



# Continuous window functions for NFFT

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Received: 8 October 2020 / Accepted: 29 April 2021 / Published online: 30 June 2021  
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## Abstract

In this paper, we study the error behavior of the nonequispaced fast Fourier transform (NFFT). This approximate algorithm is mainly based on the convenient choice of a compactly supported window function. Here, we consider the continuous Kaiser–Bessel, continuous exp-type, sinh-type, and continuous cosh-type window functions with the same support and same shape parameter. We present novel explicit error estimates for NFFT with such a window function and derive rules for the optimal choice of the parameters involved in NFFT. The error constant of a window function depends mainly on the oversampling factor and the truncation parameter. For the considered continuous window functions, the error constants have an exponential decay with respect to the truncation parameter.

**Keywords** Nonequispaced fast Fourier transform · NFFT · Error estimate · Oversampling factor · Truncation parameter · Continuous window function with compact support · Kaiser–Bessel window function

**Mathematics Subject Classification (2010)** 65T50 · 94A12 · 42A10

## 1 Introduction

The *nonequispaced fast Fourier transform* (NFFT) (see [6, 7, 16, 20] and [15, Chapter 7]) is an important generalization of the *fast Fourier transform* (FFT). The window-based approximation leads to the most efficient algorithms under different

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Dedicated to Jürgen Prestin on the occasion of his 60th birthday

Communicated by: Anna-Karin Tornberg

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approaches [8, 18]. Recently, a new class of window functions were suggested in [3] and asymptotic error estimates are given in [4]. After [20], the similarities of the window-based algorithms for NFFT became clear. Recently, we have analyzed the window-based NFFT used so far and presented the related error estimates in [17]. Now, we continue this investigation and present new error estimates for some other window functions. More precisely, we consider the continuous Kaiser–Bessel window function and two close relatives of the sinh-type window function, namely the continuous exp-type and cosh-type window functions. All these window functions have the same support and the same shape parameter. We show that these window functions are very useful for NFFT since they produce very small errors.

In this paper, we present novel explicit error estimates (3) with so-called error constants (1). The error constants of NFFT are defined by values of the Fourier transform of the window function. We show that an upper bound of Eq. 1 depends only on the oversampling factor  $\sigma > 1$  and the truncation parameter  $m \geq 2$  and decreases with exponential rate with respect to  $m$ . In numerous applications of NFFT, one uses quite often an oversampling factor  $\sigma \in [\frac{5}{4}, 2]$  and a truncation parameter  $m \in \{2, 3, \dots, 6\}$ . Therefore, we will assume that  $\sigma \geq \frac{5}{4}$ .

The outline of the paper is as follows. In Section 2, we introduce the set  $\Phi_{m, N_1}$  of continuous, even window functions with support  $[-\frac{m}{N_1}, \frac{m}{N_1}]$ , where  $m \in \mathbb{N} \setminus \{1\}$  and  $N_1 = \sigma N \in 2\mathbb{N}$  (with  $N \in 2\mathbb{N}$  and  $2m \ll N_1$ ) are fixed. We emphasize that a continuous window function  $\varphi \in \Phi_{m, N_1}$  tends to zero at the endpoints  $\pm \frac{m}{N_1}$  of its compact support  $[-\frac{m}{N_1}, \frac{m}{N_1}] \subset [-\frac{1}{2}, \frac{1}{2}]$ . In Section 3, we show that the simple rectangular window function Eq. 4 is not convenient for NFFT.

The main results of this paper are contained in Sections 4–7. For the first time, we present explicit estimates of the error constants (1) for fixed truncation parameter  $m$  and oversampling factor  $\sigma$ . In Section 4, we derive explicit error estimates for the continuous Kaiser–Bessel window function (8). In comparison, we show that the popular standard Kaiser–Bessel window function (20) has a similar error behavior as Eq. 8. A very useful continuous window function is the sinh-type window function (22) which is handled in Section 5.

The main drawback for the numerical analysis of the exp-type and cosh-type window function is the fact that an explicit Fourier transform of this window function is unknown. In Sections 6 and 7, we develop a new technique. We split the continuous exp-type/cosh-type window function into a sum  $\psi + \rho$ , where the Fourier transform of the compactly supported function  $\psi$  is explicitly known and where the compactly supported function  $\rho$  has small magnitude. Here, we use the fact that both window functions (28) and (49) are close relatives of the sinh-type window function (22) which was introduced by the authors in [17]. The Fourier transform of  $\rho$  is explicitly estimated for small as well as large frequencies, where  $\sigma$  and  $m$  are fixed. We present many numerical results so that the error constants of the different window functions can be easily compared. After this investigation, we favor the use of a continuous window function with small error constant, which can be very fast computed, such as the sinh-type, standard/continuous exp-type, or continuous cosh-type window function.

## 2 Continuous window functions for NFFT

Let  $\sigma > 1$  be an *oversampling factor*. Assume that  $N \in 2\mathbb{N}$  and  $N_1 := \sigma N \in 2\mathbb{N}$  are given. For fixed *truncation parameter*  $m \in \mathbb{N} \setminus \{1\}$  with  $2m \ll N_1$ , we introduce the open interval  $I := \left(-\frac{m}{N_1}, \frac{m}{N_1}\right)$  and the set  $\Phi_{m,N_1}$  of all continuous window functions  $\varphi : \mathbb{R} \rightarrow [0, 1]$  with the following properties:

- Each window function  $\varphi$  is even, has the support  $\bar{I}$ , and is continuous on  $\mathbb{R}$ .
- Each restricted window function  $\varphi|_{[0, m/N_1]}$  is decreasing with  $\varphi(0) = 1$ .
- For each window function  $\varphi$ , the Fourier transform

$$\hat{\varphi}(v) := \int_I \varphi(t) e^{-2\pi i vt} dt = 2 \int_0^{m/N_1} \varphi(t) \cos(2\pi vt) dt$$

is positive for all  $v \in [-N/2, N/2]$ .

Examples of continuous window functions of  $\Phi_{m,N_1}$  are the (modified) B-spline window functions, algebraic window functions, Bessel window functions, and sinh-type window functions (see [17] and [15, Chapter 7]). More examples are presented in Sections 4–7.

In the following, we denote the torus  $\mathbb{R}/\mathbb{Z}$  by  $\mathbb{T}$  and the Banach space of continuous, 1-periodic functions by  $C(\mathbb{T})$ . Let  $I_N := \{-N/2, \dots, N/2 - 1\}$  be the index set for  $N \in 2\mathbb{N}$ .

We say that a continuous window function  $\varphi \in \Phi_{m,N_1}$  is *convenient for NFFT*, if the  $C(\mathbb{T})$ -error constant

$$e_\sigma(\varphi) := \sup_{N \in 2\mathbb{N}} e_{\sigma,N}(\varphi) \tag{1}$$

with

$$e_{\sigma,N}(\varphi) := \max_{n \in I_N} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \frac{\hat{\varphi}(n + rN_1)}{\hat{\varphi}(n)} e^{2\pi i r N_1 \cdot} \right\|_{C(\mathbb{T})} \tag{2}$$

fulfills the condition  $e_\sigma(\varphi) \ll 1$  for conveniently chosen oversampling factor  $\sigma > 1$ . The norm in  $C(\mathbb{T})$  is defined by  $\|f\|_{C(\mathbb{T})} := \max_{x \in \mathbb{T}} |f(x)|$ .

Now we show that the error of the nonequispaced fast Fourier transform (NFFT) with a window function  $\varphi \in \Phi_{m,N_1}$  can be estimated by the error constant (1). The NFFT (with nonequispaced spatial data and equispaced frequencies) is an approximate, fast algorithm which computes approximately the values  $p(x_j)$ ,  $j = 1, \dots, M$ , of any 1-periodic trigonometric polynomial

$$p(x) := \sum_{k \in I_N} c_k e^{2\pi i kx}$$

at finitely many nonequispaced nodes  $x_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ ,  $j = 1, \dots, M$ , where  $c_k \in \mathbb{C}$ ,  $k \in I_N$ , are given coefficients. By the properties of the window function  $\varphi \in \Phi_{m,N_1}$ , the 1-periodic function

$$\tilde{\varphi}(x) := \sum_{k \in \mathbb{Z}} \varphi(x + k), \quad x \in \mathbb{T},$$

is continuous on  $\mathbb{T}$  and of bounded variation over  $[-\frac{1}{2}, \frac{1}{2}]$ . Then, from the Convergence Theorem of Dirichlet–Jordan (see [22, Vol. 1, pp. 57–58]), it follows that  $\tilde{\varphi}$  possesses the uniformly convergent Fourier expansion

$$\tilde{\varphi}(x) = \sum_{k \in \mathbb{Z}} \hat{\varphi}(k) e^{2\pi i k x}$$

with the Fourier coefficients

$$\hat{\varphi}(k) = \int_{\mathbb{R}} \varphi(x) e^{-2\pi i k x} dx = \int_0^1 \tilde{\varphi}(x) e^{-2\pi i k x} dx .$$

We approximate the trigonometric polynomial  $p$  by the 1-periodic function

$$s(x) := \sum_{\ell \in I_{N_1}} g_\ell \tilde{\varphi}\left(x - \frac{\ell}{N_1}\right) \in C(\mathbb{T})$$

with the coefficients

$$g_\ell := \frac{1}{N_1} \sum_{k \in I_N} \frac{c_k}{\hat{\varphi}(k)} e^{2\pi i k \ell / N_1}, \quad \ell \in I_{N_1} .$$

The computation of the values  $s(x_j)$ ,  $j = 1, \dots, M$ , is very easy, since  $\varphi$  is compactly supported. The computational cost of the algorithm is  $\mathcal{O}(N_1 \log N_1 + (2m + 1)M)$ , see [15, Algorithm 7.1] and [11] for details.

We interpret  $s - p$  as the *error function of the NFFT* which we measure in the norm of  $C(\mathbb{T})$ .

**Theorem 1** *Let  $\sigma > 1$ ,  $N \in 2\mathbb{N}$ , and  $N_1 = \sigma N \in 2\mathbb{N}$  be given. Further let  $m \in \mathbb{N} \setminus \{1\}$  with  $2m \ll N_1$ .*

*Then the error function of the NFFT can be estimated by*

$$\|s - p\|_{C(\mathbb{T})} \leq e_\sigma(\varphi) \sum_{n \in I_N} |c_n|. \tag{3}$$

The proof of Theorem 1 is based on the equality

$$s(x) - p(x) = \sum_{n \in I_N} \sum_{r \in \mathbb{Z} \setminus \{0\}} c_n \frac{\hat{\varphi}(n + rN_1)}{\hat{\varphi}(n)} e^{2\pi i (n+rN_1)x}, \quad x \in \mathbb{T} .$$

For details of the proof see [17, Lemma 2.3] and [13, Lemma 2.1].

In order to describe the behavior of  $C(\mathbb{T})$ -error constant (1), we have to study the Fourier transform of a window function  $\varphi$  with the support  $\tilde{I}$ .

### 3 Rectangular window function

In this section, we present a simple discontinuous window function which is not convenient for NFFT. Later, we will use this discontinuous window function in Remarks 1 and 3, where we estimate the  $C(\mathbb{T})$ -error constants for the standard Kaiser-Bessel window function and the original exp-type window function, respectively.

The simplest window function is the *rectangular window function*

$$\varphi_{\text{rect}}(x) := \begin{cases} 1 & x \in I, \\ \frac{1}{2} & x = \pm \frac{m}{N_1}, \\ 0 & x \in \mathbb{R} \setminus \bar{I}, \end{cases} \tag{4}$$

Now, we show that the rectangular window function (4) is not convenient for NFFT. The Fourier transform of Eq. 4 has the form

$$\hat{\varphi}_{\text{rect}}(v) = \frac{2m}{N_1} \text{sinc} \frac{2\pi m v}{N_1}, \quad v \in \mathbb{R}. \tag{5}$$

The discontinuous window function  $\varphi_{\text{rect}}$  does not belong to  $\Phi_{m, N_1}$ .

**Lemma 1** For each  $n \in I_N \setminus \{0\}$ , the Fourier series

$$\frac{n}{N_1} \sum_{r \in \mathbb{Z}} \frac{1}{r + \frac{n}{N_1}} e^{2\pi i r N_1 x}$$

converges pointwise to the  $\frac{1}{N_1}$ -periodic function of the form

$$\frac{\pi n i}{N_1} (1 - e^{-2\pi i n / N_1})^{-1} e^{-2\pi i n x}, \quad x \in (0, \frac{1}{N_1}).$$

*Proof* For fixed  $n \in I_N \setminus \{0\}$ , we consider the Fourier series of the special  $\frac{1}{N_1}$ -periodic function  $g_n(x) := e^{-2\pi i n x}$  for  $x \in (0, \frac{1}{N_1})$  with

$$g_n(0) = g_n(\frac{1}{N_1}) := \frac{1}{2} (1 + e^{-2\pi i n / N_1}).$$

For  $n = 0$ , we have  $g_0(x) = 1$ . Then, the  $r$ th Fourier coefficient of  $g_n$  reads as follows:

$$N_1 \int_0^{1/N_1} g_n(t) e^{-2\pi i r N_1 t} dt = \frac{1}{\pi i} (1 - e^{-2\pi i n / N_1}) \frac{1}{r + \frac{n}{N_1}}$$

for  $r \in \mathbb{Z}$ . By the Convergence Theorem of Dirichlet–Jordan, the Fourier series of  $g_n$  is pointwise convergent such that for each  $x \in \mathbb{R}$  and  $n \in I_N \setminus \{0\}$

$$g_n(x) = \frac{1}{\pi i} (1 - e^{-2\pi i n / N_1}) \sum_{r \in \mathbb{Z}} \frac{1}{r + \frac{n}{N_1}} e^{2\pi i r N_1 x}.$$

This completes the proof. □

**Lemma 2** The  $C(\mathbb{T})$ -error constant of the rectangular window function (4) can be estimated by

$$0.18 < \frac{1}{2} - \frac{1}{\pi} \leq e_\sigma(\varphi_{\text{rect}}) \leq \frac{1}{2} + \frac{\pi}{4} < 1.3, \tag{6}$$

i.e., the rectangular window function (4) is not convenient for NFFT.

