2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Continuous-time Errors-in-variables System Identification through Covariance Matching without Input Signal Modeling

Magnus Mossberg and Torsten Söderström

***Abstract*— The continuous-time errors-in-variables system identification problem is studied. The proposed solution is to use a covariance matching approach in which the input signal is not modeled, allowing for general types of input signals. As a consequence, no input signal parameters have to be estimated.**

I. Introduction

Consider the system

$$y_0(t) = \frac{B(p)}{A(p)} u_0(t), \tag{1}$$

where $A(p) = p^n + a_1 p^{n-1} + \ldots + a_n$ and $B(p) = b_1 p^{m-1} + \ldots + b_m$, with p denoting the differentiation operator. The discrete-time measurements $u(kh) = u_0(kh) + \tilde{u}(kh)$ and $y(kh) = y_0(kh) + \tilde{y}(kh)$, where h is the sampling interval and where $\tilde{u}(kh)$ and $\tilde{y}(kh)$ are discrete-time white noise sources, are available for $k = 1, \ldots, N$. The problem considered here is the one of estimating the parameters

$$\boldsymbol{\theta} = \begin{bmatrix} a_1 & \cdots & a_n & b_1 & \cdots & b_m \end{bmatrix}^T \tag{2}$$

from these measurements. This is the so called continuous-time errors-in-variables (EIV) problem, see for example [1]–[3], that appears in several engineering applications [4]. A survey of the discrete-time EIV problem is given in [5].

In [3] a covariance matching approach for solving the continuous-time EIV problem was considered. It was assumed that $u_0(t)$ could be described as a continuous-time ARMA process, and its parameters were estimated. The approach taken here is more general in the sense that $u_0(t)$ is only assumed to be a stationary process.

II. Preliminaries

For the stochastic process $s(t)$, let $r_s(\tau)$ be its covariance function and $\phi_s(\omega)$ denote the spectrum. Then [6]

$$r_s(\tau) = \int_{-\infty}^{\infty} e^{\mathrm{i}\omega\tau} \phi_s(\omega) d\omega \tag{3}$$

and

$$\begin{aligned} \mathrm{E}\{p^j s(t) p^k s(t)\} &= \int_{-\infty}^{\infty} (\mathrm{i}\omega)^j (-\mathrm{i}\omega)^k \phi_s(\omega) d\omega \\ &= \mathrm{i}^{j+k} (-1)^k \int_{-\infty}^{\infty} \omega^{j+k} \phi_s(\omega) d\omega. \end{aligned} \tag{4}$$

M. Mossberg is with the Department of Physics and Electrical Engineering, Karlstad University, Sweden. E-mail: Magnus.Mossberg@kau.se. His research was partially supported by Paper Surface Center at Karlstad University, Sweden.

T. Söderström is with the Division of Systems and Control, Department of Information Technology, Uppsala University, Sweden. E-mail: Torsten.Soderstrom@it.uu.se

Apparently, i^{j+k} is purely imaginary when $j+k$ is an odd integer. However, in that case the integral is zero, as the integrand becomes an odd function. Differentiation of (3) gives

$$\begin{aligned} p^\mu r_s(\tau)|_{\tau=0} &= \int_{-\infty}^{\infty} (\mathrm{i}\omega)^\mu e^{\mathrm{i}\omega\tau} \phi_s(\omega) d\omega|_{\tau=0} \\ &= \mathrm{i}^\mu \int_{-\infty}^{\infty} \omega^\mu \phi_s(\omega) d\omega \end{aligned} \tag{5}$$

and therefore

$$\mathrm{E}\{p^j s(t) p^k s(t)\} = (-1)^k p^{j+k} r_s(0). \tag{6}$$

In (5) it is assumed that the spectrum $\phi_s(\omega)$ decreases for large ω at least as $1/\omega^{2\mu+2}$, and similarly for (4).

