manuscripta math. 125, 157–167 (2008)

Mihai Mihăilescu · Vicenţiu Rădulescu

Continuous spectrum for a class of nonhomogeneous differential operators

Received: 27 June 2007 / Revised: 10 August 2007
Published online: 6 October 2007

Abstract. We study the boundary value problem $-\text{div}((|\nabla u|^{p_1(x)-2}+|\nabla u|^{p_2(x)-2})\nabla u) = \lambda|u|^{q(x)-2}u$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, λ is a positive real number, and the continuous functions p_1, p_2, and q satisfy $1 < p_2(x) < q(x) < p_1(x) < N$ and $\max_{y\in\overline{\Omega}} q(y) < \frac{Np_2(x)}{N-p_2(x)}$ for any $x \in \overline{\Omega}$. The main result of this paper establishes the existence of two positive constants λ_0 and λ_1 with $\lambda_0 \leq \lambda_1$ such that any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue, while any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of the above problem.

1. Introduction and preliminary results

In this paper we are concerned with the study of the eigenvalue problem

$$\begin{cases} -\text{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u) = \lambda|u|^{q(x)-2}u, & \text{for } x \in \Omega \\ u = 0, & \text{for } x \in \partial\Omega\,, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary, λ is a positive real number, and p_1, p_2, q are continuous functions on $\overline{\Omega}$.

The study of eigenvalue problems involving operators with variable exponents growth conditions has captured a special attention in the last few years. This is in keeping with the fact that operators which arise in such kind of problems, like the $p(x)$-Laplace operator (i.e., $\text{div}(|\nabla u|^{p(x)-2}\nabla u)$, where $p(x)$ is a continuous positive function), are not homogeneous and thus, a large number of techniques which can be applied in the homogeneous case (when $p(x)$ is a positive constant) fail in this new setting. A typical example is the Lagrange multiplier theorem, which does not apply to the eigenvalue problem

$$\begin{cases} -\text{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda|u|^{q(x)-2}u, & \text{for } x \in \Omega \\ u = 0, & \text{for } x \in \partial\Omega\,, \end{cases} \tag{2}$$

M. Mihăilescu · V. Rădulescu (✉): Department of Mathematics, University of Craiova, 200585 Craiova, Romania. e-mail: vicentiu.radulescu@math.cnrs.fr; mmihailes@yahoo.com

V. Rădulescu: Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

Mathematics Subject Classification (2000): 35D05, 35J60, 35J70, 58E05, 68T40, 76A02

DOI: 10.1007/s00229-007-0137-8

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. This is due to the fact that the associated Rayleigh quotient is not homogeneous, provided both p and q are not constant.

On the other hand, problems like (2) have been largely considered in the literature in the recent years. We give in what follows a concise but complete image of the actual stage of research on this topic.

- In the case when $p(x) = q(x)$ on $\overline{\Omega}$, Fan, Zhang and Zhao [8] established the existence of infinitely many eigenvalues for problem (2) by using an argument based on the Ljusternik-Schnirelmann critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, Fan, Zhang and Zhao showed that Λ is discrete, $\sup \Lambda = +\infty$ and they pointed out that only under special conditions, which are somehow connected with a kind of monotony of the function $p(x)$, we have $\inf \Lambda > 0$ (this is in contrast with the case when $p(x)$ is a constant; then, we always have $\inf \Lambda > 0$).
- In the case when $\min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x)$ and $q(x)$ has a subcritical growth Mihăilescu and Rădulescu [12] used the Ekeland's variational principle in order to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin.
- In the case when $\max_{x\in\overline{\Omega}} p(x) < \min_{x\in\overline{\Omega}} q(x)$ and $q(x)$ has a subcritical growth a mountain-pass argument, similar with those used by Fan and Zhang in the proof of Theorem 4.7 in [7], can be applied in order to show that any $\lambda > 0$ is an eigenvalue of problem (2).
- In the case when $\max_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x)$ it can be proved that the energy functional associated to problem (2) has a nontrivial minimum for any positive λ (see Theorem 4.3 in [7]). Clearly, in this case the result in [12] can be also applied. Consequently, in this situation there exist two positive constants $\lambda^\star$ and $\lambda^{\star\star}$ such that any $\lambda \in (0, \lambda^\star) \cup (\lambda^{\star\star}, \infty)$ is an eigenvalue of problem (2).

