Continuum Modeling of Granular Media

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This is a survey of the interesting phenomenology and the prominent regimes of granular flow, followed by a unified mathematical synthesis of continuum modeling. The unification is achieved by means of "parametric" viscoelasticity and hypoplasticity based on elastic and inelastic potentials. Fully nonlinear, anisotropic viscoelastoplastic models are achieved by expressing potentials as functions of the joint isotropic invariants of kinematic and structural tensors. These take on the role of evolutionary parameters or "internal variables," whose evolution equations are derived from the internal balance of generalized forces. The resulting continuum models encompass most of the mechanical constitutive equations currently employed for granular media. Moreover, these models are readily modified to include Cosserat and other multipolar effects. Several outstanding questions are identified as to the contribution of parameter evolution to dissipation; the distinction between quasielastic and inelastic models of material instability; and the role of multipolar effects in material instability, dense rapid flow, and particle migration phenomena. [DOI: 10.1115/1.4026242]

1 Introduction

> *"Granular materials represent a major object of human activities: as measured in tons, the first material manipulated on earth is water; the second is granular matter… This may show up in very different forms: rice, corn, powders for construction… In our supposedly modern age, we are extraordinarily clumsy with granular systems…"*
>
> *P.G. de Gennes* [1][1]

The epigrammatic quote, familiar to many in the field of granular mechanics, reflects a long-standing scientific fascination and practical interest, which is acknowledged by many others [3,4] and summarized in several monographs and review articles [5–11], with several devoted to specific applications.

As a general definition, we understand by *granular medium* a particle assembly dominated by pairwise nearest-neighbor interactions and usually limited to particles larger than ca. 1 μm, for which the direct mechanical effects of van der Waals and ordinary thermal ("Brownian") forces are negligible. This includes a large class of materials, such as cereal grains; pharmaceutical tablets and capsules; geomaterials, such as sand; and the masses of rock and ice in planetary rings.

Another important class of granular media consists of highly fractured rock masses, and it is worth recalling that a classic self-consistent (mean-field) treatment of cracked elastic solids [12] predicts that all elastic moduli should vanish at a critical crack concentration. This portends a *crack-percolation threshold* suggested by other studies [13], such that at least one crack spans the solid and thereby demarcates at least two "grains". Hence, this threshold can be regarded as the transition from a fractured but still coherent solid to a granular medium that owes its shear resistance to the combined effects of confining pressure, friction, and granular dilatancy (*vide infra*).

Of the several facets of research on granular mechanics: experiment and application; analytical and numerical micromechanics; homogenization (i.e., "upscaling" or "coarse-graining"); the mathematical classification and solution of field equations; and

phenomenological continuum models, the present review focuses almost exclusively on the latter. The review is mainly concerned with dry granular materials, or else with those completely saturated by an interstitial fluid, in which capillary forces and other forms of cohesion are largely negligible.

Strictly speaking, the above restrictions rule out applications to the fine powders, colloidal systems, and clayey soils discussed in other works [8,14], since the granular elastic modulus G_s is the only relevant stress scale. In this regard, there is an interesting and nagging question as to the microscopic origins of the stress scales assumed in various empiricisms associated with critical-state soil mechanics and hypoplasticity (e.g., Refs. [14–18]). This question may reflect a philosophical divide, separating those concerned with the relation of constitutive equations to micromechanics from those whose primary concern is correlation of data from laboratory and field tests (Ref. [16], p. 13).

It is perhaps a truism that future progress in the applied mathematics and mechanics of granular media rests on the continuing development of continuum models with plausible connection not only to experiment, but also to micromechanics. In the classical paradigm, this development would proceed through a time-honored approach of homogenization (i.e., upscaling or coarse-graining) of idealized microstructural models to obtain continuum-level constitutive models that are applicable to the boundary-value problems inspired by laboratory and field tests. The author must confess to an incomplete appreciation of the practical versus the scientific merits of the recently proposed methodology that would proceed directly from grain-scale observations of local microstructure to continuum-level numerical simulations [19].

The present survey represents an effort by the author to understand and codify in broad outline the ideas underlying much of the existing constitutive modeling of granular media. One goal is to point up certain unresolved mathematical and physical issues that seem crucial to the physical understanding and modeling of granular flows, and a brief summary of such issues is offered below in the Conclusions.

Another goal is to identify a broad class of plausible constitutive equations as generalizations of contemporary *hypoplastic* models [14,20] based on elastic and inelastic potentials that are familiar in the classical theories of elastoplasticity [21–23].

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[1]The Introduction draws heavily on Ref. [2].

With all due respect to the outstanding challenges mentioned in the review article [10] (final paragraph of Sec. 3) and the caveat expressed in a recent monograph [11] (p. 7)

> "...there does not exist a universal constitutive model to simulate the behavior of bulk solids during both rapid, transitional and slow. flow...[which] requires a fusion of ideas from solid, fluid and gas mechanics..."

it will be shown below that the above elastoplastic models can be enlarged to include models of viscoelastoplasticity that are broadly applicable to all the prominent regimes of granular flow. Following the brief review in Sec. 1.1 of phenomenology and flow regimes, a systematic exposition of such models will be presented.

There is scant effort in the present survey to cover a large body of work nominally devoted to the physics of granular media, e.g., Ref. [24], since much of it does not appear to be primarily focused on the development of the continuum models of interest here. Furthermore, while citing results from various numerical simulations, the present review does not deal with the details of numerical methods, neither the direct simulation of micromechanics by means of various distinct element ("DEM") simulations nor continuum-mechanics simulations based on finite element methods ("FEM") or related techniques.

Apart from the notions of stored and recoverable elastic energy (Helmholtz free energy) and of energy dissipation, the present article makes limited formal appeal to thermodynamics, e.g., as represented by the standard continuum entropy balance (Clausius–Duhem inequality). In the author's view, this has but marginal relevance to granular mechanics, owing to the lack of *intrinsic* kinetic energy at the grain level. Thus, the nominal "granular temperature" and the associated "$k_B T$" based on grain kinetic energy owe their existence entirely to external forcing, as indicated by the seminal paper of Haff [25]. This temperature enjoys few of the attributes of molecular-kinetic temperature, despite its utility in the pioneering works on the associated kinetic theory [26].

Without minimizing the importance of comparing theory to experiment, little effort is made in this review to assess the relative merits of various constitutive models based on their success in fitting limited experimental data or numerical simulations to models with multiple adjustable constants.

It is hoped that the more knowledgeable readers will forgive the missing citations of important work, misapprehensions about work actually cited, and finally, the occasional lapse in clarity of exposition.

1.1 Phenomenological Aspects. We consider here the important physical parameters and nondimensional groups that characterize the various regimes of granular mechanics and flows. We note that Forterre and Pouliquen [10] have given a similar review, providing more quantitative comparisons for fairly simple shearing flows, where our representative strain rate $|\mathbf{D}'|_0$ can be identified with a representative value of their shear rate $\dot{\gamma}$.

Key Parameters and Nondimensional Groups. Apart from various nondimensional parameters describing grain shape, the most prominent physical parameters for noncohesive granular media, listed in the Nomenclature, are: grain elastic (shear) modulus G_s (force/area), intrinsic grain density ρ_s (mass/volume), representative grain diameter d (length), intergranular (Coulomb) contact-friction coefficient μ_s or macroscopic counterpart μ_C, intergranular collisional coefficient of restitution e_s, interstitial ("pore") fluid viscosity η_f (force-time/area) for neutrally buoyant fluid-saturated granular media (or fluid-particle suspensions), grain volume fraction ("compacity") ϕ or "void ratio" $e = (1 - \phi)/\phi$, and confining pressure p_s (e.g., lithostatic). Of these, the last two may be regarded as parameters associated with some reference states or

else variables, either a *control variable* or an evolutionary *internal variable* in the case of ϕ.

In addition to the above, one may identify somewhat less accessible quantities, such as a symmetric second-rank (Satake) fabric tensor $\mathbf{A} \hat{=} [A_{ij}]$, and higher-order versions [23], as well as a coordination number Z or active contact-number density $n_c = 3Z\phi/\pi d^3$ (for spheres). While certain past studies show that these correlate well with void ratio e [23] (p. 535ff and Fig. 7.3.1), recent DEM studies on sphere assemblies [27,28] in the static-elastic and steady-dense flow regimes suggest a lack of correlation and the necessity of including both ϕ and Z in the constitutive model. Unfortunately, it is not clear what set of macroscopic measurements would allow for the determination of n_c, Z, or some mechanically equivalent quantity, a difficulty that becomes even more severe for nonspherical particles.

Radjai et al. [29] propose to reduce the dependence on coordination number Z to a multiplicative factor in the definition of fabric $\mathbf{A}$, which seems to be at odds with the correlation proposed by others [28].

By assigning the role of control variable to the confining pressure p_s, which we generally identify with mean compressive stress $-\text{tr}\mathbf{T}/3$, we emphasize the focus on *pressure-sensitive elastoplasticity* [23]. At the same time, we forsake a well-known "principle of determinism" [30], according to which mechanical constitutive equations must specify stress as a function of a given kinematic history. This unconventional view is reflected in the *implicit constitutive equations* studied extensively by Rajagopal [31], e.g., to describe the effect of pressure on the viscosity of nearly incompressible liquids. While this view seems to be dignified by practice in numerous stress-controlled rheological experiments [9,10] and DEM simulations [32], one can argue that such experiments involve feedback loops that actually impose the kinematics necessary to attain the desired stress states.

The above parameters define key nondimensional groups that serve to delineate various regimes of granular flow, namely, an *elasticity number* [2], *inertia number* [9,10,33,34], and *viscosity number* [2,35,36], given, respectively, by
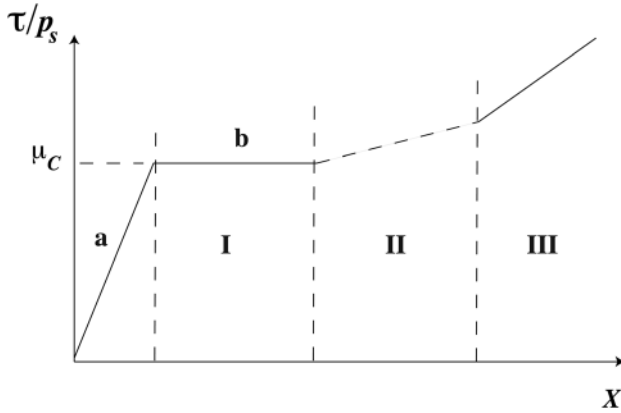
$$\mathbb{E} = G_s/p_s, \quad \mathbb{I} = |\mathbf{D}'|_0 d\sqrt{\rho_s/p_s}, \quad \mathbb{H} = \eta_s|\mathbf{D}'|_0/p_s \quad (1)$$

based on a representative shear rate $|\mathbf{D}'|_0$. In addition, we will have occasion to refer briefly below to a Knudsen number based on characteristic length scales.

As a bit of perspective and history, note that $\mathbb{I}$ is the analog of the Deborah number of non-Newtonian fluid mechanics, involving relaxation time that represents the competition between grain inertia and Coulomb friction $\mu_C p_s$. As pointed out elsewhere [10], the quantity $\mathbb{I}^2$, representing the ratio of representative granular kinetic energy $\rho_s d^2\dot{\gamma}^2$ to frictional confinement, could justifiably be designated as the "Savage number" [33]. With similar attribution to Savage, it was identified previously as "expansivity" [34], in analogy to the ratio $k_B T/p_s d^3$ in molecular gases.

It is instructive to consider the vertical density variation due to lithostatic pressure $p_s \rho_s g z$ at elevation z, so that $k_B T/p_s d^3 \sim k_B T/mgz$, where m is representative grain mass. Then, with $k_B T \sim \dot{\gamma}^2 d^2$, Boltzmann's law of the atmosphere $\rho/\rho_0 = \exp\{-mgz/k_B T\}$ translates to $\rho/\rho_0 = \exp\{-gz/d^2\dot{\gamma}^2\}$. Hence, at shear rates $\dot{\gamma} \sim 100\,s^{-1}$, the vertical e-folding of density is found to be of the order of one particle diameter for a granular solid with $d \sim 1\,\text{mm}$. This signals the gravitational collapse of a granular gas to a dense state.

For the rarefied gaseous state, we should replace d by the mean free path $\ell = d/6\phi$ in the above estimate for $k_B T$ [37], in effect replacing the shear rate $\dot{\gamma}$ by an enhanced value $\dot{\gamma}/\phi$. With this modification, the statement of Goldhirsch [37], "granular gases cannot exist on Earth without a continual supply of energy," would still as well read ".a continual and *vigorous* supply of energy".

Fig. 1 Schematic diagram of granular-flow regimes

	Regime	τ Scaling
I.	Quasistatic: (Hertz–Coulomb) elastoplastic "solid"	
a.	(Hertz) elastic	$G_s\gamma$
b.	(Coulomb) elastoplastic	μ_C
II.	Dense-rapid: viscoplastic "liquid"	$p_s f(I)$
III.	Rarified-rapid: (Bagnold) viscous "granular gas"	$\rho_s d^2 \dot\gamma^2$

The viscosity number $\mathbb{H}$, denoted elsewhere by I_v [36], is identified in a previous work [35], which also points out its relevance to the transition from fluid-saturated granular medium to dense fluid-particle suspension near the point $\mathbb{H} \approx 1$, where viscous forces are comparable to frictional forces. Note also that the Stokes or Bagnold number for fluid-particle suspensions can be identified with the quantity $\mathbb{I}^2/\mathbb{H}$, representing the magnitude of inertial to viscous forces.

In the sections immediately following, we shall focus attention on dry granular media, deferring to Sec. 3 a brief discussion of fluid-particle systems.

Granular Flow Regimes. Although granular materials are devoid of intrinsic thermal motion at the grain level, they nevertheless exhibit a certain kinship to molecular materials, characterized by states that resemble solid, liquid, and gas [10]. These states may coexist as the analogs of multiphase flow, and there are several outstanding questions as to the proper matching of solid-like immobile states with the rapidly sheared states [8].

Letting γ represent shear strain relative to a rest state, $\tau/p_s = 0, \mathbb{I} = 0$, the corresponding flow regimes are delineated in the qualitative, highly simplified sketch in Fig. 1. There, the nondimensional shear-stress/pressure ratio τ/p_s is represented as a function of a single nondimensional variable of general form

$$X \sim \mu_C \mathbb{E}\gamma/(\mathbb{E}\gamma + \mu_C) + \mathbb{I}^2 \tag{2}$$

A simpler representation, closer to the constitutive models considered below and illustrated in Fig. 5, is:

$$\tau/p_s = \mu_C + \mathbb{I}^2 \quad \text{with} \quad \gamma_E = \tau/p_s \mathbb{E} \tag{3}$$

where the first member represents the Coulomb–Bagnold interpolation reflected in certain constitutive models [38] and γ_E is elastic deformation at any stress state.

Figure 1 fails to capture the strong nonlinearity and history dependence of granular plasticity in regime Ib, a matter addressed by the constitutive models to be discussed in the following matter.

As with the liquid states of molecular systems, dense-rapid flow, represented by a nondimensional function $f(\mathbb{I})$ in Table 1, may be the most poorly understood regime of granular mechanics. It involves important phenomena, such as granular size segregation [39,40], which is touched on only briefly at the end of this review. The review by Forterre and Pouliquen [10] (Sec. 4) discusses various efforts to model the transition between regimes II and III.

Among its many other limitations, the simple schema presented above may not apply to "soft" granular materials, where the dense-rapid flow may involve elastic effects of the type suggested by the DEM simulations of Campbell [32]. This would require, in effect, a more general function $f(\mathbb{I}, \mathbb{E})$ in regime II, implied by

Fig. 5 of Ref. [32], which amounts to a plot of $\gamma_E = \tau/p_s \mathbb{E}$ versus $\mathbb{E}/\mathbb{I}^2$. (The rather large and small numerical values on the respective axes of that plot can no doubt be attributed to large values of $\mathbb{E}$, such as those estimated below.)

