Contracting with Word-of-Mouth Management

By

Yuichiro Kamada and Aniko Öry

August 2018

COWLES FOUNDATION DISCUSSION PAPER NO. 2048R2

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.yale.edu/

Contracting with Word-of-Mouth Management

Yuichiro Kamada and Aniko Öry*

First Version: July 3, 2015

This Version: August 29, 2018

Abstract

We propose a model for word of mouth (WoM) management where a firm has two tools at hand: referral rewards and offering a free contract. Current customers' incentives to engage in WoM can affect the contracting problem of a firm in the presence of positive externalities of users. Formally, we consider a classic Maskin-Riley contracting problem for the receiver of WoM where the firm can pay the senders referral rewards and a sender experiences positive externalities if the receiver adopts. A free contract can incentivize WoM because the higher adoption probability increases the expected externalities that the sender receives. We characterize the optimal incentive scheme and show when the two tools serve as substitutes and complements to each other depending on whether the market is niche and whether the product is social. We show that offering a free contract is optimal only if the fraction of premium users in the population is small, which is consistent with the observation that companies that successfully offer freemium contracts oftentimes have a high percentage of free users.

*Kamada: Haas School of Business, University of California, Berkeley, Berkeley, CA 94720, e-mail: `y.cam.24@gmail.com`; Öry: School of Management, Yale University, New Haven, CT 06511, e-mail: `aniko.oery@yale.edu`. This paper is previously circulated as a working paper titled "Encouraging Word of Mouth: Free Contracts, Referral Programs, or Both?" We thank the editor, the associate editor, and referees for excellent suggestions to improve the paper. We are grateful to Juan Escobar, Johannes Hörner, Fuhito Kojima, Vineet Kumar, Dina Mayzlin, Takeshi Murooka, Motty Perry, Klaus Schmidt, Jiwoong Shin, Philipp Strack, Steve Tadelis, Juuso Välimäki, Miguel Villas-Boas, Alex Wolitzky, Jidong Zhou, and seminar participants at the 13th Summer Institute in Competitive Strategy, Osaka University, the University of Munich (LMU), Wharton, and Yokohama National University for helpful comments. Jovian Chen, Don Hampton, Emily Her, Yi (Michelle) Lu, Valeree Simon, and Tawney Warren provided excellent research assistance.

"Cost per acquisition: \$233-\$388. For a \$99 product. Fail."

—Drew Houston, founder of Dropbox

1 Introduction

In April 2010, Dropbox announced that it would start a referral program, increase visibility of its free 2 GB option, and introduce a sharing option. All in all, this led to 2.8 million direct referral invites within 30 days (Houston, 2010). Before the change, the costs per acquisition had been more than 200 dollars for a 99 dollar product, so Dropbox was not even able to survive in the market without word of mouth (WoM). The introduction of the sharing option makes Dropbox a "social product," with which users experience positive externalities from friends using the product. Similarly, WoM was essential for the growth of another social product, Skype. The company founded in 2003 spent nothing on marketing until it was acquired by eBay when it already had 54 million registered users (Eisenmann, 2006). Both Dropbox and Skype use the so-called "freemium" (a free contract + premium contracts) strategy. However, the former combines it with a referral program, which Houston (2010) emphasizes as a way to encourage WoM, while the latter only relies on a freemium strategy.

The objective of this paper is to develop a simple model that highlights when offering a free contract and referral rewards can optimally incentivize WoM about existence of a product. Specifically, we model the incentive for old customers (senders of WoM) to talk to new customers (receivers) who are offered a menu of contracts as in Maskin and Riley (1984).[1] The firm can reward senders directly through referral rewards. A reward to the receivers via a free contract increases the likelihood of them using the product. This in turn raises the size of the expected externalities the senders receive from talking, and thus encourages WoM. All in all, the model highlights a fundamental difference between referral rewards and a "freemium" strategy when it comes to encouraging WoM. Figure 1 offers a schematic presentation of the main logic.

