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CONTINUOUS-TIME SKEWED MULTIFRACTAL PROCESSES AS A MODEL FOR FINANCIAL RETURNS

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Abstract

We present the construction of a continuous time stochastic process which has moments that satisfy an exact scaling relation, including odd order moments. It is based on a natural extension of the MRW construction described in [3]. This allows us to propose a continuous time model for the price of a financial asset that reflects most major stylized facts observed on real data, including asymmetry and multifractal scaling.

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1. Introduction

A striking feature of the prices of financial assets resides in that a certain amount of statistical properties, generally referred to as "stylized facts", appear to be universal. Indeed, it is for instance well known that for any considered asset, and for any time scale from few minutes to few months or years, the log-returns are a centered and uncorrelated time series with a heavy-tailed distribution, and that the absolute or squared log-returns time series presents long memory and persistence. We refer to [10] and [6] for a thorough review of the various statistical properties that can be observed on financial data. Formulating a probabilistic, continuous-time model that reflects most of these stylized facts is naturally of first importance, both from a theoretical and practical point of view, and has been the motivation of a considerable number of research papers.

Many recent empirical studies have also suggested that financial data share statistical properties with turbulent intermittent velocity fields [2, 6, 11]: areas of rapid and violent activities alternate with more peaceful ones, and this phenomenon repeats itself at any time-scale in the "same" way. So as to take this elaborate scale invariance

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into account, a natural approach is then to suppose that the local Hölder regularity of the underlying continuous signal is itself random, which also translates into a "multiscaling" or a "multifractal scaling" of price return fluctuations, see [13, 12] . A process X with stationary increments is said to have a multifractal scaling if it satisfies

$$\mathbb{E}\big[|X(t)|^q\big] \sim c_q t^{\zeta_q} \text{ as } t \to 0, \tag{1}$$

for all q's in some real interval, some positive constants c_q, and a scaling exponent $q \mapsto \zeta_q$ that is *nonlinear.* Since the pioneering work of Mandelbrot [15], the phenomenology of such multifractal models has provided new concepts and tools to analyze market fluctuations. It notably inspired the family of so-called "cascade" random processes that account for the main statistical properties of financial prices in an elegant and parsimonious way [7, 16] (see also [4] for a non financial approach). Moreover, these models are amenable to many analytical computations: they are easy to estimate and they lead to simple and yet very competitive solutions to the problem of conditional risk (volatility or VaR) forecasting (see [7, 2, 11]). However, although these multifractal models reproduce all the stylized facts we have already mentioned, we would like to point out one stylized fact that is not taken into account by them and actually neither by most mathematical models, namely the *leverage effect.*

The so-called leverage effect is a feature that is mostly present on stock and index prices: the variation of the log-return in the past is found to be negatively correlated to the volatility in the future (see [5]). Here, the volatility may be defined for instance as the squared or absolute log-return. So, basically, this effect quantifies the "panic" effect that takes place after a large downward move of the price which tends to increase the volatility much more than a large upward move would. Let us note that this effect induces two types of asymmetry in the price process. The first one (*time asymmetry*) is that the price process is not invariant under reversion of time. Indeed, the variation of the volatility in the past is not correlated to the variation of the log-return in the future (if it were true, then a simple arbitrage could actually be performed). The second one (*return asymmetry*) is that this effect implies a negative skewness of the distribution of log-returns. The higher the leverage effect, the higher the skewness, the higher the asymmetry of the implied volatility smile (see [6]).

Thus, it appears clearly important to incorporate these asymmetries in a probabilistic model of log-returns. Let us note that it has already been done in a non multifractal setting for instance by [5, 17, 18, 9]. However, explicitly constructing a skewed multifractal process with leverage effect has not yet been done, though we should mention two very interesting works. In the first one [19] the authors built a skewed model with some multiscaling properties and leverage effect in discrete-time. However, when the sampling pace goes to zero, eventhough the multiscaling properties converge to standard multifractal behavior (1), the skewness tends to zero so that any leverage effect unfortunately disappears. The second work [20] takes place in the setting of turbulence study and not of finance, and as such is not interested in this leverage effect.

In the present paper, we try to fill this gap, and show how one can obtain a skewed, continuous time, multifractal model for log-returns that reproduces all stylized facts mentioned in this introduction, including the leverage effect. Moreover, this modelization may be described as "parsimonious" insofar as it relies only on a very small number of scalar parameters. So as to briefly summarize our approach, let us say that we extend the so-called Multifractal Random Walk (MRW) model, which is one of the simplest multifractal, continuous "cascade" models [16, 2], by introducing some explicit correlations between log-returns and volatility.

However, by doing so, we also introduce correlations between log-returns, which is an undesirable feature when modeling the price of a financial asset. As we argue below, one can find a regime for the parameters of our model where these correlations are almost zero, while the leverage effect remain quite noticeable. Hence, the statistical properties of real financial data match rather closely those of our model, as we are able to check on simulations. It should though be noted that obtaining a continuous-time multifractal random model with leverage effect and uncorrelated increments remain an open problem and we hope that the present study is a helpful step toward solving it.

The paper is organized as follows. In Section 2, we present a brief overview of the (symmetrical) multifractal log-normal MRW model for financial data and discuss the first attempt to input asymmetry in this model [19]. In Section 3, we propose a new construction of a continuous-time, skewed, multifractal process that depends on a Hurst exponent $H > 1/2$. In Section 4, we investigate its scaling properties and the behaviour of all q-order moments (including the moments of order 3 and the skewness). In Section 5, we explicitly compute the leverage effect and discuss the choice of the parameter H, which affects both the correlation of the increments (which should be close to zero, as in the case of financial data) and the skewness and the leverage effect (which should be significantly non zero). A simulation scheme and some numerical simulations are presented in Section 6, which also contains a comparison with real data. Some computations and proofs that are used in the rest of the paper are finally postponed in the Appendix.

2. Multifractal processes

Let $X = \big(X(t),\, t \geq 0\big)$ be a real-valued stochastic process with stationary increments. For $t \geq 0$ and $\tau > 0$, we write $\delta_\tau X(t)$ for the increment $X(t+\tau) - X(t)$. When the moments of order p of X satisfy:

$$\mathbb{E}\big[|\delta_\tau X(t)|^q\big] \approx c_q \tau^{\zeta_q} \tag{2}$$

for (small) $\tau > 0$, it is usual to speak of either a *monofractal* scaling if the exponent ζ_q is a linear function of q, or a *multifractal* scaling if it is nonlinear.

2.1. The Multifractal Random Walk

In [16], two of us proposed the construction of a continuous-time stochastic random process that exhibits features quite similar to most stylized facts observed on the returns of financial assets (see [2]), including multifractal scaling, but excluding leverage effect. The Multifractal Random Walk (MRW) $X(t)$ can be defined as the continuous limit as n goes to $+\infty$ of the following discretized process:

$$X_n(t) = \sum_{k=0}^{\lfloor nt \rfloor} \varepsilon_n(k/n) e^{\omega_n(k/n)} \tag{3}$$

where the $\varepsilon_n(k/n)$'s are independent Gaussian random variables with mean equal to zero and variance equal to σ^2/n for some $\sigma > 0$, and ω_n is a Gaussian stationary process independent of the $\varepsilon_n(k)$'s. Alternatively, one can consider a continuous construction

$$X(t) = \sigma \lim \int_0^t e^{\omega_n(u)} dB(u) \quad \text{as } n \to +\infty$$

where B is a standard Brownian motion independent of ω_n, see [3]. The autocovariance of ω_n is the following:

$$\mathbb{C}\mathrm{ov}\big[\omega_n(j/n), \omega_n(k/n)\big] = \begin{cases} \lambda^2 \log \frac{nT}{|j-k|+1} & \text{if } \frac{|j-k|+1}{n} \leq T \\ 0 & \text{else,} \end{cases} \tag{4}$$

and the expectation of ω_n is such that $\mathbb{E}\big[e^{2\omega_n(\cdot)}\big] = 1$. Here, $\lambda^2 > 0$ is a parameter called intermittency coefficient, and $T > 0$ is a parameter (called integral scale) such that the scaling (2) holds exactly for all $\tau \in [0, T]$. More precisely, as shown in [3], the following relation holds for all $r \in [0, 1]$:

$$\big(X(rt),\, 0 \leq t \leq T\big) \overset{law}{=} r^{1/2} e^{\Omega_r} \big(X(t),\, 0 \leq t \leq T\big)$$

where Ω_r is a Gaussian random variable independent of X, and with expectation $-\lambda^2 \log(r^{-1})$ and variance $\lambda^2 \log(r^{-1})$. One can then deduce that the following multifractal scaling is satisfied for $\tau \in [0, T]$:

$$\mathbb{E}\big[|\delta_\tau X(t)|^q\big] = c_q \tau^{\zeta_q}$$

with $\zeta_q = q/2 - \lambda^2(q - q^2/2)$ and $c_q = \mathbb{E}\big[|X(T)|^q\big] T^{-\zeta_q}$ for all q's such that c_q is finite. It is furthermore shown in [3] that c_q is finite for all $q \in (0, \lambda^{-2})$ and $c_q = +\infty$ for all $q > \lambda^{-2}$, so that the process X does not have moments of all orders and the distribution of $X(t)$ is heavy-tailed.