*Proof* By Eq. 5 we obtain for  $n = 0$  and  $r \in \mathbb{Z}$  that

$$\frac{\hat{\varphi}_{\text{rect}}(rN_1)}{\hat{\varphi}_{\text{rect}}(0)} = \begin{cases} 1 & r = 0, \\ 0 & r \in \mathbb{Z} \setminus \{0\} \end{cases}$$

and hence

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} \frac{\hat{\varphi}_{\text{rect}}(rN_1)}{\hat{\varphi}_{\text{rect}}(0)} e^{2\pi i r N_1 x} = 0.$$

By Eq. 5 we have for  $n \in I_N \setminus \{0\}$  and  $r \in \mathbb{Z}$ ,  $\frac{\hat{\varphi}_{\text{rect}}(n+rN_1)}{\hat{\varphi}_{\text{rect}}(n)} = \frac{n}{n+rN_1}$ . Thus, we obtain by Lemma 1 that for  $x \in (0, \frac{1}{N_1})$  and  $n \in I_N \setminus \{0\}$

$$\begin{aligned} \sum_{r \in \mathbb{Z} \setminus \{0\}} \frac{\hat{\varphi}_{\text{rect}}(n+rN_1)}{\hat{\varphi}_{\text{rect}}(n)} e^{2\pi i r N_1 x} &= \frac{n}{N_1} \sum_{r \in \mathbb{Z} \setminus \{0\}} \frac{1}{r + \frac{n}{N_1}} e^{2\pi i r N_1 x} \\ &= \frac{\pi n i}{N_1} (1 - e^{-2\pi i n / N_1})^{-1} e^{-2\pi i n x} - \frac{n}{N_1}. \end{aligned}$$

Using  $|1 - e^{-2\pi i n / N_1}| = 2 |\sin \frac{\pi n}{N_1}|$ , it follows that

$$\begin{aligned} \left| \frac{\pi n i}{N_1} (1 - e^{-2\pi i n / N_1})^{-1} e^{-2\pi i n x} - \frac{n}{N_1} \right| &\geq \frac{|n|}{N_1} \left( \frac{\pi}{2} |\sin \frac{\pi n}{N_1}|^{-1} - 1 \right) \\ &= \frac{|n|}{2N_1} \frac{\pi - 2 |\sin \frac{\pi n}{N_1}|}{|\sin \frac{\pi n}{N_1}|} \geq \frac{1}{2\pi} (\pi - 2 |\sin \frac{\pi n}{\sigma N}|) \geq \frac{1}{2\pi} (\pi - 2). \end{aligned}$$

Analogously, we estimate

$$\begin{aligned} \left| \frac{\pi n i}{N_1} (1 - e^{-2\pi i n / N_1})^{-1} e^{-2\pi i n x} - \frac{n}{N_1} \right| &\leq \frac{|n|}{N_1} \left( \frac{\pi}{2} |\sin \frac{\pi n}{N_1}|^{-1} + 1 \right) \\ &= \frac{|n|}{2N_1} \frac{\pi + 2 |\sin \frac{\pi n}{N_1}|}{|\sin \frac{\pi n}{N_1}|} \leq \frac{1}{4} (\pi + 2 |\sin \frac{\pi n}{N_1}|) \leq \frac{1}{4} (\pi + 2). \end{aligned}$$

Consequently, the rectangular window function is not convenient for NFFT since the corresponding  $C(\mathbb{T})$ -error constant  $e_\sigma(\varphi_{\text{rect}})$  can be estimated by Eq. 6.  $\square$

**Lemma 3** *The Fourier transform of rectangular window function (4) has the following property:*

$$\left(1 - \frac{2}{\pi}\right) \frac{m}{N_1} \left| \text{sinc} \frac{\pi m}{\sigma} \right| \leq \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{rect}}(n+rN_1) e^{2\pi i r N_1 \cdot} \right\|_{C(\mathbb{T})} \leq \frac{3m}{N_1}. \tag{7}$$

*Proof* By Eqs. 1 and 2, we obtain

$$\max_{n \in I_N} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{rect}}(n+rN_1) e^{2\pi i r N_1 \cdot} \right\|_{C(\mathbb{T})} \geq e_\sigma(\varphi_{\text{rect}}) \min_{n \in I_N} |\hat{\varphi}_{\text{rect}}(n)|.$$

Then from Lemma 2 and Eq. 5 it follows immediately the lower estimate in Eq. 7.

Now we show the upper estimate in Eq. 7. For each  $n \in I_N \setminus \{0\}$ , the  $\frac{1}{N_1}$ -periodic Fourier series

$$\begin{aligned}
 f_n(x) &:= \sum_{r \in \mathbb{Z}} \hat{\varphi}_{\text{rect}}(n + rN_1) e^{2\pi i r N_1 x} = \frac{2m}{N_1} \sum_{r \in \mathbb{Z}} \text{sinc} \frac{2\pi m (n + rN_1)}{N_1} e^{2\pi i r N_1 x} \\
 &= \frac{1}{\pi N_1} \sin \frac{2\pi mn}{N_1} \sum_{r \in \mathbb{Z}} \frac{1}{r + \frac{n}{N_1}} e^{2\pi i r N_1 x}
 \end{aligned}$$

is pointwise convergent by Lemma 1 such that

$$f_n(x) = \begin{cases} \frac{i}{2m} \sin \frac{2\pi mn}{N_1} (1 - e^{-2\pi i n/N_1})^{-1} e^{-2\pi i n x} & x \in (0, \frac{1}{N_1}), \\ \frac{1}{4m} \sin \frac{2\pi mn}{N_1} \cot \frac{\pi n}{N_1} & x \in \{0, \frac{1}{N_1}\}. \end{cases}$$

In the case  $n = 0$ , we have  $f_0(x) := \sum_{r \in \mathbb{Z}} \text{sinc}(2\pi m r) e^{2\pi i r N_1 x} = 1$ . For arbitrary  $x \in \mathbb{R}$  and  $m \in \mathbb{N}$ , it holds obviously  $|\sin(2mx)| \leq 2m |\sin x|$  and so  $|1 - e^{-2\pi i n/N_1}| = 2 |\sin \frac{\pi n}{N_1}|$ . We obtain for  $n \in I_N \setminus \{0\}$  that

$$\max_{x \in \mathbb{R}} |f_n(x)| \leq \frac{1}{2N_1} \left| \sin \frac{2\pi mn}{N_1} \right| \left| \sin \frac{\pi n}{N_1} \right|^{-1} \leq \frac{m}{N_1}.$$

Thus for  $n \in I_N \setminus \{0\}$  and  $x \in [0, \frac{1}{N_1}]$ , we receive

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{rect}}(n + rN_1) e^{2\pi i r N_1 x} = f_n(x) - \frac{2m}{N_1} \text{sinc} \frac{2\pi mn}{N_1}$$

and hence

$$\left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{rect}}(n + rN_1) e^{2\pi i r N_1 \cdot} \right\|_{C(\mathbb{T})} \leq \frac{3m}{N_1}.$$

In the case  $n = 0$ , the above estimate is also true, since

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{rect}}(rN_1) e^{2\pi i r N_1 x} = 0.$$

This completes the proof. □

### 4 Continuous Kaiser–Bessel window function

In the following, we consider the so-called *continuous Kaiser–Bessel window function*

$$\varphi_{\text{cKB}}(x) := \begin{cases} \frac{1}{I_0(\beta)-1} \left( I_0(\beta \sqrt{1 - (\frac{N_1 x}{m})^2}) - 1 \right) & x \in I, \\ 0 & x \in \mathbb{R} \setminus I \end{cases} \tag{8}$$

with  $b := 2\pi(1 - \frac{1}{2\sigma})$ ,  $\sigma > 1$ ,  $\beta := bm$ ,  $N \in 2\mathbb{N}$ , and  $N_1 \in 2\mathbb{N}$ , see [13], where

$$I_0(x) := \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{x}{2}\right)^{2k}, \quad x \in \mathbb{R},$$

denotes the *modified Bessel function of first kind*. Furthermore, we assume that  $m \in \mathbb{N} \setminus \{1\}$  fulfills  $2m \ll N_1$ . We emphasize that the *shape parameter*  $\beta$  in Eq. 8 depends on  $m$  and  $\sigma$ . Obviously,  $\varphi_{\text{cKB}} \in \Phi_{m, N_1}$  is a continuous window function. Note that a discontinuous version (20) of Eq. 8 (see Remark 1) was considered in [9], but the  $C(\mathbb{T})$ -error constant of Eq. 20 was not determined.

By [14, p. 95] and Eq. 5, the Fourier transform of the continuous Kaiser–Bessel window function (8) has the form

$$\hat{\varphi}_{\text{cKB}}(v) = \frac{2m}{(I_0(\beta) - 1) N_1} \left( \frac{\sinh(\beta \sqrt{1 - (\frac{2\pi v}{N_1 b})^2})}{\beta \sqrt{1 - (\frac{2\pi v}{N_1 b})^2}} - \text{sinc} \frac{2\pi m v}{N_1} \right)$$

for  $|v| < \frac{N_1 b}{2\pi}$  and

$$\hat{\varphi}_{\text{cKB}}(v) = \frac{2m}{(I_0(\beta) - 1) N_1} \left( \text{sinc}(\beta \sqrt{(\frac{2\pi v}{N_1 b})^2 - 1}) - \text{sinc} \frac{2\pi m v}{N_1} \right)$$

for  $|v| \geq \frac{N_1 b}{2\pi}$ . One can show that  $\hat{\varphi}_{\text{cKB}}|_{[0, \frac{N_1 b}{2\pi})}$  is positive and decreasing such that

$$\min_{n \in I_N} \hat{\varphi}_{\text{cKB}}(n) = \hat{\varphi}_{\text{cKB}}\left(\frac{N}{2}\right) \geq \frac{2m}{(I_0(\beta) - 1) N_1} \left[ \frac{\sinh(2\pi m \sqrt{1 - 1/\sigma})}{2m\pi \sqrt{1 - 1/\sigma}} - \frac{\sigma}{\pi m} \right]. \tag{9}$$

Using the scaled frequency  $w = 2\pi m v / N_1$ , we obtain

$$\hat{\varphi}_{\text{cKB}}\left(\frac{N_1 w}{2\pi m}\right) = \frac{2m}{(I_0(\beta) - 1) N_1} \cdot \begin{cases} \left( \frac{\sinh(\sqrt{\beta^2 - w^2})}{\sqrt{\beta^2 - w^2}} - \text{sinc } w \right) & |w| < \beta, \\ \left( \text{sinc}(\sqrt{w^2 - \beta^2}) - \text{sinc } w \right) & |w| \geq \beta. \end{cases} \tag{10}$$

Note that we have  $\frac{N_1 b}{2\pi} = N_1 - \frac{N}{2} > \frac{N}{2}$  by the special choice of  $b$ .

**Lemma 4** For  $|w| \geq \beta$  we have

$$|\text{sinc}(\sqrt{w^2 - \beta^2}) - \text{sinc } w| \leq \frac{2\beta^2}{w^2}.$$

*Proof* Since the sinc-function is even, we consider only the case  $w \geq \beta$ . For  $w = \beta$ , the above inequality is true, since  $|\text{sinc } 0 - \text{sinc } \beta| \leq 1 + |\text{sinc } \beta| < 2$ . For  $w > \beta$  we obtain

$$\begin{aligned} |\text{sinc}(\sqrt{w^2 - \beta^2}) - \text{sinc } w| &= \left| \frac{\sin \sqrt{w^2 - \beta^2}}{\sqrt{w^2 - \beta^2}} - \frac{\sin w}{w} \right| \\ &\leq \frac{1}{w} |\sin \sqrt{w^2 - \beta^2} - \sin w| + |\sin \sqrt{w^2 - \beta^2}| \left( \frac{1}{\sqrt{w^2 - \beta^2}} - \frac{1}{w} \right) \\ &\leq \frac{2}{w} \left| \sin \frac{1}{2}(\sqrt{w^2 - \beta^2} - w) \right| + |\sin \sqrt{w^2 - \beta^2}| \left( \frac{1}{\sqrt{w^2 - \beta^2}} - \frac{1}{w} \right). \end{aligned}$$



From

$$\sqrt{w^2 - \beta^2} - w = w \left( 1 - \sqrt{1 - \frac{\beta^2}{w^2}} \right) = w \frac{1 - \left( 1 - \frac{\beta^2}{w^2} \right)}{1 + \sqrt{1 - \frac{\beta^2}{w^2}}} \leq \frac{\beta^2}{w} \tag{11}$$

it follows that

$$\left| \sin \frac{1}{2} (\sqrt{w^2 - \beta^2} - w) \right| \leq \frac{1}{2} (\sqrt{w^2 - \beta^2} - w) \leq \frac{\beta^2}{2w}.$$

Furthermore, we receive by Eq. 11 that

$$\begin{aligned} \left| \sin \sqrt{w^2 - \beta^2} \left( \frac{1}{\sqrt{w^2 - \beta^2}} - \frac{1}{w} \right) \right| &= \left| \sin \sqrt{w^2 - \beta^2} \right| \frac{w - \sqrt{w^2 - \beta^2}}{w \sqrt{w^2 - \beta^2}} \\ &\leq \left| \sin \sqrt{w^2 - \beta^2} \right| \frac{\beta^2}{w^2} \leq \frac{\beta^2}{w^2}. \end{aligned}$$

This completes the proof. □

In our study, we use the following:

**Lemma 5** *Let  $f : (0, \infty) \rightarrow (0, \infty)$  be a decreasing function which is integrable on each interval  $[1 - |u|, \infty)$  with arbitrary  $u \in (-1, 1)$ . Furthermore, we extend  $f$  by  $f(-x) := f(x)$  for all  $x > 0$ .*

*Then for each  $u \in (-1, 1)$ , it follows:*

$$\begin{aligned} \sum_{r \in \mathbb{Z} \setminus \{0, \pm 1\}} f(u + r) &\leq 2 \int_{1 - |u|}^{\infty} f(t) dt, \\ \sum_{r \in \mathbb{Z} \setminus \{0\}} f(u + r) &\leq 2 f(1 - |u|) + 2 \int_{1 - |u|}^{\infty} f(t) dt. \end{aligned}$$

*Proof.* For arbitrary  $u \in (-1, 1)$  and  $r \in \mathbb{N}$ , we have

$$f(u + r) \leq f(r - |u|). \tag{12}$$

Using Eq. 12, the following series can be estimated by

$$\begin{aligned} \sum_{r=2}^{\infty} f(u + r) &\leq \sum_{r=2}^{\infty} f(r - |u|), \\ \sum_{r=2}^{\infty} f(u - r) &\leq \sum_{r=2}^{\infty} f(r - |u|). \end{aligned}$$

Hence, it follows by the integral test for convergence of series that

$$\begin{aligned} \sum_{r \in \mathbb{Z} \setminus \{0, \pm 1\}} f(u + r) &= \sum_{r=2}^{\infty} f(u + r) + \sum_{r=2}^{\infty} f(u - r) \leq 2 \sum_{r=2}^{\infty} f(r - |u|) \\ &\leq 2 \int_1^{\infty} f(x - |u|) dx = 2 \int_{1 - |u|}^{\infty} f(t) dt. \end{aligned}$$

Hence, we conclude that

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} f(u+r) \leq 2 f(1-|u|) + 2 \int_{1-|u|}^{\infty} f(t) dt. \quad \square$$

We illustrate Lemma 5 for some special functions  $f$ , which we need later.