In this paper we study problem (1) under the following assumptions:

$$1 < p_2(x) < \min_{y\in\overline{\Omega}} q(y) \le \max_{y\in\overline{\Omega}} q(y) < p_1(x) \quad \forall\, x \%in \overline{\Omega} \tag{3}$$

and

$$\max_{y\in\overline{\Omega}} q(y) < p_2^\star(x) \quad \forall\, x \in \overline{\Omega}, \tag{4}$$

where $p_2^\star(x) := \frac{Np_2(x)}{N-p_2(x)}$ if $p_2(x) < N$ and $p_2^\star(x) = +\infty$ if $p_2(x) \ge N$.

Thus, the case considered here is different from all the cases studied before. In this new situation we will show the existence of two positive constants λ_0 and λ_1 with $\lambda_0 \le \lambda_1$ such that any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of problem (1) while any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (1). An important consequence of our study is that, under hypotheses (3) and (4), we have

$$\inf_{u\in W_0^{1,p_1(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\,\mathrm{d}x + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\,\mathrm{d}x}{\displaystyle\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x} > 0\,.$$

That fact is proved by using the Lagrange Multiplier Theorem. The absence of homogeneity will be balanced by the fact that assumptions (3) and (4) yield

$$\lim_{\|u\|_{p_1(x)}\to 0} \frac{\displaystyle\int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\,\mathrm{d}x + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\,\mathrm{d}x}{\displaystyle\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x} = \infty$$

and

$$\lim_{\|u\|_{p_1(x)}\to \infty} \frac{\displaystyle\int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\,\mathrm{d}x + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\,\mathrm{d}x}{\displaystyle\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x} = \infty\,,$$

where $\|\cdot\|_{p_1(x)}$ stands for the norm in the variable exponent Sobolev space $W_0^{1,p_1(x)}(\Omega)$.

We start with some preliminary basic results on the theory of Lebesgue–Sobolev spaces with variable exponent. For more details we refer to the book by Musielak [14] and the papers by Edmunds et al. [4–6], Kovacik and Rákosník [10], Mihăilescu and Rădulescu [11,13], and Samko and Vakulov [16].

Set

$$C_+(\overline{\Omega}) = \{h;\ h\in C(\overline{\Omega}),\quad h(x)>1 \quad \text{for all} \quad x\in\overline{\Omega}\}.$$

For any $h\in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $p\in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u;\ u \text{ is a measurable real-valued function such that} \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x<\infty \right\}.$$

We define on this space the *Luxemburg norm* by

$$|u|_{p(x)} = \inf\left\{\mu>0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)}\mathrm{d}x \le 1\right\}.$$

Let $L^{p'(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv\,\mathrm{d}x\right| \le \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)} \tag{5}$$

holds true.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ then the following relations hold true

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+} \tag{6}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-} \tag{7}$$

$$|u_n - u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_n - u) \to 0. \tag{8}$$

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

The space $W_0^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space. We note that if $s \in C_+(\overline{\Omega})$ and $s(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact and continuous, where $p^\star(x)$ denotes the corresponding critical Sobolev exponent, that is, $p^\star(x) := \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^\star(x) = +\infty$ if $p(x) \ge N$.

For applications of Sobolev spaces with variable exponent we refer to Acerbi and Mingione [1], Chen, Levine and Rao [2], Diening [3], Halsey [9], Ruzicka [15], and Zhikov [18].