III. Covariance matching

Define $z_0(t)$ as

$$z_0(t) = \frac{1}{A(p)} u_0(t) \tag{7}$$

which means that

$$y_0(t) = \sum_{j=1}^{m} b_j p^{m-j} z_0(t), \quad u_0(t) = \sum_{j=0}^{n} a_j p^{n-j} z_0(t), \tag{8}$$

with $a_0 = 1$. From the material in Section II, we have

$$\begin{aligned} &\mathrm{E}\{p^\mu y_0(t) p^\nu y_0(t)\} = (-1)^\nu p^{\mu+\nu} r_{y_0}(0) \\ &= (-1)^\nu \sum_{j=1}^{m}\sum_{k=1}^{m} (-1)^{m-k} b_j b_k p^{2m-j-k+\mu+\nu} r_{z_0}(0), \end{aligned} \tag{9}$$

$$\begin{aligned} &\mathrm{E}\{p^\mu u_0(t) p^\nu u_0(t)\} = (-1)^\nu p^{\mu+\nu} r_{u_0}(0) \\ &= (-1)^\nu \sum_{j=0}^{n}\sum_{k=0}^{n} (-1)^{n-k} a_j a_k p^{2n-j-k+\mu+\nu} r_{z_0}(0), \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\mathrm{E}\{p^\mu y_0(t) p^\nu u_0(t)\} = (-1)^\nu p^{\mu+\nu} r_{y_0 u_0}(0) \\ &= (-1)^\nu \sum_{j=1}^{m}\sum_{k=0}^{n} (-1)^{n-k} b_j a_k p^{m+n-j-k+\mu+\nu} r_{z_0}(0) \end{aligned} \tag{11}$$

for $\mu, \nu = 0, 1, \ldots$.

Now, (9)–(11) can be used to get a number of covariance matching equations, where $\{a_j\}_{j=1}^n$, $\{b_k\}_{k=1}^m$, and $\{p^\ell r_{z_0}(0)\}_{\ell \in L}$, with $L = \{2, 4, \ldots\}$, are regarded as unknowns. The covariance matching equations can be written as

$$\mathbf{A}(\boldsymbol{\theta})\mathbf{x} = \mathbf{c}, \tag{12}$$

978-1-4244-4524-0/09/$25.00 ©2009 AACC

where the elements of the matrix

$$\mathbf{A}(\boldsymbol{\theta}) \triangleq \begin{bmatrix} \mathbf{A}_{y_0}(\boldsymbol{\theta}) \\ \mathbf{A}_{u_0}(\boldsymbol{\theta}) \\ \mathbf{A}^{\text{even}}_{y_0 u_0}(\boldsymbol{\theta}) \\ \mathbf{A}^{\text{odd}}_{y_0 u_0}(\boldsymbol{\theta}) \end{bmatrix} \tag{13}$$

depend on $\boldsymbol{\theta}$, and where

$$\mathbf{x} = \begin{bmatrix} p^2 r_{z_0}(0) & p^4 r_{z_0}(0) & \cdots & p^\alpha r_{z_0}(0) \end{bmatrix}^T, \tag{14}$$

$$\mathbf{c} \triangleq \begin{bmatrix} \mathbf{c}^T_{y_0} & \mathbf{c}^T_{u_0} & (\mathbf{c}^{\text{even}}_{y_0 u_0})^T & (\mathbf{c}^{\text{odd}}_{y_0 u_0})^T \end{bmatrix}^T, \tag{15}$$

with

$$\mathbf{c}_{y_0} = \begin{bmatrix} p^2 r_{y_0}(0) & p^4 r_{y_0}(0) & \cdots & p^\beta r_{y_0}(0) \end{bmatrix}^T, \tag{16}$$

$$\mathbf{c}_{u_0} = \begin{bmatrix} p^2 r_{u_0}(0) & p^4 r_{u_0}(0) & \cdots & p^\gamma r_{u_0}(0) \end{bmatrix}^T, \tag{17}$$

$$\mathbf{c}^{\text{even}}_{y_0 u_0} = \begin{bmatrix} p^2 r_{y_0 u_0}(0) & p^4 r_{y_0 u_0}(0) & \cdots & p^\delta r_{y_0 u_0}(0) \end{bmatrix}^T, \tag{18}$$

$$\mathbf{c}^{\text{odd}}_{y_0 u_0} = \begin{bmatrix} p^1 r_{y_0 u_0}(0) & p^3 r_{y_0 u_0}(0) & \cdots & p^\epsilon r_{y_0 u_0}(0) \end{bmatrix}^T. \tag{19}$$

