperspaces

by

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**Abstract.** A $T_2$-space $Z$ is ultraparacompact if each open cover of $Z$ has a disjoint open refinement. In this paper we present a sequence of results which guarantee that for certain spaces $X$, the Pixley–Roy hyperspace construction has the property that for each finite $m$ and $n$, $(\mathscr{F}[X^m])^n$ is ultraparacompact. We also investigate ultrametrizability of certain PR-hyperspaces.

**1. Introduction.** This paper continues the study of the Pixley–Roy hyperspace initiated in [BFL]. Recall that for each space $X$, the space $\mathscr{F}[X]$, called the *Pixley–Roy hyperspace* of $X$, is the collection of all nonempty finite subsets of $X$ topologized by using all sets of the form

$$[F, V] = \{F' \in \mathscr{F}[X]:\ F \subset F' \subset V\}$$

as a neighborhood base at $F \in \mathscr{F}[X]$, where $V$ is allowed to be any open subset of $X$ which contains $F$. In [BFL] we proved that if $X$ is any first-countable subspace of any ordinal, then $\mathscr{F}[X]$ is metrizable. In [$L_2$] it was asserted that, for such an $X$, even $\mathscr{F}[X^2]$ is metrizable. In this paper we significantly sharpen (and simplify) both results by proving that if $X$ is any subspace of any ordinal then for each $m$, $n \leqslant \omega_0$, $(\mathscr{F}[X^m])^n$ is *ultraparacompact*, i.e., each open cover admits an open refinement which partitions the space. (Indeed, we prove a stronger, but more technical, result — see Section 2.) It follows immediately that if $X$ is a first-countable subspace of any ordinal then $(\mathscr{F}[X^m])^n$ is a Moore space (cf. [vD] or [$L_2$]) and is ultraparacompact and hence has a base of open sets which is the union of countably many subcollections, each of which is a disjoint open cover of the entire space. Spaces having such a base are called *ultrametrizable* and admit a compatible metric $d$ which satisfies a very strong triangle inequality, viz., for any points $x$, $y$ and $z$,

$$d(x, y) \leqslant \max\{d(x, z), d(z, y)\}.$$

Another result in [BFL] characterized those generalized ordered spaces $X$ built on a separable linearly ordered space (see Section 3 for definitions) for

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