Although not attempted here, the 2D graph in Fig. 1 could more appropriately be presented as a parametric or multidimensional plot, in order to represent the effect of nondimensional parameters, such as intergranular friction and initial volume fraction. In that respect, it is worth noting that Fig. 1 represents the load temperature plane in the 3D "jamming" diagram of Liu and Nagel [41], based on a paradigm rooted in equilibrium thermodynamics. Thus, if their load, temperature, and density are replaced by the proper nondimensional forms $\mathbb{I}, \tau/p_s$, and ϕ for granular media, their representation of the quasistatic regime $\mathbb{I} = 0$ is tantamount to a highly simplified yield curve of τ/p_s versus ϕ, presumably restricted to rigid, frictionless grains.

Without striving further to simplify the complex, we turn to a more-detailed discussion of the various regimes shown in Fig. 1.

The Elastic Regime. It is generally recognized that the geometric nonlinearity of Hertzian contacts should lead to nonlinear elasticity at low confining pressures p_s and to an interesting scaling of elastic moduli and elastic wave speeds with pressure. The paper by Agnolin and Roux [27] provides a comprehensive survey of previous literature and the results of their DEM simulations of the elastic moduli of sphere assemblies. They, as others before them, point out that classical theoretical predictions are based on a questionable (Cauchy) "mean-field" or "affine-motion" approximation for grains, and they give an account of previous attempts to include the effect of micromechanical grain fluctuations along with the results of their own DEM study.

Unfortunately, the DEM results in Ref. [27] are presented in terms of variables with physical dimensions, and this author has not attempted to translate them to the elasticity number $\mathbb{E}$ of the present work. At any rate, one concludes from the former that the effect of fluctuations (represented by corrections $1 + \alpha$ displayed in their Eq. (27) and Fig. 10) can result in large errors in predicted elastic modulus at small p_s, i.e., at large $\mathbb{E}$, although the exact magnitude of the error depends crucially on the model for fluctuations and the initial state of their sphere packing. Also, as the authors of this study recognize, slight departures from sphericity could have large influence on the results, leaving one to wonder about typical systems of irregular grains.

With the above reservations in mind, we resort to a rather crude estimate for the effective global shear modulus G, say, in terms of the grain modulus G_s based on Hertzian contact and the mean-field assumption. Thus, with the purely geometric estimate of the diameter a of the Hertzian contact zone given by $a/d \sim (\delta_n/d)^{1/2}$ in terms of relative normal displacement δ_n of particle pairs, estimates of normal stress, strain, contact normal force, and contact stiffness are given, respectively, by

$$\sigma_n \sim G_s \varepsilon_n, \quad \varepsilon_n \sim \delta_n/d, \quad f_n \sim G_s d^2 (\delta_n/d)^{3/2}$$
$$k_n = df_n/d\delta_n \sim G_s d(f_n/G_s d^2)^{1/3} \tag{4}$$

which, with the assumption of comparable shear and normal contact forces, leads to the mean-field estimate of global stiffness G and pressure,
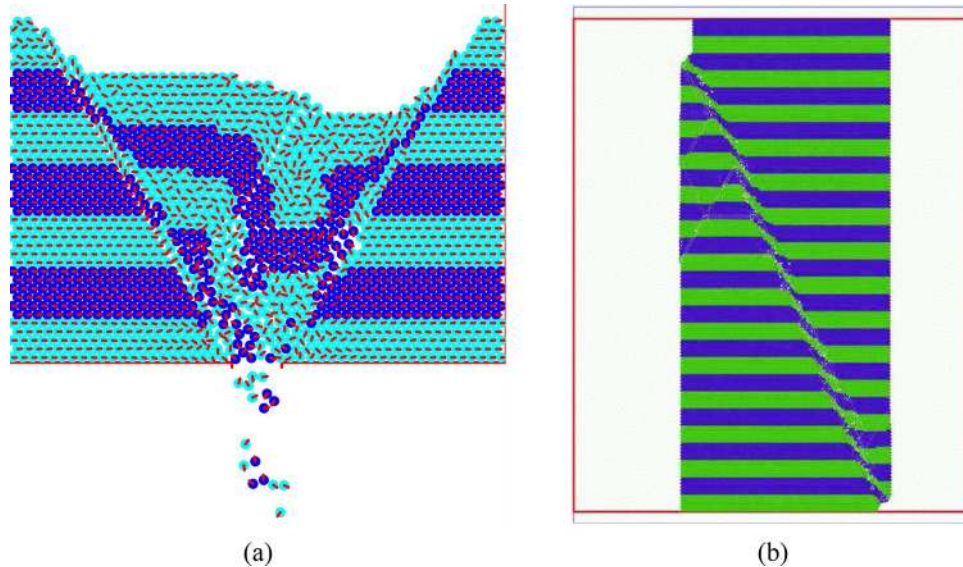
Fig. 2   2D DEM simulations (courtesy of W. Ehlers).

$$G \sim k_n/d, \quad p_s \sim f_n/d^2, \quad \text{hence} \quad G/G_s \sim \mathbb{E}^{-1/3} \qquad (5)$$

Thus, with confining pressures $p \sim 100\,\mathrm{kPa}$ and the estimate $G_s \sim 100\,\mathrm{GPa}$ (a rather stiff geomaterial), one finds $\mathbb{E} \sim 10^6$ and $G \sim 10^{-2}G_s$, amounting to a huge reduction of global stiffness due to relatively soft Hertzian contact.

In principle, we should replace G_s by G in Table 1 and $\mathbb{E} = G_s/p_s$ by $G/p_s = \mathbb{E}^{-2/3}$ in Eqs. (2)–(3). In so doing, we obtain an estimate of the limiting elastic strain for the onset of Coulomb slip (with $\mu_C \sim 1$) to be $\gamma_E \sim \mathbb{E}^{-2/3} \sim 10^{-4}$, given the above numerical value. Although admittedly crude, this provides a reasonable estimate of the small elastic range of stiff geomaterials.

Elastoplastic Regime. Given its venerable history, dating back to the classical works of Coulomb, Rankine, and others, and its perennial relevance to geomechanics, the field of elastoplasticity is without doubt the most thoroughly studied area of granular mechanics. It is therefore impossible in a brief review to give a comprehensive summary of the phenomenology and current state of theoretical understanding, much of which is discussed in recent treatises [8,23]. Here, we touch on a few salient phenomena and the theoretical issues surrounding them as background for the discussion of constitutive modeling to follow.

Figure 2 shows the results from the 2D DEM simulations of Ehlers [42] of a quasistatic hopper discharge and a biaxial com-

pression test, both of which illustrate a well-known localization of deformation into shear bands. A hallmark of granular plasticity, this localized slip (or "failure") may be implicated in dynamic "arching" with large transient stresses on bounding surfaces, such as hopper walls and structural retaining walls. Similar phenomena are implicated in large-scale landslides.

Figure 3 reveals the development of shear bands in a standard experimental quasistatic compression test on sand. The literature abounds with many interesting experimental observations (e.g., Ref. [43]) and numerical simulations of shear bands, with numerous striking examples presented in a sustained body of work by Tejchman [11,14] and Widuliński et al. [44]. Following the pioneering studies of J. Desrues, recent developments in X-ray computer tomography have led to grain-level visualization that reveal such bands via density contrasts arising from granular dilatancy [45].

The occurrence of shear bands can be viewed mathematically as material bifurcation and instability arising from loss of convexity in the underlying constitutive equations, accompanied by a *change of type* in the field equations. From a physical point of view, this can be regarded as "soft-mode" instability, analogous to that associated with thermodynamic phase transitions [46]. In the case of granular media and soils, the preference for shear-banding modes may be attributed to the relative weakness in shear compared to compression, as exhibited by the Coulomb yield surface and related empirical variants (Refs. [8] (p. 218ff) and [23,47]). This is to be contrasted with localized failure in *compaction bands* arising from micromechanical buckling instabilities and fracture in cellular materials [48] and porous sandstones [49] subject to compressive loads.

To help identify certain crucial issues surrounding elastoplastic instability in granular media, Fig. 4 presents an iconic qualitative sketch, cf. Refs. [8] (Chap. 5) and [23] (Chap. 4), of the typical stress-strain/dilatancy behavior in the axial compression of dense versus loose sands. While no numerical scale is shown on the axes, the peak stress and the change of volumetric strain from negative (contraction) to positive (dilation) typically occurs at strains of the order of a few percent (i.e., $\varepsilon \sim 0.01$–0.05) for dense sands.

Although tempting to regard the initial growth of axial compressive stress σ with strain ε as elastic in nature, the elastic regime is represented by much smaller strains (corresponding to the estimates of order 10^{-4} given above), corresponding to a nearly vertical unloading from any point on the σ–ε curve. It is therefore much more plausible that the initial stress growth represents an *almost completely dissipative* plastic "hardening" associated with compaction ($\varepsilon_V < 0$) accompanied by growth of contact
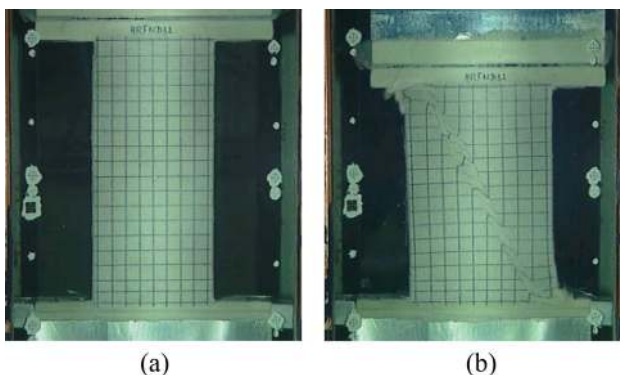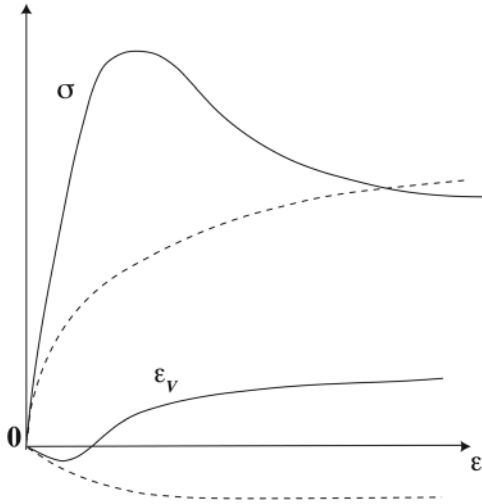

Fig. 3   Axial compression of dry Hostun sand specimen (courtesy of W. Ehlers).

Fig. 4 Schematic diagram of triaxial stress/dilatation-strain curves for initially dense (solid curves) and loose (dashed curves) sands

Fig. 5 Dashpot/spring/slide-block analog of viscoelasto-plasticity

number density n_c and contact anisotropy [18], whereas the maximum in stress can be attributed to the subsequent decrease in n_c accompanying dilation ($\varepsilon_V > 0$) [50] (p. 345ff).

The above qualitative picture is supported quantitatively by a micromechanical variant on the well-known *double-shearing* model [51] developed by Anand and Gu [52]. The model involves an assumed dilatancy-coupled plastic hardening, with eventual softening that leads to curves like that in Fig. 4. Thus, we encounter an essentially *inelastic* formation of shear bands.
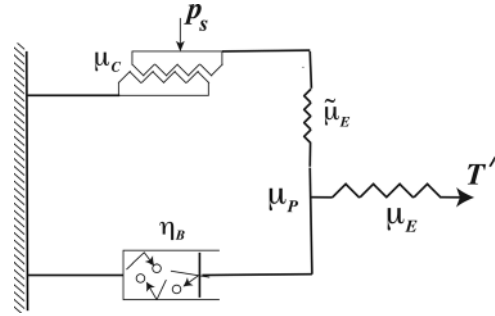
A somewhat different view of material instability is to be found in the (Hill) "second-order" work criterion, championed especially by those in the French School of Geomechanics [16,53–55], the last dealing with a diffuse "collapse" that may be associated with *seismic liquefaction*. According to the second-order work criterion, no doubt restricted to hypoplasticity, the instability is basically of the Hadamard type, associated with singularities in the acoustic tensor arising from nonconvex elasticity.

Thus, in the first interpretation above, one may attribute instability to a sharp decrease in a *dissipative secant modulus*, to be defined below, and in the second interpretation to a nonpositive *quasielastic tangent modulus*. To the extent that the first type of instability is strongly dissipative ("hyperdissipative"), it corresponds to loss of convexity of an inelastic potential, since one can show that rate-independent plasticity is marginally convex [56]. Hence, we discern two limiting forms of the loss of convexity in potential: the first involving a hyperdissipative potential and, in the second, a hyperelastic potential.

Whatever the precise nature of the material instability (a term we prefer despite Chambon's reservations on p. 161 of Ref. [16]), the seminal works of Mühlhaus and Vardoulakis [57] point out the fact that it generally calls for multipolar or "higher-gradient" constitutive models. Such models, endowed with intrinsic length scale, are necessary to regularize field equations and assign finite thickness to zones of strain localization. This suggests yet another thermodynamic analogy, namely the van der Waals–Cahn–Hilliard thermodynamic models of phase transitions [37]. The exception may be the diffuse instabilities.

Hence, to our list of important nondimensional parameters we must now add a *Knudsen number* $\mathbb{K} = \ell/L$, to be defined more precisely in Sec. 4 below. It is worth noting here that numerical simulation of Ehlers [42] cited above involves homogenization with length-scale and couple stress implicated in the finite-width shear zone shown in Fig. 2(*b*).

While the exact role of multipolar effects in granular mechanics is still unclear, the very concept gives one reason to be wary of various experiments and numerical simulations based on relatively small granular systems. In essence, the results of these studies may involve multipolar effects that may be unwittingly interpreted as peculiar nonpolar effects. Although molecular systems dominated by thermal randomization may often exhibit simple-continuum behavior on small-length scales, to expect the same of granular systems requires a considerable leap of faith.

We turn now to some simple phenomenological notions that motivate the continuum models discussed below.

Springs and Slideblocks. The mechanical model in Fig. 5 conveys a useful intuitive view of granular mechanics, although such "spring-dashpottery" is generally shunned by continuum mechanicians. There, the applied force $\mathbf{T}'$ is the analog of continuum-mechanical stress deviator and the rate of extension of the device represents deviatoric deformation rate $\mathbf{D}'$. Collins [58] proposes a related model and provides a brief history of such models. The present version, which differs somewhat from that proposed by Collins and which can readily be modified to represent pressure-independent plasticity, is motivated by the hypoplastic models to be discussed below.

The serrated slide block in Fig. 5 represents the "sawtooth model" of Rowe [8,59] and the granular "interlocking" of Taylor [50,60]. Both reflect the contribution of Reynolds dilatancy [61] to the shearing resistance of a granular medium, owing to the work of volumetric expansion against the confining pressure. Various elementary arguments (Refs. [8] (p. 96ff) and [47]) based on this simple model give the apparent friction, up to the point of instability at maximum dilation, in terms of the dilatancy angle and the actual intergranular coefficient of friction μ_s. Thus, upon amendment of the analysis in Ref. [8] (p. 96ff), one finds that the apparent angle of friction for planar deformations is given by

$$\phi_{\mu,\text{app}} = \tan^{-1} \mu_C = \phi_\mu + \phi_D, \quad \text{with} \quad \phi_\mu = \tan^{-1} \mu_s$$
$$\text{and} \quad \therefore \ |\mathbf{T}'|/p = \sqrt{3} \tan \phi_{\mu,\text{app}} \tag{6}$$

where the sawtooth angle ϕ_D is the planar equivalent of the *angle of dilatancy* defined below. A more comprehensive micromechanical treatment is given elsewhere [47].