2.2. Further extensions : towards a skewed model with leverage effect

The MRW model has been shown to have interesting applications to financial data, in particular in terms of volatility and Value at Risk forecasting, see [7, 2, 11]. However, one significant drawback of the MRW approach is that the distribution of $X(t)$ is symmetric, so that it does not reflect the skewness empirically observed on financial data. Moreover, since the two processes ε and ω are independent, it follows that the increments of the process X and the square of the increments are uncorrelated. Therefore, the model does not present the "leverage effect" observed on stocks and financial indexes prices.

The authors of [19] therefore proposed to modify the construction (3) in the following way:

$$\tilde{X}_n(t) = \sum_{k=0}^{\lfloor nt \rfloor} \varepsilon_n(k/n) e^{\omega_n(k/n) - n^\alpha \sum_{i<k} K(i/n, k/n)\varepsilon(i/n)}, \tag{5}$$

where K is a positive kernel. As they show, this enables to fairly well reproduce the leverage effect observed on the data, as well as retaining most of the nice properties of the original MRW. However, the authors note that this holds only for $n < +\infty$: indeed, the odd moments of the process vanish when n goes to $+\infty$, so that their approach only gives a discrete-time model for $t = 0,\, 1/n,\, \ldots, i/n,\, \ldots$ Another related work can be found in [20] where the authors study a skewed 3D generalization of the MRW model with applications to hydrodynamics.

In what follows, we propose an alternate, continuous-time, construction for a multifractal random walk with skewness and leverage effect. In particular, we modify the construction (3) by defining the noise ε as a *fractional* Gaussian noise with Hurst exponent H, where H is chosen in a regime where the increments of X are almost uncorrelated. Thus, our approach shares some connections with the previous works of [14] and [1] who also considered the question of constructing an MRW with a fractional noise. However, in these papers the noise ε is independent of the volatility process ω, whereas we will define them as correlated processes.

3. Construction of a continuous-time skewed MRW

3.1. Definition of the skewed process

Fix the following parameters: $\lambda \in (0, 1/2)$, $T > 0$, $\sigma > 0$, $H \in (1/2 + \lambda^2/2, 1)$. The parameters λ, T and σ are of similar nature as above, while H can be seen as a Hurst exponent as in [14] and [1]. We define a skewed multifractal random walk by:

$$X(t) = \lim X_l(t) \quad \text{as} \quad l \to 0 \tag{6}$$

where

$$X_l(t) = \int_0^t \varepsilon_l(u) e^{\omega_l(u)} du$$

and $(\varepsilon, \omega) = \Big(\big(\varepsilon_l(u), \omega_l(u)\big),\, u \in \mathbb{R},\, l \in (0, T)\Big)$ is a Gaussian process with values in $\mathbb{R}^2$ that satisfies the following properties:

Property 1. *$\big(\varepsilon_l(u), \omega_l(u)\big)$ is stationary in u, that is: for $u_1, \ldots, u_n, \tau \in \mathbb{R}$*

$$\Big(\big(\varepsilon_l(u_1), \omega_l(u_1)\big), \ldots, \big(\varepsilon_l(u_n), \omega_l(u_n)\big),\, l \in (0, T)\Big) \quad \text{and}$$

$$\Big(\big(\varepsilon_l(u_1 + \tau), \omega_l(u_1 + \tau)\big), \ldots, \big(\varepsilon_l(u_n + \tau), \omega_l(u_n + \tau)\big),\, l \in (0, T)\Big)$$

have the same law.

Property 2. *$\big(\varepsilon_l(u), \omega_l(u)\big)$ has independent increments in l, that is: for $l' < l$*

$$\Big(\big(\varepsilon_{l'}(u) - \varepsilon_l(u), \omega_{l'}(u) - \omega_l(u)\big),\, u \in \mathbb{R}\Big) \text{ and } \Big(\big(\varepsilon_l(u), \omega_l(u)\big),\, u \in \mathbb{R}\Big)$$

are independent.

Property 3. *The expectation of $\omega_l(u)$ is $-1/2\mathbb{V}\mathrm{ar}\big[\omega_l(u)\big]$ (so that $\mathbb{E}\big[e^{\omega_l(u)}\big] = 1$). The expectation of $\varepsilon_l(u)$ is zero.*

Note that the normalisation of ω is thus different from the MRW case presented in the previous section.

Property 4. *For $\tau \in \mathbb{R}$, let us write $\gamma_l^\omega(\tau)$ (resp. $\gamma_l^\varepsilon(\tau)$, $\gamma_l^{\omega\varepsilon}(\tau)$) for $\mathbb{C}\mathrm{ov}[\omega_l(u), \omega_l(u + \tau)]$ (resp. $\mathbb{C}\mathrm{ov}[\varepsilon_l(u), \varepsilon_l(u + \tau)]$, $\mathbb{C}\mathrm{ov}[\varepsilon_l(u), \omega_l(u + \tau)]$), and let us define*

$$\gamma^\omega(\tau) = \lambda^2 \max\big(\log(T/|\tau|), 0\big) \tag{7}$$

$$\gamma^{\varepsilon}(\tau) = c^{\varepsilon}\sigma^2|\tau|^{-2+2H} \tag{8}$$

$$\gamma^{\omega\varepsilon}(\tau) = c^{\omega\varepsilon}\sigma\lambda\big(\max(\tau,0)\big)^{-1+H} \tag{9}$$

for some positive constants c^{ε} and $c^{\omega\varepsilon}$ (we use the convention $0^{-1} = +\infty$). Then for fixed τ, $\gamma_l^{\omega}(\tau) \uparrow \gamma^{\omega}(\tau)$ (resp. $\gamma_l^{\varepsilon}(\tau) \uparrow \gamma^{\varepsilon}(\tau)$, $\gamma_l^{\omega\varepsilon}(\tau) \uparrow \gamma^{\omega\varepsilon}(\tau)$) as l goes to zero. Moreover, $\gamma_l^{\omega\varepsilon}(\tau)$ is zero for $l \in (0,T)$ and $\tau \leq 0$.

Property 5. *We have the following scaling equations for $l \in (0,T)$, $\tau \in [-T,T]$ and $r \in (0,1]$:*

$$\gamma_{rl}^{\omega}(r\tau) = -\lambda^2 \log(r) + \gamma_l^{\omega}(\tau),$$

$$\gamma_{rl}^{\varepsilon}(r\tau) = r^{-2+2H}\gamma_l^{\varepsilon}(\tau),$$

and

$$\gamma_{rl}^{\omega\varepsilon}(r\tau) = r^{-1+H}\gamma_l^{\omega\varepsilon}(\tau).$$

As explained below, these properties are sufficient to prove the convergence $X_l \to X$ and study the properties of X. However, we still need to justify the existence of such a process (ε,ω); this is done in Subsection 3.3 where we explicitly construct an example of (ε,ω) that satisfies the above properties. The exact values $\gamma_l^{\omega}(\tau)$, $\gamma_l^{\varepsilon}(\tau)$ and $\gamma_l^{\omega\varepsilon}(\tau)$ for $\tau \in \mathbb{R}$ and $l \in (0,T)$ as well as the constants c^{ε} and $c^{\omega\varepsilon}$ corresponding to this construction can all be found in Appendix A.

Remark: It can be immediately seen that the function γ^{ε} in Property 4 is the covariance of a fractional Brownian noise with Hurst exponent H, so that the process $\big(\int_0^t \varepsilon_l(u)du,\, t \geq 0\big)$ converges in law to a fractional Brownian motion as l goes to 0. (One could easily prove that the convergence also holds under stronger modes, however this is of little interest for our purpose here.)

3.2. Existence of X

We here prove the existence and nondegeneracy of the process X. Let us define the following condition $\mathcal{H}(p)$ on $p \geq 2$

$$\mathcal{H}(p): \quad pH - \frac{\lambda^2}{2}p(p-1) - 1 > 0. \tag{10}$$

Note that since we chose $H \in (1/2 + \lambda^2/2, 1)$, then $\mathcal{H}(2)$ is always satisfied.

We first state two useful results:

Proposition 1. *Let $(\mathcal{F}_l)_{l>0}$ be the following filtration:*

$$\mathcal{F}_l = \sigma\Big\{\big(\varepsilon_{l'}(u), \omega_{l'}(u)\big),\, u \in \mathbb{R},\, l' \geq l\Big\}.$$

Then for fixed $t > 0$, $\big(X_l(t),\, l > 0\big)$ is an $\mathcal{F}_l$-martingale.