We provide a characterization of the optimal incentive scheme for WoM, which we use to discuss *substitution* and *complementation* between the two strategies. This analysis allows us to understand

[1]Van den Bulte and Joshi (2007) consider a product diffusion model with influentials and imitators. One can think of our model as capturing the firm's strategies to encourage influentials to talk to imitators. Berger (2014) and Berger and Schwartz (2011) provide a summary of the various psychological reasons why consumers engage in WoM. Instead, we are interested in the strategic implications of different incentives for WoM.

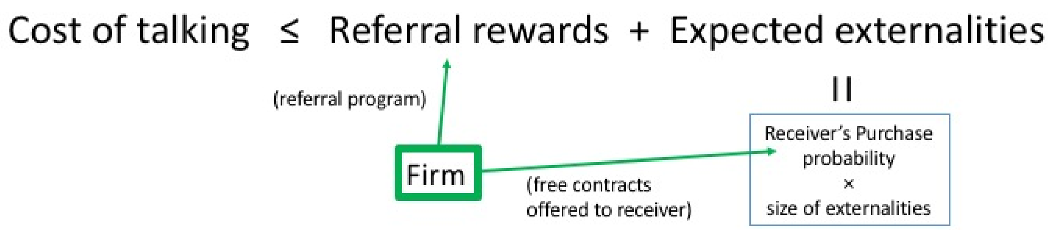


Figure 1: Schematic presentation of the sender's trade off

two fundamentally different situations that a company might be in. First, a company might already incentivize WoM with only one of the tools, but it could save costs by substituting the tool with the other one. Second, the company might not be able to incentivize WoM with any one of the two tools alone, but if it used both tools together it could successfully encourage talking as seems to be the case with Dropbox.

We focus on two market characteristics which turn out to determine whether substitution and complementation occur. First, a market can be either a *niche market* or a *mass market*, depending on the fraction of premium customers. Second, the product can be a *social product* or a *private product*, depending on the positive externalities it generates among customers. We show that substitution occurs for social products, while complementation occurs for rather private products. For referral rewards to substitute a free contract, the market needs to be mass, while a free contract substitutes referral rewards if the market is niche. This difference arises because the benefit of using a free contract is to expand the expected externalities that the senders receive. The "jump" of the expected externalities is large (and thus effective in incentivizing WoM) only when the fraction of users who would otherwise not use the product is high. For a related reason, a free contract complements referral rewards only in a very niche market, while referral rewards can complement a free contract also in a slightly more mass market.

The optimal scheme that we characterize exhibits a rich pattern of the use of referral rewards and a free contract, and the prediction is roughly consistent with what we observe in the real world.

Most notably, our findings are consistent with a paradoxical feature of the customer base of the aforementioned companies: While profits kept increasing, Dropbox faced consistently only 4% of customers actually paying for the product (*Economist*, 2012). Similarly, only 8% of the customers who are served by Skype actually pay. As we discussed, our model predicts that a free contract is useful in niche markets because it is used to boost up the expected externalities. Conditional on the fraction of "premium users" being low, referral rewards are not used for sufficiently social products, which is consistent with Skype's strategy. When the externalities are not too low or too high, referral rewards are used in conjunction with a free contract, which is consistent with Dropbox's strategy. For ride share companies like Uber, externalities would be low (it is a private product) and the share of users who would be willing to pay is high (the market is mass). Our model shows that the optimal scheme is not to use a free contract but to offer referral rewards, being consistent with Uber's strategy.

