Continuum modelling of spatial and dynamic equilibrium in a travel corridor with heterogeneous commuters—a partial differential complementarity system approach

Wang, David Zhiwei; Du, Bo

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Continuum modeling of spatial and dynamic equilibrium in a travel corridor with heterogeneous commuters - a partial differential complementarity system approach

David Z. W. Wang[*], Bo Du

School of Civil and Environmental Engineering, Nanyang Technological University, 50 Nanyang Avenue, 639798, Singapore

Abstract

This paper studies on modeling and solving spatial and dynamic equilibrium travel pattern in a travel corridor. Consider a travel corridor connecting continuously distributed commuters to the city center. The traffic is subject to flow congestion and the commuter heterogeneity is captured. The traffic flow dynamics is described by flow continuity equation and the equilibrium travel pattern is assumed to follow trip-timing condition. The continuous spatial and dynamic equilibrium travel pattern is formulated into a partial differential complementarity system, which is then solved through Godunov scheme. The proof of solution existence is provided, and a set of numerical experiments are demonstrated.

Keywords

Dynamic user equilibrium, Corridor problem, Partial differential complementarity system, Heterogeneous commuters

1. Introduction

Equilibrium dynamics of peak-hour traffic congestion has received much research attention since the pioneer work of Vickrey (1969), wherein a deterministic queueing model was developed to describe a single bottleneck. As an extension to Vickrey (1969), Newell (1987) assumed different values to queueing delay and schedule delay when considering travellers with different work starting time in the morning commute problem. The result showed that most travellers choose to arrive at workplace on time, and the queueing patterns were quite different from those in the previous studies of identical travellers. Arnott et al. (1990) investigated the commuters' route and departure time choice behaviour on a simple network following user equilibrium and system optimum principles, and the efficiency of various tolls was analysed as well. Liu and Nie (2011) extended the bottleneck model by considering route choice, heterogeneous users, no-toll equilibrium, two system optima and related transportation economics analysis. Han et al. (2011) formulated a single origin-destination dynamic user equilibrium (DUE) problem as a complementarity problem based on cell transmission model (CTM) (Daganzo, 1994, 1995), wherein two methods, maximum and average travel time, were used to estimate the travel time. Gonzales and Daganzo (2012) studied a morning commute problem with auto and public transit modes, considering commuters heterogeneity and distributed demand. It showed that public transit was a Pareto improvement and it reduced the duration of the rush hours and the overall cost. Pang et al. (2012) applied a linear complementarity system approach for a DUE model with single bottleneck, which was approximated by a time-stepping discretization scheme and solved by Lemke's algorithm. Han et al. (2013a, b) conducted a series of studies to reformulate the classic single point queue model as a partial differential equation (PDE) and obtained the explicit solution via a variational method. Yang et al. (2013) investigated how the morning commute problem with consideration of bottleneck congestion and parking space constraints simultaneously differed from the traditional bottleneck model. They determined certain proportion of the two classes (with/without reserved parking spot) of commuters that can reduce the traffic congestion as well as the total system cost. Raadsen et al. (2015) proposed an event-based algorithm to solve simplified first-order dynamic network loading problem, which could generate exact, as well as approximate solutions.

However, very few of the research works investigated the spatial and temporal dynamics of the peak-hour traffic congestion simultaneously. Newell (1988) extended the bottleneck problem to a travel corridor context by applying the Lighthill-Whitham-Richards (LWR) traffic flow regulation (Lighthill and Whitham, 1955; Richards, 1956). Arnott and DePalma (2011) studied the corridor problem by applying an analytical approach to

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[*] Corresponding author. Tel.: +65 67905281; fax: +65 67905281.
  E-mail addresses: wangzhiwei@ntu.edu.sg (David Z.W. Wang).

solve the equilibrium spatial dynamics of traffic congestion in a travel corridor. Classical flow congestion was assumed and the equilibrium travel pattern satisfied the trip-time condition. The solution of the model was determined by deriving a necessary condition of the problem solution, i.e., at each location, the departure rate was constant over the interior of the departure set. Later, DePalma and Arnott (2012) analysed a single entry corridor problem. It presented closed-form solutions for the social optimum and quasi-analytic solutions for the user optimum. Indeed, the sought-after model of solving spatial dynamics of equilibrium travel pattern and traffic congestion is very important in paving the way to solve many other interesting problems, such as optimal time-distance based road toll scheme to achieve system optimal, multi-modal travel equilibrium and transportation system design, understanding how urban density distribution affects the transportation system performance.

In this paper, we investigate a similar corridor problem as defined in Arnott and DePalma (2011). Consider a travel corridor connecting the continuum residential locations to the central business district (CBD), which is subject to traffic congestion. The equilibrium travel pattern in the travel corridor is assumed to follow trip-timing condition, i.e., commuters choose their departure times to minimize their individual travel cost. Moreover, it is assumed that the travel demand along the corridor is exogenously given. Heterogeneous commuters are considered, with different value of time and scheduling delay. This paper formulates a mathematical model to solve the spatial and dynamic equilibrium travel pattern in the corridor, basically, to answer what departure rates satisfy the equilibrium condition across the corridor? The key to address this problem lies in how to integrate the modelling of flow congestion dynamics and commuters' departure time choice into one single modelling framework. In this study, we will employ a partial differential complementarity system approach to model the corridor problem, wherein classical flow congestion model combining the equation of continuity with an assumed traffic speed-density relationship, as well as the travel time interrelation among commuters, is captured by PDEs; while complementarity conditions are applied to describe commuters' departure time choice behaviour following trip-timing condition. The differential complementarity system approach employed in this study belongs to a big class of modeling paradigm of differential variational inequalities (DVI), introduced in Pang and Stewart (2008), which was applied in many research works (Ban et al., 2012a, b; Pang et al., 2012; Wang and Du, 2013; Du and Wang, 2014; Wang and Du, 2015; Wang and Xu, 2016). The formulated model is then solved through a discretization scheme and the results are demonstrated in the numerical examples.

This study mainly aims to apply the partial differential complementarity system approach to model and solve the "corridor problem", i.e., to model and solve the spatial traffic congestion dynamics along a continuous travel corridor. In the corridor problem, it is assumed that travel demand is continuously distributed along the corridor, and travellers could enter the travel corridor through continuously distributed entry points. Therefore, the continuum modeling approach, rather than the discrete one, is applied in modeling the "corridor problem". Indeed, the corridor problem, as well as the adopted continuum modeling approach, is different from the conventional study of DUE and dynamic traffic assignment on a concrete discrete transportation network, and more suitable for the initial phase of planning in regional study, when there is insufficient data of the system for setting up a dense network for detailed analysis using the discrete modeling approach. The analysis result of the corridor problem has more policy implications on such issues as how the urban development density distribution planning may affect the corridor traffic flow congestion, which is more important question to be answered in the initial phase of corridor transportation planning. More applications of the corridor problem include the urban economics issues like better understanding the economics of traffic congestion and residential location choice behaviour, etc. To summarize, this study carries the work in Arnott and DePalma (2011) a step forward by applying partial differential complementarity system approach to modeling and solving the corridor problem. Extensions are made in the following aspects: firstly, the model formulation allows for the consideration of heterogeneous commuters in analysing the corridor problem. In reality, commuters may be classified into different groups with different preferred arrival times, values of time and scheduling delay, etc., therefore the heterogeneity assumption has to be incorporated in the model to capture the realistic equilibrium travel pattern. Secondly, the model formulation enables the adoption of well-established complementarity theory and methods to prove the equilibrium solution existence and obtain a complete numerical solution of the corridor problem. As a relevant study, in Arnott and DePalma (2011), a solution scheme is proposed for solving the corridor problem, which is based on implication of trip-timing conditions, i.e., the departure rate is constant over the interior of the departure set. However, the solution is incomplete, as was pointed out by the authors, because the equilibrium solution exists only for some specific population distribution along the road. In this study, although the model formulation is only solved in its discretized approximation form, the proof of solution existence is provided by employing complemetrarity theory and complete numerical solution is obtained. Thirdly, in considering the scheduling cost, late arrival dealy is explicitly accounted in this model. In Arnott and DePalma

(2011), only early arrival is allowed. However, late arrival is rather common in reality, and the consideration of late arrival would make the model more relastic, for example, in analyzing how congestion mitigation measure like flexible work time scheme or various work starting time scheme (late arrival is allowed) may affect the peark-hour traffic congestion.

The reminder of this paper is organized as follows. In section 2, we present the continuous-spatial-temporal equilibrium travel pattern model formulation in a travel corridor. In section 3, the continuous model is approximated through a discretization scheme, and the proof of solution existence is illustrated. Section 4 provides a set of numerical examples. Finally, section 5 concludes with short discussion.

2. Continuum spatial-temporal instantaneous dynamic user equilibrium model

Consider a travel corridor connecting the continuous residential locations to the city centre. Here, we consider the travel corridor in a one-dimensional domain. It is assumed that trip demand density along the corridor is given exogenously, vehicles are identical, and all commuters at different locations choose departure times to minimize individual travel cost to go to CBD for work, i.e., following the trip-timing condition. The preferred arrival time is given and both early arrival and late arrival are allowed with different scheduling cost penalties. Note that all the variables and parameters in this paper are nonnegative. Suppose the study horizon is $\overline{T}$, the length of corridor is L, and the location and time are indexed as x and t respectively. It should be noted that x here represents the distance between the location to the city boundary, and the boundary of the corridor is located at $x = 0$ and the CBD lies at $x = L$. Then, we develop a model to describe the continuous spatial-temporal travel equilibrium. Due to the dynamic features of traffic flow, the formulation is based on a time-space plane.

2.1. Flow propagation

For almost all $t \in [0, \overline{T}]$, $x \in [0, L]$,

$$\frac{\partial f_t^x}{\partial x} + \frac{\partial k_t^x}{\partial t} = r_t^x \tag{1}$$

$$f_t^x = k_t^x u_t^x(k_t^x) \tag{2}$$

$$u_t^x(k_t^x) = -a k_t^x + b \tag{3}$$

where

k_t^x - the traffic density at location x and time t, $0 \leq k_t^x \leq k_{max}$;

f_t^x - the traffic flow at location x and time t, $0 \leq f_t^x \leq f_{max}$;

u_t^x - the velocity of vehicle at location x and time t, $0 \leq u_t^x \leq u_{max}$;

r_t^x - the departure rate at location x and time t.