Proof. This is a straightforward application of Properties 2 and 3.

□

Proposition 2. *Let $p \geq 2$ be an integer. Then for $t \in [0, T]$,*

$$\lim_{l \to 0} \mathbb{E}\big[X_l(t)^p\big] = K(p) t^{pH - \frac{\lambda^2}{2} p(p-1)}, \tag{11}$$

where $K(p) \in (0, +\infty]$ is a constant that depends on the parameters σ^2, T, λ^2, and H, but not on t. Moreover, $K(p)$ is finite if and only if p satisfies $\mathcal{H}(p)$.

The proof of this proposition is postponed in Appendix B, where the reader will also find the value of the constant $K(p)$.

It is then easy to prove the following:

Theorem 1. *For fixed $t \geq 0$, $X_l(t)$ goes to a nondegenerate limit $X(t)$ as l goes to 0, and the convergence holds almost surely and in L^3. Moreover, the process $X = \big(X(t),\, t \geq 0\big)$ is well defined as a continuous version of the almost sure limit of $\big(X_l(t),\, t \geq 0\big)$ in the space of continuous functions.*

Proof. Applying Proposition 1 and a classical result of the theory of martingales, if for some fixed $t > 0$, the moments $\mathbb{E}\big[|X_l(t)|^p\big]$ remain bounded for some $p > 1$, we have:

$$X(t) = \lim X_l(t) \quad \text{as} \quad l \to 0$$

almost surely and in L^q for all $q \in [1, p)$. Since we choose $H > 1/2 + \lambda^2/2$ and $\lambda^2 < 1/4$, $\mathcal{H}(4)$ holds. Proposition 2 then proves the first half of the statement of the theorem.

Moreover, the same Proposition 2 shows that the Kolmogorov criterion for convergence and regularity of stochastic processes is satisfied: there exists some $a > 0$, $b > 0$ and $c > 0$ such that for any $t \in [0, T]$ and any $l > 0$ small enough,

$$\mathbb{E}\big[|X_l(t)|^a\big] \leq c t^{1+b}.$$

Indeed, we can choose $a = 4$ and $b = 4H - 6\lambda^2 - 1$. The rest of the theorem follows from a standard application of this criterion.

□

Remark: From Properties 1 and 3, X is a process with stationary increments and zero expectation.

3.3. Explicit construction of (ε, ω)

The construction of the process ω is almost the same as the one used in the definition of the symmetrical MRW in [3]. The process ε is also constructed in a similar fashion. Let us consider the "time-scale" half-plane $\mathbb{R} \times (0, +\infty)$, and define on it a 2D Gaussian white noise $P(dt', dl')$ with variance $l'^{-2} dt' \times dl'$. Then ω is obtained as:

$$\omega_l(t) = -\lambda^2/2\big(\log(T/l) + 1\big) + \lambda \int_{\mathcal{A}_l(t)} P(dt', dl'),$$

where $\mathcal{A}_l(t)$ is the conical domain

$$\mathcal{A}_l(t) = \big\{(t', l') \in \mathbb{R} \times (0, +\infty),\, l' \geq l, 0 \leq t - t' \leq \min(l', T)\big\}.$$

So as to construct ε, we now consider the domain $\mathcal{B}_l(t)$:

$$\mathcal{B}_l(t) = \{(t', l') \in \mathbb{R} \times (0, +\infty),\, l' \geq l, 0 \leq t' - t \leq l'\}$$

and define

$$\varepsilon_l(t) = \sigma \int_{\mathcal{B}_l(t)} l'^{-1+H} P(dt', dl').$$

We refer to Figure 1 for a graphical representation of $\mathcal{A}_l(t)$ and $\mathcal{B}_l(t)$. In Appendix A we give the exact first and second moments of (ε, ω), which are obtained through straightforward computations. It is then easy to check that (ε, ω) satisfies Properties 1 to 5. In particular, the constants c^ε and $c^{\omega\varepsilon}$ of Property 4 are respectively equal to $\frac{1}{(2-2H)(3-2H)}$, and $\frac{2^{2-H}-2}{(1-H)(2-H)}$.

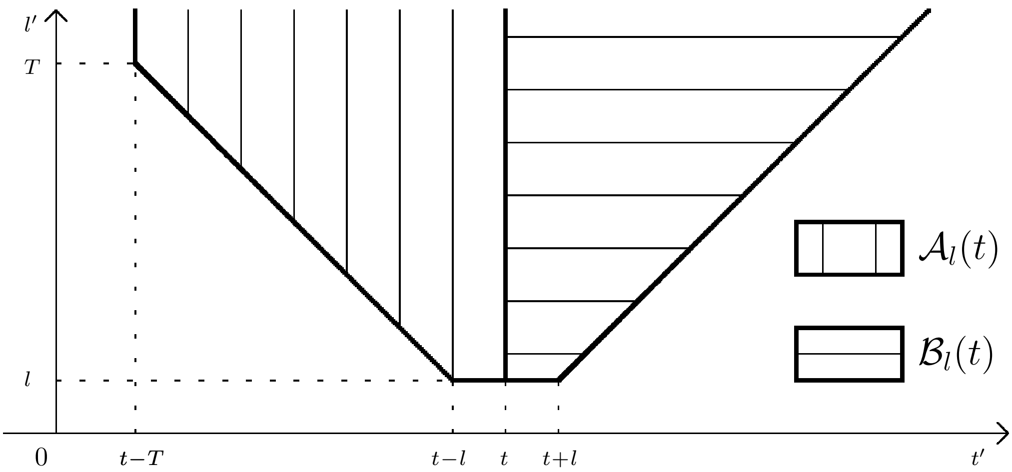

FIGURE 1: *The cones $\mathcal{A}_l(t)$ and $\mathcal{B}_l(t)$*

4. Scaling and moments of the skewed process X

The following theorem characterizes the scaling behavior of the distribution of $X(t)$:

Theorem 2. *For $r \in (0, 1]$,*

$$\big(X(rt),\, 0 \leq t \leq T\big) \stackrel{law}{=} e^{\Omega_r} r^H \big(X(t),\, 0 \leq t \leq T\big) \tag{12}$$

where $\Omega_r \sim N(-\lambda^2 \ln(r^{-1})/2, \lambda^2 \ln(r^{-1}))$ is a Gaussian constant independent of X.

Proof. First note that it follows from Properties 3 and 5 that for fixed $l > 0$ and $r \in (0, 1]$

$$\Big(\big(\varepsilon_{rl}(ru), \omega_{rl}(ru)\big),\, u \in [0, T]\Big) \stackrel{law}{=} \Big(\big(r^{-1+H}\varepsilon_l(u), \Omega_r + \omega_l(u)\big),\, u \in [0, T]\Big),$$

where Ω_r is a $N(-\lambda^2 \ln(r^{-1})/2, \lambda^2 \ln(r^{-1}))$ random variable, which is furthermore independent of (ε, ω). From this we deduce:

$$\Big(\varepsilon_{rl}(ru) \exp\big(\omega_{rl}(ru)\big),\, u \in [0, T]\Big) \stackrel{law}{=} r^{-1+H} \exp(\Omega_r) \Big(\varepsilon_l(u) \exp\big(\omega_l(u)\big),\, u \in [0, T]\Big)$$

We now consider the process $X_{rl}(rt)$:

$$\begin{aligned}\Big(X_{rl}(rt),\, t\in[0,T]\Big) &= \Big(\int_0^{rt} \varepsilon_{rl}(u)\exp\big(\omega_{rl}(u)\big)du,\, t\in[0,T]\Big)\\ &= \Big(r\int_0^{t} \varepsilon_{rl}(ru)\exp\big(\omega_{rl}(ru)\big)du,\, t\in[0,T]\Big)\\ &\overset{law}{=} r^H\exp(\Omega_r)\Big(\int_0^{t} \varepsilon_{l}(u)\exp\big(\omega_{l}(u)\big)du,\, t\in[0,T]\Big)\\ &= r^H\exp(\Omega_r)\Big(X_l(t),\, t\in[0,T]\Big).\end{aligned}$$

Taking the limit $l \to 0$ gives (12).

□

We now turn on the absolute moments of X, and show that they satisfy (2) with an exact equality.

Theorem 3. *If for some $q > 0$, there is an even integer $p > q$ such that $\mathcal{H}(p)$ is satisfied, then for $t \in [0,T]$*

$$\mathbb{E}\big[|X(t)|^q\big] = C(q)t^{qH-\frac{\lambda^2}{2}q(q-1)}, \tag{13}$$

where $C(q)$ is the positive finite constant

$$C(q) = T^{-qH+q(q-1)\lambda^2/2}\mathbb{E}\big[|X(T)|^q\big].$$

If q is moreover an integer, then

$$\mathbb{E}\big[X(t)^q\big] = K(q)t^{qH-\frac{\lambda^2}{2}q(q-1)}, \tag{14}$$

where $K(q)$ is the same as in Proposition 2. Conversely, if $\mathcal{H}(q)$ is not satisfied for some $q > 2$, $\mathbb{E}\big[|X(t)|^q\big] = +\infty$ for $t > 0$.