The positive externalities of the receiver on the sender plays a key role in our model. Externalities can be a real value of social usage , e.g., sharing documents on Dropbox, or psychological benefit from having convinced a friend to use the same product as in (Campbell et al., 2015). The sender may also benefit from the continuation value in a repeated relationship with the receiver if the receiver actually starts using the product. While we focus on the externalities generated by the receiver, naturally, the sender can also generate externalities or the receiver can become a new sender and generate externalities. We abstract away from this in our main analysis, as the focus is on the sender's incentive to talk and it would not change the essence of our analysis.[2]

The argument also requires that there is an exogenous cost of talking for the sender. There are many reasons why talking may be costly: Senders incur opportunity costs of talking (Lee et al., 2013), and/or they may feel psychological barriers. We assume that each sender wants to talk if and only if the cost of talking is smaller than the benefit. Lastly, we only consider WoM about the existence of a product. For commodities (such as online storage or phone services) and for new products and categories (such as new startups), one of the main purposes of WoM seems to be to inform the receiver of the existence. The study of WoM about the product quality or match value can be especially important in other product markets, but is abstracted away from in our model.[3]

[2] The Online Appendix discusses the case when the receiver receives externalities r as well.

[3] For example, Anderson (1998) studies WoM concerning evaluations of goods and services and more recently, Leduc et al. (2017) study WoM about quality in a network.

The paper is structured as follows. Section 2 introduces the model. Section 3 characterizes the optimal scheme when only a free contract can be used, when only referral rewards can be used, and when both tools are available (the full model). Section 4 compares the optimal schemes in those three models to identify when substitution and complementation occur. Section 5 discusses comparative statics for the full model. Finally, Section 6 concludes. The Appendix generalizes the full model in a number of ways, and provides proofs for the results. The Online Appendix discusses various extensions, robustness checks, and welfare considerations.

1.1 Related Literature

In general, WoM can be about either existence of a product, its quality or the match value. Papers in marketing that consider WoM about product existence like us include Biyalogorsky et al. (2001), Campbell et al. (2015), and Kornish and Li (2010). The view that WoM can be about quality of a product is adopted in marketing as early as Dichter (1966), and has been recently studied by Mayzlin (2006) in the context of reviews when advertising and WoM act as substitutes. The importance of information contagion relative to targeted marketing has also been studied by Manchanda et al. (2008) and Iyengar et al. (2011), among others. Recently, papers in particular on online reviews have studied the role of customer communication on match value (e.g. Chen and Xie (2008)). Godes et al. (2005) provide a detailed survey of the literature on various aspects of WoM.

We show that a firm can encourage WoM about the existence of a product indirectly through free contracts given to the receiver. This explanation complements the existing explanations in the literature on how to encourage WoM: Biyalogorsky et al. (2001) compare the benefits of price reduction and referral programs in the presence of WoM. In their model, a reduced price offered to the sender of WoM is beneficial because it makes the sender "delighted" and thereby encourages him to talk. Depending on the delight threshold, the seller should use one of the two strategies or both. In contrast, our focus is on WoM in the presence of positive externalities of talking and our model accommodates menus of contracts. In Campbell et al. (2015), senders talk in order to affect how they are perceived by the receiver of the information. The perception is better if the information is more exclusive. Thus, a firm can improve overall awareness of the product by restricting access to information (i.e., by advertising less). One could interpret the positive externalities in our model also as a reduced form of a "self-enhancement motive" as in their model. Bimpikis et al. (2016)

analyze how advertising should be targeted in the presence of WoM. Although we discuss advertising in the Online Appendix, we focus on the relative effectiveness of free contracts and referral rewards instead of advertising. Kornish and Li (2010) consider the tradeoff between referral rewards and pricing in a model where the sender cares about the receiver's surplus and the firm offers a single price. Due to the assumption of a single price, any price set by the firm generating strictly positive profits is necessarily strictly positive, so it cannot accommodate free contracts. Our model, on the other hand, allows for screening by the firm and we analyze how that interacts with referral rewards in the presence of externalities. This enables us to give predictions consistent with the strategies used by various companies.