Eq. (1) entails equation of continuity for traffic flow, stating the flow conservation law on the highway, which is indeed the key postulate of LWR theory. The LWR model is a well-known PDE, which can be solved with proper initial/boundary conditions. Eq. (2) and Eq. (3) describe the flow-velocity-density relationship. Let k_{max} represent the maximum/jam density, u_{max} denote the maximum/free-flow speed and f_{max} represent the maximum flow capacity, and for simplicity, we assume the maximum density k_{max} is identical and constant at each location along the corridor, so are the maximum speed u^{max} and maximum flow capacity f_{max}, and this assumption is easy to be relaxed without theoretical obstacles. For simplicity, we apply the Greenshields model (Greenshields, 1935), $u(k) = u_{max}(1 - \frac{k}{k_{max}})$, and express it in a succinct form as Eq. (3) with $a = \frac{u_{max}}{k_{max}} > 0$, $b = u_{max} > 0$. It is easy to obtain the maximum flow capacity f_{max} by combining Eq. (2) with Eq.

(3); the velocity at jam density is zero: $u(k_{max})=0$; the velocity is equal to the free flow speed when the density is zero: $u(0)=u_{max}$. Moreover, substitute Eq. (3) into Eq. (2) to obtain the following equation:

$$f_t^x = -a(k_t^x)^2 + bk_t^x \tag{4}$$

The standard first-order LWR model without source term is given as $\dfrac{\partial f_t^x}{\partial x} + \dfrac{\partial k_t^x}{\partial t} = 0$, which is a hyperbolic

PDE and can be solved with proper initial/boundary data as a Riemann problem. Due to the existence of shock wave in reality, a weak solution to LWR model is often applied. Mathematically, a weak solution is a function $(k,f)(x,t)$ that satisfies $\dfrac{\partial f_t^x}{\partial x} + \dfrac{\partial k_t^x}{\partial t} = r_t^x$ everywhere except at a certain $x(t)$, and $(k,f)(x,t)$ is discontinuous

but obeys the integral forms of the conservation law on $x(t)$ (Gartner et al., 2001). The complete theories of LWR model and the hyperbolic system of conservation law can be referred to Daganzo (1997); Bressan (2000); Gartner et al. (2001); Garavello and Piccoli (2006); Dafermos (2009).

2.2. Instantaneous travel time cost

In the continuum formulation of the travel corridor, traffic flow f_t^x can be regarded as the outflow from location x at time t to neighbouring downstream location $x+dx$, as well as the inflow into location $x+dx$. Therefore the inflow at location x consists of new departure r_t^x and the flow from neighbouring upstream f_t^{x-dx}, and the outflow at location x is equal to f_t^x. Considering the fact that the new departure may not be satisfied due to the flow capacity constraint, we devise a simple mechanism to capture this additional queueing delay for new departures. Thus the total travel time for users departing at time t from location x to CBD consists of three parts: travel time on highway TT_t^x, queueing time $\dfrac{Q_t^x}{f_{max}}$ and schedule delay, where Q_t^x represents the number of vehicles in the queue or queue length at location x at time t.

For almost all $t \in [0,\overline{T}]$, $x \in [0,L]$,

$$\frac{\partial TT_t^x}{\partial x} = -\frac{1}{u_t^x(k_t^x)+\sigma} \tag{5}$$

$$\frac{\partial Q_t^x}{\partial t} = \eta_t^x + r_t^x + f_t^{x-dx} - f_{max} \tag{6}$$

$$0 \le \eta_t^x \perp Q_t^x \ge 0 \tag{7}$$

The partial differential equation (5) reflects the spatial interaction of the travel time along the corridor, where TT_t^x represents the instantaneous travel time for the commuters departing from location x at time t to CBD. In this study, it is assumed that the boundary of the corridor is located at $x=0$ and the CBD lies at $x=L$, thus $\dfrac{\partial TT_t^x}{\partial x} = \lim\limits_{dx\to 0}\dfrac{TT_t^{x+dx}-TT_t^x}{dx} < 0$ and $TT_t^x = TT_t^{x+dx} + \dfrac{dx}{u_t^x(k_t^x)+\sigma}$, and it is reasonable to assume that travel time is strictly positive. σ is an extremely small and positive dummy speed value to refrain from zero denominator. To capture the queueing time for new departures, a simple point queue model is applied here: if the departure demand exceeds the maximal flow capacity, queue arises and the queue length is described as $\dfrac{\partial Q_t^x}{\partial t} = r_t^x + f_t^{x-dx} - \begin{cases} \min\{r_t^x + f_t^{x-dx}, f_{max}\}, & \text{if } Q_t^x \le 0 \\ f_{max}, & \text{if } Q_t^x > 0 \end{cases}$. An equivalent complementarity system approach as in (6) with (7) is adopted to describe the dynamics of the queue at time instance t, wherein the notation "$\perp$" means "perpendicular" (i.e., vectors $a \perp b \Leftrightarrow a^T b = 0$), and η_t^x is an auxiliary variable to determine whether there is a positive queue or not, and these functions guarantee the non-negativity of the queue length Q_t^x. Specifically, when $Q_t^x > 0$, we have $\eta_t^x = 0$ from (7), then the outflow f_t^x is at its maximal capacity f_{max} with queue length

calculated by the equation $\frac{\partial Q_t^x}{\partial t} = r_t^x + f_t^{x-dx} - f_{max}$; otherwise if no queue, then the outflow f_t^x is determined by the LWR model with the auxiliary variable η_t^x, as described in Eqs. (1) - (3). By doing so, the queueing time due to the capacity constraint is explicitly considered in the model formulation. Interested readers can refer to Ban et al. (2012a); Pang et al. (2012); Han et al. (2013c) for further analysis and the proof of the continuous dependence of this point queue model. It should be noted that, we assume an average queueing time for all the travellers at one specific location. To model a more realistic travel cost, one may need to take into account the situation that some of the travellers may not be accommodated immediately into the highway, which would be addressed in the future study.

2.3. Trip-timing condition

For almost all $t \in [0, \overline{T}]$, $x \in [0, L]$,

$$\int_0^{\overline{T}} r_w^x dw = D_x \tag{8}$$

$$0 \le r_t^x \perp \alpha(TT_t^x + \frac{Q_t^x}{f_{max}}) + \beta max(0, \tilde{t} - t - TT_t^x - \frac{Q_t^x}{f_{max}}) + \gamma max(0, t + TT_t^x + \frac{Q_t^x}{f_{max}} - \tilde{t}) - \pi^x \ge 0 \tag{9}$$

Eq. (8) ensures that integration of trip departure rates over the time horizon amounts to the given travel demand D_x at any location x. The complementarity condition (9) guarantees the trip-timing condition that no commuter can experience a lower travel cost by departing at a different time, wherein π^x denotes the minimum travel cost for commuters departing from location x, and $\tilde{t}$ denotes the preferred arrival time. The travel cost consists of both travel time cost and scheduling delay costs. Specifically, $max(0, \tilde{t} - t - TT_t^x - \frac{Q_t^x}{f_{max}})$ and $max(0, t + TT_t^x + \frac{Q_t^x}{f_{max}} - \tilde{t})$ represent the early and late arrival delay respectively, and α, β, γ denote the value of time (VOT) for travel time, early arrival delay and late arrival delay, respectively. It is reasonable to assume that $\gamma > \alpha > \beta$. It is noted that travellers' choice behaviour other than rational behaviour (like bounded rationality, Wu et al. (2013)) is not considered in this paper and could be further addressed in the future study.

Moreover, negative exponential distribution function is commonly used to describe the fact that population density, as well as travel demand density, declines from the CBD towards the city boundary. For illustration purpose, we apply Eq. (10) to depict the spatial distribution of the travel demand along the travel corridor. It should be noted that many other distribution functions can also be applied without theoretical obstacles.

$$D^x = D^0 e^{-\phi(L-x)} \tag{10}$$

where D^0 denotes the demand density at CBD and ϕ is a parameter. Eq. (10) can be substituted into Eq. (8) and expressed as a complementarity condition readily as follows:

$$0 \le \pi^x \perp \int_0^{\overline{T}} r_w^x dw - D^0 e^{-\phi(L-x)} \ge 0 \tag{11}$$

2.4. The overall model

For almost all $t \in [0, \overline{T}]$, $x \in [0, L]$, the overall continuous spatial and dynamic travel equilibrium model can be expressed as a partial differential complementarity system (PDCS) as follows:

$$\frac{\partial f_t^x}{\partial x} + \frac{\partial k_t^x}{\partial t} = r_t^x \tag{12}$$

$$\frac{\partial TT_t^x}{\partial x} = -\frac{1}{-ak_t^x + b + \sigma} \tag{13}$$

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$$\frac{\partial Q_t^x}{\partial t} = \eta_t^x + r_t^x + f_t^{x-dx} - f_{max} \tag{14}$$

$$0 \le \eta_t^x \perp Q_t^x \ge 0 \tag{15}$$

$$0 \le \pi^x \perp \int_0^{\bar{T}} r_w^x dw - D^0 e^{-\phi(L-x)} \ge 0 \tag{16}$$

$$0 \le r_t^x \perp \alpha(TT_t^x + \frac{Q_t^x}{f_{max}}) + \beta max(0, \tilde{t} - t - TT_t^x - \frac{Q_t^x}{f_{max}}) + \gamma max(0, t + TT_t^x + \frac{Q_t^x}{f_{max}} - \tilde{t}) - \pi^x \ge 0 \tag{17}$$

The above formulated PDCS model fully describes the spatial and dynamic equilibrium travel pattern in the travel corridor, and the initial and boundary conditions of the PDCS are given as $k_t^x\big|_{t=0} = 0$, $f_t^x\big|_{t=0} = 0$, $Q_t^x\big|_{t=0} = max(0, r_0^x - f_{max})$, $\forall x \in [0, L]$ and $TT_t^x\big|_{x=L} = 0$, $\forall t \in [0, \bar{T}]$.