2. The main result

Since $p_2(x) < p_1(x)$ for any $x \in \overline{\Omega}$ it follows that $W_0^{1,p_1(x)}(\Omega)$ is continuously embedded in $W_0^{1,p_2(x)}(\Omega)$. Thus, a solution for a problem of type (1) will be sought in the variable exponent space $W_0^{1,p_1(x)}(\Omega)$.

We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of problem (1) if there exists $u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u \nabla v\, dx - \lambda \int_\Omega |u|^{q(x)-2} uv\, dx = 0\,,$$

for all $v \in W_0^{1,p_1(x)}(\Omega)$. We point out that if λ is an eigenvalue of problem (1) then the corresponding eigenfunction $u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}$ is a *weak solution* of problem (1).

Define

$$\lambda_1 := \inf_{u \in W_0^{1,p_1(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\, dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\, dx}{\displaystyle\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\, dx}.$$

Our main result is given by the following theorem.

Theorem 1. *Assume that conditions* (3) *and* (4) *are fulfilled. Then* $\lambda_1 > 0$. *Moreover, any* $\lambda \in [\lambda_1, \infty)$ *is an eigenvalue of problem* (1). *Furthermore, there exists a positive constant* λ_0 *such that* $\lambda_0 \leq \lambda_1$ *and any* $\lambda \in (0, \lambda_0)$ *is not an eigenvalue of problem* (1).

Proof. Let E denote the generalized Sobolev space $W_0^{1,p_1(x)}(\Omega)$. We denote by $\|\cdot\|$ the norm on $W_0^{1,p_1(x)}(\Omega)$ and by $\|\cdot\|_1$ the norm on $W_0^{1,p_2(x)}(\Omega)$.

Define the functionals $J, I, J_1, I_1 : E \to \mathbb{R}$ by

$$J(u) = \int_\Omega \frac{1}{p_1(x)} |\nabla u|^{p_1(x)}\, \mathrm{d}x + \int_\Omega \frac{1}{p_2(x)} |\nabla u|^{p_2(x)}\, \mathrm{d}x,$$

$$I(u) = \int_\Omega \frac{1}{q(x)} |u|^{q(x)}\, \mathrm{d}x,$$

$$J_1(u) = \int_\Omega |\nabla u|^{p_1(x)}\, \mathrm{d}x + \int_\Omega |\nabla u|^{p_2(x)}\, \mathrm{d}x,$$

$$I_1(u) = \int_\Omega |u|^{q(x)}\, \mathrm{d}x.$$

Standard arguments imply that $J,\ I \in C^1(E, \mathbb{R})$ and for all $u,\ v \in E$,

$$\langle J^{'}(u), v\rangle = \int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u \nabla v\, \mathrm{d}x,$$

$$\langle I^{'}(u), v\rangle = \int_\Omega |u|^{q(x)-2} u v\, \mathrm{d}x.$$

We split the proof of Theorem 1 into four steps.

- *Step 1.* We show that $\lambda_1 > 0$.

Since for any $x \in \overline{\Omega}$ we have $p_1(x) > q^+ \geq q(x) \geq q^- > p_2(x)$ we deduce that for any $u \in E$,

$$2(|\nabla u(x)|^{p_1(x)} + |\nabla u(x)|^{p_2(x)}) \geq |\nabla u(x)|^{q^+} + |\nabla u(x)|^{q^-}$$

and

$$|u(x)|^{q^+} + |u(x)|^{q^-} \geq |u(x)|^{q(x)}.$$

Integrating the above inequalities we find

$$2 \int_\Omega (|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)})\, \mathrm{d}x \geq \int_\Omega (|\nabla u|^{q^+} + |\nabla u|^{q^-})\, \mathrm{d}x \quad \forall\, u \in E \tag{9}$$

and

$$\int_\Omega (|u|^{q^+} + |u|^{q^-})\, dx \geq \int_\Omega |u|^{q(x)}\, dx \quad \forall\, u \in E. \tag{10}$$