Proof. Propositions 1 and 2 yield that $X_l(T)$ converges in L^q to $X(T)$, so that $\mathbb{E}\big[|X(T)|^q\big]$ is finite. In the case where q is an integer, (14) is also a direct consequence of these two propositions. In the general case, we apply Theorem 2: by setting $r = t/T$, we have:

$$\mathbb{E}\big[|X(t)|^q\big] = T^{-qH+q(q-1)\lambda^2/2}\mathbb{E}\big[|X(T)|^q\big]t^{qH-\frac{\lambda^2}{2}q(q-1)}.$$

Conversely, let us suppose that $\mathbb{E}\big[|X(t)|^q\big]$ is finite for some $t \in (0,T]$ and $q > 1$. Then from the stationarity of the increments of X and a basic convexity inequality, we obtain:

$$\mathbb{E}\big[|X(t)|^q\big] > 2\mathbb{E}\big[|X(t/2)|^q\big].$$

Then, applying Theorem 2, we have $2^{1-qH+\lambda^2 q(q-1)/2} < 1$ so that $\mathcal{H}(q)$ is satisfied. This proves the result.

□

5. Modeling the asymmetry of financial data

5.1. Preliminaries

In this section, we focus on the second and third order properties of the increments of the process X, and show how, depending on the value of the parameter H, they can reflect the following stylized facts: first, there is no statistically significant correlation between two log-returns at different times. Second, there is a negative, slightly significant correlation between the past log-returns and the future squared log-returns, while the converse is false: past volatilities and future returns appear to be uncorrelated. (Note that while the former fact is universally observed on financial assets prices, the latter is mainly observed on stocks and indices prices, see [6].)

Let us examine the moment of second order of X:

Proposition 3. *For $t \in [0, T]$,*

$$\mathbb{E}\big[X(t)^2\big] = \frac{2\sigma^2 T^{\lambda^2} c^{\varepsilon}}{(2H-1-\lambda^2)(2H-\lambda^2)} t^{2H-\lambda^2}.$$

Proof. This is a simple application of Proposition 2 and of the value of $K(2)$ given in Appendix B.

□

Let us now recall that X is properly defined only in the case $H \in (1/2+\lambda^2/2, 1)$ (so that $\mathcal{H}(4)$ holds). Thus we can write $\mathbb{E}\big[X(t)^2\big] = K(2)t^{1+d}$ with $K(2)$ being the above fraction and

$$d = 2H - 1 - \lambda^2 > 0.$$

So as to obtain a satisfying model of financial data, we clearly have to place ourselves in a regime where d is small, so that $\mathbb{E}\big[X(t)^2\big]$ scales approximately as a linear function of t, and the covariance between the increments of X at different times vanishes. However, as can be seen from Proposition 3, $K(2)$ goes to $+\infty$ as d goes to 0.

In this section, we therefore study in the regime of small d the second and third order properties of the normalized process:

$$Y_d(t) = -\sigma \frac{X(t)}{\big(\mathbb{E}\big[X(1)^2\big]\big)^{1/2}}.$$

Note that we introduced a minus sign so as to reproduce the negative skewness empirically observed, and we added a d subscript to emphasize the dependence on this parameter (we will continue to use the notation $d \in (0, 1-\lambda^2)$ instead of $H \in (1/2+\lambda^2/2, 1)$). We intend to show here that for a well chosen value of d, the process Y_d reproduces the type of asymmetry observed on stocks and indices prices.

Recall that the notation $\delta_\tau Y_d(t)$ refers to the increment $Y_d(t+\tau) - Y_d(t)$. For $\tau > 0$, $k \in \mathbb{Z}$, we are interested in the following functions:

$$\rho_d^{(1)}(\tau, k) = \frac{\mathbb{E}\big[\delta_\tau Y_d(0)\delta_\tau Y_d(k\tau)\big]}{\mathbb{E}\big[\delta_\tau Y_d(0)^2\big]} \tag{15}$$

and

$$\rho_d^{(2)}(\tau,k) = \frac{\mathbb{E}\big[\delta_\tau Y_d(0)\delta_\tau Y_d(k\tau)^2\big]}{\big(\mathbb{E}\big[\delta_\tau Y_d(0)^2\big]\big)^2}, \tag{16}$$

where the normalization of $\rho^{(2)}$ has been introduced in [5] and further used in the literature, for instance in [9] or [17]. Alternatively, one could wish to examine the proper linear correlation between $\delta_\tau Y_d(0)$ and $\delta_\tau Y_d(k\tau)^2$, that is:

$$\rho_d^{(3)}(\tau,k) = \frac{\mathbb{E}\big[\delta_\tau Y_d(0)\delta_\tau Y_d(k\tau)^2\big]}{\big(\mathbb{E}\big[\delta_\tau Y_d(0)^2\big]\big)^{1/2}\big(\mathbb{E}\big[\delta_\tau Y_d(0)^4\big]\big)^{1/2}}. \tag{17}$$

Since we are dealing with correlations that empirically decays to zero after a few lags, we will restrict ourselves to the case $(|k|+1)\tau \le T$.

5.2. Behavior of Y_d in the regime of small d

We first examine the moments of Y_d as d goes to 0:

Proposition 4. *For $t \in [0,T]$, and $p \ge 2$ an even integer such that $\mathbb{E}\big[X(t)^p\big] < +\infty$, $\mathbb{E}\big[Y_d(t)^p\big]$ remains positive and bounded as d goes to 0. However, if p is odd, $\mathbb{E}\big[Y_d(t)^p\big]$ goes to zero.*

Proof. Again, this is a direct consequence of Proposition 2 and of the value of $K(p)$ given in Appendix B.

□

Remark: This suggests that a limiting process

$$Y(t) = \lim Y_d(t) \quad \text{as} \quad d \to 0$$

may exist, where the convergence is understood as a convergence in distribution. This is quite reminiscent of the study of [1] who investigated the validity of the limit

$$\lim \frac{\int_0^t e^{\omega_l}(u)dB^H(u)}{\mathbb{E}\Big[\big(\int_0^t e^{\omega_l}(u)dB^H(u)\big)^2\Big]^{1/2}} \quad \text{as} \quad l \to 0,$$

where B^H is a fractionary Brownian motion with Hurst exponent H and which is (in contrast with our setting) independent of ω. In the case $H = 1/2 + \lambda^2/2$ (that is, $d = 0$ in our notations), these authors obtained only the convergence of the moments of integer order and postulated the convergence in law. Note however that in the present work, we are not chiefly interested in the validity of the convergence $Y_d \to Y$, since the moments of order 3 of Y_d vanish, so that the limiting process Y_0 (if it exists) has a skewness equal to 0.

We now place ourselves in the regime of small but nonzero values of d, and examine the magnitude of the correlation functions $\rho_d^{(i)}$, $i = 1, 2, 3$.

Theorem 4. *For $0 < \tau < T$ and $(|k|+1)\tau \le T$, we have:*

$$\rho_d^{(1)}(\tau,k) = O(d) \text{ if } |k| \ge 1$$
$$|\rho_d^{(2)}(\tau,k)| = O(d^{1/2}) \text{ and } |\rho_d^{(3)}(\tau,k)| = O(d^{1/2}) \text{ if } k \ge 0$$
$$|\rho_d^{(2)}(\tau,k)| = O(d^{3/2}) \text{ and } |\rho_d^{(3)}(\tau,k)| = O(d^{3/2}) \text{ if } k < 0$$

as $d \to 0$.

The proof of this theorem can be found in Appendix C.

This therefore suggests that when d is of order roughly 0.01 to 0.1, $\rho_d^{(2)}(\tau,k)$ for $k \ge 0$ is significantly non zero (as it is of order $d^{1/2}$), while $\rho_d^{(2)}(\tau,k)$ for $k < 0$ and $\rho_d^{(1)}(\tau,k)$ for $k \ne 0$ are much smaller, and in practice indiscernible from the noise. We refer to Section 6.2 for a empirical discussion concerning the choice of d.

6. Numerical simulation and comparison to real data

In this section we present a numerical method for simulating the process we introduce, and we compare the leverage effect observed on simulations to the leverage effect measured on empirical data. Since the main objective of this paper was to define and study the mathematical properties of the model, we do not discuss any parameter estimation issue. This problem and a more exhaustive comparison to financial data will be the subject of a forthcoming work.