3. Discretization scheme and approximated complementarity problem

The model formulation PDCS will be solved via a discretization scheme, wherein the study horizon is divided into v subintervals indexed by $t \in \{0, 1, ..., v\}$ with equal time length $\Delta t = \bar{T}/v$, and the corridor is divided into n cells indexed by $i \in \{1, 2, ..., n\}$ with uniform length $\Delta x = L/n$. Commuters are allowed to arrive at workplaces earlier or later than their preferred arrival time $\tilde{t}$, but they are supposed to depart no later than $\tilde{t}$. As is shown in Fig.1, the sequence of cells 1 to n is the direction from the boundary of the corridor to CBD, and all commuters along the corridor travel to CBD (sink cell / cell $n+1$) for work.
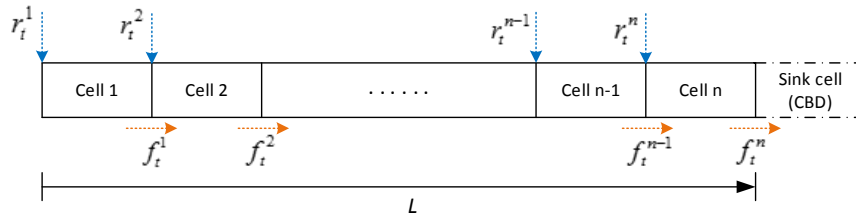

Fig. 1 A corridor with many-to-one OD pattern

To present our discretized model, we first introduce the variables and parameters used throughout this paper (shown in Table 1), and note that all of the parameters and variables are nonnegative.

Table 1 List of notations and descriptions in discretization scheme

Notation	Description
Indices	
t	Index of time intervals, $t \in \{0, 1, ..., v\}$
i	Index of cells, $i \in \{1, 2, ..., n\}$
g	Index of commuter groups, $g \in \mathbb{G} = \{1, ..., G\}$
Variables	
$f_{t,g}^i$	Traffic flow from group g in cell i at the beginning of time interval t
$k_{t,g}^i$	Traffic density from group g in cell i in time interval t
$r_{t,g}^i$	Number of vehicles from group g departing from cell i at the beginning of time interval t
TT_t^i	Travel time for commuters departing in time interval t from cell i
Q_t^i	Number of vehicles queueing in cell i in time interval t
η_t^i	Slack variable for queueing time computation in cell i in time interval t
$\theta_{t,g}^i$	Auxiliary variable for travel cost computation for group g in cell i in time interval t
π_g^i	Equilibrium cost for commuters from group g departing from cell i
Parameters	
L	Length of the corridor
n	Total number of cells
v	Total number of time intervals

G	Total number of commuter groups
D_g^0	Total travel demand along the corridor for commuters from group g
D_g^i	Travel demand for commuters from group g departing from cell i
α_g	Unit cost of travel time for group g
β_g	Unit cost of arriving early to work for group g
γ_g	Unit cost of arriving late to work for group g
$\tilde{t}_g$	Preferred arrival time for group g
a	Parameter of the velocity-density function
b	Parameter to the velocity-density function
ϕ_g	Parameter of the negative exponential distribution function for group g

3.1. Discretization scheme

One of the most commonly used and efficient method for solving the continuum formulation of the LWR model (12) is Godunov scheme (Godunov, 1959). To apply Godunov method, a regular grid is assumed and the corridor is discretized into sections/cells, then the cell average of the analytical solution is regarded as the numerical value of the solution. At each cell boundary, the resulting Riemann problem is then solved and the union of all Riemann solutions averaged over each cell to give the updated numerical solution values (Sweby, 2001). As was done in Leclercq (2007), in the discretization scheme, k_t^i and f_t^i are used to approximate the traffic density and flow respectively in cell i in time interval t. Moreover, the traffic flow f_t^i can be expressed in the following supply-demand method by applying the Godunov scheme (Lebacque, 1996).

$$f_t^i = \min\left\{UD(k_t^i), DS(k_t^{i+1})\right\} \tag{18}$$

where $UD(k_t^i)$ and $DS(k_t^{i+1})$ denote the upstream demand and downstream supply respectively, which are defined by

$$UD(k_t^i) = \begin{cases} k_t^i u_t^i, & \text{if } k_t^i \le k_c \\ f_{\max}, & \text{if } k_t^i > k_c \end{cases} \tag{19}$$

$$DS(k_t^i) = \begin{cases} f_{\max}, & \text{if } k_t^i \le k_c \\ k_t^i u_t^i, & \text{if } k_t^i > k_c \end{cases} \tag{20}$$

where k_c denotes the critical density associated with the flow capacity $f_{\max}$. The LWR model (12) can be approximated with a finite difference equation, where the density can be updated.

$$k_{t+1}^i = k_t^i + \frac{\Delta t}{\Delta x}\left(f_t^{i-1} - f_t^i\right) + \Delta t \cdot r_t^i \tag{21}$$

where the discretization scheme should satisfy the Courant-Friedrichs-Lewy (CFL) condition: $\dfrac{\Delta t}{\Delta x} \le \dfrac{1}{u_{\max}}$ (Courant et al., 1928).

Eq. (13) can be expressed in a discretized form as follows:

$$\frac{TT_t^{i+1} - TT_t^i}{\Delta x} = -\frac{1}{-ak_t^i + b + \sigma} \tag{22}$$

which can also be converted to a complementarity condition as follows:

$$0 \le TT_t^i \perp TT_t^i - TT_t^{i+1} - \frac{\Delta x}{-ak_t^i + b + \sigma} \ge 0 \tag{23}$$

It can be proved that the complementarity condition (23) is equivalent to Eq. (22), which holds only at $TT_t^i > 0$ with right side of (23) as an equality, i.e., $TT_t^i - TT_t^{i+1} - \frac{\Delta x}{-ak_t^i + b + \sigma} = 0$. Specifically, if $TT_t^i > 0$, the right side of (23) must be zero, thus we get Eq. (22); otherwise if $TT_t^i = 0$, the right side of (23) reduces to $-TT_t^{i+1} - \frac{\Delta x}{-ak_t^i + b + \sigma} \geq 0$. Since all the variables in this paper are nonnegative, thus $k_t^i \geq 0$. If $k_t^i = 0$, we have $\frac{\Delta x}{-ak_t^i + b + \sigma} > 0$ since $b > 0$ and $\sigma > 0$, therefore we can get the right side of (23) is negative, $-TT_t^{i+1} - \frac{\Delta x}{-ak_t^i + b + \sigma} < 0$, which is contradictory with the complementarity condition; if $k_t^i > 0$, it is easy to observe that $-a(k_t^i)^2 + bk_t^i \geq 0$ from Eq. (4), thus we have $-ak_t^i + b \geq 0$. Moreover, since $\sigma > 0$, thus $\frac{\Delta x}{-ak_t^i + b + \sigma} > 0$, therefore we can get the right side of (23) is negative, $-TT_t^{i+1} - \frac{\Delta x}{-ak_t^i + b + \sigma} < 0$, which is also contradictory with the complementarity condition. To summarize, we have an important positive property of travel time, $TT_t^i > 0$, and the complementarity condition (23) is equal to Eq. (22).

Similarly, Eq. (14) in discretized form is shown as:

$$\eta_{t+1}^i = \frac{Q_{t+1}^i - Q_t^i}{\Delta t} - r_{t+1}^i - f_{t+1}^{i-1} + f_{max} \tag{24}$$

which can be substituted into Eq. (15) in discretized form and multiplied by Δt as follows:

$$0 \leq Q_{t+1}^i \perp Q_{t+1}^i - Q_t^i - \Delta t \left(r_{t+1}^i + f_{t+1}^{i-1} - f_{max} \right) \geq 0 \tag{25}$$

Moreover, complementarity conditions can also be applicable to represent *max* function by adding a nonnegative auxiliary variable θ_t^i. Specifically, the complementarity condition (17) can be recast into the following conditions:

$$0 \leq r_t^i \perp \alpha(TT_t^i + \frac{Q_t^i}{f_{max}}) + \beta\theta_t^i + \gamma[\theta_t^i - (\tilde{t} - t - TT_t^i - \frac{Q_t^i}{f_{max}})] - \pi^i \geq 0 \tag{26}$$

$$0 \leq \theta_t^i \perp \theta_t^i - (\tilde{t} - t - TT_t^i - \frac{Q_t^i}{f_{max}}) \geq 0 \tag{27}$$

In the above conditions, the auxiliary variables θ_t^i is introduced to define the scheduling delays, i.e., $\theta_t^i = max(0, \tilde{t} - t - TT_t^i - \frac{Q_t^i}{f_{max}})$ and $\theta_t^i - (\tilde{t} - t - TT_t^i - \frac{Q_t^i}{f_{max}}) = max(0, t + TT_t^i + \frac{Q_t^i}{f_{max}} - \tilde{t})$. One can easily verify the equivalence between these two complementarity conditions and (17). Basically, if early arrival is triggered, i.e., $\tilde{t} - t - TT_t^i - \frac{Q_t^i}{f_{max}} > 0$, we must have $\theta_t^i > 0$ and $\theta_t^i = \tilde{t} - t - TT_t^i - \frac{Q_t^i}{f_{max}}$, and thus $\theta_t^i = max(0, \tilde{t} - t - TT_t^i - \frac{Q_t^i}{f_{max}})$ and $\theta_t^i - (\tilde{t} - t - TT_t^i - \frac{Q_t^i}{f_{max}}) = max(0, t + TT_t^i + \frac{Q_t^i}{f_{max}} - \tilde{t})$; if late arrival occurs, then $\tilde{t} - t - TT_t^i - \frac{Q_t^i}{f_{max}} < 0$ and $\theta_t^i - (\tilde{t} - t - TT_t^i - \frac{Q_t^i}{f_{max}}) > 0$, and we must have $\theta_t^i = 0$, thus $\theta_t^i = max(0, \tilde{t} - t - TT_t^i - \frac{Q_t^i}{f_{max}})$ and $\theta_t^i - (\tilde{t} - t - TT_t^i - \frac{Q_t^i}{f_{max}}) = max(0, t + TT_t^i + \frac{Q_t^i}{f_{max}} - \tilde{t})$; if arrival on time, we have $\tilde{t} - t - TT_t^i - \frac{Q_t^i}{f_{max}} = 0$ and $\theta_t^i = 0$, and the equivalence also holds.