At the point of sliding instability in the model, the stored volumetric energy must be dissipated by collapse and collisional impact, which may be assumed to occur on time scales so short that the precise value of e_c is probably immaterial. In any case, this dissipation of energy gives rise to an apparent sliding friction, *even if* $\mu_s = 0$. This idea is borne out by the numerical simulations of Peyneau and Roux [62] of frictionless sphere assemblies, for which they find vanishing dilatancy with nonzero friction. At odds with the energy balance of Reynolds [61] for frictionless particles, this rapid dissipation of stored energy is emblematic of tribological and plastic-flow processes, where, owing to topological roughness and instability *in the small* stored energy is thermalized on negligibly small time scales, giving rise to rate-independent forces *in the large*.

A second special feature of the current model, the lateral spring with constant $\tilde{\mu}_E$ converts plastic deformation into the "frozen elastic energy" of Collins [58] and also provides an elementary model of plastic work hardening. Although Fig. 5 differs from the picture of Ref. [58], both illustrate the fact that a fraction of the stored elastic energy can never be entirely recovered by the sole mechanical action of $\mathbf{T}'$. This leads to well-known complications in the thermodynamics [21] and to the history effects associated with evolutionary plasticity.

As for history effects, it is worth recalling a quote from a bygone era on the variability of the sand pressure acting on a retaining wall arising from different modes of sand deposition [43]:

> "… the angle of repose of shaken sand is a phrase without meaning. When the author was beginning these experiments he had the advantage of discussing the subject with the late Professor Clerk Maxwell, who remarked that he supposed that the 'historical element' would enter largely into the nature of the limiting equilibrium of sand. By this he meant that sand when put together in different ways would exercise different thrusts, although presenting visibly the same external appearance. The author has kept this valuable remark before him throughout and found that Maxwell's conjecture was correct. The historical element is one which essentially eludes mathematical treatment."
>
> G.H. Darwin [63]

Clearly, Maxwell anticipates the difficulties inherent in plasticity and the lack of "fading memory," and just as clearly Darwin (the son of the father of the theory of evolution), fails to anticipate *evolutionary* models of plasticity.

Quite similar depositional-history effects have been rediscovered in contemporary experiments [64] on the vertical-pressure minimum beneath sand piles (the so-called "dip in the heap"), apparently regarded by many as revolutionary. In that respect, it should be gratifying to those who labor in the field of continuum mechanics to learn that continuum plasticity models show promise of describing this phenomenon [65]. At the same time, it is clear that more work is needed on the transition between the rapid and quasistatic granular flows involved in granular deposition and the associated development of granular anisotropy and "backstress" [23].

As for rapid granular flow, the viscous dashpot, with (Bagnold) viscosity $\boldsymbol{\eta}_B$, in Fig. 5 adds a rate-dependent force associated with granular kinetic energy and collisional dissipation e_c. This leads to a form of *Bingham plasticity* discussed below and described by certain rheological models [8,38], providing a rough interpolation between regimes I and III in Fig. 1.

Dense Rapid Flows and Granular Gases. The review by Goldhirsch [37] provides a comprehensive survey of rapid (or with his qualification "rapidly sheared") granular flows, with a focus on kinetic theory that contains many citations of past work. The more recent surveys [9,10] provide a comprehensive review of experiments and constitutive modeling of simple shearing flows and also suggest the need for nonlocal models with intrinsic length scale. A more recent work [66] suggests a sensitivity to boundary conditions that may not be anticipated in Ref. [9].

Apart from its advocacy for more general constitutive models than those currently employed, the present review does not address the numerous issues and challenges in modeling the dense-rapid flow regime.

1.2 Mathematical Preliminaries. Before discussing constitutive models, we establish a bit of mathematical notation aimed at achieving a reasonably compact summary of a wide ranging body of literature, with formulae that should be fairly evident to those in theoretical and applied mechanics, particularly those

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² Viewed elsewhere [67] as the "quicksand" of continuum mechanics, in a metaphor particularly appropriate to the present setting.

versed in plasticity theory² and rheology. Those less comfortable with the essential tensor analysis may in many instances substitute scalars for tensors, with a concomitant restriction to extremely simple kinematics, such as simple shear.

Notation and Kinematics. To the extent possible, we employ direct notation (bold-face symbols) for tensors, with the conventions indicated in the Nomenclature. When necessary for clarity, we display tensor indices with lower-case Roman $i, j,\ldots$, and we denote elements of abstract arrays of variables or tensors by indices α and β, which are allowed to represent single tensor indices or groups of tensor indices, such as $\alpha = \{ij\}$. With the convention spelled out in the Nomenclature, the forms

$$\boldsymbol{\mu} = 2\mu(\boldsymbol{\delta}^{(\mathrm{s})} - \hat{\mathbf{I}} \otimes \hat{\mathbf{I}}) + 3\kappa\hat{\mathbf{I}} \otimes \hat{\mathbf{I}}, \quad \text{with} \quad \boldsymbol{\delta}^{(\mathrm{s})} \mathrel{\hat{=}} [\delta_{(ij)(kl)}]$$

$$\text{and} \quad \boldsymbol{\kappa} = \boldsymbol{\mu}^{-1} = \frac{1}{2\mu}(\boldsymbol{\delta}^{(\mathrm{s})} - \hat{\mathbf{I}} \otimes \hat{\mathbf{I}}) + \frac{1}{3\kappa}\hat{\mathbf{I}} \otimes \hat{\mathbf{I}}$$

represent isotropic fourth-rank isotropic moduli and compliances.

As what some may regard as the notational consistency of a small mind, the author prefers certain notation differing from that in standard treatises. Thus, $\mathbf{x}°$ and $\mathbf{x}(\mathbf{x}°)$, respectively, are employed for reference position and current placement of material points instead of the standard continuum-mechanics $\mathbf{X}$ and $\boldsymbol{\chi}(\mathbf{X})$, and the moduli $\boldsymbol{\mu}$ and inverse compliances $\boldsymbol{\kappa}$ are those often denoted in the standard plasticity literature [22,23] by $\mathcal{L}$ and $\mathcal{M}$, respectively. Also, $\mathbf{T}$ is employed for Cauchy ("true") stress, in deference to the large body of literature based on the landmark treatise of Truesdell and Noll [30] and in contrast to the $\boldsymbol{\sigma}$ employed in most of the engineering-mechanics and plasticity literature.

When it serves the present purposes, the commonly employed colon ":" is used for scalar products of second-rank tensors or for other operations involving a similar-ordered contraction of tensor indices. We employ conventional strain energies or free energies based on mass, but our dissipation (pseudo)potentials are based on unit volume. As the latter represent dissipation rate, the usual applications do not require time derivatives and they can simply be divided by mass density ρ or multiplied by $J = \det \mathbf{F}$ to obtain mass-based rates.

The versor or director of vectors and tensors is denoted by carets and serves to define a multidimensional form of the signum function, e.g., for vectors

$$\text{dir } \mathbf{a} = \hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}| := \mathbf{0} \quad \text{for} \quad |\mathbf{a}| = 0 \qquad (7)$$

where the norm $|()|$ is Euclidean, unless otherwise specified.

As done in much of the applied-mechanics literature, we often blur the distinction between generalized velocities and conjugate forces, e.g., deformation rate and stress, as elements of dual spaces [56], but strive whenever possible to distinguish them, respectively, by contravariant and covariant tensors indices.

In the standard literature on finite-strain elastoplasticity, heroic efforts have been expended on the development of various objective deformation measures, their associated rates, and their work-conjugate stresses. Comprehensive summaries are given by others [22,68], but, as emphasized by Casey and Naghdi [69], properly formulated theories of plasticity should be independent of the particular choice of frame-indifferent descriptors of kinematics and stress.

The present work is focused on a class of rheological models that relate Cauchy stress $\mathbf{T}$ to deformation rate $\mathbf{D}$, often referred to as the *Eulerian description* [69], which have certain advantages for numerical treatments. In line with the above remarks, we exploit the fact that various kinematic rates, representing the tangent space to the underlying material configuration space, are connected by invertible linear transformations, e.g., by nonsingular fourth-rank tensors mapping second-rank tensors into second-rank tensors.

Adhering to the Eulerian framework, we therefore treat the linear transformations connecting **D** to other rates as a kind of "mixed metric" tensor that defines bilinear pairing between generalized velocities and dual-space forces. To illustrate this, consider the left Cauchy–Green tensor $\mathbf{B} = \mathbf{FF}^{\mathrm{T}}$ as finite Eulerian strain measure and its *Jaumann* (or *Zaremba–Jaumann*) rate,

$$\overset{\circ}{\mathbf{B}} = \dot{\mathbf{B}} + \mathrm{sym}\,\mathbf{WB} = \boldsymbol{\alpha}[\mathbf{D}] = \boldsymbol{\alpha}:\mathbf{D}\hat{=}[\alpha^{ij}_{..kl}D^{kl}] = \mathrm{sym}\,\mathbf{BD} \qquad (8)$$

where we distinguish tangent space by raised indices and where **D** is a nonholonomic rate (i.e., is not equal to the rate of change of a strain tensor derived from **F**).

Now, Eq. (8) defines a linear transformation $\boldsymbol{\alpha}(\mathbf{B})[\mathbf{D}]$ identical with that in Eq. (17). The construction of the inverse

$$\boldsymbol{\gamma} \hat{=} [\gamma^{ij}_{..kl}] = \boldsymbol{\alpha}^{-1}$$

is given elsewhere [68], but it is worth noting that, since **B** is positive definite, it can also be expressed as

$$\boldsymbol{\gamma}[\mathbf{X}] = \int_0^\infty e^{-\mathbf{B}s}\mathbf{X}e^{-\mathbf{B}s}ds = \left(\int_0^\infty e^{-\mathbf{B}s} \otimes e^{-\mathbf{B}s}ds\right):\mathbf{X} \qquad (9)$$

where the matrix exponential can be expressed as a second-order tensor polynomial in **B**.

The stress power can then be written as

$$\mathbf{T}:\mathbf{D} = \mathbf{T}\cdot\overset{\circ}{\mathbf{B}} = \mathbf{T}:\boldsymbol{\gamma}[\overset{\circ}{\mathbf{B}}] = \overset{\circ}{\mathbf{B}}:\boldsymbol{\gamma}^*[\mathbf{T}] = \gamma^{ij}_{..kl}T_{ij}(\overset{\circ}{B})^{kl} \qquad (10)$$

where the asterisk denotes the dual linear transformation. This same idea is employed below to express the power of an ordered modular array $\mathcal{V} \hat{=} [v^\alpha]$ of generalized velocities with array of conjugate forces $\mathcal{T} \hat{=} [T_\alpha]$ as

$$\mathcal{T} \cdot \mathcal{V} \overset{\text{def}}{=} G^\alpha_{.\beta}T_\alpha v^\beta, \quad \text{with} \quad \mathfrak{G} \hat{=} [G^\alpha_{.\beta}] \qquad (11)$$

where $\mathfrak{G}$ is an ordered modular array of linear transformations.

We now proceed to a systematic development of parametric hypoplasticity, a development that rests heavily on classical elastoplasticity. There is a twofold justification recognized, at least tacitly, by numerous workers in the field of granular mechanics. Firstly, classical elastoplastic models involve physical concepts and principles that are directly relevant to granular elastoplasticity. Secondly, certain necessary extensions of the classical models to accommodate granular mechanics provides new vistas and challenges, e.g., pressure dependence and dilatancy, and rate and gradient effects, represented above by $\mathbb{E}^{-1}$, $\mathbb{I}$ or $\mathbb{H}$, and $\mathbb{K}$.

2 Elastoplastic Models

Nemat-Nasser [23] gives an account of the elastoplasticity of pressure-sensitive materials. The author could not recommend a more comprehensive survey of the classical plasticity of granular media than that given in Sections 4.3 and Chapter 7 of that treatise. The introductory paragraphs contain an impressive summary of the subsequent matter, which contains a wealth of ideas and information on granular plasticity and the important microstructural parameters, including void ratio e, contact number density n_c (denoted there by $\bar{n}_c$), and the (first) fabric tensor $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$ (denoted there by $\boldsymbol{\Phi}$), along with various higher-order variants.

As an extension of classical plasticity, the present review focuses on a class of generalized hypoplastic models, also based heavily on a stress-space formulation. Although the approximation of incrementally linear elastic response is suitable for many granular materials, particularly stiff geomaterials, a nonlinear elastic theory is offered here based on an *isotropic extension* of nonlinear anisotropic elasticity that employs evolutionary

parameters to describe anisotropy. The nonlinear elastic model may be appropriate to soft granular materials, such as pharmaceutical capsules and clayey soils.

The intent of this review is to provide a rather broad theoretical survey and distillation of the major results of mathematical modeling, showing the progression from elasticity to viscoelastoplasticity, with a pronounced emphasis on elastoplasticity. The approach is based heavily on parametric hypoplasticity, reflecting the view that differential equations, essentially ordinary differential equations in the material or Lagrangian description, are the natural vehicles for describing evolutionary phenomena, such as history-dependent elastoplasticity. We will occasionally designate these as Lagrangian ordinary differential equations ("LODEs").

The treatise by Kolymbas [20] and the contemporaneous collection [16], particularly the chapter by Wu and Kolymbas [70], provide an excellent overview and history of hypoplastic modeling. The more recent monograph [11] provides an account of multipolar hypoplasticity with numerous applications to shear localization.

While many proponents of hypoplasticity seem to regard it as completely distinct from classical plasticity, the present work takes pains to show that it includes classical plasticity as a special case and, moreover, that it may require some of the structure of classical plasticity in order to achieve thermodynamic admissibility.

This review is decidedly theoretical in viewpoint, and comparisons with experiments or numerical simulations are largely relegated to references. For the theoretical development, we shall rely on two important concepts. First, we shall represent evolutionary material parameters by means of *structural tensors*, i.e., collections of scalars, vectors, and higher-order tensors that serve as internal variables of a type that are familiar in the literature on plasticity and, to a lesser extent, in the literature on complex fluids. Second, and as signaled above, the discussion of rheological models is based almost entirely on fourth-rank tensor moduli or compliances, either elastic or dissipative (inelastic), representing *pseudolinear* forms connecting stress or stress-rate to deformation rate. This is close in spirit to elementary rheological models based on one-dimensional depictions of stress and deformation, and it generally allows for a rather obvious generalization to fully three-dimensional models.

For hyperelastic (or "Green-elastic") systems, the existence of the above moduli is guaranteed by the underlying strain energy function or Helmholtz free energy. On the other hand, for strongly dissipative or hyperdissipative systems, it is guaranteed by the existence of a dissipation potential or inelastic potential (dubbed "pseudopotential" by Ref. [21], p. 52, where it seems to be restricted to homogeneous forms).

The existence of an inelastic potential, assumed in countless papers and a much-cited work [71], is mathematically guaranteed by the landmark work of Edelen [56,72], which serves to extend the Onsager "force-flux" formalism to nonlinear systems by means of pseudolinear "conductance" and "resistance" matrices. Moreover, it applies not only to plasticity but to viscoplasticity as well, a fact not generally recognized in the literature on viscoplasticity.

The relevant moduli or matrices are given as Hessians of potentials or complementary (Lagrangian dual) potentials, which assign an underlying *Hessian geometric structure* [73] to energy storage or dissipation, a structure that is imparted to equilibrium thermodynamics and thermodynamic fluctuations by the metrics of Weinhold [74] and Ruppeiner [75], with obvious application to the elastic strain energy of the present work.

Thus, in both hyperelastic and hyperdissipative systems, the respective moduli serve as induced metrics connecting forces, as covariant vectors, to displacements or velocities, representing contravariant vectors in the tangent space of the underlying manifold of material configurations. This abstract concept is immediately relevant to the mechanics at hand, since the loss of convexity of the associated potentials may be viewed as a

topological transition involving nonpositive moduli and signaling the onset of the bifurcation and material instability that is such a prominent feature of granular plasticity.