By Sobolev embeddings, there exist positive constants λ_{q^+} and λ_{q^-} such that

$$\int_\Omega |\nabla u|^{q^+}\, dx \geq \lambda_{q^+} \int_\Omega |u|^{q^+}\, dx \quad \forall\, u \in W_0^{1,q^+}(\Omega) \tag{11}$$

and

$$\int_\Omega |\nabla u|^{q^-}\, dx \geq \lambda_{q^-} \int_\Omega |u|^{q^-}\, dx \quad \forall\, u \in W_0^{1,q^-}(\Omega). \tag{12}$$

Using again the fact that $q^- \leq q^+ < p_1(x)$ for any $x \in \overline{\Omega}$ we deduce that E is continuously embedded in $W_0^{1,q^+}(\Omega)$ and in $W_0^{1,q^-}(\Omega)$. Thus, inequalities (11) and (12) hold true for any $u \in E$.

Using inequalities (11), (12) and (10) it is clear that there exists a positive constant μ such that

$$\int_\Omega (|\nabla u|^{q^+} + |\nabla u|^{q^-})\, dx \geq \mu \int_\Omega |u|^{q(x)}\, dx \quad \forall\, u \in E. \tag{13}$$

Next, inequalities (13) and (9) yield

$$\int_\Omega (|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)})\, dx \geq \frac{\mu}{2} \int_\Omega |u|^{q(x)}\, dx \quad \forall\, u \in E. \tag{14}$$

By relation (14) we deduce that

$$\lambda_0 = \inf_{v \in E \setminus \{0\}} \frac{J_1(v)}{I_1(v)} > 0 \tag{15}$$

and thus,

$$J_1(u) \geq \lambda_0 I_1(u) \quad \forall\, u \in E. \tag{16}$$

The above inequality yields

$$p_1^+ \cdot J(u) \geq J_1(u) \geq \lambda_0 I_1(u) \geq \lambda_0 I(u) \quad \forall\, u \in E. \tag{17}$$

The last inequality assures that $\lambda_1 > 0$ and thus, step 1 is verified.

- *Step 2.* We show that λ_1 is an eigenvalue of problem (1).

Lemma 1. *The following relations hold true*:

$$\lim_{\|u\|\to\infty} \frac{J(u)}{I(u)} = \infty \tag{18}$$

and

$$\lim_{\|u\|\to 0} \frac{J(u)}{I(u)} = \infty. \tag{19}$$

Proof. Since E is continuously embedded in $L^{q^{\pm}}(\Omega)$ it follows that there exist two positive constants c_1 and c_2 such that

$$\|u\| \geq c_1 \cdot |u|_{q^+} \quad \forall\, u \in E \tag{20}$$

and

$$\|u\| \geq c_2 \cdot |u|_{q^-} \quad \forall\, u \in E. \tag{21}$$

For any $u \in E$ with $\|u\| > 1$ by relations (6), (10), (20), (21) we infer

$$\frac{J(u)}{I(u)} \geq \frac{\dfrac{\|u\|^{p_1^-}}{p_1^+}}{\dfrac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}} \geq \frac{\dfrac{\|u\|^{p_1^-}}{p_1^+}}{\dfrac{c_1^{-q^+}\|u\|^{q^+} + c_2^{-q^-}\|u\|^{q^-}}{q^-}}.$$

Since $p_1^- > q^+ \geq q^-$, passing to the limit as $\|u\| \to \infty$ in the above inequality we deduce that relation (18) holds true.

Next, let us remark that since $p_1(x) > p_2(x)$ for any $x \in \overline{\Omega}$, the space $W_0^{1,p_1(x)}(\Omega)$ is continuously embedded in $W_0^{1,p_2(x)}(\Omega)$. Thus, if $\|u\| \to 0$ then $\|u\|_1 \to 0$.

The above remarks enable us to affirm that for any $u \in E$ with $\|u\| < 1$ small enough we have $\|u\|_1 < 1$.