The complementarity condition (16) in a discretized form is shown as follows:

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$$0 \leq \pi^i \perp \sum_{t=0}^{v-1} r_t^i - D^0 e^{-\phi(n-i)} \geq 0 \tag{28}$$

3.2. Heterogeneous case

Heterogeneity across commuters is considered in this study mainly on the preferred arrival time, value of travel time and schedule delays. With heterogeneous commuters, the flow conservation function (21) is not suitable to determine the flow for each individual commuter group. In particular, the commuters are classified into different groups indexed by $g \in \mathbb{G} \triangleq \{1,...,G\}$. To distinguish the flow of each commuter group $f_{t,g}^i$ and the total flow $\sum_{g=1}^{G} f_{t,g}^i$, f_t^i is used to represent the aggregate flow $\sum_{g=1}^{G} f_{t,g}^i$ hereafter. Similarly, we use k_t^i and r_t^i instead of $\sum_{g=1}^{G} k_{t,g}^i$ and $\sum_{g=1}^{G} r_{t,g}^i$ respectively. In the discretized scheme, in order to obtain the heterogeneous flow $f_{t,g}^i$ among the aggregate flow f_t^i, a proportional allocation procedure is used to determine the flow fraction as follows:

$$f_{t,g}^i = \begin{cases} f_t^i \dfrac{k_{t,g}^i}{k_t^i}, & if \quad k_t^i > 0 \\ 0, & otherwise \end{cases} \tag{29}$$

Thus, for each commuter group g, Eq. (21) can be approximated in a discretized form as:

$$k_{t+1,g}^i = k_{t,g}^i + \frac{\Delta t}{\Delta x}\left(f_{t,g}^{i-1} - f_{t,g}^i\right) + \Delta t \cdot r_{t,g}^i \tag{30}$$

These flow evolvement equations can be readily solved with given departure pattern and initial/boundary conditions. Moreover, Eq. (4) in discretized form is shown as follows, which regulates the total flow in an aggregated form.

$$f_t^i = -a\left(k_t^i\right)^2 + bk_t^i \tag{31}$$

Based on Eq. (29), the disaggregated flow can be derived as follows:

$$f_{t,g}^i = -ak_{t,g}^i k_t^i + bk_{t,g}^i \tag{32}$$

Let $\boldsymbol{f} \triangleq \left\{ f_{t,g}^i \big|_{t=0,1,...,v-1;\ i=1,...,n;\ g=1,...,G} \right\}$, $\boldsymbol{k} \triangleq \left\{ k_{t,g}^i \big|_{t=0,1,...,v-1;\ i=1,...,n;\ g=1,...,G} \right\}$ and $\boldsymbol{r} \triangleq \left\{ r_{t,g}^i \big|_{t=0,1,...,v-1;\ i=1,...,n;\ g=1,...,G} \right\}$, then with Eqs. (18)-(20), (29) and the initial conditions, $k_{0,g}^i = 0$ and $f_{0,g}^i = 0$, $\forall i=1,...,n;\ g=1,...,G$, one can observe that the traffic flow $f_{t,g}^i$ and traffic density $k_{t,g}^i$ are determined by the departure rate $\boldsymbol{r}$. Moreover, traffic flow $f_{t,g}^i$ can also be regarded as a function of $\boldsymbol{k}$ as well. These properties are formally stated in the following lemmas with proof.

Lemma 1. *Traffic flow $f_{t,g}^i(\boldsymbol{k})$ is continuous in $\boldsymbol{k}$ for each $t = 0,1,...,v-1;\ i=1,...,n;\ g=1,...,G$.*

Proof. Since the flow propagation equation (30) is linear, its continuity is obvious. As for (29), it can be derived that $f_{t,g}^i(\boldsymbol{k})$ is continuous in $\boldsymbol{k}$ when $k_t^i > 0$. Moreover, with Eqs. (18)-(20) and $u_t^i \leq u_{max}$, we have

$$0 \leq f_t^i \frac{k_{t,g}^i}{k_t^i} = \frac{k_{t,g}^i}{k_t^i}\min\left\{UD(k_t^i), DS(k_t^{i+1})\right\} \leq k_t^i u_t^i \leq k_t^i u_{max}, \quad \forall t=0,1,...,v-1;\ i=1,...,n;\ g=1,...,G$$

therefore it is easy to get that

$$\lim_{\left(k_t^i \to 0\right)} f_t^i \frac{k_{t,g}^i}{k_t^i} = 0, \quad \forall t=0,1,...,v-1;\ i=1,...,n;\ g=1,...,G$$

This concludes the proof. □

Lemma 2. *Traffic density* $k_{t,g}^i(\boldsymbol{r})$ *and flow* $f_{t,g}^i(\boldsymbol{r})$ *are continuous in* $\boldsymbol{r}$ *for each* $t=0,1,...,v-1$; $i=1,...,n$ *and* $g=1,...,G$.

Proof. In terms of the linearity of Eq. (30), it suffices to show that $k_{t,g}^i(\boldsymbol{r})$ is continuous in $\boldsymbol{r}$. It has been demonstrated that $f_{t,g}^i(\boldsymbol{k})$ is continuous in $\boldsymbol{k}$ by Lemma 1, therefore it is ready to derive that $f_{t,g}^i(\boldsymbol{r})$ is also continuous in $\boldsymbol{r}$. This concludes the proof. □

Lemma 3. $TT_t^i(\boldsymbol{r})$ *is continuous in* $\boldsymbol{r}$ *for each* $t=0,1,...,v-1$; $i=1,...,n$.

Proof. As discussed in previous analysis in this study, we have $-ak_t^i+b+\sigma>0$, therefore $\dfrac{1}{-ak_t^i+b+\sigma}$ is continuous in $\boldsymbol{k}$. Moreover, $k_{t,g}^i(\boldsymbol{r})$ has been proved to be continuous in $\boldsymbol{r}$, thus it is straightforward to derive that $\dfrac{1}{-ak_t^i+b+\sigma}$ is continuous in $\boldsymbol{r}$ as well. Therefore, the travel time $TT_t^i(\boldsymbol{r})$ is continuous in $\boldsymbol{r}$. This concludes the proof. □

As mentioned in previous content, in this study, the travel time function represents the time on highway, which does not includes the queueing time. Interested readers can refer to Ban et al. (2012a); Pang et al. (2012); Han et al. (2013c) for further analysis and the proof of the continuous dependence of the queueing time function.

3.3. The overall complementarity model

Now the PDCS model can be presented in standard complementarity problem (CP) form, *CP(q,M)*, to find a vector of variables $\boldsymbol{z}$ satisfying

$$0\le z \perp F(z)\ge 0 \tag{33}$$

As we have discussed in Lemma 1 and Lemma 2, the traffic density $k_{t,g}^i$ can be written as a function of the departure rate $\boldsymbol{r}$, denoted by $k_{t,g}^i(\boldsymbol{r})$, so can the aggregate traffic density $k_t^i(\boldsymbol{r})$. Similarly we also have traffic flow as a function of departure rate in both disaggregate and aggregate form: $f_{t,g}^i(\boldsymbol{r})$, $f_t^i(\boldsymbol{r})$. For all $t=0,1,...,v-1$; $i=1,...,n$; $g=1,...,G$, the overall model with heterogeneity assumptions in this study is given as follows with variables as $\boldsymbol{z}=\{\boldsymbol{r},\boldsymbol{\theta},\boldsymbol{Q},\boldsymbol{TT},\boldsymbol{\pi}\}\triangleq\left\{r_{t,g}^i,\theta_{t,g}^i,Q_t^i,TT_t^i,\pi_g^i\Big|_{t=0,1,...,v-1;\ i=1,...,n;\ g=1,...,G}\right\}$.

$$
\begin{aligned}
&0\le TT_t^i \perp TT_t^i - TT_t^{i+1} - \frac{\Delta x}{-ak_t^i(\boldsymbol{r})+b+\sigma}\ge 0\\[4pt]
&0\le Q_{t+1}^i \perp Q_{t+1}^i - Q_t^i - \Delta t\left(r_{t+1}^i + f_{t+1}^{i-1}(\boldsymbol{r}) - f_{\max}\right)\ge 0\\[4pt]
&0\le \pi_g^i \perp \sum_{t=0}^{v-1} r_{t,g}^i - D_g^0 e^{-\phi_g(n-i)}\ge 0\\[4pt]
&0\le r_{t,g}^i \perp (\alpha_g+\gamma_g)TT_t^i + \frac{(\alpha_g+\gamma_g)}{f_{\max}}Q_t^i + (\beta_g+\gamma_g)\theta_{t,g}^i + \gamma_g(t-\tilde{t}_g) - \pi_g^i\ge 0\\[4pt]
&0\le \theta_{t,g}^i \perp \theta_{t,g}^i + TT_t^i + \frac{Q_t^i}{f_{\max}} + t - \tilde{t}_g\ge 0
\end{aligned}
\tag{34}
$$

wherein the initial traffic density, traffic flow and queueing size in any cell at the beginning of time $t=0$ is set as $k_{0,g}^i=0$, $f_{0,g}^i=0$, $Q_0^i=max(0,r_0^i-f_{\max})$, $\forall i=1,...,n$; $g=1,...,G$ and the travel time, $TT_t^{n+1}=0$, $\forall t=1,...,v$ guarantees that travel time for passengers departing from CBD is zero. Some other properties can be derived from the CP model (34), which are formally stated as follows.

Lemma 4. *Suppose there exist a solution to the CP model (34), then it must hold that*

$$\sum_{t=0}^{\nu-1} r_{t,g}^i = D_g^0 e^{-\phi_g(n-i)}, \quad \forall i=1,...,n; \ g=1,...,G$$

Proof. Suppose there exists a i and a g such that $\sum_{t=0}^{\nu-1} r_{t,g}^i - D_g^0 e^{-\phi_g(n-i)} > 0$, hence we can derive $\pi_g^i = 0$ from the third complementarity condition in (34), and there must exist a t such that $r_{t,g}^i > 0$. Based on the fourth complementarity condition in (34), we have $\alpha_g(TT_t^i + \frac{Q_t^i}{f_{\max}}) + \beta_g \theta_{t,g}^i + \gamma_g(\theta_{t,g}^i + TT_t^i + \frac{Q_t^i}{f_{\max}} + t - \tilde{t}_g) - \pi_g^i = 0$.