Most of the other theoretical literature on WoM has focused on mechanical processes of communication in networks. This literature mostly focuses on how characteristics of the social network affect a firm's optimal advertising and pricing strategy. Campbell (2012) analyzes the interaction of advertising and pricing. Galeotti (2010) is concerned with optimal pricing when agents without information search for those with information. Galeotti and Goyal (2009) show that advertising can become more effective in the presence of WoM (i.e., WoM and advertising are complements) as well as that it can be less effective (i.e., WoM and advertising are substitutes). All of these papers consider information transmission processes in which once a link is formed between two agents, they automatically share information. Related to the network literature, Goldenberg et al. (2001) study WoM across weak versus strong ties in a network.

WoM is also analyzed empirically in the marketing literature. Godes and Mayzlin (2009) and Aral and Walker (2011), among others, study other aspects of WoM management, e.g., who to target (e.g., loyal customers, opinion leaders, etc.) and how to facilitate the referral process by offering personalized referral features or automated broadcast notifications. Schmitt et al. (2011) study how valuable referred customers are in the data. Katona et al. (2011) analyze how diffusion is affected by the network structure and characteristics of direct neighbors.

There is also a literature on contracting models in the presence of network effects. Besides the critical difference that our focus is on how the firm can optimally affect incentives to talk, there is a subtle difference in the optimal contracts. Csorba (2008) analyzes a contracting model in which the more the other buyers use the product, the higher the utility from using the product is.[4] He

[4] See Segal (2003) for a seminal work on this literature. See also Hahn (2003).

shows that an optimal contract scheme introduces a distortion at the top because a reduction of the quantity offered to low types should decrease the value of the product to high types. Unlike in his model, we have no distortions at the top in the optimal contract scheme. The reason is that receivers do not receive externalities from each other, and that we consider quantity-independent externalities rather than assuming that the total quantity consumed generates externalities. We discuss the implications of quantity-dependent externalities in the Online Appendix. We do not consider the case of externalities between receivers themselves given our focus on the sender's incentives to talk. Introducing such a feature would not change the qualitative results on the optimal incentive schemes to encourage WoM. The modeling difference leads to the difference in terms of applications. When the focus is on receivers generating externalities to each other, the model would be suitable for the analysis of, for example, social networks such as LinkedIn or Facebook. In such a context, a recent working paper by Shi et al. (2017) considers a static model of product line design without WoM when free users generate positive externalities on all premium users. When the firm can manipulate the amount of externalities enjoyed by customers conditional on the user type, freemium contracts can arise as an optimal strategy. In contrast, in our model, there is no manipulation of the size of externalities and the price of the low-type contracts must be zero because the surplus from selling to the low types is negative. Even so, the monopolist sells contracts with positive quantities for free to the low types because those free contracts encourage WoM which attracts premium users.

The marketing literature has offered multiple views regarding the role of WoM in the presence of advertising, depending on the context. Campbell (2012) and Joshi and Musalem (2017) show that WoM and advertising are complements to each other in their model, while Campbell et al. (2015) and Hollenbeck et al. (2017) show they are substitutes. In the model of Fainmesser et al. (2018), they can be either substitutes or complements depending on what information is disclosed. In our model, we show in the Online Appendix that WoM substitutes advertising. This is because advertising increases the probability of the receiver already knowing the product before the sender's talking, which necessitates the firm to pay a higher referral reward conditional on the receiver starting to use the product due to the sender talking. Hence, advertising may be detrimental to the firm in the presence of WoM. This rationale for substitutes hinges on the firm offering referral rewards, and is a an insight new to the literature.

While the focus of this paper is not to add another rationale for freemium strategies, it is important to note the connection to the literature on "freemium" strategies. The literature has identified various other reasons: (i) free contracts may be useful in penetration of customers or information transmission about the quality of the product to them, which can induce their upgrade,[5] (ii) the firm may hope that the free users will refer someone who will end up using the premium version,[6] (iii) free products attract attention of customers and prevent them from purchasing the competitors' products, and (iv) the increased number of customers due to free contracts raises the advertising revenue or sales of data.[7] None of these reasons pertains to the senders' incentives. Instead, our focus (with regards to free contracts) is on how free contracts help firms to manage senders' incentives. Thus, instead of convoluting our model with these other aspects of free contracts, we aim to *isolate* the effect of the tradeoff that the senders of information face. Similarly, we do not intend to create a "complete" model that incorporates all conceivable features that are relevant for firms' decision making. Instead, the goal of this paper is to understand how the incentives for WoM can be managed. Our simplification allows us to isolate the factors pertaining to the encouragement of WoM and to examine the tradeoffs involved.