*Example 1* For the function  $f(x) = x^{-\mu}$ ,  $x > 0$ , with  $\mu > 1$ , Lemma 5 provides that for each  $u \in (-1, 1)$  it holds

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} |u+r|^{-\mu} \leq 2(1-|u|)^{-\mu} + \frac{2(1-|u|)^{1-\mu}}{\mu-1}. \quad (13)$$

Especially for  $\mu = 2$ , it follows that

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} |u+r|^{-2} \leq \frac{4-2|u|}{(1-|u|)^2} \leq \frac{4}{(1-|u|)^2}. \quad (14)$$

For the function  $f(x) = e^{-ax}$ ,  $x > 0$ , with  $a > 0$ , we obtain by Lemma 5 that for each  $u \in (-1, 1)$ ,

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} e^{-a|u+r|} \leq \left(2 + \frac{2}{a}\right) e^{a|u|-a}. \quad (15)$$

Choosing the function  $f(x) = \frac{1}{ax} e^{-\sqrt{ax}}$ ,  $x > 0$ , with  $a > 0$ , Lemma 5 implies that for each  $u \in (-1, 1)$ ,

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} \frac{1}{a|u+r|} e^{-\sqrt{a|u+r|}} \leq \frac{2}{a(1-|u|)} e^{-\sqrt{a-a|u|}} + \frac{4}{a} E_1(\sqrt{a(1-|u|)}), \quad (16)$$

where  $E_1$  denotes the exponential integral  $E_1(x) := \int_x^{\infty} \frac{1}{t} e^{-t} dt$ ,  $x > 0$ .

**Lemma 6** For all  $n \in I_N$  we estimate

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\varphi}_{\text{cKB}}(n+rN_1)| \leq \frac{8m}{(I_0(\beta) - 1) N_1}.$$

*Proof* By Eq. 10 and Lemma 4, we obtain for all frequencies  $|v| \geq \frac{N_1 b}{2\pi} = N_1 - \frac{N}{2}$  that

$$|\hat{\varphi}_{\text{cKB}}(v)| \leq \frac{mb^2 N_1}{2(I_0(bm) - 1) \pi^2 v^2}.$$

Thus, we have for each  $n \in I_N$

$$\begin{aligned} \sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\varphi}_{\text{cKB}}(n+rN_1)| &\leq \frac{b\beta}{2(I_0(\beta) - 1) N_1 \pi^2} \sum_{r \in \mathbb{Z} \setminus \{0\}} \left(r + \frac{n}{N_1}\right)^{-2} \\ &\leq \frac{2b\beta}{(I_0(\beta) - 1) N_1 \pi^2} \left(1 - \frac{|n|}{N_1}\right)^{-2} \leq \frac{2b\beta}{(I_0(\beta) - 1) N_1 \pi^2} \left(1 - \frac{1}{2\sigma}\right)^{-2}, \end{aligned}$$

since by Eq. 14 it holds for all  $n \in I_N$ ,

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} \left(r + \frac{n}{N_1}\right)^{-2} \leq 4 \left(1 - \frac{|n|}{N_1}\right)^{-2} \leq 4 \left(1 - \frac{1}{2\sigma}\right)^{-2}.$$

By the special choice of  $b = 2\pi \left(1 - \frac{1}{2\sigma}\right)$ , we obtain the above inequality. □

From Lemma 6, it follows immediately that for all  $n \in I_N$ ,

$$\left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{cKB}}(n + rN_1) e^{2\pi i r N_1 \cdot} \right\|_{C(\mathbb{T})} \leq \sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\varphi}_{\text{cKB}}(n + rN_1)| \leq \frac{8m}{(I_0(\beta) - 1) N_1}. \tag{17}$$

**Theorem 2** *Let  $b = 2\pi \left(1 - \frac{1}{2\sigma}\right)$ ,  $\sigma > 1$ ,  $\beta = bm$ ,  $N \in 2\mathbb{N}$ , and  $N_1 = \sigma N \in 2\mathbb{N}$ . Furthermore,  $m \in \mathbb{N} \setminus \{1\}$  with  $2m \ll N_1$  is given.*

*Then, the  $C(\mathbb{T})$ -error constant of the continuous Kaiser–Bessel window function (8) can be estimated by*

$$e_\sigma(\varphi_{\text{cKB}}) \leq 16m\pi \sqrt{1 - 1/\sigma} \left[ e^{2\pi m \sqrt{1 - 1/\sigma}} - e^{-2\pi m \sqrt{1 - 1/\sigma}} - 4\sqrt{\sigma^2 - \sigma} \right]^{-1}. \tag{18}$$

Note that for  $\sigma \geq \frac{5}{4}$ , it holds  $2\pi \sqrt{1 - 1/\sigma} \geq 2\pi/\sqrt{5}$  and hence

$$e^{-2\pi m \sqrt{1 - 1/\sigma}} \leq e^{-2\pi m/\sqrt{5}} < 0.06^m. \tag{19}$$

*Proof* By the definition (1) of the  $C(\mathbb{T})$ -error constant, it holds

$$e_\sigma(\varphi_{\text{cKB}}) = \sup_{N \in 2\mathbb{N}} e_{\sigma, N}(\varphi_{\text{cKB}})$$

with

$$\begin{aligned} e_{\sigma, N}(\varphi_{\text{cKB}}) &= \max_{n \in I_N} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \frac{\hat{\varphi}_{\text{cKB}}(n + rN_1)}{\hat{\varphi}_{\text{cKB}}(n)} e^{2\pi i r N_1 \cdot} \right\|_{C(\mathbb{T})} \\ &\leq \frac{1}{\min_{n \in I_N} \hat{\varphi}_{\text{cKB}}(n)} \max_{n \in I_N} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{cKB}}(n + rN_1) e^{2\pi i r N_1 \cdot} \right\|_{C(\mathbb{T})}, \end{aligned}$$

where it holds (9), i.e.,

$$\begin{aligned} \hat{\varphi}_{\text{cKB}}\left(\frac{N}{2}\right) &\geq \frac{1}{(I_0(\beta) - 1) N_1 \pi \sqrt{1 - 1/\sigma}} \left[ \sinh(2\pi m \sqrt{1 - 1/\sigma}) - 2\sqrt{\sigma^2 - \sigma} \right] \\ &= \frac{1}{2(I_0(\beta) - 1) N_1 \pi \sqrt{1 - 1/\sigma}} \left[ e^{2\pi m \sqrt{1 - 1/\sigma}} - e^{-2\pi m \sqrt{1 - 1/\sigma}} - 4\sqrt{\sigma^2 - \sigma} \right]. \end{aligned}$$

Thus, from Eq. 17, it follows the assertion (18). □

*Remark 1* As in [9] and [4], we consider also the *standard Kaiser–Bessel window function*

$$\varphi_{\text{KB}}(x) := \begin{cases} \frac{1}{I_0(\beta)} I_0(\beta \sqrt{1 - (\frac{N_1 x}{m})^2}) & x \in I, \\ \frac{1}{2I_0(\beta)} & x = \pm \frac{m}{N_1}, \\ 0 & x \in \mathbb{R} \setminus \bar{I} \end{cases} \tag{20}$$

with the shape parameter  $\beta = mb = 2\pi m (1 - \frac{1}{2\sigma})$ ,  $\sigma > 1$ ,  $N \in 2\mathbb{N}$ , and  $N_1 \in 2\mathbb{N}$ . Furthermore, we assume that  $m \in \mathbb{N} \setminus \{1\}$  fulfills  $2m \ll N_1$ . This window function possesses jump discontinuities at  $x = \pm \frac{m}{N_1}$  with very small jump height  $I_0(\beta)^{-1}$ , such that Eq. 20 is “almost continuous.” The Fourier transform of Eq. 20 is even and reads by [14, p. 95] as follows:

$$\hat{\varphi}_{\text{KB}}(v) = \begin{cases} \frac{2m}{I_0(\beta) N_1} \frac{\sinh(\beta \sqrt{1 - (\frac{2\pi v}{N_1 b})^2})}{\beta \sqrt{1 - (\frac{2\pi v}{N_1 b})^2}} & |v| < \frac{N_1 b}{2\pi}, \\ \frac{2m}{I_0(\beta) N_1} \text{sinc}(\beta \sqrt{(\frac{2\pi v}{N_1 b})^2 - 1}) & |v| \geq \frac{N_1 b}{2\pi}. \end{cases}$$

Thus,  $\hat{\varphi}_{\text{KB}}|_{[0, \frac{N_1 b}{2\pi}]}$  is positive and decreasing such that

$$\min_{n \in I_N} \hat{\varphi}_{\text{KB}}(n) = \hat{\varphi}_{\text{KB}}(\frac{N}{2}) \geq \frac{1}{I_0(\beta) N_1 \pi} \frac{\sinh(2\pi m \sqrt{1 - 1/\sigma})}{\sqrt{1 - 1/\sigma}}.$$

Splitting  $\varphi_{\text{KB}}$  in the form

$$\varphi_{\text{KB}}(x) = (1 - \frac{1}{I_0(\beta)}) \varphi_{\text{cKB}}(x) + \frac{1}{I_0(\beta)} \varphi_{\text{rect}}(x), \quad x \in \mathbb{R},$$

we can estimate for all  $n \in I_N$ ,

$$\begin{aligned} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{KB}}(n + rN_1) e^{2\pi i r N_1} \right\|_{C(\mathbb{T})} &\leq (1 - \frac{1}{I_0(\beta)}) \sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\varphi}_{\text{cKB}}(n + rN_1)| \\ &\quad + \frac{1}{I_0(\beta)} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{rect}}(n + rN_1) e^{2\pi i r N_1} \right\|_{C(\mathbb{T})}. \end{aligned}$$

Then, from Lemma 6 and Eq. 7, it follows that

$$\max_{n \in I_N} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{KB}}(n + rN_1) e^{2\pi i r N_1} \right\|_{C(\mathbb{T})} \leq \frac{11m}{I_0(\beta) N_1}.$$

Consequently, we obtain the following estimate of the  $C(\mathbb{T})$ -error constant of Eq. 20

$$\begin{aligned} e_\sigma(\varphi_{\text{KB}}) &\leq 11m\pi \sqrt{1 - \frac{1}{\sigma}} [\sinh(2\pi m \sqrt{1 - 1/\sigma})]^{-1} \\ &= 22\pi m \sqrt{1 - \frac{1}{\sigma}} [e^{2\pi m \sqrt{1 - 1/\sigma}} - e^{-2\pi m \sqrt{1 - 1/\sigma}}]^{-1}. \end{aligned}$$

For  $\sigma \geq \frac{5}{4}$ , we sustain by Eq. 19 that

$$e_\sigma(\varphi_{KB}) \leq 22\pi m \sqrt{1 - \frac{1}{\sigma}} \left[ e^{2\pi m \sqrt{1 - 1/\sigma}} - 0.06^m \right]^{-1}. \tag{21}$$

### 5 sinh-type window function

For fixed shape parameter  $\beta = bm$  with  $m \in \mathbb{N} \setminus \{1\}$ ,  $b = 2\pi \left(1 - \frac{1}{2\sigma}\right)$ , and oversampling factor  $\sigma \geq \frac{5}{4}$ , we consider the sinh-type window function

$$\varphi_{\sinh}(x) := \begin{cases} \frac{1}{\sinh \beta} \sinh(\beta \sqrt{1 - (N_1 x/m)^2}) & x \in I, \\ 0 & x \in \mathbb{R} \setminus I. \end{cases} \tag{22}$$

Obviously,  $\varphi_{\sinh} \in \Phi_{m, N_1}$  is a continuous window function which was introduced by the authors in [17] (with another shape parameter  $\beta = 4m$ ). Let  $N \geq 8$  be an even integer. For a sampling factor  $\sigma \geq \frac{5}{4}$  and  $b = 2\pi \left(1 - \frac{1}{2\sigma}\right)$ , we form  $N_1 = \sigma N \in 2\mathbb{N}$  and the shape parameter  $\beta = bm$ , where  $m \in \mathbb{N} \setminus \{1\}$  with  $2m \ll N_1$ .