Since $\pi_g^i = 0$, then $\alpha_g(TT_t^i + \frac{Q_t^i}{f_{\max}}) + \beta_g \theta_{t,g}^i + \gamma_g(\theta_{t,g}^i + TT_t^i + \frac{Q_t^i}{f_{\max}} + t - \tilde{t}_g) = 0$. Moreover, we have $\theta_{t,g}^i \geq 0$ and $\theta_{t,g}^i + TT_t^i + \frac{Q_t^i}{f_{\max}} + t - \tilde{t}_g \geq 0$ based on the last complementarity condition in (34), and $Q_t^i \geq 0$, $TT_t^i > 0$ as discussed in previous analysis in this study, thus $\alpha_g(TT_t^i + \frac{Q_t^i}{f_{\max}}) + \beta_g \theta_{t,g}^i + \gamma_g(\theta_{t,g}^i + TT_t^i + \frac{Q_t^i}{f_{\max}} + t - \tilde{t}_g) > 0$. This is a contradiction. $\square$

In addition, the boundedness properties of $f_{t,g}^i$ and $k_{t,g}^i$ will be derived from Eqs. (18)-(20), (29)-(30) and the initial conditions, $k_{0,g}^i = 0$ and $f_{0,g}^i = 0$, $\forall i=1,...,n; \ g=1,...,G$. The following lemmas with proofs are given as follows.

Lemma 5. *The boundedness of $f_{t,g}^i$ holds for each $t=0,1,...,\nu-1; \ i=1,...,n \ and \ g=1,...,G$.*

Proof. Obviously, based on Eqs. (18)-(20) and (29), it is easy to derive the following condition:

$$f_{t,g}^i \leq \min\{UD(k_t^i), DS(k_t^{i+1})\} \leq f_{\max}$$

for all $t=0,1,...,\nu-1; \ i=1,...,n \ and \ g=1,...,G$. This concludes the proof. $\square$

Lemma 6. *The boundedness of $k_{t,g}^i$ holds for each $t=0,1,...,\nu-1; \ i=1,...,n \ and \ g=1,...,G$.*

Proof. Based on the flow propagation equation (30), we have

$$k_{t+1,g}^i = k_{t,g}^i + \frac{\Delta t}{\Delta x}\left(f_{t,g}^{i-1} - f_{t,g}^i\right) + \Delta t \cdot r_{t,g}^i \leq k_{t,g}^i + \frac{\Delta t}{\Delta x} f_{t,g}^{i-1} + \Delta t \cdot r_{t,g}^i$$

for all $t=0,1,...,\nu-1; \ i=1,...,n \ and \ g=1,...,G$. From Lemma 4, we can find that $r_{t,g}^i$ is bounded. With $f_{t,g}^i \leq f_{\max}$ from Lemma 5, we have

$$k_{t,g}^i \leq k_{t-1,g}^i + \frac{\Delta t}{\Delta x} f_{\max} + \Delta t \cdot D_g^0 \leq k_{t-2,g}^i + \frac{2\Delta t}{\Delta x} f_{\max} + 2\Delta t \cdot D_g^0 \leq \frac{\nu\Delta t}{\Delta x} f_{\max} + \nu\Delta t \cdot D_g^0$$

This concludes the proof. $\square$

3.4. Solution existence of the formulation

As was done in Han et al. (2011); Pang et al. (2012), we apply Lemma 7 as follows to analyse the solution existence of CP model (34), and one can refer to Facchinei and Pang (2003) for the proof of Lemma 7 and more complete theories of complementarity problem.

Lemma 7. *(Facchinei and Pang, 2003) Let $F : R^n \to R^n$ be a continuous function, if there exists a constant $d > 0$ such that all the solutions of the CP: $0 \leq z \perp F(z) + \tau z \geq 0, \forall \tau > 0$ satisfy $\|z\| \leq d$, then the CP: $0 \leq z \perp F(z) \geq 0$ has a solution.*

As indicated by Lemma 7, the solution to the original CP $0 \leq z \perp F(z) \geq 0$ exists if the solutions to the extended CP $0 \leq z \perp F(z) + \tau z \geq 0$ are uniformly bounded. Using Lemma 7, we will show the following Theorem 1 holds.

Theorem 1. *The CP model (34):* $0 \leq z \perp F(z) \geq 0$ *has a solution.*

Proof. Based on the above analysis, we can apply Lemma 7 to prove the solution existence in Theorem 1. Since the continuity has been demonstrated in the previous content, now it is ready to prove the uniform boundedness. Suppose there exists a sequence of positive scalars $\{\lambda^l\}_{l=1}^{\infty}$ and a sequence of vectors $\{TT^l, Q^l, \pi^l, r^l, \theta^l\}_{l=1}^{\infty}$ satisfying for each $l = 1,...,\infty$. For all $t = 0,1,...,v-1$; $i = 1,...,n$; $g = 1,...,G$, we have

$$0 \leq TT_t^{i,l} \perp TT_t^{i,l} - TT_t^{i+1,l} - \frac{\Delta x}{-ak_t^{i,l}(r^l) + b + \sigma} + \lambda^l TT_t^{i,l} \geq 0 \tag{35}$$

$$0 \leq Q_{t+1}^{i,l} \perp Q_{t+1}^{i,l} - Q_t^{i,l} - \Delta t\left(r_{t+1}^{i,l} + f_{t+1}^{i-1,l}(r^l) - f_{\max}\right) + \lambda^l Q_{t+1}^{i,l} \geq 0 \tag{36}$$

$$0 \leq \pi_g^{i,l} \perp \sum_{t=0}^{v-1} r_{t,g}^{i,l} - D_g^0 e^{-\phi_g(n-i)} + \lambda^l \pi_g^{i,l} \geq 0 \tag{37}$$

$$0 \leq r_{t,g}^{i,l} \perp (\alpha_g + \gamma_g)TT_t^{i,l} + \frac{(\alpha_g + \gamma_g)}{f_{\max}} Q_t^{i,l} + (\beta_g + \gamma_g)\theta_{t,g}^{i,l} + \gamma_g(t - \tilde{t}_g) - \pi_g^{i,l} + \lambda^l r_{t,g}^{i,l} \geq 0 \tag{38}$$

$$0 \leq \theta_{t,g}^{i,l} \perp \theta_{t,g}^{i,l} + TT_t^{i,l} + \frac{Q_t^{i,l}}{f_{\max}} + t - \tilde{t}_g + \lambda^l \theta_{t,g}^{i,l} \geq 0 \tag{39}$$

where $r_t^{i,l}$ denotes $\sum_{g=1}^{G} r_{t,g}^{i,l}$, $k_t^{i,l}(r^l)$ represents $\sum_{g=1}^{G} k_{t,g}^{i,l}(r^l)$ and $f_{t+1}^{i-1,l}(r^l)$ represents $\sum_{g=1}^{G} f_{t+1,g}^{i-1,l}(r^l)$ for simplicity.

Boundedness of $\left\{(\pi_g^{i,l})_{i=1,...,n;\ g=1,...,G}\right\}_{l=1}^{\infty}$

Suppose that this is not true for some $i \in \{1,...,n\}$ and $g \in \{1,...,G\}$. We may assume without loss of generality that

$$\lim_{l \to \infty} \pi_g^{i,l} = \infty$$

and $\pi_g^{i,l} > 0$ for all $l = 1,...,\infty$. It follows from (37) that

$$\sum_{t=0}^{v-1} r_{t,g}^{i,l} - D_g^0 e^{-\phi_g(n-i)} + \lambda^l \pi_g^{i,l} = 0$$

for all $l = 1,...,\infty$. It suffices to deduce that $\pi_g^{i,l} = \dfrac{D_g^0 e^{-\phi_g(n-i)} - \sum_{t=0}^{v-1} r_{t,g}^{i,l}}{\lambda^l} \leq \dfrac{D_g^0}{\lambda^l}$. This is a contradiction, so the boundedness of $\left\{(\pi_g^{i,l})_{i=1,...,n;\ g=1,...,G}\right\}_{l=1}^{\infty}$ follows.

Boundedness of $\left\{(r_{t,g}^{i,l})_{t=0,1,...,v-1;\ i=1,...,n;\ g=1,...,G}\right\}_{l=1}^{\infty}$

Suppose that this is not true for some $t \in \{0,1,...,v-1\}$, $i \in \{1,...,n\}$, $g \in \{1,...,G\}$. We may assume without loss of generality that

$$\lim_{l \to \infty} r_{t,g}^{i,l} = \infty$$

and $r_{t,g}^{i,l} > 0$ for all $l = 1,...,\infty$. It follows from (38) that

$$(\alpha_g + \gamma_g)TT_t^{i,l} + \frac{(\alpha_g + \gamma_g)}{f_{\max}} Q_t^{i,l} + (\beta_g + \gamma_g)\theta_{t,g}^{i,l} + \gamma_g(t - \tilde{t}_g) - \pi_g^{i,l} + \lambda^l r_{t,g}^{i,l} = 0$$

for all $l = 1,...,\infty$. It suffices to deduce that

$$r_{t,g}^{i,l} = \frac{\gamma_g(\tilde{t}_g - t) + \pi_g^{i,l} - (\alpha_g + \gamma_g)TT_t^{i,l} - \dfrac{(\alpha_g + \gamma_g)}{f_{max}}Q_t^{i,l} - (\beta_g + \gamma_g)\theta_{t,g}^{i,l}}{\lambda^l} \leq \frac{\gamma_g \tilde{t}_g + \pi_g^{i,l}}{\lambda^l}$$

Note that $\pi_g^{i,l}$ has been proved to be bounded, thus this is a contradiction, so the boundedness of $\left\{(r_{t,g}^{i,l})_{t=0,1,...,v-1;\ i=1,...,n;\ g=1,...,G}\right\}_{l=1}^{\infty}$ follows.

Boundedness of $\left\{(\theta_{t,g}^{i,l})_{t=0,1,...,v-1;\ i=1,...,n;\ g=1,...,G}\right\}_{l=1}^{\infty}$

Suppose that this is not true for some $t \in \{0,1,...,v-1\}$, $i \in \{1,...,n\}$, $g \in \{1,...,G\}$. We may assume without loss of generality that

$$\lim_{l\to\infty}\theta_{t,g}^{i,l} = \infty$$

and $\theta_{t,g}^{i,l} > 0$ for all $l = 1,...,\infty$. It follows from (39) that

$$\theta_{t,g}^{i,l} + TT_t^{i,l} + \frac{Q_t^{i,l}}{f_{max}} + t - \tilde{t}_g + \lambda^l \theta_{t,g}^{i,l} = 0$$

for all $l = 1,...,\infty$. It suffices to deduce that $\theta_{t,g}^{i,l} = \dfrac{\tilde{t}_g - TT_t^{i,l} - \dfrac{Q_t^{i,l}}{f_{max}} - t}{1 + \lambda^l} \leq \dfrac{\tilde{t}_g}{1 + \lambda^l}$. This is a contradiction, so the boundedness of $\left\{(\theta_{t,g}^{i,l})_{t=0,1,...,v-1;\ i=1,...,n;\ g=1,...,G}\right\}_{l=1}^{\infty}$ follows.