6.1. The simulation scheme

We propose in this section to approximate the increments

$$\delta_\tau X(k\tau) = \lim_{l \to 0} \int_{k\tau}^{(k+1)\tau} \varepsilon_l(u) e^{\omega_l(u)} du$$

of the process X (for $k \in \mathbb{N}$ and $\tau > 0$) by Riemann sums. If the parameter $d = 2H - 1 - \lambda^2$ defined in the previous section is large enough, this is easily done, however if this parameter is small, then some extra difficulties must be taken care of. This comes mainly from the fact that the approximation

$$1/n \sum_{k=1}^{n} (k/n)^{-1+d} \approx \int_0^1 u^{-1+d} du$$

is valid only in a regime $n \gg e^{1/d}$ which might be unfeasible in practice.

We set $(l_n,\, n \in \mathbb{N})$ as:

$$l_n = \left(d(1 - d/2 - \lambda^2/2)\right)^{1/(1-d)} n^{-1},$$

and

$$\delta_\tau \tilde{X}_{1/n}(k\tau) = n^{-1} \sum_{j=\lfloor k\tau n \rfloor}^{\lfloor n(k+1)\tau \rfloor - 1} \varepsilon_{l_n}(j/n) e^{\omega_{l_n}(j/n)},$$

where (ε, ω) are as in Section 3.3. We then have the following result:

Theorem 5. *For $k \in \mathbb{N}$ and $\tau > 0$, $\delta_\tau \tilde{X}_{1/n}(k\tau)$ converges to $\delta_\tau \tilde{X}(k\tau)$ in L^2 as $n \to +\infty$. Moreover, let us write r_n for*

$$r_n = \frac{\left|\mathbb{E}\left[(\delta_\tau X(k\tau))^2\right] - \mathbb{E}\left[(\delta_\tau \tilde{X}_{1/n}(k\tau))^2\right]\right|}{\mathbb{E}\left[(\delta_\tau X(k\tau))^2\right]}.$$

Then r_n is of order dn^{-d}: that is for fixed n, r_n/d is bounded as $d \to 0$, and for fixed d, $n^d r_n$ is bounded as $d \to +\infty$.

Proof. With no loss of generality, we suppose that $\tau = 1$. The exact value of $\mathbb{E}\left[(\delta_1 X(k))^2\right]$ can be found in Proposition 3; we rewrite it as

$$\mathbb{E}\left[(\delta_1 X(k))^2\right] = 2\sigma^2 T^{\lambda^2} c^\varepsilon \Big(n^{-d}/d + \int_{1/n}^1 u^{-1+d} du - \int_0^1 u^d du \Big).$$

In order to compute $\mathbb{E}\left[(\delta_1 \tilde{X}_{1/n}(k))^2\right]$, we use the relation

$$\varepsilon_{l_n}(k_1/n)\varepsilon_{l_n}(k_2/n) e^{\omega_{l_n}(k_1/n)+\omega_{l_n}(k_2/n)} = \frac{\partial^2}{\partial x_1 \partial x_2}\bigg|_{x_1=x_2=0} e^{\omega_{l_n}(k_1/n)+\omega_{l_n}(k_2/n)+x_1\varepsilon_{l_n}(k_1/n)+x_2\varepsilon_{l_n}(k_2/n)}.$$

This and the values of the covariance functions given in Appendix A yield

$$\mathbb{E}\left[(\delta_1 \tilde{X}_{1/n}(k))^2\right] = \frac{\sigma^2 T^{\lambda^2} c^\varepsilon}{n} \Big((2 - d - \lambda^2) l_n^{-1+d} + 2\sum_{k=1}^{n-1} (1 - k/n)(k/n)^{-1+d} \Big).$$

Using a first order Taylor expansion gives the result for r_n.

Moreover, going along the same lines, one easily obtains the exact value of $\mathbb{E}\left[\delta_1 X(k) \delta_1 \tilde{X}_{1/n}(k)\right]$. This allows to check that $\mathbb{E}\left[(\delta_1 X(k) - \delta_1 \tilde{X}_{1/n}(k))^2\right]$ goes to 0 as n goes to $+\infty$.

□

Remark: If l_n is not chosen as the value that we specify above, but instead as a more generic value like $l_n = 1/n$, then r_n will be of order n^{-d} which may decrease very slowly to 0 for small d.

Theorem 5 shows that we can well approximate the increments of the process X through the discrete process $\tilde{X}_{1/n}$. This last process is easily simulated with the help of some efficient procedures for the simulation of stationary Gaussian random fields like the one proposed by [8], which is based on Fast Fourier Transforms.

6.2. Numerical results and comparisons to empirical data

Let us illustrate previous results on some numerical simulations. We choose the following values for the parameters: $\sigma^2 = 1$, $\lambda^2 = 0.04$ and $T = 200$, which are usual values for modeling financial data with the MRW model (see [2]). The parameter d has been chosen to vary from 0.01 to 0.3, i.e., H varies from 0.525 to 0.67. In all the reported results, we have set $\tau = 1$, $N = 5000$, $n = 500$ and we performed averages over 100 realizations of the process. For each value of d, each realization of the sequence

$\delta_1\tilde{X}_{1/n}(0),\ldots,\delta_1\tilde{X}_{1/n}(N-1)$ has been simulated using the techniques described above. We then approximate $\delta_1 Y_d(k)$ by

$$\delta_1 Y_d(k) \approx -\frac{\delta_1\tilde{X}_{1/n}(k)}{\left(\overline{\left(\delta_1\tilde{X}_{1/n}\right)^2}\right)^{1/2}},$$

$\overline{\left(\delta_1\tilde{X}_{1/n}\right)^2}$ being the empirical means of the squared increments $\delta_1\tilde{X}_{1/n}(0),\ldots,\delta_1\tilde{X}_{1/n}(N-1)$.

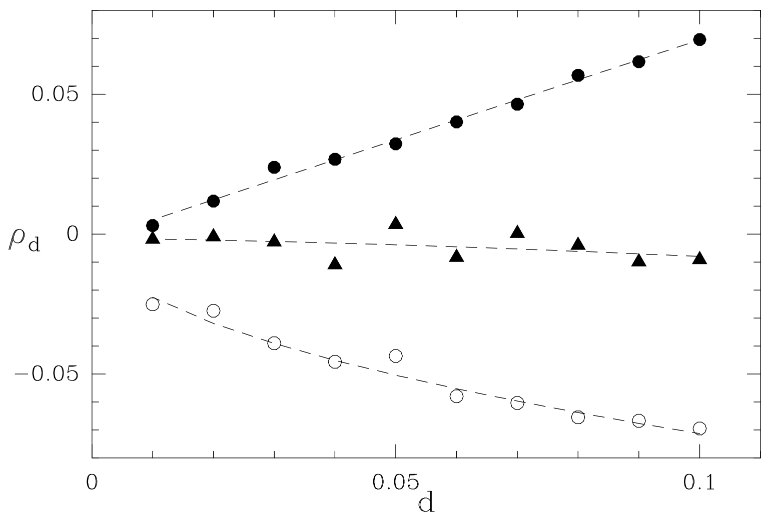

FIGURE 2: *Values $\rho_d^{(1)}(1)$ ($\bullet$), $\rho_d^{(3)}(1)$ ($\circ$), and $\rho_d^{(3)}(-1)$ ($\blacktriangle$) for $d = 0.01$ to $d = 0.10$, and their respective adjustment to a fit $c_1 d$, $c_2 d^{1/2}$, and $c_3 d^{3/2}$ (dashed lines).*

In Figure 2, we check the dependency of the correlation functions $\rho_d^{(1)}$ and $\rho_d^{(3)}$ (defined in (15) and (17)) on the parameter d. Recall that we obtained $\rho_d^{(1)}(k) = O(d)$ for all $|k| \geq 1$, $\rho_d^{(3)}(k) = O(d^{1/2})$ for all $k \geq 0$, and $\rho_d^{(3)}(k) = O(d^{3/2})$ for all $k \leq -1$. This is well confirmed by our simulations.

In Figure 3 we plot the auto-correlations $\rho_d^{(1)}(k)$ of the return series as a function of the lag k for $d = 0.01$, 0.05 and 0.1. We see that after a few lags all series are almost uncorrelated; but it is only for d small enough ($d \leq 0.05$) that the first lag correlation is inside the 95% confidence interval of a series of $N = 5000$ uncorrelated random variables. Since financial returns are well known to be uncorrelated (or very weakly correlated), the parameter d should probably be chosen below the value 0.05.

In Figure 4 we report the estimation of the leverage effect on our simulated series. We estimate $\rho_d^{(3)}(k)$ as a function of k for 3 values of d. For comparison purpose, we also plot the correlation that we measure on real data, namely the daily quotation of 5 stock indices. More precisely we considered the CAC40, DAX, FTSE100, S&P500, and Dow-Jones index daily series from 1990/12/03 to 2010/02/15 and averaged the empirical correlations over the 5 indices, so as to reduce the noise. We confirm our previous computations: the estimated function $\rho_d^{(3)}(k)$ on our simulation exhibits a strong asymmetry and is clearly negative for positive lags k. Moreover, we see that as d increases the leverage effect indeed becomes stronger. The curves we obtain seem quite similar to the effect observed on stock index returns.