In the sections to follow, we shall deal with anisotropic elastic response that depends on the history of plastic deformation. As anticipated above, we assume that both intrinsic and evolutionary anisotropy can be represented by a discrete set of evolutionary internal variables. In so doing, we set aside a long-standing issue as to the modification of material symmetry by inelastic deformation and related processes, including proposals for modifications of Noll's well-known classification [30,67,76–79].

2.1 From Anisotropic Elasticity to Hypoelasticity. Before treating plastic deformation, we consider the representation of anisotropic nonlinear elasticity. The main purpose of this section is to show how a special form of evolutionary elastic anisotropy leads to a general form of hypoelasticity summarized by Eq. (23). One goal is to reveal certain shortcomings of hypoplasticity as a model of elastoplasticity before proceeding to the more plausible hypoplasticity. Those who are content to accept Eq. (23) as an evident extension of standard hypoelasticity may wish to skip directly to that relation.

Otherwise, recall that, given the *deformation gradient* $\mathbf{F} = \partial \mathbf{x}(\mathbf{x}^\circ)/\partial \mathbf{x}^\circ$ at a material point $\mathbf{x}^\circ$ [30], the frame-indifferent (Helmholtz) strain energy (per unit mass) of a *hyperelastic or Green-elastic* solid is given by a scalar-valued function $\psi_E^\circ(\mathbf{C})$ of the *right Cauchy–Green strain* $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Moreover, this function is invariant under the material symmetry group $\mathcal{G}$, a subset of the full orthogonal group $\mathcal{O}(3)$, i.e.,

$$\psi_E^\circ(\mathbf{Q}_\mathcal{G}\mathbf{C}\mathbf{Q}_\mathcal{G}^T) = \psi_E^\circ(\mathbf{C}), \quad \forall \mathbf{Q}_\mathcal{G} \in \mathcal{G} \subseteq \mathcal{O}(3) \quad (12)$$

It appears possible for many crystalline symmetry groups, although perhaps not all [80], to obtain an isotropic extension $\check{\psi}_E$ of Eq. (12) represented by an isotropic function of $\mathbf{C}$ and a set of vectors and second-rank tensors designated as structural tensors, which, to be definite, are assumed to be independent of $\mathbf{C}$. To make the point, we focus attention on the special case of a single vector $\mathbf{b}^\circ$ and an independent second-rank tensor $\mathbf{A}^\circ$, with

$$\psi_E^\circ(\mathbf{C}) = \check{\psi}_E(\mathbf{C}, \mathbf{A}^\circ, \mathbf{b}^\circ)$$
$$= \check{\psi}_E(\mathbf{Q}\mathbf{C}\mathbf{Q}^T, \mathbf{Q}\mathbf{A}^\circ, \mathbf{Q}\mathbf{b}^\circ\mathbf{Q}^T), \quad \forall \mathbf{Q} \in \mathcal{O}(3) \quad (13)$$

Hence, $\check{\psi}_E$ must reduce to a function of the well-known [81] joint isotropic scalar invariants of $\mathbf{C}$, $\mathbf{A}^\circ$, and $\mathbf{b}^\circ$.

Making use of Eq. (12), it is easy to show that

$$\check{\psi}_E(\mathbf{C}, \mathbf{Q}_\mathcal{G}\mathbf{A}^\circ\mathbf{Q}_\mathcal{G}^T, \mathbf{Q}_\mathcal{G}\mathbf{b}^\circ) = \check{\psi}_E(\mathbf{C}, \mathbf{A}^\circ, \mathbf{b}^\circ), \quad \forall \mathbf{Q}_\mathcal{G} \in \mathcal{G} \quad (14)$$

i.e., $\check{\psi}_E$ qua function of $\mathbf{b}^\circ$ and $\mathbf{A}^\circ$ depends only on their joint scalar invariants under $\mathcal{G}$.

We may take $\mathbf{A}^\circ = \mathbf{I}$ and $\mathbf{b}^\circ = \mathbf{0}$ for a fully isotropic reference state. On the other hand, if $\mathbf{A}^\circ = \mathbf{I}$ and $\mathbf{b} \neq \mathbf{0}$, we obtain transverse isotropy with axis of symmetry $\mathbf{b}^\circ$ or, if $\mathbf{A}^\circ = \mathbf{A}^\circ T \neq \mathbf{I}$ and $\mathbf{b}^\circ = \mathbf{0}$, orthotropic symmetry, with three symmetry axes represented by the principal directions of $\mathbf{A}^\circ$, according to Xiao et al. [80]. Degeneracy (confluence of two eigenvalues) in one invariant subspace of $\mathbf{A}^\circ$ leads once again to transverse isotropy. When there is no dependence on $\mathbf{A}^\circ$ or $\mathbf{b}^\circ$, we obtain full isotropy, with $\mathcal{G} \equiv \mathcal{O}(3)$.

The above structural tensors may be identified with various representations of *texture* or *fabric* of anisotropic bone, polycrystalline metals, and granular materials [28,78,82–85]. By considering a set of vectors $\mathbf{b}$ defining lattice parameters, Rubin [86] has proposed a theory to remove indeterminacies inherent to the standard elastoplastic decomposition discussed below.

From Eq. (13), we may obtain an isotropic extension involving the *left Cauchy–Green strain* $\mathbf{B}$,

$$\check{\psi}_E(\mathbf{C}, \mathbf{A}^\circ, \mathbf{b}^\circ) = \check{\psi}_E(\mathbf{R}^T\mathbf{B}\mathbf{R}, \mathbf{A}^\circ, \mathbf{b}^\circ) = \Psi_E(\mathbf{B}, \mathbf{A}, \mathbf{b})$$
$$= \Psi_E(\mathbf{Q}\mathbf{B}\mathbf{Q}^T, \mathbf{Q}\mathbf{A}\mathbf{Q}^T, \mathbf{Q}\mathbf{b}), \quad \forall \mathbf{Q} \in \mathcal{O}(3), \quad (15)$$
$$\text{where } \mathbf{B} = \mathbf{F}\mathbf{F}^T, \ \mathbf{A} = \mathbf{F}\mathbf{A}^\circ\mathbf{F}^T, \ \mathbf{b} = \mathbf{F}\mathbf{b}^\circ$$

It is clear that Eq. (15) is frame indifferent, which dictates the symmetry of the second-rank tensor

$$\partial_\mathcal{B}\Psi_E = (2\mathbf{B}\partial_\mathbf{B} + \partial_\mathcal{X})\Psi_E,$$
$$\text{where } \partial_\mathcal{X} = 2\mathbf{A}^T\partial_\mathbf{A} + \mathbf{b} \otimes \partial_\mathbf{b}, \quad (16)$$
$$\mathcal{B} = \{\mathbf{B}, \mathcal{X}\}, \quad \text{and} \quad \mathcal{X} = \{\mathbf{A}, \mathbf{b}\}$$

defining the linear operator such that $\delta\Psi_E = \partial_\mathcal{B}\Psi_E[\delta\mathcal{B}]$. Anticipating the notation for the arrays of internal variables discussed below, we have distinguished the control variable $\mathbf{B}$ from the microstructural tensors $\mathcal{X}$.

The requirement that $\mathbf{A}^\circ$ and $\mathbf{b}^\circ$ be material constants leads to the following evolution equations involving linear forms in $\mathbf{D}$:

$$\overset{\circ}{\mathbf{A}} = \boldsymbol{\alpha}[\mathbf{D}] = \mathbf{D}\mathbf{A} + \mathbf{D}\mathbf{A}$$
$$\text{and} \quad \overset{\circ}{\mathbf{b}} = \boldsymbol{\beta}[\mathbf{D}] = \mathbf{D}\mathbf{b} \quad (17)$$

where the first equation is also satisfied by $\mathbf{B}$.

As discussed below, Eq. (17) must in general be replaced for more general evolution equations for $\mathbf{A}$ and $\mathbf{b}$, regarded as a special case of a more general set of internal variables. However, as a preliminary consideration, we treat the special case of the imbedded microstructure defined by Eq. (17) in order to identify conjugate elastic forces and stress rate associated with that relation.

Thus, if we assume that all structural tensors are enslaved to the velocity gradient $\mathbf{L}$, the expression for the conservative stress power follows from the appropriate limit of the standard Clausius–Duhem inequality [87]:

$$\mathbf{T} : \mathbf{D} = \rho\dot{\Psi}_E \quad (18)$$

and yields the following expression for the Cauchy stress:

$$\mathbf{T} = \mathsf{t}(\mathcal{B}) = \rho\partial_\mathcal{B}\Psi_E(\mathcal{B}) \quad (19)$$

where the derivative is defined in Eq. (16).

The Jaumann stress rate is then given by Eqs. (17)–(19) as the linear function of $\mathbf{D}$,

$$\overset{\circ}{\mathbf{T}} = \overset{\circ}{\mathsf{t}}(\mathcal{B})[\mathbf{D}] = \mathsf{t}_\mathbf{B} \circ \boldsymbol{\alpha}[\mathbf{D}] + \mathsf{t}_\mathbf{A} \circ \boldsymbol{\alpha}[\mathbf{D}] + \mathsf{t}_\mathbf{b} \circ \boldsymbol{\beta}[\mathbf{D}]$$
$$= \mathsf{t}_\mathbf{B}[\mathbf{D}\mathbf{B} + \mathbf{B}\mathbf{D}] + \mathsf{t}_\mathbf{A}[\mathbf{D}\mathbf{A} + \mathbf{A}\mathbf{D}] + \mathsf{t}_\mathbf{b}[\mathbf{D}\mathbf{b}] \quad (20)$$

where the in-line "∘" denotes functional product and $\mathsf{t}_X[]$ the linear function defining the partial derivative $\partial\mathsf{t}/\partial X$.

Assuming invertibility of Eq. (19) with respect to the argument $\mathbf{B}$, we have

$$\mathcal{B} = \mathsf{b}(\mathcal{S}) = \mathsf{t}^{-1}(\mathcal{S}), \quad \text{with} \quad \mathcal{S} = \{\mathbf{T}, \mathcal{X}\} \quad (21)$$

and, hence, the stress rate in terms of *tangent modulus*,

$$\overset{\circ}{\mathbf{T}} = \boldsymbol{\mu}_E(\mathcal{S})[\mathbf{D}], \quad \text{where} \quad \boldsymbol{\mu}_E(\mathcal{S}) = (\overset{\circ}{\mathsf{t}} \circ \mathsf{b})(\mathcal{S}) \quad (22)$$

Despite the frequent claims for the merits of particular objective rates, one may obtain any rate of the form

$$\overset{\triangledown}{\mathbf{T}} = \overset{\circ}{\mathbf{T}} + \boldsymbol{\lambda}(\mathcal{S})[\mathbf{D}]$$

where $\lambda(\mathcal{S})$ is an isotropic function of $\mathcal{S}$, by the substitution $\mu_E \to \mu_E - \lambda$. Hence, apart from possible issues associated with numerical integration, the fundamental issue would seem to be the mathematical properties of the modulus μ_E and not of the derivative.

The relations in Eqs. (17) and (22) are subsumed in the rate-independent LODEs,

$$\overset{\circ}{\mathcal{S}} = \mathfrak{M}(\mathcal{S})[\mathbf{D}] \qquad (23)$$

As a generalized form of *hypoelasticity* [30], this represents a dynamical system, with linearity in the control variable $\mathbf{D}$. Since this system governs the evolution of stress as well as the structural parameters $\mathcal{X}$, the previously proposed [85] designation *parametric hypoelasticity* seems appropriate. Up to an initial condition on $\mathcal{S}$, the system in Eq. (23) satisfies the principle of determinism [30,76].

We could have begun with the first equality in Eq. (21) and the first relation in Eq. (17) with $\mathbf{A} = \mathbf{B}$ to obtain the inverse form of Eq. (23) involving a compliance $\mathfrak{R} = \mathfrak{M}^{-1}$.

The above derivation is an anisotropic version of that employed to derive isotropic hypoelasticity from isotropic hyperelasticity [88,89]. As for the converse, we recall Bernstein's [89] integrability conditions for the reduction of the isotropic form of Eq. (23), with $\mathcal{S} \equiv \mathbf{T}$, $\mathfrak{M} \equiv \mu(\mathbf{T})$, to the isotropic form of Eq. (12), with $\mathcal{G} = \mathcal{O}(3)$. It would no doubt be more difficult and largely of theoretical interest to establish similar conditions for the reduction of the anisotropic form in Eq. (23), involving Eqs. (17) and (22), to the form in Eq. (19).

A somewhat more practical but theoretically more difficult issue is the thermodynamic admissibility of hypoelastic models [85,90,91], which *inter alia* requires that stress work be nonnegative on periodic cycles of deformation. Unfortunately, there is no guarantee of thermodynamic admissibility of the general form in Eq. (23), in contrast to various constitutive equations built on plausible physical models in the field of classical plasticity [71].

This raises another important question as to the relation of isotropic hypoelasticity to classical incremental theories of plasticity. Various attempts [30,92–94] have been made to establish an equivalence-based special or singular dependence of $\mu(\mathbf{T})$ on $\mathbf{T}$, but it is doubtful that a strictly linear form in $\mathbf{D}$ suffices. This is shown explicitly by the classical formula for nonassociative, perfect (i.e., nonhardening) plasticity involving an elastic region $\mathcal{E}$ in stress space and a plastic potential $\varphi_P(\mathbf{T})$ [23],

$$\overset{\circ}{\mathbf{T}} = \mu(\mathbf{T}):\mathbf{D}, \quad \text{with} \quad \mu = \mu_E - \frac{(\mu_E:\mathbf{M}) \otimes (\mathbf{N}:\mu_E)}{\mathbf{N}:\mu_E:\mathbf{M}} H,$$

where

$$H = \begin{cases} 1 & \text{for} \quad \mathbf{T} \in \partial\mathcal{E} \quad \text{and} \quad \mathbf{N}:\mu:\mathbf{D} \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad (24)$$

where $\mathbf{N}(\mathbf{T})$ and $\mathbf{M}(\mathbf{T})$ are the respective normals to $\partial\mathcal{E}$ and to the level surfaces of $\varphi_P(\mathbf{T})$. It is clear that the (Kuhn–Tucker [95]) inequality condition on $\mathbf{D}$ in Eq. (24) cannot be represented by a strictly linear form $\mu(\mathbf{T})[\mathbf{D}]$.

Removing the inequality from Eq. (24), one does indeed obtain a hypoelastic model, but it permits plastic deformation on unloading from the yield surface. Tacitly assumed in a previous work by the present author [85], this does not seem to be ruled out by any general physical principle and may be a good approximation for many materials. However, it is less than evident that one can account for history-dependent effects, such as work hardening, within the standard hypoelastic framework.

By contrast, and as discussed next, the constitutive equation in Eq. (24), as well as a well-known version involving work hardening [23], is easily represented by generalized hypoplasticity. Moreover, it appears that the so-called "pseudoelasticity" [96] can readily be incorporated into the same framework.

In closing here, we remark that, within the strictly elastic framework, the individual partial derivatives in Eq. (16) can be viewed as conjugate forces for the microstructural tensors, a matter discussed in more detail below.

2.2 From Hypoelasticity to Hypoplasticity. Following a long-standing body of work on the hypoplasticity of particulate media [34,67,84,85], we show how hypoplasticity can be based on *pseudolinear forms* that are derivable from potential functions. Although the resulting hypoplastic model is less general than those obtained by converting viscoelastic-fluid models of the "rate type" to a rate-independent form [97,98], it offers a much more direct way to construct variants on classical models that exhibit proper dissipative behavior. Moreover, the model is readily extended to include higher-order stress rates [85].