Boundedness of $\left\{(Q_t^{i,l})_{t=0,1,...,v-1;\ i=1,...,n}\right\}_{l=1}^{\infty}$

Let $t_* \in \{0,1,...,v-1\}$ be the smallest t such that $Q_{t_*+1}^{i,l}$ is unbounded. We may assume that

$$\lim_{l\to\infty}Q_{t_*+1}^{i,l} = \infty$$

and $Q_{t_*+1}^{i,l} > 0$ for all $l = 1,...,\infty$. It follows from (36) that

$$Q_{t_*+1}^{i,l} - Q_{t_*}^{i,l} - \Delta t\left(r_{t_*+1}^{i,l} + f_{t_*+1}^{i-1,l}(r^l) - f_{max}\right) + \lambda^l Q_{t_*+1}^{i,l} = 0$$

for all $l = 1,...,\infty$. This implies that

$$(1 + \lambda^l)Q_{t_*+1}^{i,l} - Q_{t_*}^{i,l} = \Delta t\left(r_{t_*+1}^{i,l} + f_{t_*+1}^{i-1,l}(r^l) - f_{max}\right)$$

for all $l = 1,...,\infty$. Note that the boundedness of $f_{t_*+1}^{i-1,l}(r^l)$ and $r_{t_*+1}^{i,l}$ have been proved, therefore we can conclude that $Q_{t_*}^{i,l}$ is also unbounded. This contradicts the definition of index t_*, so the boundedness of $\left\{(Q_t^{i,l})_{t=0,1,...,v-1;\ i=1,...,n}\right\}_{l=1}^{\infty}$ follows.

Boundedness of $\left\{(TT_t^{i,l})_{t=0,1,...,v-1;\ i=1,...,n}\right\}_{l=1}^{\infty}$

Let $i_* \in \{1,...,n\}$ be the smallest i (note that the sequence of cell 1 to n is the direction from the corridor boundary to CBD, i.e., $TT_t^1 > TT_t^2 > \cdots > TT_t^n$, $\forall t = 0,1,...,v-1$) such that $TT_t^{i,l}$ is unbounded. We may assume that

$$\lim_{l\to\infty}TT_t^{i_*,l} = \infty$$

and $TT_t^{i_*,l} > 0$ for all $l = 1,...,\infty$, thus,

$$TT_t^{i_*,l} - TT_t^{i_*+1,l} - \frac{\Delta x}{-ak_t^{i_*,l}(\mathbf{r}^l) + b + \sigma} + \lambda^l TT_t^{i_*,l} = 0$$

for all $l = 1,...,\infty$. This implies that

$$\left(1 + \lambda^l\right)TT_t^{i_*,l} - TT_t^{i_*+1,l} = \frac{\Delta x}{-ak_t^{i_*,l}(\mathbf{r}^l) + b + \sigma}$$

for all $l = 1,...,\infty$. Due to the boundedness of $k_t^{i_*,l}(\mathbf{r}^l)$ as stated in previous analysis, we can conclude that $TT_t^{i_*+1,l}$ is also unbounded. This contradicts the definition of index i_*. Therefore the boundedness of $TT_t^{i,l}$ holds readily. This concludes the whole proof. □

4. Numerical experiments

In this section, a set of numerical examples are presented with an idealized travel corridor for three different scenarios: homogeneous commuters with no late arrival, homogeneous commuters allowing for late arrival and heterogeneous commuters. A corridor with length of 10km is adopted here for illustration, which connects the city centre with residential locations. The corridor is divided into 10 sections, and the sequence of section 1 to 10 is the direction from the boundary of the corridor to CBD. The time horizon is set between 8:00am to 8:40am, which is divided into 100 intervals with each time interval 0.4min. The free flow speed is set at 60km/h and the maximum density is given at 150veh/km.

### 4.1. Scenario I: homogeneous commuters with no late arrival

In this scenario, homogeneous users are assumed to be uniformly distributed along the corridor (i.e., parameter $\phi = 0$ in Eq. (10)) with travel demand of 150 veh at each section, and the preferred arrival time is set at 8:28am, the 70th time interval. No late arrival is allowed for all commuters along the corridor. One interesting finding is that, when the ratio β/α (VOT for early arrival time/travel time) becomes small, there exists the phenomenon of constant departure rate at some sections, and the smaller the ratio is the longer time range the constant departure rate occupies. For example, Fig. 2 shows the departure rate over time at each location along the corridor at a small ratio β/α=0.03 (α=S\$1, β=S\$0.03), wherein the blue solid line denotes the departure rate, and specially, the red dashed line represents the constant departure rate.

It is easy to observe that, the commuters located further toward the corridor boundary depart much earlier than those living closer to CBD, and the time interval with positive departure rate reduces from the locations around corridor boundary towards CBD. As stated in previous content, another interesting finding is that, except for the last two sections (9 and 10), the departure rate at each location is constant at most of the departure time interval (red dashed line in Fig. 2), which is consistent with the conclusion in Arnott and DePalma (2011) that, for no late corridor problem, the departure rate is constant at each location along the corridor. However, one can also observe the "discontinuity" of the departure rates at many locations, which is indeed allowed and induced by the complementarity conditions formulation applied in this study. Despite that the model formulation in this study is only solved to its form of discretized approximation, one can still observe from the numerical results that the discontinuity may occur in the solution. However, in Arnott and DePalma (2011), the solution derivation is based on the strict assumption that the differential variable, $\hat{T}(x,a)$, is not only continuous but also differentiable everywhere. This could be the assumption that somehow "over-constrains" the problem, as was pointed out in Arnott and DePalma (2011), and produces incomplete solution, which only exists for some special type of population density distribution. In the subsequent section, we will demonstrate that our model formulation is able to handle various types of population density distribution.

Fig. 3 illustrates the evolution of traffic density over time at each location along the corridor. The general trend is that, as time elapses at each location along the corridor, traffic density rises from zero to a certain level and then keeps constant for some time interval due to the constant departure rate as showed in previous results. It is easy to observe that the peaks of density are higher at locations closer to CBD, and positive traffic density occurs earlier at locations further away from CBD.
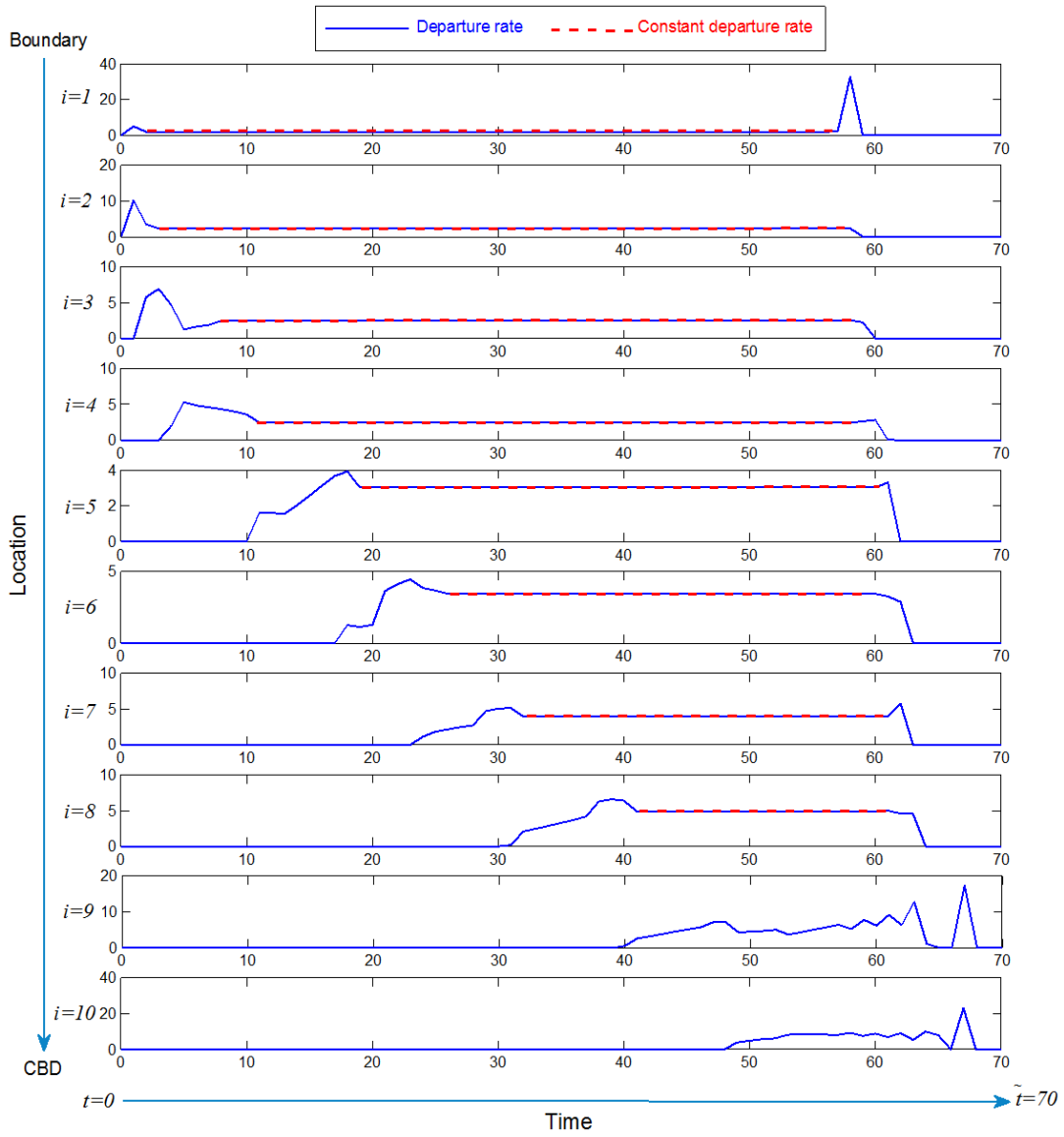
Fig. 2 Departure rate over time at each location along the corridor
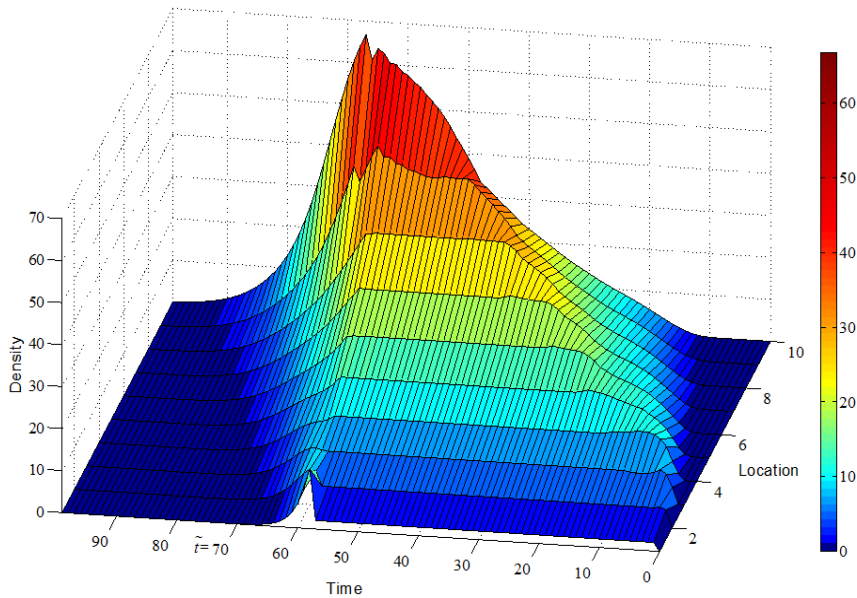
Fig. 3 Traffic density at each location over time along the corridor