Substituting  $t = N_1 x/m$ , we determine the even Fourier transform

$$\hat{\varphi}_{\sinh}(v) = \int_{\mathbb{R}} \varphi_{\sinh}(x) e^{-2\pi i vx} dx = \frac{2m}{N_1 \sinh \beta} \int_0^1 \sinh(\beta \sqrt{1 - t^2}) \cos \frac{2\pi m vt}{N_1} dt.$$

Using the scaled frequency  $w := 2\pi m v/N_1$ , the Fourier transform of Eq. 22 reads by [14, p. 38] as follows:

$$\hat{\varphi}_{\sinh}\left(\frac{N_1 w}{2\pi m}\right) = \frac{\pi m \beta}{N_1 \sinh \beta} \begin{cases} (\beta^2 - w^2)^{-1/2} I_1(\sqrt{\beta^2 - w^2}) & w \in (-\beta, \beta), \\ 1/4 & w = \pm\beta, \\ (w^2 - \beta^2)^{-1/2} J_1(\sqrt{w^2 - \beta^2}) & w \in \mathbb{R} \setminus [-\beta, \beta]. \end{cases} \tag{23}$$

By the power series expansion of the modified Bessel function  $I_1$ , we obtain for  $w \in (-\beta, \beta)$ ,

$$(\beta^2 - w^2)^{-1/2} I_1(\sqrt{\beta^2 - w^2}) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{4^k k! (k+1)!} (\beta^2 - w^2)^k$$

such that  $\hat{\varphi}_{\sinh}(v)$  is decreasing for  $v \in [0, N_1 \left(1 - \frac{1}{2\sigma}\right)]$ . The frequency  $v = \frac{N}{2}$  corresponds to the scaled frequency  $w = \frac{\pi m}{\sigma} < \beta = 2\pi m \left(1 - \frac{1}{2\sigma}\right)$ . Hence, we have

$$\begin{aligned} \min_{n \in I_N} \hat{\varphi}_{\sinh}(n) &= \hat{\varphi}_{\sinh}\left(\frac{N}{2}\right) \\ &= \frac{m\pi\beta}{N_1 \sinh \beta} (\beta^2 - \frac{\pi^2 m^2}{\sigma^2})^{-1/2} I_1\left(\sqrt{\beta^2 - \frac{\pi^2 m^2}{\sigma^2}}\right) \\ &= \frac{\beta}{2N_1 \sinh \beta} \left(1 - \frac{1}{\sigma}\right)^{-1/2} I_1\left(2\pi m \sqrt{1 - \frac{1}{\sigma}}\right). \end{aligned}$$

By  $m \geq 2$  and  $\sigma \geq \frac{5}{4}$ , we maintain

$$2\pi m \sqrt{1 - \frac{1}{\sigma}} \geq 4\pi \sqrt{1 - \frac{1}{\sigma}} \geq x_0 := \frac{4\pi}{\sqrt{5}}.$$

By the inequality (see [1] or [17, Lemma 3.3])

$$\sqrt{2\pi x_0} e^{-x_0} I_1(x_0) \leq \sqrt{2\pi x} e^{-x} I_1(x), \quad x \geq x_0,$$

we preserve for all  $x \geq x_0$ ,

$$I_1(x) \geq \sqrt{x_0} e^{-x_0} I_1(x_0) x^{-1/2} e^x > \frac{2}{5} x^{-1/2} e^x$$

and hence

$$\hat{\varphi}_{\sinh}\left(\frac{N}{2}\right) \geq \frac{\beta}{5 N_1 \sqrt{2\pi m} \sinh \beta} \left(1 - \frac{1}{\sigma}\right)^{-3/4} e^{2\pi m \sqrt{1-1/\sigma}}. \tag{24}$$

Now we estimate  $\hat{\varphi}_{\sinh}\left(\frac{N_1 w}{2\pi m}\right)$  for  $|w| > \beta = bm$ . For  $|w| > \beta$ ,  $\sigma \geq \frac{5}{4}$ , and  $N \geq 8$ , it holds

$$\frac{N_1 |w|}{2\pi m} > N_1 \left(1 - \frac{1}{2\sigma}\right) = N \left(\sigma - \frac{1}{2}\right) \geq \frac{3N}{4} \geq 6.$$

By an inequality of the Bessel function  $J_1$  (see [12] or [17, Lemma 3.2]), we have for all  $x \geq 6$ ,

$$|J_1(x)| \leq \frac{1}{\sqrt{x}}.$$

Using Eq. 23, it follows that for  $|w| > \beta$ ,

$$\begin{aligned} \left|\hat{\varphi}_{\sinh}\left(\frac{N_1 w}{2\pi m}\right)\right| &= \frac{\beta \pi m}{N_1 \sinh \beta} (w^2 - \beta^2)^{-1/2} |J_1(\sqrt{w^2 - \beta^2})| \\ &\leq \frac{\beta \pi m}{N_1 \sinh \beta} (w^2 - \beta^2)^{-3/4} \end{aligned}$$

i.e., for  $|v| > N_1 \left(1 - \frac{1}{2\sigma}\right)$ ,

$$|\hat{\varphi}_{\sinh}(v)| \leq \frac{\sqrt{\pi m}}{\sqrt{2} N_1 \sinh \beta} \left(\frac{v^2}{N_1^2} - \left(1 - \frac{1}{2\sigma}\right)^2\right)^{-3/4} \left(1 - \frac{1}{2\sigma}\right). \tag{25}$$

For each  $n \in I_N$  with  $|n \pm N_1| > N_1 \left(1 - \frac{1}{2\sigma}\right)$ , we obtain by Eq. 23 and  $|J_1(x)| \leq \frac{1}{2} |x|$  that

$$|\hat{\varphi}_{\sinh}(n \pm N_1)| \leq \frac{\pi m \beta}{2 N_1 \sinh \beta}. \tag{26}$$

In the case  $-\frac{N}{2} + N_1 = N_1 \left(1 - \frac{1}{2\sigma}\right)$ , we get by Eq. 23 that

$$\hat{\varphi}_{\sinh}\left(-\frac{N}{2} + N_1\right) = \frac{\pi m \beta}{4 N_1 \sinh \beta},$$

i.e., we can use Eq. 26 for all  $n \in I_N$ .

For all  $n \in I_N$  and  $r \in \mathbb{Z} \setminus \{0, \pm 1\}$ , we have by  $\sigma \geq \frac{5}{4}$  that

$$\left|\frac{n}{N_1} + r\right| \geq 2 - \frac{1}{2\sigma} \geq 2 - \frac{2}{5} = \frac{8}{5}.$$

Since the decreasing function  $h : [\frac{8}{5}, \infty) \rightarrow \mathbb{R}$ ,

$$h(x) := x^{3/2} \left(x^2 - \left(1 - \frac{1}{2\sigma}\right)^2\right)^{-3/4} = \left(1 - \left(1 - \frac{1}{2\sigma}\right)^2 x^{-2}\right)^{-3/4},$$

is bounded by

$$h\left(\frac{8}{5}\right) = \left(1 - \frac{25}{64} \left(1 - \frac{1}{2\sigma}\right)^2\right)^{-3/4} < \left(1 - \frac{25}{64}\right)^{-3/4} < \frac{3}{2},$$

we receive for all  $n \in I_N$  and  $r \in \mathbb{Z} \setminus \{0, \pm 1\}$ ,

$$\begin{aligned} |\hat{\varphi}_{\sinh}(n + rN_1)| &\leq \frac{\sqrt{\pi m}}{\sqrt{2} N_1 \sinh \beta} \left(\left(r + \frac{n}{N_1}\right)^2 - \left(1 - \frac{1}{2\sigma}\right)^2\right)^{-3/4} \\ &\leq \frac{3\sqrt{\pi m}}{2\sqrt{2} N_1 \sinh \beta} \left|r + \frac{n}{N_1}\right|^{-3/2}. \end{aligned}$$

Therefore, we get by Eq. 13 that for all  $n \in I_N$ ,

$$\begin{aligned} \sum_{r \in \mathbb{Z} \setminus \{0, \pm 1\}} |\hat{\varphi}_{\text{Bessel}}(n + rN_1)| &\leq \frac{3\sqrt{\pi m}}{2\sqrt{2} N_1 \sinh \beta} \sum_{r \in \mathbb{Z} \setminus \{0, \pm 1\}} \left|r + \frac{n}{N_1}\right|^{-3/2} \\ &\leq \frac{3\sqrt{2\pi m}}{N_1 \sinh \beta} \left(1 - \frac{1}{2\sigma}\right)^{-1/2} \end{aligned}$$

and hence by Eq. 26,

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\varphi}_{\text{Bessel}}(n + rN_1)| \leq \frac{1}{N_1 \sinh \beta} \left[\pi m \beta + 3\sqrt{2\pi m} \left(1 - \frac{1}{2\sigma}\right)^{-1/2}\right]. \tag{27}$$

Then, from Eqs. 23 and 26, it follows that

$$\begin{aligned} e_{\sigma, N}(\varphi_{\sinh}) &\leq \frac{1}{\hat{\varphi}_{\sinh}(N/2)} \max_{n \in I_N} \sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\varphi}_{\text{Bessel}}(n + rN_1)| \\ &\leq \left[5\pi m \sqrt{2\pi m} + 3\left(1 - \frac{1}{2\sigma}\right)^{-3/2}\right] \left(1 - \frac{1}{\sigma}\right)^{3/4} e^{-2\pi m \sqrt{1-1/\sigma}}. \end{aligned}$$

Using  $5\pi \sqrt{2\pi} < 40$ , we summarize:

**Theorem 3** Let  $N \in 2\mathbb{N}$ ,  $N \geq 8$ , and  $\sigma \geq \frac{5}{4}$  be given, where  $N_1 = \sigma N \in 2\mathbb{N}$ . Furthermore, let  $m \in \mathbb{N} \setminus \{1\}$  with  $2m \ll N_1$  and  $\beta = 2\pi m \left(1 - \frac{1}{2\sigma}\right)$ .

Then, the  $C(\mathbb{T})$ -error constant of the  $\sinh$ -type window function (22) can be estimated by

$$e_{\sigma}(\varphi_{\sinh}) \leq \left[40 m^{3/2} + 3\left(1 - \frac{1}{2\sigma}\right)^{-3/2}\right] \left(1 - \frac{1}{\sigma}\right)^{3/4} e^{-2\pi m \sqrt{1-1/\sigma}},$$

i.e., the  $\sinh$ -type window function (22) is convenient for NFFT.

For a fixed oversampling factor  $\sigma \geq \frac{5}{4}$ , the  $C(\mathbb{T})$ -error constant of Eq. 22 decays exponentially with the truncation parameter  $m \geq 2$ . On the other hand, the computational cost of NFFT increases with respect to  $m$  (see [15, pp. 380–381]) such that  $m$  should be not too large. For  $\sigma = 2$  and  $m = 4$ , we obtain  $e_{\sigma}(\varphi_{\sinh}) \leq 3.7 \cdot 10^{-6}$ .

### 6 Continuous exp-type window function

For fixed shape parameter  $\beta = bm$  with  $m \in \mathbb{N} \setminus \{1\}$ ,  $b = 2\pi \left(1 - \frac{1}{2\sigma}\right)$ , and oversampling factor  $\sigma \geq \frac{5}{4}$ , we consider the *continuous exp-type window function*

$$\varphi_{\text{cexp}}(x) := \begin{cases} \frac{1}{e^\beta - 1} \left( e^{\beta \sqrt{1 - (N_1 x/m)^2}} - 1 \right) & x \in I, \\ 0 & x \in \mathbb{R} \setminus I. \end{cases} \tag{28}$$

Obviously,  $\varphi_{\text{cexp}} \in \Phi_{m, N_1}$  is a continuous window function. Note that a discontinuous version of this window function was suggested in [3, 4]. A corresponding error estimate for the NFFT was proved in [4], where an asymptotic value of its Fourier transform was determined for  $\beta \rightarrow \infty$  by saddle point integration. We present new explicit error estimates for fixed shape parameter  $\beta$  of moderate size.

In the following, we present a new approach to an error estimate for the NFFT with the continuous exp-type window function (28). Unfortunately, the Fourier transform of Eq. 28 is unknown analytically. Therefore, we represent (28) as sum

$$\varphi_{\text{cexp}}(x) = \psi(x) + \rho(x), \tag{29}$$

where the Fourier transform of  $\psi$  is known and where the correction term  $\rho$  has small magnitude  $|\rho|$ . We choose

$$\psi(x) := \begin{cases} \frac{2}{e^\beta - 1} \sinh(\beta \sqrt{1 - (N_1 x)^2/m^2}) & x \in I, \\ 0 & x \in \mathbb{R} \setminus I \end{cases}$$

and

$$\rho(x) := \begin{cases} \frac{1}{e^\beta - 1} \left( e^{-\beta \sqrt{1 - (N_1 x)^2/m^2}} - 1 \right) & x \in I, \\ 0 & x \in \mathbb{R} \setminus I. \end{cases}$$

The Fourier transform of  $\psi$  reads as follows (see [14, p. 38]):

$$\hat{\psi}(v) = \frac{2\pi m\beta}{(e^\beta - 1) N_1} \begin{cases} \frac{1}{2\pi m} \left( \left(1 - \frac{1}{2\sigma}\right)^2 - \frac{v^2}{N_1^2} \right)^{-1/2} I_1 \left( 2\pi m \sqrt{\left(1 - \frac{1}{2\sigma}\right)^2 - \frac{v^2}{N_1^2}} \right) & |v| < N_1 \left(1 - \frac{1}{2\sigma}\right), \\ \frac{1}{4} & v = \pm N_1 \left(1 - \frac{1}{2\sigma}\right), \\ \frac{1}{2\pi m} \left( \frac{v^2}{N_1^2} - \left(1 - \frac{1}{2\sigma}\right)^2 \right)^{-1/2} J_1 \left( 2\pi m \sqrt{\frac{v^2}{N_1^2} - \left(1 - \frac{1}{2\sigma}\right)^2} \right) & |v| > N_1 \left(1 - \frac{1}{2\sigma}\right). \end{cases} \tag{30}$$

Since  $\rho$  is even and  $\rho|_{[0, \frac{m}{N_1}]}$  is increasing, we have  $0 \geq \rho(x) \geq \rho(0) = \frac{1}{e^\beta - 1} (e^{-\beta} - 1) = -e^{-\beta}$  (Table 1).

Since  $\rho$  has small absolute values in the small support  $\bar{I}$ , the Fourier transform  $\hat{\rho}$  is small too and it holds

$$|\hat{\rho}(v)| = \left| \int_I \rho(x) e^{-2\pi i v x} dx \right| \leq \frac{2m}{N_1} e^{-\beta}.$$



**Table 1** Upper bounds of the function  $-\rho$  for  $m = 2, 3, \dots, 6$  and  $\sigma = 2$

$m$	$-\rho(0) = e^{-\beta}$
2	$8.06 \cdot 10^{-5}$
3	$7.24 \cdot 10^{-7}$
4	$6.51 \cdot 10^{-9}$
5	$5.85 \cdot 10^{-11}$
6	$5.25 \cdot 10^{-13}$

Substituting  $t = N_1x/m$ , we determine the Fourier transform

$$\begin{aligned} \hat{\varphi}_{\text{cexp}}(v) &= \int_I \varphi_{\text{cexp}}(x) e^{-2\pi i vx} dx = \int_I \psi(x) e^{-2\pi i vx} dx + \int_I \rho(x) e^{-2\pi i vx} dx \\ &= \frac{2m}{(e^\beta - 1) N_1} \left[ 2 \int_0^1 \sinh(\beta\sqrt{1-t^2}) \cos\left(\frac{2\pi mvt}{N_1}\right) dt \right. \\ &\quad \left. + \int_0^1 (e^{-\beta\sqrt{1-t^2}} - 1) \cos\left(\frac{2\pi mvt}{N_1}\right) dt \right]. \end{aligned}$$

For simplicity, we introduce the scaled frequency  $w := 2\pi mv/N_1$  such that

$$\begin{aligned} \hat{\varphi}_{\text{cexp}}\left(\frac{N_1 w}{2\pi m}\right) &= \frac{2m}{(e^\beta - 1) N_1} \left[ 2 \int_0^1 \sinh(\beta\sqrt{1-t^2}) \cos(wt) dt \right. \\ &\quad \left. + \int_0^1 (e^{-\beta\sqrt{1-t^2}} - 1) \cos(wt) dt \right]. \end{aligned} \tag{31}$$

From [14, p. 38], it follows that

$$\begin{aligned} &\int_0^1 \sinh(\beta\sqrt{1-t^2}) \cos(wt) dt \\ &= \frac{\pi\beta}{2} \begin{cases} (\beta^2 - w^2)^{-1/2} I_1(\sqrt{\beta^2 - w^2}) & w \in (-\beta, \beta), \\ 1/4 & w = \pm\beta, \\ (w^2 - \beta^2)^{-1/2} J_1(\sqrt{w^2 - \beta^2}) & w \in \mathbb{R} \setminus [-\beta, \beta], \end{cases} \end{aligned}$$

where  $I_1$  denotes the modified Bessel function and  $J_1$  the Bessel function of first order. Therefore, we consider

$$\begin{aligned} \hat{\rho}\left(\frac{N_1 w}{2\pi m}\right) &= \frac{2m}{(e^\beta - 1) N_1} \int_0^1 (e^{-\beta\sqrt{1-t^2}} - 1) \cos(wt) dt \\ &= \frac{m}{(e^\beta - 1) N_1} \int_{-1}^1 (e^{-\beta\sqrt{1-t^2}} - 1) e^{iwt} dt \end{aligned} \tag{32}$$

as correction term of Eq. 31. Now, we estimate the integral

$$\mathcal{I}(w) := \int_{-1}^1 (e^{-\beta\sqrt{1-t^2}} - 1) e^{iwt} dt \tag{33}$$

by complex contour integrals.