As one step in the rational construction of hypoplasticity, we consider the representation of a dissipative process by the pseudolinear forms involving *secant moduli*,

$$\mathbf{T} = \boldsymbol{\eta}(\mathbf{D}):\mathbf{D} \quad \text{or} \quad \mathbf{D} = \boldsymbol{\varphi}(\mathbf{T}):\mathbf{T}, \quad \text{with} \quad \boldsymbol{\varphi} = \boldsymbol{\eta}^{-1} \qquad (25)$$

represented by viscosity $\boldsymbol{\eta}$ and fluidity $\boldsymbol{\varphi}$. Here, we have suppressed notation for dependence of these quantities on the parameters $\mathcal{X}$. For rate-independent (rigid) plasticity, we take [67]

$$\boldsymbol{\eta} = \boldsymbol{\mu}_P/|\mathbf{D}| \quad \text{and} \quad \mathbf{T} = \boldsymbol{\mu}_P(\hat{\mathbf{D}}):\hat{\mathbf{D}} \quad \text{or} \quad \hat{\mathbf{D}} = \boldsymbol{\kappa}_P(\mathbf{T}):\mathbf{T}$$
$$\text{with} \quad \boldsymbol{\kappa}_P = \boldsymbol{\mu}_P^{-1} \quad \text{and} \quad |\boldsymbol{\kappa}_P(\mathbf{T}):\mathbf{T}| = 1 \qquad (26)$$

where $\hat{\mathbf{D}} = \mathbf{D}/|\mathbf{D}|$ and the modulus and compliance are positive-definite and invertible as indicated. The third relation in Eq. (26) is equivalent to the classical *plastic flow rule*, as discussed below.

The relations in Eqs. (25) and (26) can also be derived from a dissipation potential [56,72]. Thus, given the convex *stress potential* $\varphi_P(\mathbf{T})$ (energy/volume), one can construct [56] a function $\Psi_P(\mathbf{T})$ such that

$$\mathbf{D} = \partial_{\mathbf{T}}\varphi_P = \boldsymbol{\varphi}:\mathbf{T} = (\partial_{\mathbf{T}}^2\Psi_P):\mathbf{T}$$
$$\text{with} \quad \varphi_P = \mathbf{T}:\partial_{\mathbf{T}}\Psi_P - \Psi_P, \quad \text{where} \quad \partial_{\mathbf{T}}^2 = \partial_{\mathbf{T}} \otimes \partial_{\mathbf{T}} \qquad (27)$$
$$\text{and} \quad \therefore \quad \boldsymbol{\kappa}_P = \partial_{\mathbf{T}}^2\Psi_P/|\partial_{\mathbf{T}}^2\Psi_P:\mathbf{T}|$$

This expression for the compliance $\boldsymbol{\kappa}_P$ represents the normalization required by the final member of Eq. (26), a condition that is misconstrued in a previous work [85] as a yield condition.

By contrast, a recent work [56] shows that rate-independence implies marginal convexity of φ_P, which can be associated with the differentiable surface of a bounded region in stress space of the form

$$|\mathbf{T}| = r(\hat{\mathbf{T}}) \quad \text{with} \quad r \in \mathbf{C}^1 \qquad (28)$$

If this surface is regarded as a yield surface, then Eq. (27) represents the standard *associative flow rule* of classical plasticity [22,23]. That is, the flow rule is a mathematical consequence of strong dissipation and convexity, *requiring no further physical or mathematical postulates*. As an exception to this state of affairs, the idealized conical yield surfaces often attributed to granular plasticity cannot be represented in the above form. This reflects an extreme form of "vertex plasticity" and leads to a breakdown of associative plasticity, with appropriate representations given below.

In order to attribute dissipation solely to plastic deformation, we must replace $\mathbf{D}$ in Eqs. (26) and (27) by a *plastic strain rate* $\mathbf{D}_P$, such as that obtained by the classical additive (tangent-space) decomposition, with associated (Kröner–Lee) product decomposition in configuration space [22,79],

$$\mathbf{L} = \mathbf{L}_E + \mathbf{L}_P \quad \text{and} \quad \mathbf{F} = \mathbf{F}_E \mathbf{F}_P$$
$$\text{with} \quad \mathbf{D}_E = \text{sym}\mathbf{L}_E = \mathbf{D} - \mathbf{D}_P, \quad \mathbf{W} = \text{skw}\mathbf{L}_E,$$
$$\mathbf{D}_P = \text{sym}\mathbf{L}_P, \quad \mathbf{W}_P = \text{skw}\mathbf{L}_P \tag{29}$$
$$\dot{\mathbf{F}}_E = \mathbf{L}_E \mathbf{F}_E \quad \text{and} \quad \dot{\mathbf{F}}_P = (\mathbf{F}_E^{-1}\mathbf{L}_P\mathbf{F}_E)\mathbf{F}_P$$

where $\mathbf{F}_P$ represents the plastic deformation from the reference state to a stress-free intermediate state and $\mathbf{F}_E$ represents the current elastic deformation from that state. The present definition of plastic velocity gradient $\mathbf{L}_P$ differs inconsequentially from that employed by others, and it is evident that Eq. (29) represents a time-integrable system for $\mathbf{F}_E$ and $\mathbf{F}_P$ given $\mathbf{L} = \mathbf{D} + \mathbf{W}$, $\mathbf{L}_P = \mathbf{D}_P + \mathbf{W}_P$, and hence, $\mathbf{D}_E = \mathbf{D} - \mathbf{D}_P$ at a fixed material point $\mathbf{x}^{\circ}$.

In principle, we should substitute $\mathbf{L}_P$ for $\mathbf{D}$ in Eq. (25), thereby allowing for a contribution of *plastic spin* $\mathbf{W}_P$ to dissipation. However, as discussed below, we shall eventually express $\mathbf{W}_P$ as a pseudolinear form in $\mathbf{D}_P$ based on an internal moment balance, leading to a result like Eq. (25).

To recall the major postulates underlying classical (incremental) plasticity models, we make use of elastoplastic moduli to retrace the principal steps in the formulation. For this purpose, we consider the special case of perfectly plastic materials, where the moduli are isotropic functions of $\mathbf{T}$ in the absence of history effects, such as work hardening.

Classical Postulates and Internal Moments. The additivity of elastic and plastic rates in the first member of Eq. (29) represents the *first postulate of classical elastoplasticity* (by which we refer to the incremental form relating stress increment linearly to plastic strain increment). Since $\mathbf{L}_E$ and $\mathbf{L}_P$ represent separate and generally independent kinematical degrees of freedom, one should, as advocated by Gurtin and Anand [79], respect the principle of virtual work and assign distinct conjugate stresses $\mathbf{T}_E$ and $\mathbf{T}_P$ with stress power,

$$\mathcal{P} = \mathbf{T}_E : \mathbf{D}_E + \mathbf{T}_P : \mathbf{L}_P \tag{30}$$

These same authors [79] appeal to the notion of an intrinsic force balance involving stresses $\mathbf{T}_E$ and $\mathbf{T}_P$ together with external force fields and inertial effects. In the absence of the latter two effects, their analysis for isotropic materials leads to the *second postulate of classical elastoplasticity*,

$$\mathbf{T}_P = \mathbf{T}_E = \rho \mathbf{B} \partial_{\mathbf{B}} \psi_E \equiv \mathbf{T} = \mathbf{T}^{\mathrm{T}} \quad \text{for} \quad |\mathbf{D}_P| > 0 \tag{31}$$

It is plausible that the condition in Eq. (31) can also be obtained from simultaneous extrema of elastic and inelastic potentials under quasistatic condition with fixed $\mathbf{L}$ and $\mathbf{W}$ viz.,

$$\partial_{\mathbf{L}_P} \psi_P : \delta \mathbf{L}_P + \lambda \partial_{\mathcal{B}} \psi_E \cdot \dot{\mathcal{B}}[\delta \mathbf{D}_E] = 0$$
$$\text{with} \quad \delta \mathbf{L} = \delta \mathbf{D}_E + \delta \mathbf{L}_P = \mathbf{0} \tag{32}$$

where λ is a Lagrange multiplier. We note that external forces derivable from a potential could also be included in Eq. (32).

Various works on plasticity appear to conclude that relations like Eq. (31) represent a functional connection between potentials. The notion of a single potential motivates several "thermomechanical" models of granular elastoplasticity, notably by Collins [58,99], who traces the history and assumes additive elastic and plastic contributions to a single potential. By contrast, the elastic and inelastic potentials are treated here as completely independent.

While not pursued further in the present paper, the joint extrema of independent potentials is an alternative notion that appears worthy of further investigation, as it might possibly be extended to networks or composite media containing both dissipative and elastic parts. At any rate, and as discussed further below,

Eq. (31) is restricted to simple (i.e., *nonpolar*) materials, since it rules out the multipolar effects discussed below in Sec. 4. Such effects, which lead to asymmetric stress and associated couple stress, may arise in strongly inhomogeneous deformations also discussed below.

Within the framework of nonpolar materials, the present modeling allows for an internal moment balance, e.g., of the type that is known to occur in viscous suspensions of anisometric freely rotating particles [100]. In particular, the dissipative moment balance requires that

$$\mathbf{0} = \text{skw}\mathbf{T}_P = \text{skw}\partial_{\mathbf{L}_P}\psi_P$$
$$= \eta_P^{(\times)} : \mathbf{D}_P + \zeta_P^{(\times)} : \mathbf{W}_P \tag{33}$$

where $(\ )^{(\times)}$ indicates skew symmetrization of a pseudolinear form and $\boldsymbol{\zeta}_P^{(\times)}$ represents a "spin" viscosity or plasticity. In principle, this relation and Eq. (25) yield a solution for spin,

$$\mathbf{W}_P = \boldsymbol{\omega}_P[\mathbf{D}_P] = \boldsymbol{\omega}_P \circ \boldsymbol{\varphi} : \mathbf{T} \quad \text{with} \quad \boldsymbol{\omega}_P = \boldsymbol{\zeta}_P^{(\times)-1} \circ \boldsymbol{\eta}_P^{(\times)} \tag{34}$$

where $\boldsymbol{\varphi}$ is the fluidity tensor in Eq. (25). This or other specification of viscoplastic spin is essential to the integration of Eq. (29) in models with microstructural orientations determined by the inelastic deformation $\mathbf{F}_P$.

Isotropic Hypoplasticity. To obtain the simplest version of hypoplasticity, we consider the isotropic form of Eq. (22) combined with Eq. (26), where $\mathbf{D}_E$ and $\mathbf{D}_P$ are substituted, respectively, for $\mathbf{D}$. This yields

$$\overset{\circ}{\mathbf{T}} = \boldsymbol{\mu}_E(\mathbf{T})[(\mathbf{D} - |\mathbf{D}_P|\mathbf{M})], \quad \text{with} \quad \mathbf{M}(\mathbf{T}) = \boldsymbol{\kappa}_P : \mathbf{T} \tag{35}$$

where $\mathbf{M}$ is up to an algebraic sign equal to tensor appearing in Eq. (24) and $\boldsymbol{\mu}_E(\mathbf{T})$ is an isotropic tensor function of $\mathbf{T}$.

It remains to specify $|\mathbf{D}_P|$, and rate-independence dictates a relation of the form

$$|\mathbf{D}_P| = \vartheta(\mathbf{T}, \hat{\mathbf{D}})|\mathbf{D}| \quad \text{and} \quad \therefore \mathbf{D}_E = (\boldsymbol{\delta} - \vartheta\mathbf{M} \otimes \hat{\mathbf{D}}) : \mathbf{D} \tag{36}$$

where the second equation is one of the relations essential to the integration of Eq. (29) discussed above at the end of the preceding section.

Up to an algebraic sign, $|\mathbf{D}_P|$ is equal to quantities denoted variously in the standard plasticity literature by λ [101] or by $\dot{\lambda}$ as "consistency parameter" [102], quantities which take on both positive and negative values. In the present representation, this parameter is nonnegative, whereas the unit tensor $\mathbf{M} = \boldsymbol{\kappa}_P[\mathbf{T}]$ representing the direction $\mathbf{D}_P/|\mathbf{D}_P|$, may reverse sign depending on the direction of $\mathbf{T}$.

It is obvious that Eqs. (35) and (36) represent a special case of the most basic hypoplastic model, with stress rate given as a homogeneous form of degree one in deformation rate,

$$\overset{\circ}{\mathbf{T}} = \boldsymbol{\nu}(\mathbf{T}, \hat{\mathbf{D}}) : \mathbf{D} = \boldsymbol{\mu}(\mathbf{T}) : \mathbf{D} - |\mathbf{D}|\mathbf{K}(\mathbf{T}),$$
$$\text{where} \quad \boldsymbol{\nu} = \boldsymbol{\mu}(\mathbf{T}) - \mathbf{K}(\mathbf{T}) \otimes \hat{\mathbf{D}} \tag{37}$$

To recover classical elastoplasticity, one must interpret ϑ in Eq. (36) as an "inelastic-clock" function [85], a notion embodied in the general treatment of Pipkin and Rivlin [103]. This can be expressed in terms of a timelike variable usually identified as plastic strain in $(0, t)$,

$$\wp_P = \int_0^t |\mathbf{D}_P| dt = \int_0^t \vartheta|\mathbf{D}| dt \tag{38}$$

Thus, as a modification of the hypoelastic form proposed elsewhere [85], we take

$$\vartheta = \vartheta_P H, \quad \text{where} \quad \vartheta_P = (\mathbf{N} : \boldsymbol{\mu}_E : \hat{\mathbf{D}})/(\mathbf{N} : \boldsymbol{\mu}_E : \mathbf{M})$$

$$H = \begin{cases} 1, & \text{for} \quad \mathbf{T} \in \partial\mathcal{E} \quad \text{and} \quad \mathbf{N} : \boldsymbol{\mu}_E[(\hat{\mathbf{D}} - \vartheta_P \mathbf{M})] \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad (39)$$

in Eq. (35) to yield the result in Eq. (24). The expression for ϑ_P in Eq. (39) reflects the assumption that stress changes elastically along the yield surface [22,23].

It is apparent from the preceding analysis that the present version of the classical theory is closed by specification of *three scalar functions*, two potentials, and an inelastic-clock function, respectively, ψ_E, ψ_P, and ϑ. The function ϑ is obviously necessary to describe plastic yield.

The usual treatments (e.g., Ref. [23]) postulate various models for a yield function $f(\mathbf{T}, \mathcal{X})$, such that a relation $f = 0$ defines the boundary of the elastic region $\partial\mathcal{E}$, with $\mathbf{N} = \partial_{\mathbf{T}} f / |\partial_{\mathbf{T}} f|$ representing its unit normal appearing in Eq. (39). According to a recent analysis [85], one should always identify the plastic potential φ_P with the yield function whenever the yield surface is a bounding surface having the form in Eq. (28). As done above for hypoelasticity, we return to the problem of describing the evolution of microstructure.

2.3 Parametric Hypoplasticity.

If one assumes once again that microstructure can be described by structural tensors of various orders, $\mathbf{b}, \ldots, \mathbf{A}, \mathbf{B}, \ldots$, the essential problem is to prescribe a set of LODEs for their evolution. We may of course require similar equations for scalars such as $\phi, Z, \ldots$, and this totality of all such evolutionary parameters or internal variables together with the LODEs governing their evolution serve to define parametric hypoplasticity [85].

While one may postulate phenomenological evolution equations, e.g., as done elsewhere for fluid-particle suspension [84], an effort is made here to show how such equations might follow from a postulated internal balance of dissipative and elastic forces. This of course raises an interesting question as to the microstructural origins and physical significance of the dissipative forces associated with the rate of change of structural tensors.

A second issue is the choice of a convenient representation of the structural tensors, and we cast them here into a form that facilitates the expression of their rates of change in terms of Jaumann rates. While other objective rates could be employed, these would simply change the form of various hypoplastic moduli to be obtained below.