Fig. 4 shows the travel time at each location over time along the corridor, and one can find that at each location travel time raises as positive departure rates occur and then keeps a constant growth rate to a peak, before it drops until all the commuters at this location complete the trips to CBD.
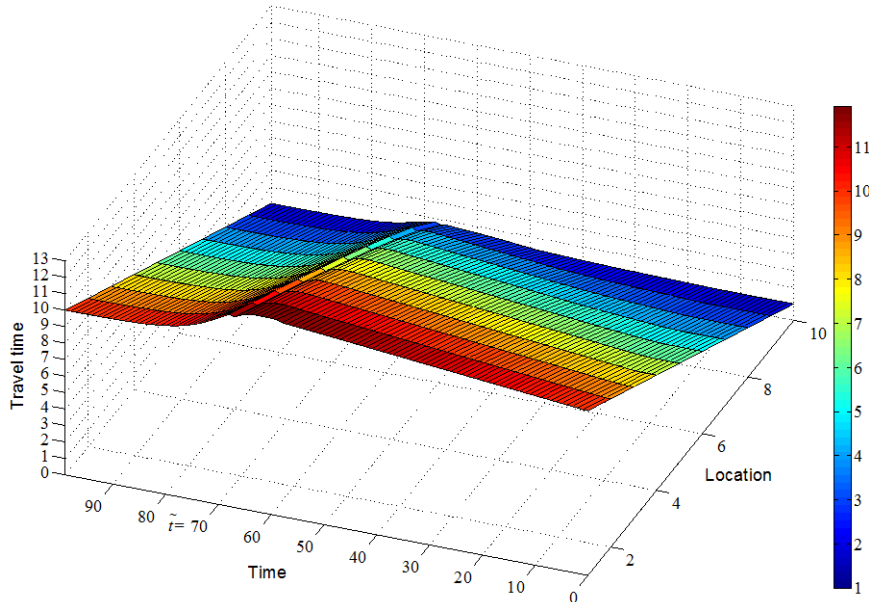
Fig. 4 Travel time at each location over time along the corridor

Next, to figure out how the ratio β/α affects the departure pattern, a comparison of the departure sets with different values of travel time and values of early arrival delay time is shown in Fig. 5. Departure set depicts the trajectory of the first and last departure time at each location along the corridor. For comparison, the VOT for travel time remains at constant S\$1 as stated previously, while the values of early arrival delay varying, and their ratios are shown clearly in Fig. 5. In Fig. 5, 'width' at $i=1$ denotes the range of departure set at this location, i.e., how many time intervals the departure set covers. For example, at the first case $\beta/\alpha=0.03$, the departure time range covers 58 time intervals at location 1. It is easy to find that as the ratio β/α raises, the width of departure set becomes narrower, which is consistent with the result in Arnott and DePalma (2011). Moreover, as the ratio β/α raises, the upper boundary (solid line) varies slightly, but the lower boundary (dashed line) rises up significantly. It is easy to understand that commuters would prefer earlier departure to avoid traffic congestion and minimize the travel cost if the unit early arrival penalty is low, however, with higher value of early arrival delay, commuters depart late to avoid early arrival and the departure set is squeezed for this situation.
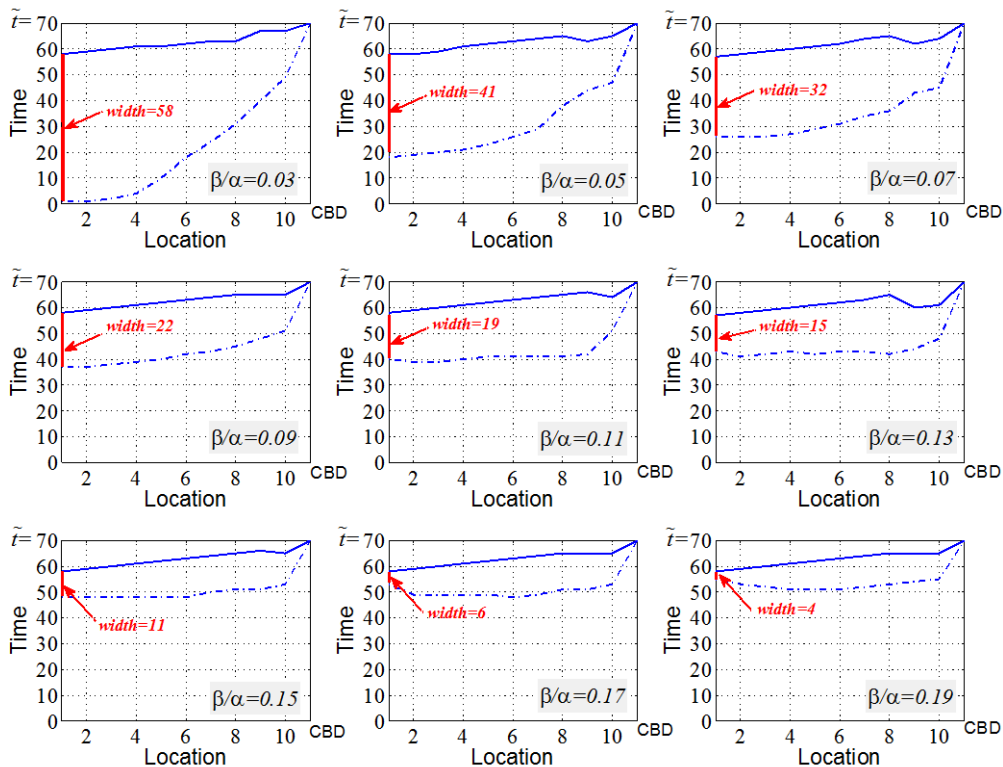
Fig. 5 Comparison of departure sets at each location along the corridor

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4.2. Scenario II: homogeneous commuters allowing for late arrival

In this numerical test, late arrival is allowed for commuters to consider more general cases of scheduling costs, and the VOTs of travel time, early arrival time and late arrival time delay are set at S\$1, S\$0.4 and S\$1.5 respectively. Moreover, different types of demand density distribution are compared in this section to demonstrate that the model results are indicative in answering such macroscopic urban planning questions as how the urban density distribution pattern may affect the traffic congestion spatial dynamics in a travel corridor. First of all, negative exponential distribution is considered here with the parameters $\phi = 0.1$, $D^0 = 150$. The equilibrium travel pattern in terms of departure rates over time at each location along the corridor is shown in Table 2 (the time intervals with zero departure rates are omitted in the table). Compared to the scenario I with no late arrival and uniform density distribution, the departure time range is shorter at all locations along the corridor. It can be interpreted by that fact that, higher unit early arrival penalty and late arrival penalty render the commuters no incentives to depart earlier or later to avoid congestion and most of the commuters would select departure time in a more concentrated time interval.

Table 2 Departure rate over time at each location

Time	Location									
	1	2	3	4	5	6	7	8	9	10
58	0	0	0	0	0	0	0	0	0	0
59	23.485	37.500	24.203	9.344	0	0	0	0	0	10.903
60	37.500	29.899	24.000	28.444	33.856	33.336	33.340	32.122	37.500	37.500
61	0	0	26.285	20.854	22.211	24.000	25.351	25.350	25.752	24.000
62	0	0	0	23.680	34.913	16.934	17.699	17.698	17.923	16.934
63	0	0	0	0	0	26.277	13.169	13.169	13.309	12.688
64	0	0	0	0	0	0	21.563	10.232	10.327	9.908
65	0	0	0	0	0	0	0	24.239	30.914	7.975
66	0	0	0	0	0	0	0	0	0	30.091
67	0	0	0	0	0	0	0	0	0	0
68	0	0	0	0	0	0	0	0	0	0
69	0	0	0	0	0	0	0	0	0	0
70 ($\tilde{t}$)	0	0	0	0	0	0	0	0	0	0