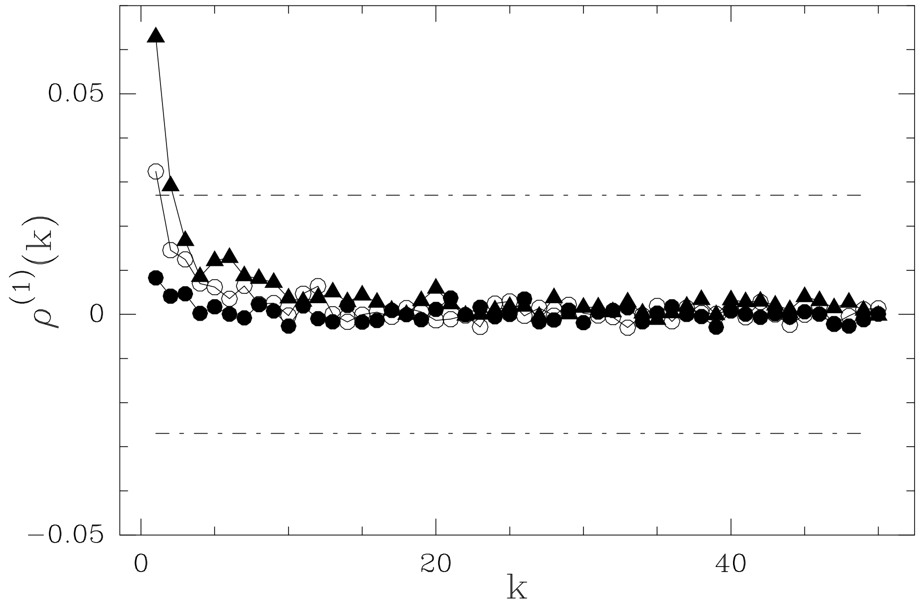

FIGURE 3: *Correlations $\rho_d^{(1)}(k)$ for $1 \le k \le 50$ with $d = 0.01$ (•), $d = 0.05$ (○) and $d = 0.10$ (▲). The dashed lines represent the 95% interval for uncorrelated random variables for a series of size 5000.*

Finally, in Figure 5, we present another way to assess the leverage effect. Indeed, the construction of our model directly suggests that there should exist a negative correlation between past returns and the *logarithm* of future volatilities, and this correlation should behave as a power-law of the time lag. That is, if we denote by $p(t)$ the log-price of an asset, then the following relation is expected:

$$C^{\omega\varepsilon}(k) \equiv -\mathbb{C}\mathrm{orr}[\delta_\tau p(0), 2\log(|\delta_\tau p(k\tau)|)] \sim ck^{-\alpha}$$

for $k \ge 1$, some constant $c > 0$, and some exponent $\alpha \in (0, 1)$. From the definition of $\gamma^{\omega\varepsilon}$, we expect to find in our model $\alpha \approx -1 + H = (-1 + \lambda^2 + d)/2$ which is close to 1/2. Figure 5(a) shows that this is indeed the case. We have

$$C^{\omega\varepsilon}(k) \simeq \gamma^{\omega\varepsilon}(k) \sim ck^{(-1+\lambda^2+d)/2}.$$

In the case of real data, as illustrated in Figure 5(b) we observed that a power-law with an exponent $\alpha \simeq 0.48$ provides of good fit of the data.

Appendix A. Covariances of ω and ε

Through straightforward computations, it is possible to obtain the following covariance functions for (ε, ω): Fix $0 < l' < l < T$ and $u, \tau \in \mathbb{R}$. The process ω has following

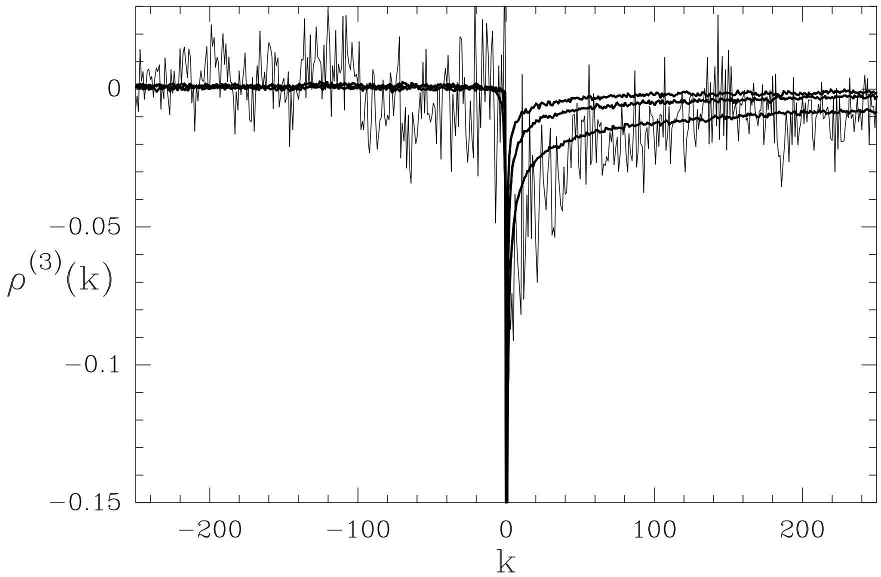

FIGURE 4: *Correlations $\rho_d^{(3)}(k)$ for $|k| \leq 250$. Thin solid lines represent, from top to bottom, $d = 0.03$, $d = 0.1$ and $d = 0.30$. The noisy curve corresponds to real data estimated on a basket of 5 indices and is shown for comparison.*

expectation:

$$\mathbb{E}\big[\omega_l(u)\big] = -\frac{\lambda^2}{2}\left(\log\Big(\frac{T}{l}\Big) + 1\right)$$

and covariance:

$$\mathbb{C}\mathrm{ov}[\omega_l(u), \omega_{l'}(u+\tau)] = \gamma_l^{\omega}(\tau)$$

with

$$\begin{cases} \gamma_l^{\omega}(\tau) &= \lambda^2\left(\log\Big(\frac{T}{l}\Big) + 1 - \frac{\tau}{l}\right) \text{ if } |\tau| \leq l; \\ \gamma_l^{\omega}(\tau) &= \lambda^2 \log\Big(\frac{T}{\tau}\Big) \text{ if } l \leq |\tau| \leq T; \\ \gamma_l^{\omega}(\tau) &= 0 \text{ if } T \leq |\tau|. \end{cases}$$

The process ε has zero expectation and satisfies

$$\mathbb{C}\mathrm{ov}[\varepsilon_l(u), \varepsilon_{l'}(u+\tau)] = \gamma_l^{\varepsilon}(\tau),$$

where $\gamma_l^{\varepsilon}(\tau)$ is defined as:

$$\begin{cases} \gamma_l^{\varepsilon}(\tau) &= \frac{\sigma^2}{(2-2H)(3-2H)}|\tau|^{-2+2H} \text{ if } l \leq |\tau|; \\ \gamma_l^{\varepsilon}(\tau) &= \sigma^2\big(\frac{1}{2-2H} - \frac{1}{3-2H}\frac{|\tau|}{l}\big)l^{-2+2H} \text{ if } 0 \leq |\tau| \leq l. \end{cases}$$

Finally, the covariance between ε and ω is given by

$$\mathbb{C}\mathrm{ov}[\varepsilon_l(u), \omega_{l'}(u+\tau)] = \mathbb{C}\mathrm{ov}[\varepsilon_{l'}(u), \omega_l(u+\tau)] = \gamma_l^{\omega\varepsilon}(\tau),$$