Thus, given that

$$\mathbf{X} = \mathbf{V}_E \mathbf{R}_E \mathbf{X}_P \mathbf{R}_E^{\mathrm{T}} \mathbf{V}_E \quad \text{for} \quad \mathbf{X} = \mathbf{A}, \mathbf{B}, \quad \text{and} \quad \mathbf{b} = \mathbf{V}_E \mathbf{R}_E \mathbf{b}_P$$

it is easy to show that the strain energy in Eq. (15) can be expressed as an isotropic function of a new set of variables,

$$\begin{aligned}
\Psi_E &= \psi_E(\mathbf{B}_E, \mathbf{Y}, \mathbf{H}, \mathbf{A}, \mathbf{b}) \\
&= \psi_E(\mathbf{Q}\mathbf{B}_E\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{Y}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{H}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{b}) \\
&\quad \forall \mathbf{Q} \in \mathcal{O}(e), \\
&\quad \text{where}
\end{aligned}$$

$$\mathbf{Y} = \mathbf{Q}_E \mathbf{B}_P \mathbf{Q}_E^{\mathrm{T}}, \quad \mathbf{H} = \mathbf{R}_E \mathbf{Q}_E^{\mathrm{T}}, \quad \mathbf{A} = \mathbf{Q}_E \mathbf{A}_P \mathbf{Q}_E^{\mathrm{T}}, \quad \mathbf{b} = \mathbf{Q}_E \mathbf{b}_P$$

$$\text{with} \quad \dot{\mathbf{Q}}_E = \mathbf{W}\mathbf{Q}_E, \quad \overset{\circ}{\mathbf{H}} = (\boldsymbol{\Omega}_E - \mathbf{W})\mathbf{H}, \quad \text{and} \quad \boldsymbol{\Omega}_E = \dot{\mathbf{R}}_E \mathbf{R}_E^{\mathrm{T}}$$

(40)

Here, $\mathbf{R}_E$ is the finite rotation in the polar decomposition of $\mathbf{F}_E$ and, for convenience, we have introduced the *mean rotation* $\mathbf{Q}_E$ generated by the spin $\mathbf{W}$. Also, subscript P refers once again to the intermediate plastic state, and the definitions of $\mathbf{A}$ and $\mathbf{b}$ have

been changed in an obvious way, such that, for constant $\mathbf{A}_P$ and $\mathbf{b}_P$, their Jaumann rate vanishes.

The objective orthogonal tensor $\mathbf{H} = \mathbf{R}_E \mathbf{Q}_E^{\mathrm{T}}$ represents the competing effects of mean and finite rotation on structural elastic energy. Its Jaumann rate shown in Eq. (40) has the hypoelastic form, since the difference in spin $(\boldsymbol{\Omega}_E - \mathbf{W})$ is linear in the elastic deformation rate $\mathbf{D}_E$ [68]. Moreover, it makes a contribution $\mathbf{H}^{\mathrm{T}} \partial_{\mathbf{H}} \psi_E : (\boldsymbol{\Omega}_E - \mathbf{W})$ to the rate of elastic energy storage. However, since one can only demand symmetry of a complete derivative like that defined in Eq. (16), and since $\mathrm{skw}\mathbf{H}^{\mathrm{T}} \partial_{\mathbf{H}} \psi_E$ represents but one contribution to an elastic torque, it must vanish in the absence of external torques.

The net result of the representation in Eq. (40) is the addition of two structural tensors having the character of $\mathbf{A}$, and we can include it in a general array of such tensors, with strain energy given by the frame-indifferent form

$$\breve{\Psi}_E = \psi_E(\mathbf{B}_E, \mathcal{X}), \quad \text{where} \quad \mathcal{X} = \{\mathbf{A}, \ldots, \mathbf{b}, \ldots\} \quad (41)$$

with $\mathbf{B}_E = \mathbf{I}$, $\mathbf{A} = \mathbf{A}_P, \ldots \mathbf{b} = \mathbf{b}_P, \ldots$ in the relaxed plastic state, where we take $\mathbf{H}_P = \mathbf{I}$.

One obtains a form appropriate to linear elasticity on replacing $\mathbf{B}_E$ by

$$\mathbf{E} = \frac{1}{2}(\mathbf{B}_E - \mathbf{I}), \quad \text{with} \quad \overset{\circ}{\mathbf{E}} = \mathbf{D} + \mathrm{sym}\,\mathbf{E}\mathbf{D} \quad (42)$$

the Eulerian form of a standard Lagrangian strain. The terms in $\mathbf{E}\mathbf{D}$ are neglected in linear elastostatics but, to the extent they are important in the present setting, can be incorporated into the definition of the hypoelastic modulus.

We recall that the assumption of linear elastic response from plastically deformed states, assumed in much of the classical plasticity literature, also represents a useful approximation for the plasticity of stiff granular materials. It leads to a simpler model that is strictly dissipative, except for small strain intervals involving elastic loading or unloading [85].

In a relaxed plastic state, the quantities $\mathbf{A}_P, \ldots, \mathbf{b}_P, \ldots$ may be treated as constant in any elastic deformation $\mathbf{D}_P \equiv \mathbf{0}$ from that state. In such deformations, only $\mathbf{B}_E$ and $\mathbf{H}$ have nonvanishing Jaumann rates and Eq. (20) is replaced by

$$\overset{\circ}{\mathbf{T}} \equiv \overset{\circ}{\mathbf{t}} = \mathbf{t}_{\mathbf{B}_E}[\mathbf{D}_E \mathbf{B}_E + \mathbf{B}_E \mathbf{D}_E] + \mathbf{t}_{\mathbf{H}}[(\boldsymbol{\Omega}_E - \mathbf{W})\mathbf{H}] \quad (43)$$

where $\mathbf{t}_X$ are derivatives of the type defined in Eq. (20), and the term in $(\boldsymbol{\Omega}_E - \mathbf{W})$ is also linear in $\mathbf{D}_E$.

For a material that is isotropic in some virgin state, with $\mathbf{A}^\circ = \mathbf{I}, \ldots, \mathbf{b}^\circ = \mathbf{0}, \ldots$, plastic deformation can lead to anisotropy in the current state. If, however, ψ_E depends only on the isotropic invariants of $\mathbf{A}, \ldots, \mathbf{b}, \ldots$, then isotropic states remain isotropic. For example,

$$\frac{1}{2}|\mathbf{A} - \mathbf{I}| = \frac{1}{2}[\mathrm{tr}(\mathbf{A}^2) - 2\mathrm{tr}(\mathbf{A}) + 3]^{1/2}, \quad \text{with} \quad \mathbf{A} = \mathbf{H}\mathbf{B}_P\mathbf{H}^{\mathrm{T}}$$

represents the plastic strain in Eq. (38).

Internal Balances. The array $\mathcal{X}$ in Eq. (41) represents a special case of general set of internal variables $\{\xi^\alpha\}$ envisaged in numerous previous works [71,87,104,105] and summarized by others [21,22]. The indices $\alpha = 1, 2, \ldots$ are understood to be in one-to-many correspondence with the indices individual tensors in the set $\mathcal{X}$. When enslaved to the global kinematics, they are qualified elsewhere [106] as *passive* and dubbed "parameters". Here, we consider the possibility of an internal balance providing the necessary evolution equations.

Whenever the free energy ψ_E depends on these variables, the partial derivatives of the isotropically extended elastic potential by $\psi_E(\mathbf{B}_E, \mathcal{X})$

$$f_\alpha^E = \rho\partial_{\mathring{\xi}_E^\alpha}\psi_E, \quad \text{i.e.,} \quad \mathcal{F}^E = \{f_\alpha^E\} = \rho\partial_{\mathcal{X}}\psi_E \tag{44}$$

can be treated as conjugate thermodynamic forces, as done in the works cited above.

Certain investigators [87,104] treat the forces in Eq. (44) as strictly dissipative, and others [21,22,71,105] require that they also satisfy certain equilibrium-thermodynamic (Maxwell) relations. In the present work, we assume only that:

(1) Generalized forces may be assigned to internal variables [79,107]
(2) Conservative forces should generally be distinguished from dissipative forces by means of independent potentials, and
(3) Generalized velocities may be decomposed according to the classical incremental elastoplastic rule

$$\mathcal{U} = \mathcal{U}_E + \mathcal{U}_P \tag{45}$$

where

$$\mathcal{U} = \{u^\alpha\}, \quad u^\alpha = \mathring{\xi}^\alpha, \quad \text{i.e.,} \quad \mathcal{U} = \mathring{\mathcal{X}} \tag{46}$$

represent Jaumann rates as generalized velocities. While Eq. (45) is by no means the most general decomposition of velocities, it suffices to yield the hypoplastic forms adopted in this review.

To indicate their role as dual spaces and also to avoid unsightly juxtaposition of indices P, E, and α, β,..., we employ oppositely disposed indices to represent the corresponding velocities u^α and forces f_α, with elastic and plastic power given by forms like Eq. (11).

Then, by means of the inelastic potential $\psi_P(\mathbf{D}_P, \mathcal{U}_P)$, we can express strictly dissipative forces in terms of dissipative velocities as

$$f_\alpha^P = \partial_{u_P^\alpha}\psi_P \quad \text{or} \quad \mathcal{F}^P = \partial_{\mathcal{U}_P}\psi_P \tag{47}$$

provided there exist no additive "gyroscopic" or "powerless" forces of the kind admitted by Edelen's general theory. In this case, Eq. (47) represents a *strongly dissipative* or *hyperdissipative* material [56] assumed in this review.

Enlarging the set of generalized velocities and conjugate forces to include both internal and "control" variables [21], with

$$\mathcal{V} = \{v^\alpha\} = \{\mathbf{D}, \mathcal{U}\} = \{D^{ij}, u^\alpha\}$$
$$\text{and} \quad \mathcal{T} = \{T_\alpha\} = \{\mathbf{T}, \mathcal{F}\} = \{T_{ij}, f_\alpha\} \tag{48}$$

and assuming that $\psi_P(\mathcal{V}_P)$ is a convex function, we can express Eq. (47), as a pseudolinear (generalized Onsager) form [56] involving symmetric positive matrices as generalized secant moduli depending generally on $\mathcal{V}_P$ or $\mathcal{T}^P$,

$$T_\alpha^P = R_{\alpha\beta}^P v_P^\beta, \quad \text{i.e.,} \quad \mathcal{T}^P = \mathfrak{R}^P[\mathcal{V}_P] \quad \text{or} \quad \mathcal{V}_P = \mathfrak{L}_P[\mathcal{T}^P],$$
$$\text{with} \quad \mathfrak{R}^P = (\mathfrak{L}_P)^{-1} \tag{49}$$

Excluding the special case of rate-independent forces, where the inelastic potential ψ_P is homogeneous degree-one in rate, one can write the above moduli as

$$\mathfrak{R}^P = (\mathfrak{L}_P)^{-1} = \partial_{\mathcal{V}_P}^2\Phi_P, \quad \text{with} \quad \psi_P = \mathcal{V}_P\cdot\partial_{\mathcal{V}_P}\Phi_P - \Phi_P \tag{50}$$

A formula for Φ_P in terms of ψ_P and a dual form are given elsewhere [56].

In the exceptional case of rate-independent forces, we can simply write

$$\mathcal{V}_P = \mathfrak{L}_P[\mathcal{T}^P] = |\mathbf{D}_P|\mathfrak{N}_P[\mathcal{T}^P] \tag{51}$$

Strictly Dissipative Systems. In the case where all elastic effects are absent, we have $\mathcal{V}_E \equiv \mathbf{0}, \mathcal{V} \equiv \mathcal{V}_P$, and Eq. (49) gives an expression for the generalized velocity $\mathcal{U}$ and, hence, according to Eq. (46), an evolution equation for the internal variables $\mathcal{X}$ in terms of $\mathbf{D}$ or $\mathbf{T}$. This fact is ignored in certain treatments of evolutionary anisotropy [28,84], where an evolution equation for a fabric tensor $\mathbf{A}$ is postulated without allowing for the possible dependence of dissipation on the rate of change of $\mathbf{A}$. This appears tantamount to assuming a special form for dissipation, which may or may not be justified by the underlying physics. At any rate, it is a matter worthy of further investigation.

Viscoelastoplastic Systems. In the case of systems endowed with elastic energy, one may generalize Eq. (31), invoking internal force balances between forces in Eqs. (44) and (47), to obtain the following generalization of Eq. (31),

$$\mathcal{F}^E = \mathcal{F}^P = \mathcal{F} \quad \text{and} \quad \therefore \mathcal{T}^E = \mathcal{T}^P = \mathcal{T} \quad \text{for} \quad \mathcal{V}_P \neq \mathbf{0} \tag{52}$$

with inelastic velocities given by

$$\mathcal{V}_P = \mathfrak{L}_P[\mathcal{T}] = \rho\mathfrak{L}_P[\partial_{\mathcal{B}}\psi_E] \tag{53}$$

In order to obtain the evolution equations from Eqs. (45) and (46), one must further specify the elastic velocities

$$\mathcal{V}_E = \{\boldsymbol{\alpha}_E[\mathbf{D}_E], ..., \boldsymbol{\beta}_E[\mathbf{D}_E], ...\} \tag{54}$$

However, if one adopts the representation in Eq. (40), assuming that elastic torques associated with $\mathbf{H}$ vanish and that all remaining internal variables have the character of the structural tensors $\mathbf{A}$ and $\mathbf{b}$, then all their rates $\boldsymbol{\alpha}_E, ..., \boldsymbol{\beta}_E, ...$ in Eq. (54) may be assumed to vanish, so that $\mathcal{V}_E \equiv \mathbf{0}, \mathcal{V} \equiv \mathcal{V}_P$, and the above balance between dissipative and conservative elastic forces yields the evolution equation for internal variables,

$$\mathring{\mathcal{X}} = \mathcal{U}_P(\mathcal{T}), \quad \text{i.e.,} \quad \mathring{\xi}^\alpha = u^\alpha = \mathfrak{L}_P^{\alpha\beta}T_\beta = \mathcal{U}_P^\alpha(T_\beta),$$
$$\text{with}$$
$$\mathcal{T} = \{T_\beta\} = \rho\partial_{\mathcal{B}}\psi_E = \rho(2\mathbf{B}_E\partial_{\mathbf{B}_E} + \partial_{\mathcal{X}})\psi_E \tag{55}$$

the latter relation representing an elastic form like that introduced in Eq. (16).

By taking $\mathbf{D} = \mathbf{D}_P$ in Eq. (27), one obtains the evolution equation for stress

$$\mathring{\mathbf{T}} = \boldsymbol{\mu}_E(\mathcal{T})[\mathbf{D} - \boldsymbol{\varphi}(\mathcal{T}):\mathbf{T}] = \boldsymbol{\mu}_E(\mathcal{T})[\mathbf{D}] - \boldsymbol{\mu}_E:\boldsymbol{\varphi}(\mathcal{T}):\mathbf{T} \tag{56}$$

and it is clear that Eqs. (55) and (56) represent a set of *viscoelastic* evolution equations for $\mathcal{S} = \{\mathbf{T}, \mathcal{X}\}$ with $\mathbf{D}$ as control variable.

In the case of rate-independent plasticity, we take

$$\mathfrak{L}_P^{\alpha\beta} = \vartheta|\mathbf{D}|\mathfrak{N}_P^{\alpha\beta}, \quad \boldsymbol{\varphi} = \vartheta|\mathbf{D}|\boldsymbol{\kappa}_P, \quad \text{and} \quad \therefore \mathring{\mathcal{S}} = \mathfrak{M}(\mathcal{S}, \hat{\mathbf{D}})[\mathbf{D}] \tag{57}$$

where $\vartheta(\mathcal{S})$ is an inelastic-clock function of the type discussed above.