**Lemma 7** For each  $w \in \mathbb{R}$  with  $|w| \geq \beta = bm$ ,  $m \in \mathbb{N} \setminus \{1\}$ , we have

$$|\mathcal{I}(w)| \leq \left(2 + \frac{2}{e}\right) \beta \sqrt[4]{5} |w|^{-5/4} + 4 |w|^{-1} e^{-\sqrt{|w|}} + (2e^{-\sqrt{2\beta}} + 1) e^{-|w|}.$$

*Proof* Here, we use the same technique as in [4, Lemma 10], where the integral

$$\int_{-1}^1 (e^{\beta\sqrt{1-t^2}} - 1) e^{iwt} dt$$

for  $|w| \geq \beta^4$  was estimated. Since  $\mathcal{I}(w) = \mathcal{I}(-w)$ , we consider only the case  $w \geq \beta$ . Let  $C_1$  be the line segment from  $-1$  to  $-1 + i$ . Furthermore,  $C_2$  is the line segment from  $-1 + i$  to  $1 + i$ ,  $C_3$  is the line segment from  $1 + i$  to  $1$ , and  $C_4$  is the line segment from  $1$  to  $-1$ . Since the principal square root function is holomorphic in  $\mathbb{C}$  except the nonpositive real axis, the complex function

$$g(z) := (e^{-\beta\sqrt{1-z^2}} - 1) e^{izw}$$

is holomorphic in  $\mathbb{C}$  except the set  $(-\infty, -1] \cup [1, \infty)$ . Hence,  $g$  is holomorphic on the interior of the closed curve  $C := C_1 \cup C_2 \cup C_3 \cup C_4$  and continuous in  $D \cup C$ , since a simple calculation shows that

$$\lim_{\substack{z \rightarrow -1 \\ z \in D}} g(z) = \lim_{\substack{z \rightarrow 1 \\ z \in D}} g(z) = 0.$$

Then, the stronger form of Cauchy’s Integral Theorem (see [5]) provides

$$\mathcal{I}(w) = \mathcal{I}_1(w) + \mathcal{I}_2(w) + \mathcal{I}_3(w) \tag{34}$$

with the contour integrals

$$\mathcal{I}_k(w) = \int_{C_k} (e^{-\beta\sqrt{1-z^2}} - 1) e^{izw} dz, \quad k = 1, 2, 3.$$

Note that  $\mathcal{I}_3(w)$  is the complex conjugate of  $\mathcal{I}_1(w)$  such that  $|\mathcal{I}_3(w)| = |\mathcal{I}_1(w)|$ .

The line segment  $C_2$  can be parametrized by  $z = t + i$ ,  $t \in [-1, 1]$  such that

$$\mathcal{I}_2(w) = e^{-w} \int_{-1}^1 (e^{-\beta\sqrt{2-t^2-2it}} - 1) e^{iwt} dt$$

and hence

$$|\mathcal{I}_2(w)| \leq e^{-w} \int_{-1}^1 |e^{-\beta\sqrt{2-t^2-2it}}| dt + 2e^{-w}.$$

Then, we have  $|e^{-\beta\sqrt{2-t^2-2it}}| = e^{-\beta \operatorname{Re}\sqrt{2-t^2-2it}}$ . Since for  $t \in [-1, 1]$ ,

$$|2 - t^2 - 2it| = \sqrt{t^2 + 4} \in [2, \sqrt{5}],$$

we obtain the estimate  $\sqrt{2} \leq \operatorname{Re}\sqrt{2 - t^2 - 2it} \leq \sqrt[4]{5}$  for  $t \in [-1, 1]$ .

Note that  $\operatorname{Re}\sqrt{2 - t^2 - 2it} > 0$  for all  $t \in [-1, 1]$ . Thus, we have

$$|\mathcal{I}_2(w)| \leq 2e^{-w} (e^{-\sqrt{2\beta}} + 1).$$

A parametrization of the line segment  $C_1$  is  $z = -1 + it$ ,  $t \in [0, 1]$ , such that

$$\mathcal{I}_1(w) = i e^{-iw} \int_0^1 (e^{-\beta\sqrt{2it+t^2}} - 1) e^{-wt} dt.$$

For  $w \geq \beta > 0$ , we split  $\mathcal{I}_1(w)$  into the sum of two integrals

$$\begin{aligned} \mathcal{I}_{1,0}(w) &:= i e^{-iw} \int_0^{w^{-1/2}} (e^{-\beta \sqrt{2it+t^2}} - 1) e^{-wt} dt, \\ \mathcal{I}_{1,1}(w) &:= i e^{-iw} \int_{w^{-1/2}}^1 (e^{-\beta \sqrt{2it+t^2}} - 1) e^{-wt} dt. \end{aligned}$$

Since  $2 \leq |\sqrt{2i+t}| \leq \sqrt[4]{5}$ ,  $t \in [0, 1]$ , the integral  $\mathcal{I}_{1,0}(w)$  is bounded in magnitude by

$$|\mathcal{I}_{1,0}(w)| \leq \max_{t \in [0, w^{-1/2}]} (1 - e^{-\beta \sqrt{t} \sqrt[4]{5}}) \int_0^{w^{-1/2}} e^{-wt} dt.$$

From

$$\int_0^{w^{-1/2}} e^{-wt} dt = \frac{1 - e^{-\sqrt{w}}}{w} \leq \frac{1}{w}$$

and

$$\max_{t \in [0, w^{-1/2}]} (1 - e^{-\beta \sqrt{t} \sqrt[4]{5}}) = 1 - e^{-\beta \sqrt[4]{5/w}} \leq \beta \sqrt[4]{\frac{5}{w}}$$

it follows that

$$|\mathcal{I}_{1,0}(w)| \leq (1 + e^{-1}) \beta \sqrt[4]{5} w^{-5/4}.$$

Above, we have used the simple inequality  $1 - e^{-x} \leq x$  for  $x \geq 0$ .

Finally, we estimate the integral  $\mathcal{I}_{1,1}(w)$  as follows:

$$|\mathcal{I}_{1,1}(w)| \leq \int_{w^{-1/2}}^1 e^{-\beta \sqrt{t} \operatorname{Re} \sqrt{2i+t} - wt} dt + \int_{w^{-1/2}}^1 e^{-wt} dt.$$

From  $\operatorname{Re} \sqrt{2i+t} > 0$  for all  $t \in [0, 1]$  and

$$\int_{w^{-1/2}}^1 e^{-wt} dt = w^{-1} (e^{-\sqrt{w}} - e^{-w}) \leq w^{-1} e^{-\sqrt{w}}$$

it follows that

$$|\mathcal{I}_{1,1}(w)| \leq \int_{w^{-1/2}}^1 e^{-wt} dt + w^{-1} e^{-\sqrt{w}} \leq 2 w^{-1} e^{-\sqrt{w}}.$$

Thus, we receive for  $w \geq \beta$ ,

$$\begin{aligned} |\mathcal{I}(w)| &\leq |\mathcal{I}_1(w)| + |\mathcal{I}_2(w)| + |\mathcal{I}_3(w)| = 2 |\mathcal{I}_1(w)| + |\mathcal{I}_2(w)| \\ &\leq 2 |\mathcal{I}_{1,0}(w)| + 2 |\mathcal{I}_{1,1}(w)| + |\mathcal{I}_2(w)| \\ &\leq \left(2 + \frac{2}{e}\right) \beta \sqrt[4]{5} w^{-5/4} + 4 w^{-1} e^{-\sqrt{w}} + (2 e^{-\sqrt{2}\beta} + 1) e^{-w}. \end{aligned}$$

This completes the proof. □

*Remark 2* In the proof of Lemma 7, it was shown that for real  $w$  with  $|w| \geq \beta$ , the contour integral  $\mathcal{I}_1(w)$  can be estimated by

$$|\mathcal{I}_1(w)| \leq \left(1 + \frac{1}{e}\right) \beta \sqrt[4]{5} |w|^{-5/4} + 2 |w|^{-1} e^{-\sqrt{|w|}}.$$

Splitting the integral  $\mathcal{I}_1(w)$  in the form

$$\mathcal{I}_1(w) = i e^{-i w} \left( \int_0^{(\log \sqrt{w})/w} + \int_{(\log \sqrt{w})/w}^1 \right) (e^{-\beta \sqrt{2it+t^2}} - 1) e^{-wt} dt,$$

this yields the better estimate for  $|w| \geq \beta$ ,

$$|\mathcal{I}_1(w)| \leq \left( \frac{1}{\sqrt{2}} \beta \sqrt[4]{5} \sqrt{\log |w|} + 2 \right) |w|^{-3/2}.$$

**Lemma 8** *Let  $\beta = bm$  be given. Then for each  $w \in \mathbb{R}$ , it holds the estimate*

$$-\text{sinc } w - \gamma(m, \sigma) \leq \int_0^1 (e^{-\beta \sqrt{1-t^2}} - 1) \cos(wt) dt \leq -\text{sinc } w + \gamma(m, \sigma)$$

with the small positive constant

$$\gamma(m, \sigma) := \int_0^1 e^{-\beta \sqrt{1-t^2}} dt.$$

*Proof* For each  $w \in \mathbb{R}$ , we have

$$\begin{aligned} \left| \int_0^1 (e^{-\beta \sqrt{1-t^2}} - 1) \cos(wt) dt + \text{sinc } w \right| &= \left| \int_0^1 e^{-\beta \sqrt{1-t^2}} \cos(wt) dt \right| \\ &\leq \int_0^1 e^{-\beta \sqrt{1-t^2}} dt = \gamma(m, \sigma) \end{aligned}$$

such that

$$\max_{w \in \mathbb{R}} \left| \int_0^1 (e^{-\beta \sqrt{1-t^2}} - 1) \cos(wt) dt + \text{sinc } w \right| \leq \gamma(m, \sigma),$$

see Table 2. □

Using Eq. 31 and Lemma 8, we receive for  $w \in (-\beta, \beta)$ ,

$$\hat{\varphi}_{\text{cexp}}\left(\frac{N_1 w}{2\pi m}\right) \geq \frac{2m}{(e^\beta - 1) N_1} \left[ \pi \beta (\beta^2 - w^2)^{-1/2} I_1(\sqrt{\beta^2 - w^2}) \right. \tag{35}$$

$$\left. - \text{sinc } w - \gamma(m, \sigma) \right]. \tag{36}$$

**Table 2** Maximum error

$\gamma(m, \sigma)$ between	$m$	$\gamma(m, 2)$
$\int_0^1 (e^{-\beta \sqrt{1-t^2}} - 1) \cos(wt) dt$	2	$1.17 \cdot 10^{-2}$
and $-\text{sinc } w$ for	3	$5.08 \cdot 10^{-3}$
$m = 2, 3, \dots, 6$ and $\sigma = 2$	4	$2.84 \cdot 10^{-3}$
	5	$1.81 \cdot 10^{-3}$
	6	$1.25 \cdot 10^{-3}$

The function  $h : [0, \beta] \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} h(w) &:= \pi\beta (\beta^2 - w^2)^{-1/2} I_1(\sqrt{\beta^2 - w^2}) - \operatorname{sinc} w \\ &= \frac{\pi\beta}{2} \sum_{s=0}^{\infty} \frac{(\beta^2 - w^2)^s}{2^{2s} s! (s + 1)!} - \operatorname{sinc} w \end{aligned}$$

has the derivative

$$h'(w) = -\pi\beta \sum_{s=1}^{\infty} \frac{(\beta^2 - w^2)^{s-1}}{2^{2s} (s - 1)! (s + 1)!} - \frac{d}{dw} \operatorname{sinc} w. \tag{37}$$

Obviously, it holds  $h'(0) = 0$ , and  $h'(w) < 0$  for  $w \in (0, \pi]$ . For  $w \in [\pi, \beta]$  and  $m \geq 2$ , we obtain

$$\left| \frac{d}{dw} \operatorname{sinc} w \right| \leq \frac{1}{\pi^2} + \frac{1}{\pi}$$

and hence by Eq. 37,

$$h'(w) < -\frac{\pi\beta}{8} + \frac{1}{\pi^2} + \frac{1}{\pi} < 0.$$

Thus,  $h(w)$  is decreasing with respect to  $w \in [0, \beta]$ .