Next, we want to figure out how urban population density distribution affects the equilibrium departure pattern and the transport system performance. By doing so, one can understand that, what urban density distribution may lead to the best performance for individual commuters and the whole transportation system. More specifically, given the same total population, how the urban density distribution should be planned to accommodate the population while ascertaining best transport system performance. Besides the negative exponential distribution, other different types of distribution functions can also be applied in this model formulation. For illustration purpose, here we compare the uniform, linear and negative exponential distribution patterns of the travel demand along the corridor. For fair comparison, the total travel demand along the corridor is kept identical with previous numerical experiments. A simple computation is conducted to find the optimal parameters for the exponential and linear distribution functions so that their total travel cost minimized. By varying ϕ in a practically reasonable range ($\phi = 0.02, 0.04, ..., 0.2$), we find the negative exponential distribution ($\phi = 0.14$) with lowest total travel cost along the corridor. Similarly, as for the linear distribution function $D_i = Ai + B$, $A \geq 0$, $B \geq 0$, with given identical total demand, we have $0 \leq B \leq 99.6$, a simple scenario study (calculation of the total travel cost with each $B = 0, 10, ..., 90$) is conducted to find the best parameter $A = 16.298$ and $B = 10$ to induce the lowest total travel cost. Therefore, with best negative exponential and linear distribution patterns, we demonstrate the comparison among these three distribution types, as shown in Fig. 6.
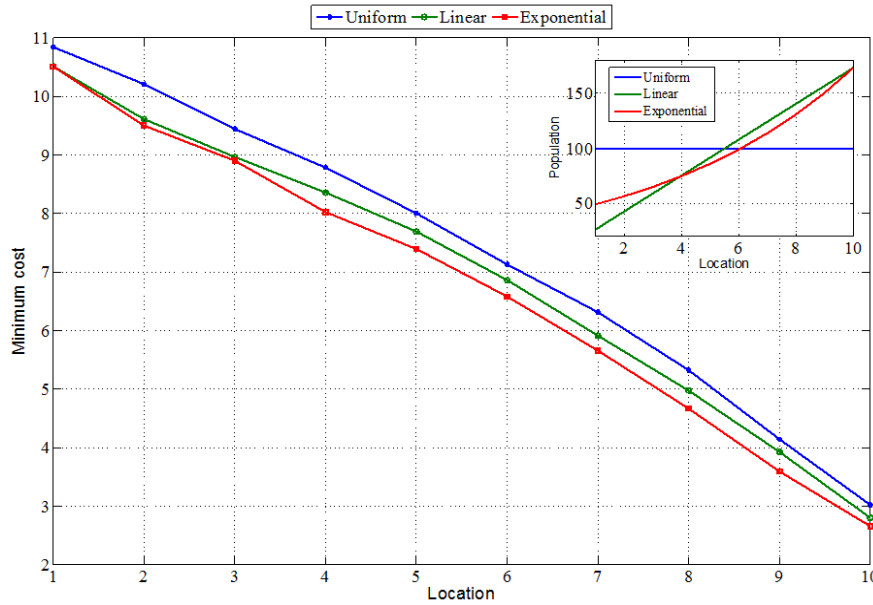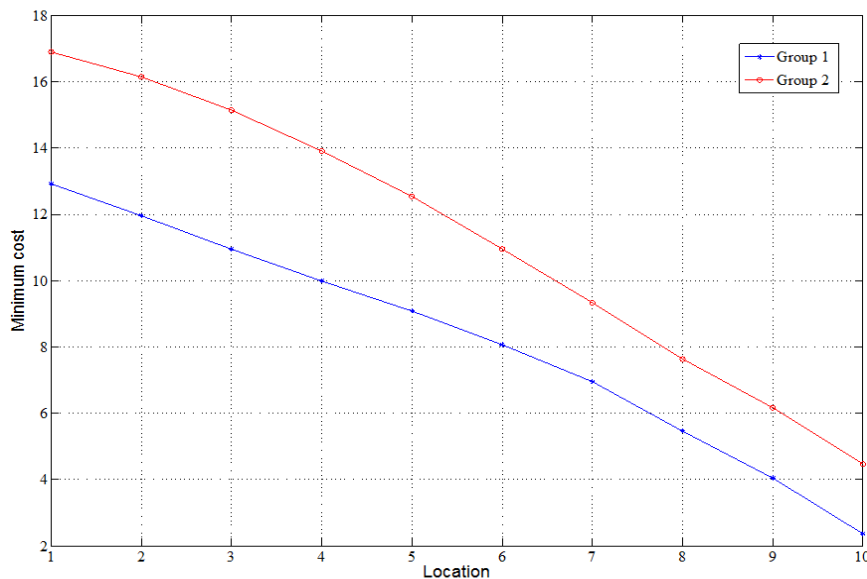
Fig. 6 Comparison of equilibrium costs along the corridor with different travel demand distributions

Fig. 6 compares the equilibrium costs along the corridor with different urban density distributions. One can easily find that, for this specific problem setting, the negative exponential distribution leads to lower equilibrium cost at each location along the corridor, as well as the lowest total travel cost, and uniform distribution results in a higher total travel cost compared to the other two types of distribution. Generally, with the aid of the model formulation and solution, one can evaluate how the urban density distribution may affects the transport system performance by conducting similar sensitivity analysis, which is vital in answering the urban planners' question on how urban density should be planned to ensure best transportation system performance. Moreover, it also demonstrate that the model formulate is suitable for various types of distribution functions.

4.3. Scenario III: heterogeneous commuters allowing for late arrival

In this scenario, heterogeneity is considered with two group commuters for illustration. For illustration purpose, travel demands are given as uniform distribution ($D_g^i = 100$ veh, $i=1,2,...,10$, $g=1,2$) and identical for two group commuters, and the heterogeneity is mainly on the VOTs and different preferred arrival time. Users in group 1 have lower unit VOTs, and later preferred arrival time than those in group 2. The preferred arrival time for group 1 and group 2 commuters are set at the 70th and 50th time interval respectively, and the other heterogeneous parameter values are given as: $\alpha_1 = 1\,\text{S\$}/\min$, $\beta_1 = 0.4\,\text{S\$}/\min$, $\gamma_1 = 1.5\,\text{S\$}/\min$, $\alpha_2 = 1.5\,\text{S\$}/\min$, $\beta_2 = 0.8\,\text{S\$}/\min$, $\gamma_2 = 2\,\text{S\$}/\min$.

Fig.7 Comparison of minimum costs along the corridor between different user groups

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Fig. 7 demonstrates the minimum costs of two group commuters along the corridor at equilibrium status. It is obvious that commuters from different groups have different equilibrium travel costs at each location. Specifically, the equilibrium travel cost for group 2 commuters is higher at each location along the corridor as they have larger VOTs and earlier preferred arrival time.

Table 3 Departure rates over time at each location

	Time	Location									
		1	2	3	4	5	6	7	8	9	10
Group 1 $\tilde{t}_1 = 70$	51	0	0	0	0	0	0	0	0	0	0
	52	0	0	0	0	0	0	0	0	2.861	13.826
	53	0	0	0	0	0	0	0	0	35.792	21.096
	54	0	0	0	0	0	0	0	9.370	17.652	11.217
	55	0	0	0	0	0	0	0	5.135	10.000	4.649
	56	0	0	0	0	0	0	8.430	8.239	6.219	4.890
	57	24.999	18.007	13.576	0	0	4.802	0	7.787	0	4.069
	58	37.498	28.161	30.572	31.813	19.145	0	0	9.455	0	4.252
	59	37.503	19.255	20.534	21.172	22.840	24.876	21.357	11.275	3.799	4.281
	60	0	14.123	15.128	15.291	16.267	17.432	15.397	8.940	3.334	3.722
	61	0	20.454	11.161	11.580	12.272	13.002	11.704	7.279	2.941	3.268
	62	0	0	9.029	9.206	9.785	10.122	9.236	6.373	2.627	2.893
	63	0	0	0	10.940	7.589	8.099	7.496	5.115	2.356	2.585
	64	0	0	0	0	12.103	6.682	15.971	4.413	2.592	2.325
	65	0	0	0	0	0	4.691	0	16.618	9.826	2.094
	66	0	0	0	0	0	10.293	0	0	0	1.286
	67	0	0	0	0	0	0	3.215	0	0	13.544
	68	0	0	0	0	0	0	3.460	0	0	0
	69	0	0	0	0	0	0	3.733	0	0	0
	70	0	0	0	0	0	0	0	0	0	0
Group 2 $\tilde{t}_2 = 50$	36	0	0	0	0	0	0	0	0	0	0
	37	24.990	17.996	21.456	16.271	4.809	8.862	1.912	0	0	0
	38	37.503	28.162	30.635	29.388	31.127	27.352	5.254	1.045	0	0
	39	37.507	53.842	45.323	19.910	20.968	23.170	24.897	27.325	5.401	0
	40	0	0	2.586	34.431	15.124	16.424	17.434	22.069	24.336	9.786
	41	0	0	0	0	27.971	12.363	13.000	15.688	17.502	15.439
	42	0	0	0	0	0	11.829	10.167	11.347	13.076	11.781
	43	0	0	0	0	0	0	7.043	1.071	0.439	9.316
	44	0	0	0	0	0	0	20.293	7.588	0.697	9.451
	45	0	0	0	0	0	0	0	13.868	38.550	9.447
	46	0	0	0	0	0	0	0	0	0	34.780
	47	0	0	0	0	0	0	0	0	0	0
	48	0	0	0	0	0	0	0	0	0	0
	49	0	0	0	0	0	0	0	0	0	0
	50	0	0	0	0	0	0	0	0	0	0

The departure rates of these two group users are shown in Table 3 (the time intervals with zero departure rates are omitted in the table). One can note that group 1 users have wider departure time choice than group 2 users. It is because group 1 commuters have lower VOTs, and they may choose early or late arrival to avoid traffic congestion. The model capability of incorporating commuter heterogeneity is practically useful and important in enabling the answer of transport managers' question that how the peak-hour congestion mitigation measures like flexible work time scheme and various work starting time scheme may affect commuters' departure time choice and how effective these schemes could be in containing peak-hour congestion.

5. Conclusion

In this study, a traffic corridor with many-to-one OD pattern and heterogeneous commuters is considered and the spatial and dynamic equilibrium with traffic congestion is modelled by applying a partial differential complementarity system (PDCS) approach. To solve the PDCS formulation numerically, a discretization scheme is applied to transform the PDCS model into a complementarity problem formulation. Solution existence is proved and a series of numerical studies are conducted to verify the model validity. Numerical examples are conducted to show how values of time and scheduling delay, as well as the population density distribution, may affect the equilibrium departure pattern. There are several limitations to the current model. For

example, to capture the queueing effect, we assume an average queueing time for all the travellers at one specific location, which is more suitable for the case with a relative low level of travel demand. To develop a more realistic model that can handle more congested scenario, it may be necessary to distinguish the accommodated demand from the desired demand as some travel demand may not be accommodated immediately into the highway. These limitations would be addressed in the future study. Moreover, although the continuum model is formulated, the numerical solution of this model is based on the discretization scheme. Besides, instantaneous travel time estimation, rather than real time one, is applied in this study to simplify the formulation. These issues are also supposed to be further addressed in the future study.

Acknowledgements

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