The odd order derivatives of $r_{y_0}(0)$ and $r_{u_0}(0)$ are not interesting to consider in the system of equations since they are zero and carry no information. The number ξ of unknowns to be estimated is $\xi = \dim\{\boldsymbol{\theta}\} + \dim\{\mathbf{x}\}$, where dim stands for dimension, and the number of equations $\kappa = \dim\{\mathbf{c}\}$. This means that the inequality $\xi \leqslant \kappa$ can be seen as a necessary identifiability condition. The structure of $\mathbf{A}(\boldsymbol{\theta})$ as well as the fulfillment of the identifiability condition are illustrated in the following example.

Example. Consider the second order case with $n = m = 2$, where the choices $\beta = 4$, $\gamma = 2$, $\delta = 4$, and $\epsilon = 3$ are made in (16)–(19). This means that $\alpha = 6$ in (14) and that

$$\mathbf{A}_{y_0}(\boldsymbol{\theta}) = \begin{bmatrix} b_2^2 & -b_1^2 & 0 \\ 0 & b_2^2 & -b_1^2 \end{bmatrix}, \tag{20}$$

$$\mathbf{A}_{u_0}(\boldsymbol{\theta}) = \begin{bmatrix} a_2^2 & -a_1^2 + 2a_2 & 1 \end{bmatrix}, \tag{21}$$

$$\mathbf{A}^{\text{even}}_{y_0 u_0}(\boldsymbol{\theta}) = \begin{bmatrix} b_2 a_2 & b_2 - b_1 a_1 & 0 \\ 0 & b_2 a_2 & b_2 - b_1 a_1 \end{bmatrix}, \tag{22}$$

$$\mathbf{A}^{\text{odd}}_{y_0 u_0}(\boldsymbol{\theta}) = \begin{bmatrix} b_1 a_2 - b_2 a_1 & b_1 & 0 \\ 0 & b_1 a_2 - b_2 a_1 & b_1 \end{bmatrix} \tag{23}$$

in (13). In this case, the number of unknowns ξ equals the number of equations κ, so the identifiability condition is fulfilled. □

The values of $p^{\mu+\nu} r_{y_0}(0)$, $p^{\mu+\nu} r_{u_0}(0)$, and $p^{\mu+\nu} r_{y_0 u_0}(0)$ are to be estimated from the measured data, giving the vector

$$\hat{\mathbf{c}} = \begin{bmatrix} \hat{\mathbf{c}}^T_{y_0} & \hat{\mathbf{c}}^T_{u_0} & (\hat{\mathbf{c}}^{\text{even}}_{y_0 u_0})^T & (\hat{\mathbf{c}}^{\text{odd}}_{y_0 u_0})^T \end{bmatrix}^T. \tag{24}$$

Note that the estimate $\hat{\mathbf{c}}$ is consistent in the case with *noisy data* as long as it is not based on $r_y(0)$ and $r_u(0)$. The estimation can be done in at least two ways as described next for $p^{\mu+\nu} r_y(0)$.

1) Estimate in a natural way the covariance function $r_y(\tau)$ as

$$\hat{r}_y(\ell h) = \frac{1}{N} \sum_{k=1}^{N} y(kh) y(kh + \ell h). \tag{25}$$

From these estimates for $\ell = 1, 2, \ldots$, make some numerical differentiation to get estimates of the derivative of $r_y(\tau)$ at $\tau = 0$. Avoid using $\ell = 0$ in order to avoid bias due to the measurement noise.