The last member of Eq. (57) represents a general statement of *parametric hypoplasticity*, which requires specification of an elastic potential, an inelastic potential, and a yield condition represented by ϑ.

In closing here, we observe that it is a relatively easy matter to derive an alternative version of hypoplasticity, in which the mixed variable $\mathcal{S} = \{\mathbf{T}, \mathcal{X}\}$ is replaced by the generalized force $\mathcal{T} = \{\mathbf{T}, \mathcal{F}\}$ in the final relation of Eq. (57).

Rigid Grains and Reynolds Dilatancy. In the case of rigid grains, we encounter a system devoid of characteristic stress or

time scales. This implies that the stress strain-rate relation must be homogeneous of degree zero in both quantities, representing cones in the respective dual spaces. In this singular limit, plastic dissipation is homogeneous degree one in either or stress or strain rate. Hence, the convex duality between stress and strain potentials breaks down, since the Lagrangian dual of a homogeneous potential of degree one must vanish identically. However, as anticipated above, the concept of dissipative modulus and compliance still holds [85], with

$$\mathbf{T} = p\boldsymbol{\mu}_C(\hat{\mathbf{D}}_P) : \hat{\mathbf{D}}_P \quad \text{and} \quad \hat{\mathbf{D}}_P = p^{-1}\boldsymbol{\kappa}_C : \mathbf{T} \qquad (58)$$

involving a nondimensional *Coulomb friction tensor* and its inverse.

In the case of rigid grains, the dilatancy constraint and, hence, the granular mass balance can be expressed, respectively, as [85]

$$\mathbf{D}_P : \hat{\mathbf{I}} \equiv \frac{1}{\sqrt{3}}\mathrm{tr}\mathbf{D}_P = \tan\phi_D|\mathbf{D}'_P| = \vartheta\sin\phi_D|\mathbf{D}|,$$
$$\text{with} \quad \hat{\mathbf{I}} = \mathbf{I}/\sqrt{3} \quad \text{and} \quad \dot{\phi} = -\mathrm{tr}\mathbf{D}_P\phi \qquad (59)$$

where $\phi_D = \phi_D(\boldsymbol{\mathcal{X}})$ is the *dilatancy angle* (unfortunately shown as the complementary angle in Ref. [85]).

The final equation in Eq. (59) represents one member of the set of evolution equations for $\boldsymbol{\mathcal{X}}$, namely, that governing granular volume fraction ϕ or void ratio $e = (1 - \phi)/\phi$. The constraint represents a cone in strain-rate space [85], with meridional plane depicted in Fig. 6, where $\mathbf{D}_P$ is represented by symbol $\mathbf{D}$. For simplicity, the cone is depicted as symmetric about the isotropic axis defined by $\hat{\mathbf{I}}$, although that need not be the case when there is strain-induced anisotropy and backstress, such as that often attributed to kinematic hardening [23].

The figure also illustrates the principle of constrained materials [30], according to which the stress in a material subject to internal constraints is determined only up to an additive *reactive stress* $\mathbf{T}^\flat$, which does no work in any deformation compatible with those constraints. The present author and coworkers [85] (and references therein) have long advocated the generalization to generally nonholonomic and evolutionary constraints of the form $\lambda(\mathbf{D}_P, \boldsymbol{\mathcal{X}}) = 0$.

Following a previous analysis [85], the foregoing principle may be stated as

$$\mathbf{T} = \mathbf{T}^\sharp + \mathbf{T}^\flat, \quad \text{where} \quad \mathbf{T}^\flat : \mathbf{D}_P = 0 \quad \text{for} \quad \lambda(\mathbf{D}_P, \boldsymbol{\mathcal{X}}) = 0 \quad (60)$$

where $\mathbf{T}^\sharp$ may be called the *coactive stress* (since it is the covector or dual of $\mathbf{D}$ and since the term *active* employed in previous works [85] conflicts with other usages.) It follows that $\mathbf{T}^\flat$ must be given
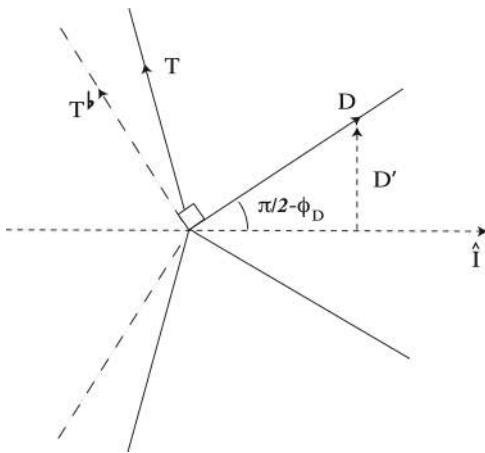
as a projection onto the tangent space of the constraint surface, i.e., as a vector orthogonal $\partial_\mathbf{D}\lambda$.

When the dilatancy constraint is given by the cone in Eq. (59), the required projection must have the general form [85]

$$\mathbf{T}^\flat = p(\boldsymbol{\delta} - \hat{\mathbf{D}}_P \otimes \hat{\mathbf{D}}_P) : \boldsymbol{\Lambda},$$
$$\text{with} \quad \hat{\mathbf{D}}_P = \cos\phi_D\hat{\mathbf{I}} + \sin\phi_D\widehat{\mathbf{D}}'_P \qquad (61)$$

where $\widehat{\mathbf{D}}'_P = \mathrm{dir}\mathbf{D}'_P$, p is the pressure, and $\boldsymbol{\Lambda}(\widehat{\mathbf{D}}'_P, \boldsymbol{\mathcal{X}})$ is a nondimensional tensor serving as Lagrange multiplier.

Now, it follows from a previous analysis [56] that Eq. (61) is given by the nondissipative part of the gradient of a dissipation potential ψ_P that is homogeneous degree one in $\mathbf{D}_P$, as guaranteed by rate-independence of stress, provided that $\boldsymbol{\Lambda}$ is symmetric. If so, then the same function represents the Edelen potential, giving the total stress in Eq. (60) as $\mathbf{T} = \partial_{\mathbf{D}_P}\psi_P$. Hence, provided the tensor $\boldsymbol{\Lambda}$ in Eq. (61) is symmetric, there exists no gyroscopic contribution to reaction stress.

The above results are in accord with conventional granular-plasticity models based on separate yield function and plastic potential, such as that considered by Ref. [23] (Sec. 4.3). However, in keeping with the stress-space formulation of this and similar works, it is necessary to restate the principle of constrained materials in a rather obvious dual form that is not shown here.

In closing this section, we point out that the *coefficient of dilatancy* α [85], the *dilatancy parameter* $\partial_p G$ of Nemat-Nasser [23] (Sec. 4.3.1), and the above dilatancy angle are related by

$$\alpha = -\sqrt{2}\partial_p(G/J) = \sqrt{3}\cot\phi_D \qquad (62)$$

since his stress potential g is expressed in the current notation as $g(\mathbf{T}, \boldsymbol{\mathcal{X}}) = J\varphi_P(\mathbf{T}, \boldsymbol{\mathcal{X}})/\sqrt{2}$, where φ_P is the stress potential of the present work.

In the model explored by Nemat-Nasser, $G = G(p, e, \gamma)$ and $\rho^{-1} = J = J(e)$, where γ is an accumulated plastic strain whose definition depends on the normal $\mathbf{N}$ to an evolving yield surface. However, in the case of rigid grains without stress scale, the function G must be linear in pressure p, such that e is independent of p. This is to be contrasted with all phenomenological models that involve dependence of e on p/p_0, where p_0 represents a stress scale whose micromechanical significance is generally unclear.

It should be emphasized that dilatancy and the cones in Fig. 6 are in general evolutionary and depend on parameters, such as particle volume fraction, accumulated plastic strain, and fabric. Also, we recall that the dilatancy vanishes asymptotically at a so-called critical state, which generally depends on the deformation history. Such asymptotic states are central to various "critical-state" ideas, including the recent model of "barodesy" [108].

3 Viscoplasticity

The formulation set forth above applies to a broad class of viscoelastic or viscoelastoplastic materials, for which we can identify both elastic potential or free energy and an inelastic potential that generally depend on various internal variables.

For example, viscoplastic fluids with "Bingham" yield behavior are described by a pseudolinear form proposed several years ago [67] as a special case of Eq. (26),

$$\mathbf{T} = \partial_\mathbf{D}\psi_P(\mathbf{D}) = \boldsymbol{\eta}_P[\mathbf{D}] = \boldsymbol{\mu}_P(\hat{\mathbf{D}})[\hat{\mathbf{D}}] + \boldsymbol{\eta}(\mathbf{D})[\mathbf{D}] \qquad (63)$$

which we now know can be derived from an inelastic potential ψ_P, with fluidity $\varphi_P = \boldsymbol{\eta}_P^{-1}$ given by a dual potential $\varphi_P(\mathbf{T})$. This general model can include viscous effects arising from the interstitial fluid in a granular medium or the suspending fluid in fluid-particle suspensions.

One may readily arrange Eq. (63) to a form more appropriate to granular media, with $\mathbf{T}'/p$ expressed as a function of a

Fig. 6 2D version of dilatancy constraint (after Ref. [85])

nondimensional deviatoric deformation rate $\mathbf{D}'/|\mathbf{D}'|_0$ and the dimensionless parameters ϕ, $\mathbb{I}$, and $\mathbb{H}$. One may also include the elasticity number $\mathbb{E}$ provided the sole effect of granular elasticity is to modify the dissipative response, as perhaps suggested by the work of Campbell [32] discussed above in Sec. 1.1. The evolution of granular volume fraction ϕ must generally be specified by a mass balance involving granular dilatancy [85].

By means of an isotropic extension of the type discussed in Sec. 2.1, we may express Eq. (63) as an isotropic tensor polynomial in $\mathbf{D}'$ and, e.g., a second-rank fabric tensor $\mathbf{A}$, as has been done for the special case of Stokesian suspensions [35,84]. Such forms are most readily derived from a inelastic potential ψ_P in Eq. (63) that depends on the isotropic scalar invariants of $\mathbf{D}'$ and $\mathbf{A}$, invariants whose general form is well known [81].

Thus, one may trivially construct an inelastic potential [56] for the plasticity model proposed by Sun and Sundaresan [28],

$$\mathbf{T}'/p = \mu(\phi, Z, \mathbf{A}:\widehat{\mathbf{D}}')\widehat{\mathbf{D}}' \qquad (64)$$

a generalization of the quasistatic limit of the model of Forterre and Pouliquen [10] cited below in Eq. (65), which in turn is a special case of the isotropic model proposed several years ago [34].

For the isotropic case, $\mathbf{A} = \mathbf{I}$, the above tensor polynomials reduce to second-degree polynomials, which may be employed to derive various conical yield surfaces by the method given elsewhere [34]. Again, such representations are most easily and reliably obtained from the derivative of an inelastic potential depending on the isotropic invariants $\mathrm{tr}\,\mathbf{D}'^2 = |\mathbf{D}'|^2$ and $\mathrm{tr}\,\mathbf{D}'^3$ or $\det\mathbf{D}'$.

This isotropic model includes, of course, the elementary isotopic form proposed by Forterre and Pouliquen [10]

$$\mathbf{T}' = p\mu_P(\phi, \mathbb{I}|\mathbf{D}'|/|\mathbf{D}'|_0)\widehat{\mathbf{D}}' \qquad (65)$$

which gives the classical Drucker–Prager circular yield cone and exhibits no viscometric normal stress, in contrast to the general isotropic model [34] or various anisotropic models depending on a fabric tensor [28,84]. In effect, Eq. (65) describes rate-dependent plastic creep for elastically stiff materials, corresponding to infinite μ_E in Fig. 5.

The existence of an intrinsic time scale in Eq. (65) allows for the prediction of nonflat velocity profiles, i.e., for departures from pluglike plastic flow. One can achieve similar results with multipolar models involving a length scale, as discussed below.

With simplified models, such as Eqs. (64) and (65), one should not expect to describe three-dimensional rheological effects, such as secondary circulations and free-surface deflections in open-channel flows [109,110] or the related "negative rod climbing" observed in Couette flows of dense suspensions [111]. It is even less clear that one should expect a unified description of such effects, let alone a unification of granular mechanics and suspension rheology [36], without more comprehensive models. Such models, exemplified by the general form in Eq. (63), may suggest at the very least what effect is being ignored by the overly simplified versions.

4 Multipolar Effects

For strongly inhomogeneous systems, such as the inhomogeneous shear field associated with shear bands, we expect to encounter departures from the response of the classical "simple" (i.e., nonpolar) material devoid of intrinsic length scale. The situation is generally characterized by nonnegligible magnitudes of a Knudsen number $\mathbb{K} = \ell/L$, where ℓ is a characteristic microscale and L is a characteristic macroscale, e.g., that defined by $L^{-1} \sim \nabla \log |()| \sim k$, with ∇ and k representing gradients or Fourier wave numbers of macroscopic fields (). Setting aside rarified granular gases, where ℓ should be identified with mean free path [37], we focus on dense systems.

In the case of elastoplasticity, the comprehensive review by Tejchman [14] of empirical parameters for soils suggests a microscopic scale of $\ell \sim 5\text{–}10d_{50}$, where d_{50} is median grain diameter. This scale is most plausibly associated with the length of the ubiquitous *force chains* in static granular assemblies. Thanks to the seminal work of Radjai et al. [112], it is now generally accepted that these represent the microscopic force network that supports granular shear stress.

It is safe to say that the micromechanics determining the length of force chains is still poorly understood. Recent works [113] attribute it to an Euler buckling instability, already postulated several years ago to explain departures from the Hertzian scaling displayed in Eqs. (4) and (5) [114]. Should the buckling mechanism prove robust, it might also suggest a source of mesoscopic or macroscopic instability implicated in shear-band formation. If so, this would establish a kinship to the buckling instabilities involved in the compaction bands discussed in Sec. 1.1, which in some cases may be attributed to the loss of elastic convexity, also discussed in that section.

Whatever the physical origins, a microscopic length scale serves as a parameter in various *enhanced continuum* models, including various *micropolar, micromorphic*, and higher-gradient models [115], all of which represent a form of *weak nonlocality* that we refer to by the blanket term *multipolar*. Here, we consider some of the simpler Cosserat models, often referred to as micropolar, and then a more general hypoplastic version.

Viscoplastic Forms. The seminal paper of Lippmann [107] provides a model of (perfect) Cosserat plasticity that treats plastic spin as Cosserat rotation. It has been applied to by Mohan et al. [116] to provide a length-scale resolution of wall slip in the quasistatic granular flow in channels. As noted in Sec. 3, this is resolved in a different manner by the time scale inherent to viscoplastic models.

The success of a recently proposed nonlocal model [117] in fitting steady various experimental velocity may point up the necessity of micropolar viscoplastic models having both length and time scales to mediate the transition between the quasistatic and dense-rapid flow regimes, especially in flows involving large-velocity gradients in thin layers. We note that the model of Ref. [117], which is formulated in terms of fluidity, can be written in the present notation as a simpler form of the tensorial model

$$\mathbf{D} = \boldsymbol{\varphi}:\mathbf{T} = \mathbb{I}|\mathbf{D}|_0\boldsymbol{\varphi}_P, \quad \text{with} \quad \ell^2\nabla^2\boldsymbol{\varphi}_P = \boldsymbol{\varphi}_P - \boldsymbol{\varphi}_P^\circ \qquad (66)$$

in which $\boldsymbol{\varphi}$ is replaced by a scalar. Here, the fluidity $\boldsymbol{\varphi}_P^\circ$ is nondimensional, with limiting value $\boldsymbol{\varphi}_P^\circ$ for $\ell = 0$ defined by a variant of the local viscoplastic model in Eq. (65). The ratio ℓ/d is given as a function of the stress ratio $|\mathbf{T}'|/p$, which tends to infinity as this ratio approaches the associated Drucker–Prager yield surface. Thus, yielding is defined by "jamming" at a diverging microscopic correlation length. Otherwise, the term involving ℓ in Eq. (66) obviously imparts a diffuse width to shear zones, and in this respect, the model bears a certain kinship to gradient-plasticity models involving similar diffusive effects [118]. At any rate, the model of Ref. [117] seems to offer a considerable improvement on the "nonlocal" model of Ref. [119], from which this author has been unable to disentangle a definitive multipolar constitutive equation.