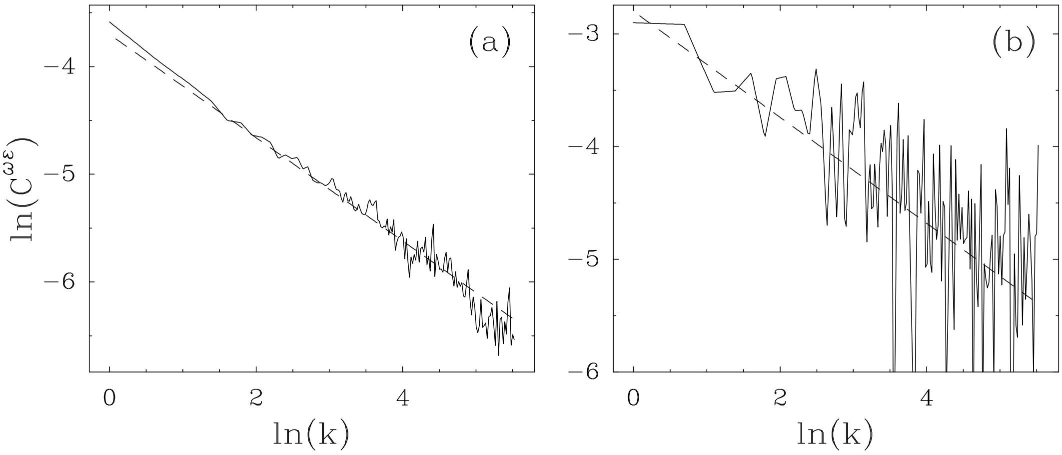

FIGURE 5: *Plot of* $C^{\omega\varepsilon} = -\mathbb{C}orr[\delta_1 p(0), 2\log(|\delta_1 p(k\tau)|)]$ *for* $|k| \le 50$ *in log-log scale. (a) Estimated correlation for 100 realizations of a Skewed MRW with* $\lambda^2 = 0.04$, $T = 250$ *and* $d = 0.03$. *In dashed line the expected power-law behavior* $ck^{(-1+\lambda^2+d)/2} \sim k^{-0.48}$ *is represented. (b) Same graph for the mean correlations over 5 indices. The same power-law behavior as in (a) has been plotted for comparison purpose.*

where $\gamma_l^{\omega\varepsilon}(\tau)$ is defined as:

$$\begin{cases} \gamma_l^{\omega\varepsilon}(\tau) &= 0 \ \text{ if } \ \tau < 0; \\ \gamma_l^{\omega\varepsilon}(\tau) &= \frac{\lambda\sigma}{2-H}\tau l^{-2+H} \ \text{ if } \ 0 \le \tau \le l; \\ \gamma_l^{\omega\varepsilon}(\tau) &= \lambda\sigma\Big(\frac{2}{1-H}l^{-1+H} - \frac{1}{2-H}\frac{\tau}{l}l^{-1+H} - \frac{2}{(1-H)(2-H)}\tau^{-1+H}\Big) \ \text{ if } \ l \le \tau \le 2l; \\ \gamma_l^{\omega\varepsilon}(\tau) &= \frac{\lambda\sigma}{(1-H)(2-H)}\Big(2^{2-H} - 2\Big)\tau^{-1+H} \ \text{ if } \ 2l \le \tau \le T; \\ \gamma_l^{\omega\varepsilon}(\tau) &= \frac{\lambda\sigma}{(1-H)(2-H)}\Big((2^{2-H}-1)\tau^{-1+H} - T^{-1+H}\Big) \ \text{ if } \ T \le \tau \le 2T; \\ \gamma_l^{\omega\varepsilon}(\tau) &= \frac{\lambda\sigma}{(1-H)(2-H)}\Big((\tau - T)^{-1+H} - \tau^{-1+H}\Big) \ \text{ if } \ \tau \ge 2T. \end{cases}$$

Appendix B. Proof of Proposition 2

We begin by evaluating the moment $\mathbb{E}\big[X_l(t)^p\big]$, for $l \in (0, T)$, $t \ge 0$, $p \ge 2$. From Fubini's theorem, we have

$$\mathbb{E}\Big[\Big(\int_0^t \varepsilon_l(u)e^{\omega_l(u)}du\Big)^p\Big] = \int_0^t \dots \int_0^t du_1 \dots du_p \mathbb{E}\big[\varepsilon_l(u_1)\dots\varepsilon_l(u_p)e^{\omega_l(u_1)+\dots+\omega_l(u_p)}\big]. \tag{18}$$

We are going to compute the right-hand side using the following relation:

$$\varepsilon_l(u_1)\dots\varepsilon_l(u_p)e^{\omega_l(u_1)+\dots+\omega_l(u_p)} = \frac{\partial^p}{\partial x_1 \dots \partial x_p}\Big|_{x_1=\dots=x_p=0} e^{\omega_l(u_1)+\dots+\omega_l(u_p)+x_1\varepsilon_l(u_1)+\dots+x_p\varepsilon_l(u_p)}.$$

Permuting expectation and differentiation, we have to differentiate p times

$$\mathbb{E}\Big[e^{\omega_l(u_1)+\dots+\omega_l(u_p)+x_1\varepsilon_l(u_1)+\dots+x_p\varepsilon_l(u_p)}\Big] = \exp(S_p(x_1,\dots,x_p)).$$

The term $S_p = S_p(x_1, \dots, x_p)$ can be evaluated as

$$S_p = \sum_{1\le i<j\le p} \gamma_l^{\omega}(u_i - u_j) + \sum_{1\le i,j\le p} x_j \gamma_l^{\omega\varepsilon}(u_i - u_j) + \sum_{1\le i<j\le p} x_i x_j \gamma_l^{\varepsilon}(u_i - u_j) + \frac{1}{2}\sum_{i=1}^{p} x_i^2 \gamma_l^{\varepsilon}(0)$$

where we used Property 3. From Property 4 $\gamma_l^{\omega\varepsilon}(u_i - u_j)$ is non zero if and only if $u_i > u_j$; however this will not be used in what follows: we do not keep track of the order of the u_i's in order to avoid introducing notations that would be of no use to this proof.

We will however need the following definitions: for $i,\, j = 1, \dots, p$

$$\begin{aligned} D_i &= D_i(x_1, \dots, x_p) \\ &= \frac{\partial}{\partial x_i} S_p(x_1, \dots, x_p), \end{aligned}$$

$$\begin{aligned} D_{i,j} &= D_{i,j}(x_1, \dots, x_p) \\ &= \frac{\partial}{\partial x_j} D_i(x_1, \dots, x_p), \end{aligned}$$

and for $1 \le n \le p$

$$\begin{aligned} R_n &= R_n(x_1, \dots, x_p) \\ &= \frac{\partial^n}{\partial x_1 \dots x_n} e^{S_p(x_1, \dots, x_p)}. \end{aligned}$$

Also, for $1 \le n \le p$ and $0 \le m \le \lfloor n/2 \rfloor$, we define $E_{m,n}$ to be the set of all partitions P of $\{1, \dots, n\}$ into $n - m$ subsets such that m of these subsets have two elements and the other $n - 2m$ subsets have one element:

$$E_{m,n} = \Big\{ P = \big\{\{a_1, a_2\}, \dots, \{a_{2m-1}, a_{2m}\}, \{a_{2m+1}\}, \dots \{a_n\}\big\},\ \{a_1, \dots, a_n\} = \{1, \dots, n\} \Big\}.$$

Then by differentiating iteratively, one can see that

$$R_n = \sum_{m=0}^{\lfloor n/2 \rfloor} \sum_{P\in E_{m,n}} D_{a_1,a_2} \dots D_{a_{2m-1},a_{2m}} D_{a_{2m+1}} \dots D_{a_n} e^{S_p}. \tag{19}$$

Indeed, the formula is clearly true for $n = 1$. Moreover, using the fact that for $1 \le a_i, a_{i+1} \le n$

$$\frac{\partial}{\partial x_{n+1}} D_{a_i,a_{i+1}} = 0,$$

we have for each $P = \big\{\{a_1, a_2\}, \dots, \{a_{2m-1}, a_{2m}\}, \{a_{2m+1}\}, \dots \{a_n\}\big\}$ in $E_{m,n}$:

$$\begin{aligned} &\frac{\partial}{\partial x_{n+1}} D_{a_1,a_2} \dots D_{a_{2m-1},a_{2m}} D_{a_{2m+1}} \dots D_{a_n} e^{S_p} \\ &= \sum_{k=2m+1}^{p} (D_{a_1,a_2} \dots D_{a_{2m-1},a_{2m}} D_{a_{2m+1}} \dots D_{a_{k-1}} D_{a_k,n+1} D_{a_{k+1}} \dots D_{a_n} \\ &\quad + D_{a_1,a_2} \dots D_{a_{2m-1},a_{2m}} D_{a_{2m+1}} \dots D_{a_n} D_{n+1}) e^{S_p}. \end{aligned}$$

We therefore obtain (19) after summing over P and m.