For  $\beta = bm$  and  $v = \frac{N_1 w}{2\pi m} \in (-N_1(1 - \frac{1}{2\sigma}), N_1(1 - \frac{1}{2\sigma}))$ , the inequality (35) implies

$$\begin{aligned} \hat{\varphi}_{\text{cexp}}(v) &\geq \frac{2m}{(e^\beta - 1) N_1} \left[ 2 \left( \left(1 - \frac{1}{2\sigma}\right)^2 - \frac{v^2}{N_1^2} \right)^{-1/2} I_1 \left( 2\pi m \sqrt{\left(1 - \frac{1}{2\sigma}\right)^2 - \frac{v^2}{N_1^2}} \right) \right. \\ &\quad \left. - \operatorname{sinc} \frac{2\pi m v}{N_1} - \gamma(m, \sigma) \right]. \end{aligned}$$

Consequently, the Fourier transform  $\hat{\varphi}_{\text{cexp}}(v)$  is positive and decreasing for

$$v \in \left[ 0, N_1 \left(1 - \frac{1}{2\sigma}\right) \right).$$

Hence, we obtain

$$\begin{aligned} \min_{n \in I_N} \hat{\varphi}_{\text{cexp}}(n) &= \hat{\varphi}_{\text{cexp}}\left(\frac{N}{2}\right) \\ &\geq \frac{2m}{(e^\beta - 1) N_1} \left[ \frac{b}{2} \left(1 - \frac{1}{\sigma}\right)^{-1/2} I_1 \left( 2\pi m \sqrt{1 - \frac{1}{\sigma}} \right) - \operatorname{sinc} \frac{\pi m}{\sigma} - \gamma(m, \sigma) \right] \\ &\geq \frac{2m}{(e^\beta - 1) N_1} \left[ \frac{b}{2} \left(1 - \frac{1}{\sigma}\right)^{-1/2} I_1 \left( 2\pi m \sqrt{1 - \frac{1}{\sigma}} \right) - 1 - \gamma(m, \sigma) \right]. \end{aligned}$$

From  $m \geq 2$  and  $\sigma \geq \frac{5}{4}$ , it follows that

$$2\pi m \sqrt{1 - \frac{1}{\sigma}} \geq 4\pi \sqrt{1 - \frac{1}{\sigma}} \geq x_0 := \frac{4\pi}{\sqrt{5}}.$$

Hence, by the inequality (see [1] or [17, Lemma 3.3])

$$\sqrt{2\pi x_0} e^{-x_0} I_1(x_0) \leq \sqrt{2\pi x} e^{-x} I_1(x), \quad x \geq x_0,$$

we sustain that

$$I_1(x) \geq \sqrt{x_0} e^{-x_0} I_1(x_0) x^{-1/2} e^x, \quad x \geq x_0.$$

Thus, for  $x = 2\pi m \sqrt{1 - 1/\sigma}$ , we get the estimate

$$I_1(2\pi m \sqrt{1 - \frac{1}{\sigma}}) \geq \frac{2}{5\sqrt{2\pi m}} \left(1 - \frac{1}{\sigma}\right)^{-1/4} e^{2\pi m \sqrt{1 - 1/\sigma}}.$$

Thus, we obtain the following

**Lemma 9** *Let  $N \in 2\mathbb{N}$  and  $\sigma \geq \frac{5}{4}$  be given, where  $N_1 = \sigma N \in 2\mathbb{N}$ . Furthermore, let  $m \in \mathbb{N}$  with  $2m \ll N_1$ ,  $\beta = bm$ , and  $b = 2\pi(1 - \frac{1}{2\sigma})$ .*

*Then, we have*

$$\hat{\phi}_{\text{cexp}}\left(\frac{N}{2}\right) \geq \frac{2m}{(e^\beta - 1)N_1} \left[ \frac{b}{5\sqrt{2\pi m}} \left(1 - \frac{1}{\sigma}\right)^{-3/4} e^{2\pi m \sqrt{1 - 1/\sigma}} - 1 - \gamma(m, \sigma) \right].$$

By Eq. 2, the constant  $e_{\sigma, N}(\hat{\phi}_{\text{cexp}})$  can be estimated as follows:

$$e_{\sigma, N}(\hat{\phi}_{\text{cexp}}) \leq \frac{1}{\hat{\phi}_{\text{cexp}}(N/2)} \max_{n \in I_N} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\phi}_{\text{cexp}}(n + rN_1) e^{2\pi i r N_1} \right\|_{C(\mathbb{T})},$$

where it holds by Eq. 29,

$$\left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\phi}_{\text{cexp}}(n + rN_1) e^{2\pi i r N_1} \right\|_{C(\mathbb{T})} \leq \sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\psi}(n + rN_1)| + \sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\rho}(n + rN_1)|.$$

**Lemma 10** *Let  $N \in 2\mathbb{N}$  and  $\sigma \geq \frac{5}{4}$  be given, where  $N_1 = \sigma N \in 2\mathbb{N}$ . Furthermore, let  $m \in \mathbb{N}$  with  $2m \ll N_1$ ,  $\beta = bm$ , and  $b = 2\pi(1 - \frac{1}{2\sigma})$ .*

*Then, it holds for all  $n \in I_N$ ,*

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\psi}(n + rN_1)| \leq \frac{\beta}{(e^\beta - 1)N_1} \left[ 2\pi m + \frac{10}{\sqrt{2\pi m}} \left(1 - \frac{1}{2\sigma}\right)^{-1/2} \right].$$

*Proof* For all  $n \in I_N$  and  $r \in \mathbb{Z} \setminus \{0\}$  with  $(n, r) \neq (-\frac{N}{2}, 1)$ , it follows that

$$|n + rN_1| > N_1 - \frac{N}{2} = N_1 \left(1 - \frac{1}{2\sigma}\right)$$

and hence by  $\sigma \geq \frac{5}{4}$ ,

$$\left| \frac{n}{N_1} + r \right| > 1 - \frac{1}{2\sigma} \geq \frac{3}{5}.$$

Thus, by Eq. 30, we receive

$$\begin{aligned} & \hat{\psi}(n + rN_1) \\ = & \frac{\beta}{(e^\beta - 1)N_1} \left( \left(\frac{n}{N_1} + r\right)^2 - \left(1 - \frac{1}{2\sigma}\right)^2 \right)^{-1/2} J_1 \left( 2\pi m \sqrt{\left(\frac{n}{N_1} + r\right)^2 - \left(1 - \frac{1}{2\sigma}\right)^2} \right). \end{aligned}$$

In the case  $(n, r) = (-\frac{N}{2}, 1)$ , from Eq. 30, it follows that

$$\hat{\psi}\left(-\frac{N}{2} + N_1\right) = \hat{\psi}\left(N_1\left(1 - \frac{1}{2\sigma}\right)\right) = \frac{\pi m \beta}{2(e^\beta - 1)N_1}.$$

Now, we estimate the sum

$$\begin{aligned} \sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\psi}(n + rN_1)| &= |\hat{\psi}(n + N_1)| + |\hat{\psi}(n - N_1)| \\ &+ \sum_{r \in \mathbb{Z} \setminus \{0, \pm 1\}} |\hat{\psi}(n + rN_1)|. \end{aligned}$$

For  $n \in I_N$  and  $r \in \mathbb{Z} \setminus \{0, \pm 1\}$ , we have by  $\sigma \geq \frac{5}{4}$ ,

$$\left| \frac{n}{N_1} + r \right| \geq 2 - \frac{1}{2\sigma} \geq \frac{7}{5}$$

and therefore by  $m \geq 2$ ,

$$\begin{aligned} 2\pi m \sqrt{\left(\frac{n}{N_1} + r\right)^2 - \left(1 - \frac{1}{2\sigma}\right)^2} &\geq 4\pi \sqrt{\left(2 - \frac{1}{2\sigma}\right)^2 - \left(1 - \frac{1}{2\sigma}\right)^2} \\ &= 4\pi \sqrt{3 - \frac{1}{\sigma}} \geq 4\pi \sqrt{\frac{11}{5}} > 6. \end{aligned}$$

By an inequality for the Bessel function  $J_1(x)$  (see [12] or [17, Lemma 3.2]), it holds for all  $x \geq 6$ ,

$$|J_1(x)| \leq \frac{1}{\sqrt{x}}$$

such that for  $x = 2\pi m \sqrt{\left(r + n/N_1\right)^2 - \left(1 - 1/(2\sigma)\right)^2}$ ,

$$\begin{aligned} &|J_1(2\pi m \sqrt{\left(\frac{n}{N_1} + r\right)^2 - \left(1 - \frac{1}{2\sigma}\right)^2})| \\ &\leq \frac{1}{\sqrt{2\pi m}} \left(\left(\frac{n}{N_1} + r\right)^2 - \left(1 - \frac{1}{2\sigma}\right)^2\right)^{-1/4}. \end{aligned}$$

The decreasing function  $h : [\frac{5}{4}, \infty) \rightarrow \mathbb{R}$  defined by

$$h(x) := x^{3/2} \left(x^2 - \left(1 - \frac{1}{2\sigma}\right)^2\right)^{-3/4} = \left(1 - \left(1 - \frac{1}{2\sigma}\right)^2 x^{-2}\right)^{-3/4},$$

is bounded by

$$h\left(\frac{5}{4}\right) = \left(1 - \frac{16}{25} \left(1 - \frac{1}{2\sigma}\right)^2\right)^{-3/4} \leq \left(1 - \frac{16}{25}\right)^{-3/4} = \left(\frac{5}{3}\right)^{3/2} < \frac{5}{2}.$$

Thus, we receive for all  $n \in I_N$  and  $r \in \mathbb{Z} \setminus \{0, \pm 1\}$ ,

$$\begin{aligned} &\left(\left(\frac{n}{N_1} + r\right)^2 - \left(1 - \frac{1}{2\sigma}\right)^2\right)^{-1/2} |J_1(2\pi m \sqrt{\left(\frac{n}{N_1} + r\right)^2 - \left(1 - \frac{1}{2\sigma}\right)^2})| \\ &\leq \frac{1}{\sqrt{2\pi m}} \left(\left(\frac{n}{N_1} + r\right)^2 - \left(1 - \frac{1}{2\sigma}\right)^2\right)^{-3/4} \leq \frac{5}{2\sqrt{2\pi m}} \left|\frac{n}{N_1} + r\right|^{-3/2}. \end{aligned}$$

Hence, for all  $n \in I_N$ , we obtain by Eq. 13 that

$$\begin{aligned} \sum_{r \in \mathbb{Z} \setminus \{0, \pm 1\}} |\hat{\psi}(n + rN_1)| &\leq \frac{5\beta}{2\sqrt{2\pi m} (e^\beta - 1) N_1} \sum_{r \in \mathbb{Z} \setminus \{0, \pm 1\}} \left| \frac{n}{N_1} + r \right|^{-3/2} \\ &\leq \frac{10\beta}{\sqrt{2\pi m} (e^\beta - 1) N_1} \left(1 - \frac{1}{2\sigma}\right)^{-1/2}. \end{aligned}$$

Now, we estimate  $\hat{\psi}(v)$  for  $v = n \pm N_1, n \in I_N$ . For  $v = -\frac{N}{2} + N_1 = N_1(1 - \frac{1}{2\sigma})$ , it holds by Eq. 30,

$$\hat{\psi}\left(-\frac{N}{2} + N_1\right) = \hat{\psi}\left(N_1\left(1 - \frac{1}{2\sigma}\right)\right) = \frac{\pi m \beta}{2(e^\beta - 1) N_1}. \tag{38}$$

For all the other  $v = n \pm N_1 \neq -\frac{N}{2} + N_1, n \in I_N$ , we have

$$|n \pm N_1| > \left(1 - \frac{1}{2\sigma}\right) N_1$$

such that by Eq. 30,  $\hat{\psi}(n \pm N_1)$  reads as follows:

$$\frac{\beta}{(e^\beta - 1) N_1} \left( \left(1 \pm \frac{n}{N_1}\right)^2 - \left(1 - \frac{1}{2\sigma}\right)^2 \right)^{-1/2} J_1\left(2\pi m \sqrt{\left(1 \pm \frac{n}{N_1}\right)^2 - \left(1 - \frac{1}{2\sigma}\right)^2}\right).$$

Since

$$2\pi m \sqrt{\left(1 \pm \frac{n}{N_1}\right)^2 - \left(1 - \frac{1}{2\sigma}\right)^2} > 0$$

can be small for  $n \in I_N$ , we estimate the Bessel function  $J_1(x), x \geq 0$ , by Poisson’s integral (see [21, p. 47]):

$$|J_1(x)| = \frac{x}{\pi} \left| \int_0^\pi \cos(x \cos t) (\sin t)^2 dt \right| \leq \frac{x}{\pi} \int_0^\pi (\sin t)^2 dt = \frac{x}{2},$$

such that

$$|\hat{\psi}(n \pm N_1)| \leq \frac{\pi m \beta}{(e^\beta - 1) N_1}.$$

By Eq. 38, this estimate of  $|\hat{\psi}(n \pm N_1)|$  is valid for all  $n \in I_N$ . This completes the proof. □

Now for arbitrary  $n \in I_N$ , we have to estimate the series

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\rho}(n + rN_1)|.$$

By Eq. 32 and Lemma 7, we obtain for any  $v \in \mathbb{R} \setminus \{0\}$ ,

$$\begin{aligned} |\hat{\rho}(v)| &\leq \frac{2m}{(e^\beta - 1) N_1} \left[ \left(1 + \frac{1}{e}\right) \frac{\beta}{2\pi m} \sqrt[4]{\frac{5}{2\pi m}} \left(\frac{|v|}{N_1}\right)^{-5/4} \right. \\ &\quad \left. + \frac{N_1}{\pi m |v|} e^{-\sqrt{2\pi m} |v|/N_1} + (e^{-\sqrt{2}\beta} + \frac{1}{2}) e^{-2\pi m |v|/N_1} \right]. \end{aligned}$$



Thus, we obtain that

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\rho}(n + rN_1)| \leq \frac{2m}{(e^\beta - 1) N_1} \left[ \left(1 + \frac{1}{e}\right) \frac{\beta}{2\pi m} \sqrt[4]{\frac{5}{2\pi m}} S_1(n) + 2 S_2(n) + \left(e^{-\sqrt{2}\beta} + \frac{1}{2}\right) S_3(n) \right]$$

with

$$S_1(n) := \sum_{r \in \mathbb{Z} \setminus \{0\}} \left| \frac{n}{N_1} + r \right|^{-5/4}, \tag{39}$$

$$S_2(n) := \sum_{r \in \mathbb{Z} \setminus \{0\}} \frac{1}{2\pi m |r + n/N_1|} e^{-\sqrt{2\pi m} |r + n/N_1|}, \tag{40}$$

$$S_3(n) := \sum_{r \in \mathbb{Z} \setminus \{0\}} e^{-2\pi m |r + n/N_1|}. \tag{41}$$