2 Model

We present a simple model using a specific functional form to illustrate our main points. The model and the results are generalized in the Appendix in a number of ways. Proofs of the results presented in the main text follow from the general results presented in the Appendix. In the main text of the paper we focus on explaining the implications of the results without going into technical details.

A monopolist seller produces a product at marginal cost $c = 0.2$ and zero fixed cost. There are two customers, the *sender* (he) and the *receiver* (she). The sender already knows about the existence of the product, while the receiver does not. The receiver is either a high type (H) with

[5] Formally, we rule out this effect by assuming that, after learning about the existence of the product, each customer has a fixed valuation to (and information about) the product that does not change over time.

[6] A recent working paper by Ajorlou et al. (2015) builds a social-network model that highlights this effect. Lee et al. (2015) empirically analyze the trade-off between growth and monetization under the use of freemium strategies, in which the value of a free customer is determined by upgrade as in (i) and the free users' referrals as in (ii).

[7] See Shapiro and Varian (1998) for (i)-(iii) and Lambrecht and Misra (2016) for (iv).

probability α or a low type (L) with probability $1-\alpha$.[8] The H-type receives $10\sqrt{q}$ for consuming quantity/quality $q \geq 0$ of the product. Meanwhile, the L-type receives $\sqrt{q}$ for consuming q. We can also interpret q as the quality of the product.[9] Each receiver incurs a fixed installation cost of $I = 3$ if the consumed quantity is strictly positive ($q > 0$). Thus, the net benefit of consuming a quantity $q > 0$ is $10\sqrt{q} - 3$ and $\sqrt{q} - 3$ for the H- and L-type receiver, respectively. The type is private information to the receiver.[10]

The game consists of three stages. First, the seller offers a *scheme* $((p_L, q_L), (p_H, q_H), R)$, which consists of a menu of contracts $((p_L, q_L), (p_H, q_H)) \in (\mathbb{R}_+ \times \mathbb{R}_+)^2$ as well as the referral rewards $R \in \mathbb{R}_+$.[11] Here, for each $\theta = H, L$, p_θ is a price offered to the θ-type buyer, and q_θ is a quantity offered to that buyer. The referral rewards are a payment from the seller to the sender that are made if the sender talks to the receiver. The rewards are assumed to be paid irrespective of the subsequent purchase behavior by the receiver. In the Appendix, we allow for the possibility that the seller makes the referral rewards conditional on the receiver's purchase behavior and show that such conditioning does not increase the seller's profit. In the second stage, observing the menu offered by the seller, the sender decides whether to talk to the receiver or not.Third, the receiver makes a purchase decision if and only if the sender has talked to the receiver.

The objective of the receiver is to choose the contract that maximizes her surplus (as in Maskin and Riley (1984)). The sender incurs a constant cost $\xi = 10$ of talking.The benefit of taking is the sum of two components. The first component is the referral rewards paid by the seller. The second is the *expected externalities*, which can be calculated as follows: If the receiver purchases and uses the product, then the sender experiences the externalities of level $r \geq 0$.

Hence, if the sender expects that both types buy the product, then the expected externalities are r. If instead he expects that only the H-type uses the product, then the expected externalities are αr.[12] The seller's objective is to maximize the expected profit from the receiver net of the

[8] Although the Introduction discussed the "share" of each type, here we consider probability because there is only one receiver in this simple setup. This assumption is generalized in the Appendix.