2) Make a numerical differentiation of the measured signal and use the so computed estimates to determine the derivative of $r_y(\tau)$ at $\tau = 0$. For example,

$$\hat{\dot{y}}(kh) = \frac{1}{h}\big(y(kh + h) - y(kh)\big), \tag{26}$$

$$\widehat{p^2 r_y(0)} = -\frac{1}{N} \sum_{k=1}^{N} \hat{\dot{y}}^2(kh). \tag{27}$$

The first alternative is most likely preferable in situations when the variance of the measurement noise is high.

Note that $r_{y_0 u_0}(0)$ could be included in (18) since it is possible to estimate with high accuracy in spite of the measurement noises. This would give an extra equation but also an additional unknown since $r_{z_0}(0)$ would appear in (14).

Based on the system of equations (12), the estimator

$$\{\hat{\boldsymbol{\theta}}, \hat{\mathbf{x}}\} = \arg \min_{\boldsymbol{\theta}, \mathbf{x}} J(\boldsymbol{\theta}, \mathbf{x}), \tag{28}$$

where

$$J(\boldsymbol{\theta}, \mathbf{x}) = ||\hat{\mathbf{c}} - \mathbf{A}(\boldsymbol{\theta})\mathbf{x}||^2_{\mathbf{Q}}, \tag{29}$$

for $\boldsymbol{\theta}$ and $\mathbf{x}$ is suggested. Here, $\mathbf{Q}$ is a symmetric and positive definite weighting matrix. From the separable least squares problem (28),

$$\hat{\mathbf{x}} = \big(\mathbf{A}^T(\boldsymbol{\theta})\mathbf{Q}\mathbf{A}(\boldsymbol{\theta})\big)^{-1} \mathbf{A}^T(\boldsymbol{\theta})\mathbf{Q}\hat{\mathbf{c}} \tag{30}$$

and

$$\hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} V(\boldsymbol{\theta}), \tag{31}$$

where

$$V(\boldsymbol{\theta}) = \hat{\mathbf{c}}^T\mathbf{Q}\hat{\mathbf{c}} - \hat{\mathbf{c}}^T\mathbf{Q}\mathbf{A}(\boldsymbol{\theta})\big(\mathbf{A}^T(\boldsymbol{\theta})\mathbf{Q}\mathbf{A}(\boldsymbol{\theta})\big)^{-1}\mathbf{A}^T(\boldsymbol{\theta})\mathbf{Q}\hat{\mathbf{c}}. \tag{32}$$

Future work includes numerical and statistical investigations of the proposed method.

REFERENCES

[1] S. Sagara, Z. J. Yang, and K. Wada, "Identification of continuous systems from noisy sampled input-output data," in *Proc. 9th IFAC/IFORS Symp. System Identification and Parameter Estimation*, Budapest, Hungary, July 8–12 1991, pp. 1622–1627.

[2] T. Söderström, E. K. Larsson, K. Mahata, and M. Mossberg, "Using continuous-time modeling for errors-in-variables identification," in *Proc. 14th IFAC Symp. System Identification*, Newcastle, Australia, March 29–31 2006, invited session.

[3] M. Mossberg, "Analysis of a covariance matching method for continuous-time errors-in-variables identification," in *Proc. 46th IEEE Conf. on Decision and Control*, New Orleans, LA, December 12–14 2007, pp. 5511–5515.

[4] S. Van Huffel and P. Lemmerling, Eds., *Total Least Squares and Errors-in-variables Modelling: Analysis, Algorithms, and Applications*. Dordrecht, The Netherlands: Kluwer, 2002.

[5] T. Söderström, "Errors-in-variables methods in system identification," *Automatica*, vol. 43, no. 6, pp. 939–958, June 2007, survey paper.

[6] T. Söderström, *Discrete-time Stochastic Systems*, 2nd ed. London, U.K.: Springer-Verlag, 2002.