On the other hand, since (4) holds true we deduce that $W_0^{1,p_2(x)}(\Omega)$ is continuously embedded in $L^{q^{\pm}}(\Omega)$. It follows that there exist two positive constants d_1 and d_2 such that

$$\|u\|_1 \geq d_1 \cdot |u|_{q^+} \quad \forall\, u \in W_0^{1,p_2(x)}(\Omega) \tag{22}$$

and

$$\|u\|_1 \geq d_2 \cdot |u|_{q^-} \quad \forall\, u \in W_0^{1,p_2(x)}(\Omega). \tag{23}$$

Thus, for any $u \in E$ with $\|u\| < 1$ small enough, relations (7), (10), (22), (23) imply

$$\frac{J(u)}{I(u)} \geq \frac{\dfrac{\int_\Omega |\nabla u|^{p_2(x)}\,dx}{p_2^+}}{\dfrac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}} \geq \frac{\dfrac{\|u\|_1^{p_2^+}}{p_2^+}}{\dfrac{d_1^{-q^+}\|u\|_1^{q^+} + d_2^{-q^-}\|u\|_1^{q^-}}{q^-}}.$$

Since $p_2^+ < q^- \leq q^+$, passing to the limit as $\|u\| \to 0$ (and thus, $\|u\|_1 \to 0$) in the above inequality we deduce that relation (19) holds true. The proof of Lemma 1 is complete. □

Lemma 2. *There exists* $u \in E \setminus \{0\}$ *such that* $\frac{J(u)}{I(u)} = \lambda_1$.

Proof. Let $\{u_n\} \subset E \setminus \{0\}$ be a minimizing sequence for λ_1, that is,

$$\lim_{n\to\infty} \frac{J(u_n)}{I(u_n)} = \lambda_1 > 0. \tag{24}$$

By relation (18) it is clear that $\{u_n\}$ is bounded in E. Since E is reflexive it follows that there exists $u \in E$ such that, up to a subsequence, $\{u_n\}$ converges weakly to u in E. On the other hand, similar arguments as those used in the proof of Lemma 3.4 in [11] show that the functional J is weakly lower semi-continuous. Thus, we find

$$\liminf_{n\to\infty} J(u_n) \geq J(u). \tag{25}$$

By the compact embedding theorem for spaces with variable exponent and assumption $1 \leq \max_{y\in\overline{\Omega}} q(y) < p_1(x)$ for all $x \in \overline{\Omega}$ (see (3)) it follows that E is compactly embedded in $L^{q(x)}(\Omega)$. Thus, $\{u_n\}$ converges strongly in $L^{q(x)}(\Omega)$. Then, by relation (8) it follows that

$$\lim_{n\to\infty} I(u_n) = I(u). \tag{26}$$

Relations (25) and (26) imply that if $u \not\equiv 0$ then

$$\frac{J(u)}{I(u)} = \lambda_1.$$

Thus, in order to conclude that the lemma holds true it is enough to show that u is not trivial. Assume by contradiction the contrary. Then u_n converges weakly to 0 in E and strongly in $L^{q(x)}(\Omega)$. In other words, we will have

$$\lim_{n\to\infty} I(u_n) = 0. \tag{27}$$

Letting $\epsilon \in (0, \lambda_1)$ be fixed by relation (24) we deduce that for n large enough we have

$$|J(u_n) - \lambda_1 I(u_n)| < \epsilon I(u_n),$$

or

$$(\lambda_1 - \epsilon) I(u_n) < J(u_n) < (\lambda_1 + \epsilon) I(u_n).$$

Passing to the limit in the above inequalities and taking into account that relation (27) holds true we find

$$\lim_{n\to\infty} J(u_n) = 0.$$

That fact combined with relation (8) implies that actually u_n converges strongly to 0 in E, i.e. $\lim_{n\to\infty}\|u_n\| = 0$. By this information and relation (19) we get

$$\lim_{n\to\infty}\frac{J(u_n)}{I(u_n)} = \infty,$$

and this is a contradiction. Thus, $u \not\equiv 0$. The proof of Lemma 2 is complete. □