The inequalities (13), (15), and (16) imply that

$$S_1(n) \leq \left(\frac{4\sigma}{2\sigma - 1} + 8\right) \left(1 - \frac{1}{2\sigma}\right)^{-1/4}, \tag{42}$$

$$S_2(n) \leq \frac{2\sigma}{(2\sigma - 1) \pi m} e^{-\sqrt{2\pi m - \pi m/\sigma}} + \frac{2}{\pi m} E_1(\sqrt{2\pi m - \pi m/\sigma}), \tag{43}$$

$$S_3(n) \leq \left(2 + \frac{1}{\pi m}\right) e^{-2\pi m + \pi m/\sigma}. \tag{44}$$

Thus, we obtain the following

**Lemma 11** *Let  $N \in 2\mathbb{N}$  and  $\sigma \geq \frac{5}{4}$  be given, where  $N_1 = \sigma N \in 2\mathbb{N}$ . Furthermore, let  $m \in \mathbb{N} \setminus \{1\}$  with  $2m \ll N_1$ ,  $\beta = bm$ , and  $b = 2\pi \left(1 - \frac{1}{2\sigma}\right)$ .*

*Then, for each  $n \in I_N$ , it holds the estimate*

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\rho}(n + rN_1)| \leq \frac{1}{(e^\beta - 1) N_1} \left[ \left(1 + \frac{1}{e}\right) \frac{\beta}{\pi} \sqrt[4]{\frac{5}{2\pi m}} \left(\frac{4\sigma}{2\sigma - 1} + 8\right) \left(1 - \frac{1}{2\sigma}\right)^{-1/4} + \frac{8\sigma}{(2\sigma - 1) \pi} e^{-\sqrt{2\pi m - \pi m/\sigma}} + \frac{8}{\pi} E_1(\sqrt{2\pi m - \pi m/\sigma}) + \left(e^{-\sqrt{2}\beta} + \frac{1}{2}\right) \left(4m + \frac{2}{\pi}\right) e^{-2\pi m + \pi m/\sigma} \right].$$

Hence, from Lemmas 10 and 11, it follows that

$$\max_{n \in I_N} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{cexp}}(n + rN_1) e^{2\pi i r N_1} \right\|_{C(\mathbb{T})} \leq \frac{\beta}{(e^\beta - 1) N_1} b(m, \sigma) \tag{45}$$

with the constant

$$\begin{aligned}
 b(m, \sigma) := & 2\pi m + \frac{10}{\sqrt{2\pi m}} \left(1 - \frac{1}{2\sigma}\right)^{-1/2} \\
 & + \left(1 + \frac{1}{e}\right) \frac{1}{\pi} \sqrt[4]{\frac{5}{2\pi m}} \left(\frac{4\sigma}{2\sigma - 1} + 8\right) \left(1 - \frac{1}{2\sigma}\right)^{-1/4} \\
 & + \frac{8\sigma}{(2\sigma - 1)\pi\beta} e^{-\sqrt{2\pi m - \pi m/\sigma}} + \frac{8}{\pi\beta} E_1(\sqrt{2\pi m - \pi m/\sigma}) \\
 & + \frac{4\pi m + 2}{\beta\pi} \left(e^{-\sqrt{2}\beta} + \frac{1}{2}\right) e^{-2\pi m + \pi m/\sigma}. \tag{46}
 \end{aligned}$$

Using Lemma 9, we obtain by

$$e_{\sigma, N}(\varphi_{\text{cexp}}) \leq \frac{1}{\hat{\varphi}_{\text{cexp}}(N/2)} \max_{n \in I_N} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{cexp}}(n + rN_1) e^{2\pi i r N_1 \cdot} \right\|_{C(\mathbb{T})}$$

the following:

**Theorem 4** *Let  $N \in 2\mathbb{N}$  and  $\sigma \geq \frac{5}{4}$  be given, where  $N_1 = \sigma N \in 2\mathbb{N}$ . Furthermore, let  $m \in \mathbb{N}$  with  $2m \ll N_1$ ,  $\beta = bm$ , and  $b = 2\pi \left(1 - \frac{1}{2\sigma}\right)$ .*

*Then, the  $C(\mathbb{T})$ -error constant of the continuous exp-type window function (28) can be estimated by*

$$e_{\sigma}(\varphi_{\text{cexp}}) \leq \frac{\beta b(m, \sigma)}{2m} \left[ \frac{b}{5\sqrt{2\pi m}} \left(1 - \frac{1}{\sigma}\right)^{-3/4} e^{2\pi m \sqrt{1-1/\sigma}} - 1 - \gamma(m, \sigma) \right]^{-1} \tag{47}$$

*In other words, the continuous exp-type window function (28) is convenient for NFFT.*

Note that for  $\sigma \in \left[\frac{5}{4}, 2\right]$  and  $m \geq 2$ , it holds by Eq. 46,

$$\frac{\beta b(m, \sigma)}{2m} = \pi b(m, \sigma) \left(1 - \frac{1}{2\sigma}\right) \leq \frac{3\pi^2}{2} m + b_0 < 15m + b_0$$

with  $b_0 < 17$ .

In order to compute the Fourier transform  $\hat{\varphi}$  of window function  $\varphi \in \Phi_{m, N_1}$ , we approximate this window function by numerical integration. In our next numerical examples, we apply the following method. Since the window function  $\varphi \in \Phi_{m, N_1}$  is even and supported in  $\left[-\frac{m}{N_1}, \frac{m}{N_1}\right]$ , we have

$$\begin{aligned}
 \hat{\varphi}(v) &= \int_{\mathbb{R}} \varphi(x) e^{-2\pi i v x} dx = \frac{m}{N_1} \int_{-1}^1 \varphi\left(\frac{m}{N_1} t\right) e^{-2\pi i m v t / N_1} dt \\
 &= \frac{2m}{N_1} \int_0^1 \varphi\left(\frac{m}{N_1} t\right) \cos \frac{2\pi m v t}{N_1} dt.
 \end{aligned}$$

We evaluate the last integral using a global adaptive quadrature [19] for  $\hat{\varphi}(k)$ ,  $k = 0, \dots, N$ . In general, these values can be precomputed; see [2, 10].

*Remark 3* As in [3, 4], we can consider also the original exp-type window function

$$\varphi_{\text{exp}}(x) := \begin{cases} \exp(\beta\sqrt{1 - (N_1x/m)^2} - \beta) & x \in I, \\ \frac{1}{2} e^{-\beta} & x = \pm \frac{m}{N_1}, \\ 0 & x \in \mathbb{R} \setminus \bar{I} \end{cases} \quad (48)$$

with the shape parameter  $\beta = mb = 2\pi m(1 - \frac{1}{2\sigma})$ ,  $\sigma \geq \frac{5}{4}$ ,  $N \in 2\mathbb{N}$ , and  $N_1 = \sigma N \in 2\mathbb{N}$ . Furthermore, we assume that  $m \in \mathbb{N} \setminus \{1\}$  fulfills  $2m \ll N_1$ . This window function possesses jump discontinuities at  $x = \pm \frac{m}{N_1}$  with very small jump height  $e^{-\beta}$ , such that Eq. 48 is “almost continuous.”

We split (48) in the form

$$\varphi_{\text{exp}}(x) = (1 - e^{-\beta}) \varphi_{\text{cexp}}(x) + e^{-\beta} \varphi_{\text{rect}}(x), \quad x \in \mathbb{R},$$

with the window functions (28) and (4). Then, the Fourier transform of Eq. 48 reads as follows:

$$\hat{\varphi}_{\text{exp}}(v) = (1 - e^{-\beta}) \hat{\varphi}_{\text{cexp}}(v) + e^{-\beta} \hat{\varphi}_{\text{rect}}(v), \quad v \in \mathbb{R},$$

By Lemma 9, it follows that

$$\begin{aligned} \hat{\varphi}_{\text{exp}}\left(\frac{N}{2}\right) &= \min_{n \in I_N} \hat{\varphi}_{\text{exp}}(n) \\ &= \min_{n \in I_N} \left[ (1 - e^{-\beta}) \hat{\varphi}_{\text{cexp}}(n) + e^{-\beta} \hat{\varphi}_{\text{rect}}(n) \right] \\ &\geq \frac{2m}{N_1 e^\beta} \left[ \frac{b}{5\sqrt{2\pi m}} \left(1 - \frac{1}{\sigma}\right)^{-3/4} e^{2\pi m\sqrt{1-1/\sigma}} - \gamma(m, \sigma) \right]. \end{aligned}$$

Using Eqs. 45 and 7, we estimate for all  $n \in I_N$ ,

$$\begin{aligned} &\left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{exp}}(n + rN_1) e^{2\pi i r N_1 \cdot} \right\|_{C(\mathbb{T})} \leq (1 - e^{-\beta}) \sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\varphi}_{\text{cexp}}(n + rN_1)| \\ &+ e^{-\beta} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\text{rect}}(n + rN_1) e^{2\pi i r N_1 \cdot} \right\|_{C(\mathbb{T})} \\ &\leq \frac{1}{N_1 e^\beta} (\beta b(m, \sigma) + 3m). \end{aligned}$$

Thus, we obtain

$$e_\sigma(\varphi_{\text{exp}}) \leq \left( \frac{\beta b(m, \sigma)}{2m} + \frac{3}{2} \right) \left[ \frac{b}{5\sqrt{2\pi m}} \left(1 - \frac{1}{\sigma}\right)^{-3/4} e^{2\pi m\sqrt{1-1/\sigma}} - \gamma(m, \sigma) \right]^{-1}.$$

Thus, the discontinuous window function (48) possesses a similar  $C(\mathbb{T})$ -error constant as the continuous exp-type window function (28).

### 7 Continuous cosh-type window function

For fixed shape parameter  $\beta = bm = 2\pi m(1 - \frac{1}{2\sigma})$  and oversampling factor  $\sigma \geq \frac{5}{4}$ , we consider the *continuous cosh-type window function*

$$\varphi_{\text{cosh}}(x) := \begin{cases} \frac{1}{\cosh \beta - 1} \left( \cosh \left( \beta \sqrt{1 - \left( \frac{N_1 x}{m} \right)^2} \right) - 1 \right) & x \in I, \\ 0 & x \in \mathbb{R} \setminus I. \end{cases} \tag{49}$$

Obviously,  $\varphi_{\text{cosh}} \in \Phi_{m, N_1}$  is a continuous window function. Note that recently a discontinuous version of this window function was suggested in [3, Remark 13]. But up to now, a corresponding error estimate for the related NFFT was unknown. Now we show that the  $C(\mathbb{T})$ -error constant  $e_\sigma(\varphi_{\text{cosh}})$  can be estimated by a similar upper bound as  $e_\sigma(\varphi_{\text{cexp}})$  in Theorem 4. Thus, the window functions (28) and (49) possess the same error behavior with respect to the NFFT.

In the following, we use the same technique as in Section 6. Since the Fourier transform of Eq. 49 is unknown analytically, we represent Eq. 49 as the sum

$$\varphi_{\text{cosh}}(x) = \psi_1(x) + \rho_1(x),$$

where the Fourier transform of  $\psi_1$  is known and where the correction term  $\rho_1$  has small magnitude  $|\rho_1|$ . We choose

$$\psi_1(x) := \begin{cases} \frac{1}{\cosh \beta - 1} \sinh(\beta \sqrt{1 - (N_1 x)^2/m^2}) & x \in I, \\ 0 & x \in \mathbb{R} \setminus I \end{cases}$$

and

$$\rho_1(x) := \begin{cases} \frac{1}{\cosh \beta - 1} \left( e^{-\beta \sqrt{1 - (N_1 x)^2/m^2}} - 1 \right) & x \in I, \\ 0 & x \in \mathbb{R} \setminus I. \end{cases}$$

Since  $\rho_1$  is even and  $\rho_1|_{[0, \frac{m}{N_1}]}$  is increasing, we have

$$0 \geq \rho_1(x) \geq \rho_1(0) = \frac{e^{-\beta} - 1}{\cosh \beta - 1} = -\frac{2}{e^\beta - 1}.$$

Since  $\rho_1$  has small values in the compact support  $\bar{I}$ , the Fourier transform  $\hat{\rho}_1$  is small too and it holds

$$|\hat{\rho}_1(v)| = \left| \int_I \rho_1(x) e^{2\pi i vx} dx \right| \leq \frac{4m}{(e^\beta - 1) N_1}.$$

Substituting  $t = N_1 x/m$ , we determine the Fourier transform

$$\begin{aligned} \hat{\varphi}_{\text{cosh}}(v) &= \int_I \varphi_{\text{cosh}}(x) e^{-2\pi i vx} dx = \int_I \psi_1(x) e^{-2\pi i vx} dx + \int_I \rho_1(x) e^{-2\pi i vx} dx \\ &= \frac{2m}{(\cosh \beta - 1) N_1} \left[ \int_0^1 \sinh(\beta \sqrt{1 - t^2}) \cos \frac{2\pi mvt}{N_1} dt \right. \\ &\quad \left. + \int_0^1 (e^{-\beta \sqrt{1 - t^2}} - 1) \cos \frac{2\pi mvt}{N_1} dt \right]. \end{aligned}$$

For simplicity, we introduce the scaled frequency  $w := 2\pi m v / N_1$  such that

$$\hat{\varphi}_{\text{exp}}\left(\frac{N_1 w}{2\pi m}\right) = \frac{2m}{(\cosh \beta - 1) N_1} \left[ \int_0^1 \sinh(\beta \sqrt{1-t^2}) \cos(wt) dt + \int_0^1 (e^{-\beta \sqrt{1-t^2}} - 1) \cos(wt) dt \right].$$

From [14, p. 38], it follows that

$$\begin{aligned} & \int_0^1 \sinh(\beta \sqrt{1-t^2}) \cos(wt) dt \\ &= \frac{\pi \beta}{2} \begin{cases} (\beta^2 - w^2)^{-1/2} I_1(\sqrt{\beta^2 - w^2}) & w \in (-\beta, \beta), \\ 1/4 & w = \pm \beta, \\ (w^2 - \beta^2)^{-1/2} J_1(\sqrt{w^2 - \beta^2}) & w \in \mathbb{R} \setminus [-\beta, \beta], \end{cases} \end{aligned}$$

where  $I_1$  denotes the modified Bessel function and  $J_1$  the Bessel function of first order.