Finally, by taking $n = p$, we have

$$\frac{\partial^p}{\partial x_1 \dots \partial x_p} e^{S_p} = \sum_{m=0}^{\lfloor p/2 \rfloor} \sum_{P \in E_{m,p}} D_{a_1,a_2} \dots D_{a_{2m-1},a_{2m}} D_{a_{2m+1}} \dots D_{a_p} e^{S_p}. \qquad (20)$$

We now evaluate this expression for $x_1 = \dots = x_p = 0$. Since

$$D_{a_i}(0, \dots, 0) = \sum_{b=1}^{p} \gamma_l^{\omega\varepsilon}(u_b - u_{a_i}),$$

$$D_{a_i,a_{i+1}}(0, \dots, 0) = \gamma_l^{\varepsilon}(u_{a_i} - u_{a_{i+1}}),$$

and

$$e^{S_p(0,\dots,0)} = e^{\sum_{1 \le i < j \le p} \gamma_l^{\omega}(u_i - u_j)},$$

we can express the moment $\mathbb{E}\big[X_l(t)^p\big]$ as:

$$\sum_{m=0}^{\lfloor p/2 \rfloor} \sum_{P \in E_{m,p}} \sum_{b_1=1}^{p} \dots \sum_{b_{p-2m}=1}^{p} \int_0^t \dots \int_0^t du_1 \dots du_p \, f_l^{m,P,b_1,\dots,b_{p-2m}}(u_1, \dots, u_p) \qquad (21)$$

where

$$f_l^{m,P,b_1,\dots,b_{p-2m}}(u_1, \dots, u_p) = \gamma_l^{\varepsilon}(u_{a_1} - u_{a_2}) \dots \gamma_l^{\varepsilon}(u_{a_{2m-1}} - u_{a_{2m}}) \\ \times \gamma_l^{\omega\varepsilon}(u_{b_1} - u_{a_{2m+1}}) \dots \gamma_l^{\omega\varepsilon}(u_{b_{p-2m}} - u_{a_p}) e^{\sum_{1 \le i < j \le p}^{p} \gamma_l^{\omega}(u_i - u_j)}.$$

Define $\Gamma_p(m)$ as:

$$\Gamma_p(m) = \sigma^p \lambda^{p-2m} T^{\lambda^2 p(p-1)/2} (c^{\varepsilon})^m (c^{\omega\varepsilon})^{p-2m}.$$

Then from Property 4, each $f_l^{m,P,b_1,\dots,b_{p-2m}} \uparrow f^{m,P,b_1,\dots,b_{p-2m}}$ as $l \to 0$, where $f^{m,P,b_1,\dots,b_{p-2m}}$ is the following:

$$f^{m,P,b_1,\dots,b_{p-2m}}(u_1, \dots, u_p) = \Gamma_p(m) |u_{a_1} - u_{a_2}|^{-2+2H} \dots |u_{a_{2m-1}} - u_{a_{2m}}|^{-2+2H} \\ \times (u_{b_1} - u_{a_{2m+1}})_+^{-1+H} \dots (u_{b_{p-2m}} - u_{a_p})_+^{-1+H} \prod_{1 \le i < j \le p} |u_i - u_j|^{-\lambda^2},$$

which is integrable if and only if

$$-1 + pH - \frac{p(p-1)}{2} \lambda^2 > 0.$$

Applying the monotone convergence theorem gives the result, the constant $K(p)$ being:

$$\sum_{m=0}^{\lfloor p/2 \rfloor} \Gamma_m(p) \sum_{P \in E_{m,p}} \sum_{b_1=1}^{p} \dots \sum_{b_{p-2m}=1}^{p} \int_0^1 \dots \int_0^1 du_1 \dots du_p |u_{a_1} - u_{a_2}|^{-2+2H} \dots \\ \times |u_{a_{2m-1}} - u_{a_{2m}}|^{-2+2H} (u_{b_1} - u_{a_{2m+1}})_+^{-1+H} \dots (u_{b_{p-2m}} - u_{a_p})_+^{-1+H} \prod_{1 \le i < j \le p} |u_i - u_j|^{-\lambda^2}.$$

Appendix C. Proof of Theorem 4

C.1. Behaviour of $\rho^{(1)}$

Here and in the following, we will use the identity:

$$\varepsilon_l(u_1)\dots\varepsilon_l(u_p)e^{\omega_l(u_1)+\dots+\omega_l(u_p)} = \frac{\partial^p}{\partial x_1\dots\partial x_p}\bigg|_{x_1=\dots=x_p=0} e^{\omega_l(u_1)+\dots+\omega_l(u_p)+x_1\varepsilon_l(u_1)+\dots+x_p\varepsilon_l(u_p)}. \tag{22}$$

Applying this identity for $p=2$, Property 4, and the monotone convergence theorem yields:

$$\mathbb{E}\big[\delta_\tau X(0)\delta_\tau X(k\tau)\big] = c^\varepsilon\sigma^2T^{\lambda^2}\int_0^\tau du_1\int_{k\tau}^{(k+1)\tau}du_2|u_2-u_1|^{-1+d}.$$

Note that from the value of c^ε given in Section 3.3, c^ε does depend on d but is approximately $1/2$ for small d. It is easy enough to compute the integral above, which gives: for $k=0$

$$\mathbb{E}\big[\delta_\tau X(0)^2\big] = \frac{2c^\varepsilon\sigma^2T^{\lambda^2}}{d(1+d)}\tau^{1+d}, \tag{23}$$

and for $|k|>0$

$$\mathbb{E}\big[\delta_\tau X(0)\delta_\tau X(k\tau)\big] = \frac{c^\varepsilon\sigma^2T^{\lambda^2}}{d(1+d)}\big(|k+1|^{1+d}+|k-1|^{1+d}-2|k|^{1+d}\big)\tau^{1+d}.$$

It follows that for $|k|>0$, the correlation $\rho_d^{(1)}(\tau,k)$ is of order d when d is small. More precisely, for $|k|=1$:

$$\rho_d^{(1)}(\tau,k)\sim d\log(2) \qquad \text{as } d\to 0.$$

and for $|k|>1$:

$$\rho_d^{(1)}(\tau,k)\sim \frac{d}{2}\big(|k|\log(1-1/k^2)+\log(1+2/(|k|-1))\big) \qquad \text{as } d\to 0.$$

C.2. Behavior of $\rho^{(2)}$ and $\rho^{(3)}$

From Proposition 4, it is enough to prove the result for $\rho^{(2)}$. Going along the same line as above, we get

$$\mathbb{E}\big[\delta_\tau X(0)\delta_\tau X(k\tau)^2\big] = c^\varepsilon c^{\omega\varepsilon}\sigma^3\lambda T^{3\lambda^2}\tau^{(3-3\lambda^2+3d)/2}\sum_{i_1,i_2,i_3}$$
$$\int_0^1 du_1\int_k^{k+1}du_2\int_k^{k+1}du_3|u_{i_1}-u_{i_2}|^{-1+d}(u_{i_2}-u_{i_3})_+^{(-1+d+\lambda^2)/2}|u_{i_3}-u_{i_1}|^{-\lambda^2},$$

the sum being taken on all permutations i_1, i_2, i_3 of the set $\{1,2,3\}$. Note that depending on the sign of k and the permutation, it may be the case that u_{i_2} lies in an interval lower than u_{i_3}, so that the corresponding integral is zero. Also note that from the value of c^ε and $c^{\omega\varepsilon}$ given in Section 3.3, the product $c^\varepsilon c^{\omega\varepsilon}$ is approximately 0.55 for small d.

Taking into account the range of possible values for $d > 0$ and λ^2, the integrals above are clearly finite. Moreover, as d goes to zero, only the integral $I_d(k)$

$$I_d(k) = \int_0^1 du_1 \int_k^{k+1} du_2 \int_k^{k+1} du_3 |u_2 - u_3|^{-1+d} (u_3 - u_1)_+^{(-1+d+\lambda^2)/2} |u_1 - u_2|^{-\lambda^2}$$

(and the one where we permute u_2 and u_3, which is much obviously the same) does explode for $k \geq 0$, while in the case $k < 0$, $I_d(k)$ is exactly zero so that the moment $\mathbb{E}\big[\delta_\tau X(0)\delta_\tau X(k\tau)^2\big]$ remains bounded. For $k \geq 2$, we have the following bounds:

$$(k+1)^{(-1+d-\lambda^2)/2} \int_0^1 du \int_0^1 dv |u-v|^{-1+d} \leq I_d(k) \leq (k+1)^{(-1+d-\lambda^2)/2} \int_0^1 du \int_0^1 dv |u-v|^{-1+d}$$

from which we get

$$I_d(k) \sim c(k) d^{-1} \quad \text{as} \quad d \to 0$$

where $c(k)$ are some positive constants such that

$$2(k+1)^{(-1-\lambda^2)/2} \leq c(k) \leq 2(k-1)^{(-1-\lambda^2)/2}.$$

For $k = 0$, 1 it can be similarly shown that

$$I_d(k) \sim c(k) d^{-1} \quad \text{as} \quad d \to 0$$

where $c(0)$, $c(1)$ are some positive constants. From this, we may write: for $k < 0$

$$|\rho_d^{(2)}(\tau, k)| = O(d^{3/2}) \quad \text{as} \quad d \to 0$$

and for $k \geq 0$,

$$\rho_d^{(2)}(\tau, k) \sim -\left(\frac{c^\varepsilon T^{3\lambda^2}}{2}\right)^{1/2} c^{\omega\varepsilon} \lambda\, c(k) d^{1/2} \quad \text{as} \quad d \to 0.$$

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