By Lemma 2 we conclude that there exists $u \in E\setminus\{0\}$ such that

$$\frac{J(u)}{I(u)} = \lambda_1 = \inf_{w\in E\setminus\{0\}}\frac{J(w)}{I(w)}. \tag{28}$$

Then, for any $v \in E$ we have

$$\frac{d}{d\epsilon}\frac{J(u+\epsilon v)}{I(u+\epsilon v)}\,|_{\epsilon=0} = 0\ .$$

A simple computation yields

$$\begin{aligned}&\int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u\nabla v\ \mathrm{d}x\cdot I(u) - J(u)\\ &\quad\cdot\int_\Omega |u|^{q(x)-2}uv\ \mathrm{d}x = 0 \quad \forall\ v \in E.\end{aligned} \tag{29}$$

Relation (29) combined with the fact that $J(u) = \lambda_1 I(u)$ and $I(u) \neq 0$ implies the fact that λ_1 is an eigenvalue of problem (1). Thus, step 2 is verified.

- *Step 3.* We show that any $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem (1).

  Let $\lambda \in (\lambda_1, \infty)$ be arbitrary but fixed. Define $T_\lambda : E \to \mathbb{R}$ by

$$T_\lambda(u) = J(u) - \lambda I(u).$$

Clearly, $T_\lambda \in C^1(E, \mathbb{R})$ with

$$\langle T_\lambda^{'}(u), v\rangle = \langle J^{'}(u), v\rangle - \lambda\langle I^{'}(u), v\rangle, \quad \forall\ u \in E.$$

Thus, λ is an eigenvalue of problem (1) if and only if there exists $u_\lambda \in E\setminus\{0\}$ a critical point of T_λ.

With similar arguments as in the proof of relation (18) we can show that T_λ is coercive, i.e. $\lim_{\|u\|\to\infty} T_\lambda(u) = \infty$. On the other hand, as we have already remarked, similar arguments as those used in the proof of Lemma 3.4 in [11] show that the functional T_λ is weakly lower semi-continuous. These two facts enable us to apply Theorem 1.2 in [17] in order to prove that there exists $u_\lambda \in E$ a global minimum point of T_λ and thus, a critical point of T_λ. In order to conclude that step 4 holds true it is enough to show that u_λ is not trivial. Indeed, since $\lambda_1 = \inf_{u\in E\setminus\{0\}}\frac{J(u)}{I(u)}$ and $\lambda > \lambda_1$ it follows that there exists $v_\lambda \in E$ such that

$$J(v_\lambda) < \lambda I(v_\lambda),$$

or

$$T_\lambda(v_\lambda) < 0.$$

Thus,

$$\inf_E T_\lambda < 0$$

and we conclude that u_λ is a nontrivial critical point of T_λ, or λ is an eigenvalue of problem (1). Thus, step 3 is verified.

- *Step 4.* Any $\lambda \in (0, \lambda_0)$, where λ_0 is given by (15), is not an eigenvalue of problem (1).

Indeed, assuming by contradiction that there exists $\lambda \in (0, \lambda_0)$ an eigenvalue of problem (1) it follows that there exists $u_\lambda \in E \setminus \{0\}$ such that

$$\langle J^{'}(u_\lambda), v\rangle = \lambda\langle I^{'}(u_\lambda), v\rangle \quad \forall\, v \in E.$$

Thus, for $v = u_\lambda$ we find

$$\langle J^{'}(u_\lambda), u_\lambda\rangle = \lambda\langle I^{'}(u_\lambda), u_\lambda\rangle,$$

that is,

$$J_1(u_\lambda) = \lambda I_1(u_\lambda).$$

The fact that $u_\lambda \in E \setminus \{0\}$ assures that $I_1(u_\lambda) > 0$. Since $\lambda < \lambda_0$, the above information yields

$$J_1(u_\lambda) \geq \lambda_0 I_1(u_\lambda) > \lambda I_1(u_\lambda) = J_1(u_\lambda).$$

Clearly, the above inequalities lead to a contradiction. Thus, step 4 is verified.