To the extent that the plasticity and viscoplasticity models of the preceding paragraphs are strongly dissipative, they are once more susceptible to representation by a inelastic potential, which can be constructed from the corresponding dissipation functions by Edelen's method [56]. Moreover, both must correspond to the dissipative limit of more general hypoplastic models of a type that may be necessary to describe evolutionary effects occurring in transient flows.

Cosserat Hypoplasticity. The monograph of Tejchman [11] provides a comprehensive summary of a fairly general form of

Cosserat hypoplasticity, which merits only a slight elaboration here. Suffice it to note that it is subsumed by the hypoplastic form in Eq. (57) by enlarging the set of variables to $\mathcal{S} = \{\Sigma, \mathbf{M}, \mathcal{X}\}$, where the second-rank tensor Σ represents an asymmetric stress, with $\mathbf{T} = \mathrm{sym}\Sigma$ denoting the symmetric Cauchy stress conjugate to $\mathbf{D}$; $\mathbf{S} = \mathrm{skw}\Sigma$ a skew-symmetric stress, whose vector $\mathbf{s}$ is conjugate to the vector of Cosserat spin $\mathbf{w}$; and $\mathbf{M}$ the second-rank moment stress conjugate to the torsion $\nabla \mathbf{w}$. Thus, the expression for stress power is

$$\mathcal{P} = \mathbf{T}:\mathbf{D} + \mathbf{s}\cdot\mathbf{w} + \mathbf{M}:\nabla\mathbf{w} \qquad (67)$$

As opposed to the internal balances discussed above, Cosserat mechanics is subject to linear and angular momentum balances that follow from virtual work principles based on internal power in Eq. (67) and related boundary conditions.

We note that the Cosserat balances can be cast into a compact complex-variable form, such that Cosserat mechanics reduces to *complex Cauchy mechanics* [106]. We do not address the issue as to whether plastic spin is generally to be regarded as internal variable or as an extrinsic variable subject to control at bounding surfaces.

A Restricted Cosserat Model. The microscopic origins of the skew-symmetric stress $\mathbf{S}$ is one of the more puzzling and controversial aspects of the quasistatic mechanics of granular media. The most convincing micromechanical models [120] require granular contact moments in order to obtain continuum level stress asymmetry in the absence of body couples. However, the relatively small dimensions of the typical Hertzian or quasi-Hertzian contact zone, given, e.g., by the estimate in Eq. (5) as $a/d \sim \mathbb{E}^{-1/3}$, suggests that contact moments, which scale as a/d relative to the moments associated with contact forces, should be entirely negligible.

While the theoretical analysis of Ref. [121] (and later defense thereof [122]) as well as simulations like those of Ref. [42] show weakly asymmetric stress in confined regions, other analyses [115] indicate that the static moment balance on individual particles should rule out stress asymmetry *in regions of any size*, at least according to the standard formula for the macroscopic stress.

There are at least two plausible explanations for the above state of affairs:

(1) The standard formula for stress may break down for small samples [115] or
(2) Dynamically unstable microscopic states may lead to a dissipative, rate-independent asymmetric stress, analogous to the dissipative stress revealed by the simulations of Peyneau and Roux [62] discussed above.

Neglecting the stress asymmetry, one obtains a restricted form of Cosserat mechanics, in which grains may be construed to rotate as quasistatic "idlers" devoid of rotational couple, as is the case of the torque-free particles in a viscous fluid referred to in conjunction with the moment balance in Eq. (33). It this case, the moment stress $\mathbf{M}$ may still be nonzero, satisfying the static moment balance

$$\nabla\cdot\mathbf{M} = \mathbf{0} \qquad (68)$$

as a companion to static Cauchy-stress equilibrium. A forthcoming paper [51] undertakes a study of the field equations for the above model (which that author has informally characterized as "Cosserat-B" in informal communications).

Although not contributing directly to stress power, the mean particle rotation must nevertheless be taken into account whenever the effects on texture or fabric arising from mean orientation of (nonspherical) particles is important. Moreover, this rotation must be specified by constitutive equations for particle spin like that in Eq. (25). This is analogous to the tracking of plastic spin [82] as

an orientational effect, even though no asymmetric stress is involved.

Particle Migration. Various models of particle migration in suspensions [84] or size segregation in granular media [39,40,123] involve various diffusionlike terms that suggest multipolar effects. While some models [40] involve gravitational driven sedimentation opposed by diffusional remixing, others [123] show a direct effect of gradients in shearing akin to those found in suspensions [84]. It is perhaps significant that many granular size-segregation effects are associated with dense flow in thin layers, which again suggests the likelihood of Knudsen number or multipolar effects.

Whatever the origins of particle migration, it can probably be treated as a strictly dissipative process, implying that it also can be represented as a generalized velocity in a dissipation potential. If so, then this suggests a convenient way of formulating properly invariant constitutive relations [84].

Relevance to Material Instability. The limitations of space and time do not allow for a meaningful review of instabilities in granular flow. As pointed out by Ref. [8], there is a bewildering variety of such instabilities. These range from the shear-banding instabilities in quasistatic flow discussed above in the Introduction, to gravitational layering in moderately dense rapid flow [124], to clustering instabilities in granular gases [37].

To clarify matters, the author advocates a distinction between *material or constitutive instability*, representing the instability of homogeneous states in the absence of boundary influences [46] and the dynamical or geometric instability that occurs in materially stable media, such as elastic buckling, inertial instability of viscous flows, etc. With this distinction, it becomes much easier to assess the importance of multipolar and other effects.

Pioneering works on the subject [57] reveal the multipolar effects on elastoplastic instability, not only on postbifurcation features, such as the width of shear bands, but also on the material instability itself. This represents an interesting and challenging area for further research based on the parametric viscoelastic or hypoplastic models of the type discussed above. The general question is whether and how the length scales that lend dimensions to spatially patterned states enter into the initial instability giving rise to those states.

As discussed in the review [46] and the monograph [8] (Sect. 8.4), the linear instability of homogeneous base states of a nonpolar fluid model is reducible to a set of ODEs, whose stability matrix is given as a polynomial in disturbance wave number with time-dependent matrix coefficients. As indicated in Ref. [46], the above representation should carry over to multipolar models, with higher-order polynomials in wave number representing the effects of higher spatial gradients. This offers the possibility of greatly simplified analytical or numerical studies of linear instabilities, but such techniques have yet to be applied to elastoplastic media. It is possible that weakly nonlinear stability analyses can also be carried out for the simpler constitutive models in order to assess the effects of disturbance amplitude.

5 Conclusions and Questions

As summarized in the abstract, a survey has been given of certain salient phenomenological aspects of granular flow, along with a unified mathematical synthesis of current continuum models. Parametric viscoelasticity and hypoplasticity provide a unified description based on elastic and inelastic potentials, like those assumed in the classical models of plasticity. The inelastic potentials follow without assumption from the work of Edelen, and they apply to viscoplasticity as well as ordinary plasticity.

To achieve a fully nonlinear anisotropic viscoelastoplasticity, it is assumed that the elastic or inelastic potentials may, by isotropic extension, be expressed in terms of the joint isotropic invariants

of elastic strain or inelastic strain rate and an appropriate set of structural tensors. The latter represent evolutionary internal variables or parameters, whose evolution equations are based on an assumed internal balance of generalized forces derived from the elastic and inelastic potentials.

The resulting continuum models encompass most of the special constitutive models that have been proposed for granular media, and these models are readily modified to include Cosserat and other multipolar effects.

Several conclusions arise from the present study:

(1) Hypoelasticity does not encompass classical plasticity, whereas the latter is easily represented by means of parametric hypoplasticity.
(2) Most existing models of granular rheology are based on highly simplified versions of hypoplasticity that are incapable of describing certain phenomena, such as noncircular yield cones and viscometric normal stress.
(3) Many features of plasticity and granular plasticity follow mathematically from the existence and convexity of inelastic potentials, requiring no special physical arguments.
(4) Loss of convexity in potentials can give rise to both inelastic and elastic material instabilities.
(5) The simplest forms of material instability should be susceptible to linear stability analyses based on the general models discussed in this review.

And several major questions may be identified, including:

(1) What is the physical significance of the characteristic stress scales appearing in various phenomenological models of granular media and soils?
(2) Are the typical elastoplastic instabilities in granular media quasielastic or inelastic in nature? To what extent are they modified by the multipolar effects defining the spatial scales of the resulting patterns?
(3) Does the rate of change of internal variables, such as granular volume fraction and fabric, contribute significantly to dissipation, requiring their inclusion in the inelastic potentials, with assignment of internal forces?
(4) What class of composite media if any are driven quasistatically to simultaneously minimize both elastic and inelastic potentials?
(5) Are multipolar models required to model dense rapid flows in thin layers and phenomena, such as particle migration and granular segregation?

It is hoped that this review will serve as motivation and perhaps as a guide to further work on these and related questions.

Acknowledgment

Nomenclature
General

$\mathbf{a} \otimes \mathbf{b}, \mathbf{a} \otimes \mathbf{A}, \mathbf{A} \otimes \boldsymbol{\beta}, \ldots =$ tensor products

$\mathbf{a} \bullet \mathbf{b}, \mathbf{A} \bullet \mathbf{B}, \ldots =$ bilinear pairing based on given metric

$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}, |\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A}}, \ldots =$ Euclidean norms

$\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}\mathbf{B}^{\mathrm{T}}) =$ Euclidean scalar product and contraction

$\widehat{\mathbf{A}'} = \mathrm{dir}(\mathbf{A}') = \mathrm{dir}(\mathrm{dev}\mathbf{A}) =$ director of the deviator

$\mathbf{Ab}, \ldots =$ second-rank tensor as linear map $R^3 \to R^3$

bl. and bu. $=$ bold lower case and upper case in this list

bl. Fraktur $\mathfrak{f}, \mathfrak{g}, \ldots =$ tensor-valued functions

bl. Greek or Hebrew $\boldsymbol{\alpha}, \boldsymbol{\beta}, \ldots \aleph, \ldots =$ tensors of rank > 2

bl. Roman $\mathbf{a}, \mathbf{b}, \ldots =$ vectors (rank-1 tensors) $\in R^3$

bu. Fraktur $\mathfrak{A}[\quad], \mathfrak{B}[\quad], \ldots =$ linear maps of arrays

bu. Roman or Greek $\mathbf{A}, \mathbf{B}, \Gamma, \ldots =$ rank-2 tensors $\in GL(3)$

bu. script $\mathcal{A}, \mathcal{B}, \ldots =$ abstract arrays of tensors

$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} =$ (right and left) Cauchy–Green strains

$\det \mathbf{A}, \mathrm{sym}\,\mathbf{A}, \text{ and } \mathrm{skw}\,\mathbf{A} =$ determinant, sym., and skew parts

$\mathbf{D} = \mathrm{sym}\mathbf{L}, \dot{\gamma} = |\mathbf{D}'|, \mathbf{W} = \mathrm{skw}\mathbf{L} =$ straining, shearing, spin

$\mathbf{F} = \partial \mathbf{x}/\partial \mathbf{x}°, J = \det \mathbf{F}, \mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \nabla \mathbf{v}^{\mathrm{T}} =$ kinematic tensors

$\mathbf{I}, \mathbf{AB}, \text{ and } \mathbf{A}^{\mathrm{T}}, \ldots =$ second-rank identity, product, and transpose

$\mathrm{tr}(\mathbf{A}) = \mathbf{I} : \mathbf{A}, \mathbf{A}' = \mathrm{dev}(\mathbf{A}) = \mathbf{A} - (\mathrm{tr}\mathbf{A}/3)\mathbf{I} =$ trace and deviator

$\mathbf{T}, p = -\frac{1}{3}\mathrm{tr}\mathbf{T}' \text{ and } \tau = |\mathbf{T}'| =$ Cauchy stress, pressure, and shear

$\mathbf{U} = \sqrt{\mathbf{C}}, \mathbf{V} = \sqrt{\mathbf{B}}, \mathbf{R} = \mathbf{F}\mathbf{V}^{-1} = \mathbf{U}^{-1}\mathbf{F} =$ stretch and rotation

$\mathbf{x} \text{ or } \mathbf{x}(\mathbf{x}°, t) =$ spatial position or material placement

$\boldsymbol{\alpha}[\quad], \boldsymbol{\beta}[\quad], \ldots =$ associated linear or pseudolinear forms

$\boldsymbol{\alpha}° \boldsymbol{\beta} = (\boldsymbol{\alpha}° \boldsymbol{\beta})[(\quad)], \ldots =$ product of compatible maps

$\boldsymbol{\delta} \hat{=} [\delta_{ijkl}] = [\delta_{ik}\delta_{jl}] =$ fourth-rank identity: $GL(3) \to GL(3)$

$()^{(\mathrm{s})} \text{ and } ()^{(\times)} =$ symmetrization and skew symmetrization

$() \hat{=} [()_{ijk\ldots}] \hat{=} [()_{ij\ldots}^{kl\ldots}] =$ components on given basis

$\widehat{()} = \mathrm{dir}() = ()/|()| =$ director (or "versor") of tensors

$()^{\doteq} d_t() = (\partial/\partial t)_{\mathbf{x}°}() =$ (Lagrangian) material rates

$()^{\doteq} (\partial_t + \mathbf{v} \cdot \nabla)(), \mathbf{v} = \dot{\mathbf{x}} =$ Eulerian material rates

$()° \text{ and } \mathbf{x}° =$ referential value and material-point position

$\partial_t = (\partial/\partial t)_{\mathbf{x}} =$ (Eulerian) time rates of change

Special

$d, G_s =$ representative grain diameter and elastic modulus

$e_c, p_s =$ collisional restitution coeff. and confining pressure

$\mathbb{E}, \mathbb{I}, \mathbb{H}, \mathbb{K} =$ elastic, inertial, viscous, and Knudsen numbers

$\mathfrak{L}, \mathfrak{R} =$ generalized conductance and resistance matrices

$\mathfrak{M} =$ generalized hypoelastic or hypoplastic modulus

$\mathbf{M}, \mathbf{N} =$ normals to plastic potential and yield surfaces

$Z, \mathbf{A}$ = granular coordination number and fabric tensor

$\gamma, \dot{\gamma}, |\mathbf{D}'|_0$ = shear, shear rate, and characteristic shear rate

$\boldsymbol{\eta} = \boldsymbol{\varphi}^{-1}$ = fourth-rank viscosity as inverse fluidity tensor

η_f, ϕ_μ = interstitial viscosity and granular friction angle

$\boldsymbol{\kappa}[()] = \boldsymbol{\kappa} : ()$ = fourth-rank compliance: $GL(3) \rightarrow GL(3)$

$\boldsymbol{\mu}_E = \boldsymbol{\kappa}_E^{-1}, \boldsymbol{\mu}_P = \boldsymbol{\kappa}_P^{-1}$ = elastic and plastic modulus and inverse

$\boldsymbol{\mu}[()] = \boldsymbol{\mu} : ()$ = fourth-rank modulus as map $GL(3) \rightarrow GL(3)$

ρ_s, μ_s = grain density and intergranular friction coefficient

ϕ, n_c = granular volume fraction and contact density

ψ and φ = Helmholtz or Edelen potential and dual

$()_E$ and $()_P$ = elastic and plasticoviscous contributions

ϑ, ϑ_P = plastic "clock functions"

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