Using Lemma 8, we receive for  $w \in (-\beta, \beta)$ ,

$$\hat{\varphi}_{\text{cosh}}\left(\frac{N_1 w}{2\pi m}\right) \geq \frac{2m}{(\cosh \beta - 1) N_1} \left[ \frac{\pi \beta}{2} (\beta^2 - w^2)^{-1/2} I_1(\sqrt{\beta^2 - w^2}) - \text{sinc } w - \gamma(m, \sigma) \right].$$

This means for  $\beta = bm$  and  $v = \frac{N_1 w}{2\pi m} \in (-N_1(1 - \frac{1}{2\sigma}), N_1(1 - \frac{1}{2\sigma}))$ ,

$$\begin{aligned} \hat{\varphi}_{\text{cosh}}(v) &\geq \frac{2m}{(\cosh \beta - 1) N_1} \left[ \frac{b}{4} \left( \left(1 - \frac{1}{2\sigma}\right)^2 - \frac{v^2}{N_1^2} \right)^{-1/2} I_1\left(2\pi m \sqrt{\left(1 - \frac{1}{2\sigma}\right)^2 - \frac{v^2}{N_1^2}}\right) \right. \\ &\quad \left. - \text{sinc } \frac{2\pi m v}{N_1} - \gamma(m, \sigma) \right]. \end{aligned}$$

Since the function  $h : [0, \beta) \rightarrow \mathbb{R}$  defined by

$$h(w) := \frac{\pi \beta}{2} (\beta^2 - w^2)^{-1/2} I_1(\sqrt{\beta^2 - w^2}) - \text{sinc } w > 0$$

is decreasing and

$$\lim_{w \rightarrow \beta-0} h(w) = \frac{\pi \beta}{4} - \text{sinc } \beta \geq \pi^2 \left(1 - \frac{1}{2\sigma}\right) - 1 \geq \frac{3\pi^2}{5} - 1,$$

the Fourier transform  $\hat{\varphi}_{\text{cosh}}(v)$  is positive and decreasing for  $v \in [0, N_1(1 - \frac{1}{2\sigma}))$  too. Hence, we obtain

$$\begin{aligned} \min_{n \in I_N} \hat{\varphi}_{\text{cosh}}(n) &= \hat{\varphi}_{\text{cosh}}\left(\frac{N}{2}\right) \\ &\geq \frac{2m}{(\cosh \beta - 1) N_1} \left[ \frac{b}{4} \left(1 - \frac{1}{\sigma}\right)^{-1/2} I_1\left(2\pi m \sqrt{1 - \frac{1}{\sigma}}\right) - \text{sinc } \frac{\pi m}{\sigma} - \gamma(m, \sigma) \right] \\ &\geq \frac{2m}{(\cosh \beta - 1) N_1} \left[ \frac{b}{4} \left(1 - \frac{1}{\sigma}\right)^{-1/2} I_1\left(2\pi m \sqrt{1 - \frac{1}{\sigma}}\right) - 1 - \gamma(m, \sigma) \right]. \end{aligned}$$

From  $m \geq 2$  and  $\sigma \geq \frac{5}{4}$ , it follows that

$$2\pi m \sqrt{1 - \frac{1}{\sigma}} \geq 4\pi \sqrt{1 - \frac{1}{\sigma}} \geq x_0 := \frac{4\pi}{\sqrt{5}}.$$

By the inequality (see [1] or [17, Lemma 3.3])

$$\sqrt{2\pi x_0} e^{-x_0} I_1(x_0) \leq \sqrt{2\pi x} e^{-x} I_1(x), \quad x \geq x_0,$$

we sustain that

$$I_1(x) \geq \sqrt{x_0} e^{-x_0} I_1(x_0) x^{-1/2} e^x > \frac{2}{5} x^{-1/2} e^x, \quad x \geq x_0.$$

Hence, for  $x = 2\pi m \sqrt{1 - 1/\sigma}$ , we get the estimate

$$I_1(2\pi m \sqrt{1 - \frac{1}{\sigma}}) > \frac{\sqrt{2}}{5\sqrt{\pi m}} \left(1 - \frac{1}{\sigma}\right)^{-1/4} e^{2\pi m \sqrt{1 - 1/\sigma}}.$$

Thus, we obtain the following:

**Lemma 12** *Let  $N \in 2\mathbb{N}$  and  $\sigma \geq \frac{5}{4}$  be given, where  $N_1 = \sigma N \in 2\mathbb{N}$ . Furthermore, let  $m \in \mathbb{N}$  with  $2m \ll N_1$ ,  $\beta = bm$ , and  $b = 2\pi(1 - \frac{1}{2\sigma})$ .*

*Then we have*

$$\hat{\phi}_{\cosh}\left(\frac{N}{2}\right) \leq \frac{2m}{(\cosh \beta - 1) N_1} \left[ \frac{\sqrt{\pi}}{5\sqrt{2m}} \left(1 - \frac{1}{2\sigma}\right) \left(1 - \frac{1}{\sigma}\right)^{-3/4} e^{2\pi m \sqrt{1 - 1/\sigma}} - 1 - \gamma(m, \sigma) \right].$$

By Eq. 2, the constant  $e_{\sigma, N}(\hat{\phi}_{\cosh})$  can be estimated as follows:

$$\begin{aligned} e_{\sigma, N}(\hat{\phi}_{\cosh}) &\leq \frac{1}{\min_{n \in I_N} \hat{\phi}_{\cosh}(n)} \max_{n \in I_N} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\phi}_{\cosh}(n + rN_1) e^{2\pi i r N_1} \right\|_{C(\mathbb{T})} \\ &= \frac{1}{\hat{\phi}_{\cosh}(N/2)} \max_{n \in I_N} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\phi}_{\cosh}(n + rN_1) e^{2\pi i r N_1} \right\|_{C(\mathbb{T})}, \end{aligned}$$

where it holds

$$\begin{aligned} &\left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\phi}_{\cosh}(n + rN_1) e^{2\pi i r N_1} \right\|_{C(\mathbb{T})} \\ &\leq \sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\psi}_1(n + rN_1)| + \sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\rho}_1(n + rN_1)|. \end{aligned} \tag{50}$$

Analogously to Lemma 10, we get

**Lemma 13** *Let  $N \in 2\mathbb{N}$  and  $\sigma \geq \frac{5}{4}$  be given, where  $N_1 = \sigma N \in 2\mathbb{N}$ . Furthermore, let  $m \in \mathbb{N}$  with  $2m \ll N_1$ ,  $\beta = bm$ , and  $b = 2\pi(1 - \frac{1}{2\pi})$ .*

*Then, it holds for all  $n \in I_N$ ,*

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\psi}_1(n + rN_1)| \leq \frac{\beta}{(\cosh \beta - 1) N_1} \left[ 2\pi m + \frac{10}{\sqrt{2\pi m}} \left(1 - \frac{1}{2\sigma}\right)^{-1/2} \right].$$

Finally, for any  $n \in I_N$ , we estimate the sum

$$\sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\rho}_1(n + rN_1)|.$$

Analogously to Lemma 11, we obtain

**Lemma 14** *Let  $N \in 2\mathbb{N}$  and  $\sigma \geq \frac{5}{4}$  be given, where  $N_1 = \sigma N \in 2\mathbb{N}$ . Furthermore, let  $m \in \mathbb{N}$  with  $2m \ll N_1$ ,  $\beta = bm$ , and  $b = 2\pi(1 - \frac{1}{2\sigma})$ .*

*Then, for all  $n \in I_N$ ,*

$$\begin{aligned} & \sum_{r \in \mathbb{Z} \setminus \{0\}} |\hat{\rho}_1(n + rN_1)| \\ & \leq \frac{1}{(\cosh \beta - 1) N_1} \left[ \left(1 + \frac{1}{e}\right) \frac{\beta}{\pi} \sqrt[4]{\frac{5}{2\pi m}} \left(\frac{4\sigma}{2\sigma - 1} + 8\right) \left(1 - \frac{1}{2\sigma}\right)^{-1/4} \right. \\ & \quad + \frac{8\sigma}{(2\sigma - 1)\pi} e^{-\sqrt{2\pi m - \pi m/\sigma}} + \frac{8}{\pi} E_1(\sqrt{2\pi m - \pi m/\sigma}) \\ & \quad \left. + \left(e^{-\sqrt{2}\beta} + \frac{1}{2}\right) \left(4m + \frac{2}{\pi}\right) e^{-2\pi m + \pi m/\sigma} \right]. \end{aligned}$$

From Lemmas 13 and 14, it follows that by Eq. 50,

$$\left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\cosh}(n + rN_1) e^{2\pi i r N_1 \cdot} \right\|_{C(\mathbb{T})} \leq \frac{\beta b(m, \sigma)}{(\cosh \beta - 1) N_1} \tag{51}$$

with the constant (46).

Using

$$e_{\sigma, N}(\varphi_{\cosh}) \leq \frac{1}{\hat{\varphi}_{\cosh}(N/2)} \max_{n \in I_N} \left\| \sum_{r \in \mathbb{Z} \setminus \{0\}} \hat{\varphi}_{\cosh}(n + rN_1) e^{2\pi i r N_1 \cdot} \right\|_{C(\mathbb{T})},$$

it follows from Eq. 51 and Lemma 12:

**Theorem 5** *Let  $N \in 2\mathbb{N}$  and  $\sigma \geq \frac{5}{4}$  be given, where  $N_1 = \sigma N \in 2\mathbb{N}$ . Furthermore, let  $m \in \mathbb{N}$  with  $2m \ll N_1$ ,  $\beta = bm$ , and  $b = 2\pi(1 - \frac{1}{2\sigma})$ .*

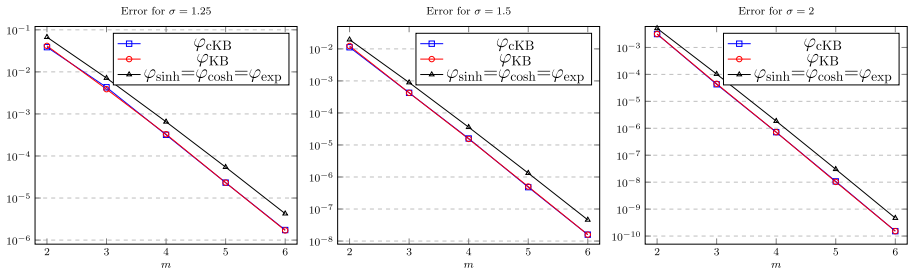
*Then, the  $C(\mathbb{T})$ -error constant of the continuous cosh-type window function (49) can be estimated by*

$$e_{\sigma}(\varphi_{\cosh}) \leq \frac{\beta b(m, \sigma)}{2m} \left[ \frac{\sqrt{\pi}}{5\sqrt{2m}} \left(1 - \frac{1}{2\sigma}\right) \left(1 - \frac{1}{\sigma}\right)^{-3/4} e^{2\pi m \sqrt{1-1/\sigma}} - 1 - \gamma(m, \sigma) \right]^{-1},$$

*i.e., the continuous cosh-type window function (28) is convenient for NFFT.*

## 8 Conclusion

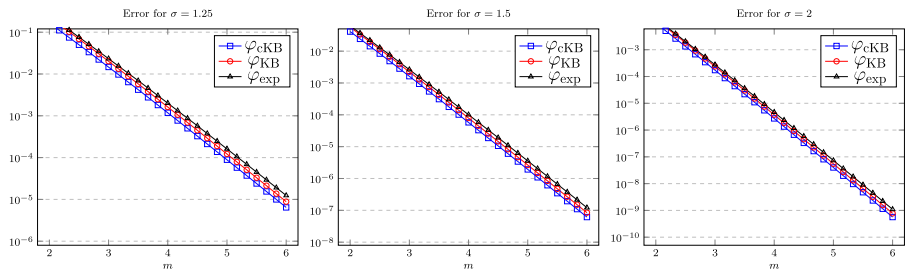
In this paper, we prefer the use of continuous, compactly supported window functions for NFFT (with nonequispaced spatial data and equispaced frequencies). Such window functions simplify the algorithm for NFFT, since the truncation error of



**Fig. 1** The constants  $e_\sigma(\varphi)$  of the different window functions with shape parameter  $\beta = \pi m(2 - 1/\sigma)$  for  $\sigma \in \{1.25, 1.5, 2\}$  and  $m \in \{2, 3, 4, 5, 6\}$  based on numerical experiments

NFFT vanishes. Furthermore, such window functions can produce very small errors of NFFT. Examples of such window functions are the continuous Kaiser-Bessel window function (8), continuous exp-type window function (28), sinh-type window function (22), and continuous cosh-type window function (49) which possess the same support and shape parameter. For these window functions, we present novel explicit error estimates for NFFT and we derive rules for the convenient choice of the truncation parameter  $m \geq 2$  and the oversampling parameter  $\sigma \geq \frac{5}{4}$ . The main tool of this approach is the decay of the Fourier transform  $\hat{\varphi}(v)$  of  $\varphi \in \Phi_{m, N_1}$  for  $|v| \rightarrow \infty$ . A rapid decay of  $\hat{\varphi}$  is essential for small error constants. Unfortunately, the Fourier transform of certain window function  $\varphi$ , such as Eqs. 28 and 49, is unknown analytically. Therefore, we propose a new technique and split  $\varphi$  into a sum of two compactly supported functions  $\psi$  and  $\rho$ , where the Fourier transform  $\hat{\psi}$  is explicitly known and where  $|\rho|$  is sufficiently small. Furthermore, it is shown that the standard Kaiser-Bessel window function and original exp-type window function, which have jump discontinuities with very small jump heights at the endpoints of their support, possess a similar error behavior as the corresponding continuous window functions.

In summary, the  $C(\mathbb{T})$ -error constant of the continuous/standard Kaiser-Bessel window function is of best order  $\mathcal{O}(m e^{-2\pi m \sqrt{1-1/\sigma}})$ . For the sinh-type, continuous/original exp-type, and continuous cosh-type window functions, the corresponding  $C(\mathbb{T})$ -error constants are of order  $\mathcal{O}(m^{3/2} e^{-2\pi m \sqrt{1-1/\sigma}})$ . Nevertheless, our numerical results show that all window functions proposed here yield a very similar



**Fig. 2** The constants  $e_\sigma(\varphi)$  of the different window functions with shape parameter  $\beta = \pi m(2 - 1/\sigma)$  for  $\sigma \in \{1.25, 1.5, 2\}$  and  $m \in \{2, 3, 4, 5, 6\}$  based on the error estimates of Eqs. 18, 21 and 47.



error (see Fig. 1). The presented error estimates are very precise (see Fig. 2). Thus, we can recommend the use of all these window functions.

**Acknowledgements** The authors would like to thank Stefan Kunis and Elias Wegert for several fruitful discussions on the topic. The authors are grateful to the anonymous referees for helpful comments and suggestions.

**Funding** Open Access funding enabled and organized by Projekt DEAL. The first author received funding from the Deutsche Forschungsgemeinschaft (German Research Foundation) – Project-ID 416228727 – SFB 1410.

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