By steps 2, 3 and 4 we deduce that $\lambda_0 \leq \lambda_1$. The proof of Theorem 1 is now complete. □

Remark 1. At this stage we are not able to deduce whether $\lambda_0 = \lambda_1$ or $\lambda_0 < \lambda_1$. In the latter case an interesting question concerns the existence of eigenvalues of problem (1) in the interval $[\lambda_0, \lambda_1)$. We propose to the reader the study of these open problems.

Acknowledgments. The authors have been supported by Grants CNCSIS 79/2007 "Degenerate and Singular Nonlinear Processes" and CNCSIS 589/2007 "Analysis and Control of Nonlinear Differential Systems".

References

[1] Acerbi, E., Mingione, G.: Gradient estimates for the $p(x)$-Laplacean system. J. Reine Angew. Math. **584**, 117–148 (2005)

[2] Chen, Y., Levine, S., Rao, R.: Functionals with $p(x)$-growth in image processing. Department of Mathematics and Computer Science, Duquesne University, Technical Report 2004-01, available at www.mathcs.duq.edu/~sel/CLR05SIAPfinal.pdf

[3] Diening, L.: Theoretical and numerical results for electrorheological fluids. Ph.D. thesis, University of Frieburg, Germany (2002)

[4] Edmunds, D.E., Lang, J., Nekvinda, A.: On $L^{p(x)}$ norms. Proc. Roy. Soc. Lond. Ser. A **455**, 219–225 (1999)

[5] Edmunds, D.E., Rákosník, J.: Density of smooth functions in $W^{k,p(x)}(\Omega)$. Proc. Roy. Soc. Lond. Ser. A **437**, 229–236 (1992)

[6] Edmunds, D.E., Rákosník, J.: Sobolev embedding with variable exponent. Stud. Math. **143**, 267–293 (2000)

[7] Fan, X.L., Zhang, Q.: Existence of solutions for $p(x)$-Laplacian Dirichlet problem. Nonlinear Anal. **52**, 1843–1852 (2003)

[8] Fan, X., Zhang, Q., Zhao, D.: Eigenvalues of $p(x)$-Laplacian Dirichlet problem. J. Math. Anal. Appl. **302**, 306–317 (2005)

[9] Halsey, T.C.: Electrorheological fluids. Science **258**, 761–766 (1992)

[10] Kováčik, O., Rákosník, J.: On spaces $L^{p(x)}$ and $W^{1,p(x)}$. Czechoslov. Math. J. **41**, 592–618 (1991)

[11] Mihăilescu, M., Rădulescu, V.: A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. Proc. Roy. Soc. A Math. Phys. Eng. Sci. **462**, 2625–2641 (2006)

[12] Mihăilescu, M., Rădulescu, V.: On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent. Proc. Am. Math. Soc. **135**, 2929–2937 (2007)

[13] Mihăilescu, M., Rădulescu, V.: Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: an Orlicz-Sobolev space setting. J. Math. Anal. Appl. **330**, 416–432 (2007)

[14] Musielak, J.: Orlicz spaces and modular spaces. Lecture Notes in Mathematics, Vol. 1034. Springer, Berlin (1983)

[15] Ruzicka, M.: Electrorheological fluids: modeling and mathematical theory. Springer, Berlin (2002)

[16] Samko, S., Vakulov, B.: Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators. J. Math. Anal. Appl. **310**, 229–246 (2005)

[17] Struwe, M.: Variational methods: applications to nonlinear partial differential equations and Hamiltonian systems. Springer, Heidelberg (1996)

[18] Zhikov, V.: Averaging of functionals in the calculus of variations and elasticity. Math. USSR Izv. **29**, 33–66 (1987)