[9] Interpreting q as quality would make a difference if we had learning about quality in the model, where using different contracts may result in different ex-post valuations.

[10] It is not crucial for our results that the sender does not know the type of the receiver, while it is important that the seller knows less about the type of the receiver than the sender does, which we view as a reasonable assumption.

[11] In the generalized model in the Appendix, we allow for negative prices as well and we show that they cannot be optimal if we distinguish between purchase and consumption of the product.

[12] The assumed functional form of the payoff functions implies that there is no possibility of only the L-type using the product.

referral rewards, subject to the following participation constraints (PC)

$$(10\sqrt{q_H} - 3) - p_H \geq 0 \quad \text{and} \quad (\sqrt{q_L} - 3) - p_L \geq 0,^{13}$$

and incentive compatibility (IC) conditions for the two types

$$(10\sqrt{q_H} - 3) - p_H \geq (10\sqrt{q_L} - 3) - p_L \quad \text{and} \quad (\sqrt{q_L} - 3) - p_L \geq (\sqrt{q_H} - 3) - p_H,$$

as well as the incentive compatibility for the sender[14]

$$\xi \leq R + \begin{cases} r & \text{if the sender expects that both types buy} \\ \alpha r & \text{if the sender expects that only the } H\text{-type buys} \end{cases}.$$

In order to be able to formally state our results, we denote the (non-empty) set of optimal schemes (i.e., maximizing the seller's profit) given parameters (α, r) to this problem by

$$\mathcal{S}(\alpha, r) \subseteq (\mathbb{R}_+ \times \mathbb{R}_+)^2 \times \mathbb{R}_+.^{15}$$

3 Optimal Scheme

In this section, we characterize the optimal scheme for the model described in Section 2. Before doing so, we first consider two benchmark models, in which either referral rewards or a free contract is not allowed. We analyze these benchmark models in order to later compare them with the full model. This helps us understand the role as substitutes or complements of referral rewards and a free contract in the optimal scheme. Note that these cases are also interesting in themselves to understand 1) for which parameters a firm can incentivize WoM solely with referral rewards and 2) for which parameters it can incentivize WoM solely with free contracts.

Section 3.1 considers a benchmark model in which using referral rewards is prohibited and

[13] An implicit assumption in the participation constraints is that the outside option generates zero surplus. The result that the price for the L-type buyer is 0 still holds (although the quantity offered is adjusted accordingly) even if the outside option generates a positive surplus.

[14] We assume the sender has already purchased the product so there is no additional revenue from the sender.

[15] Existence is proven in a more general environment in the Appendix. We will also introduce notations $\mathcal{S}^{\text{NR}}(\alpha, r)$ and $\mathcal{S}^{NF}(\alpha, r)$, and one can show by analogous proofs that those are also nonempty.

characterizes the optimal scheme. Section 3.2 then characterizes the optimal scheme for the model in which using a free contract is prohibited. Finally, Section 3.3 characterizes the optimal scheme for the full model. Although we will not be detailed about the derivation in Section 3.3, we give rather detailed explanation in Sections 3.1 and 3.2 as they provide some relevant intuition in a very simple setting.

3.1 Benchmark without Referral Reward

First, we consider the situation where the referral rewards R are exogenously set to be equal to 0. We call this model the *no rewards model.* The set of optimal schemes in the no rewards model given parameters (α, r) is denoted by

$$\mathcal{S}^{\mathrm{NR}}(\alpha, r) \subseteq (\mathbb{R}_+ \times \mathbb{R}_+)^2 \times \{0\}.$$

Fix an optimal scheme $((p_L^*, q_L^*), (p_H^*, q_H^*), 0) \in \mathcal{S}^{\mathrm{NR}}$. Notice that, if the L-type uses quantity q of the product, then her value is nonnegative only if $\sqrt{q} - 3 \geq 0$, and the L-type's marginal benefit from using the product is $\frac{1}{2\sqrt{q}}$. This implies that the marginal benefit is at most $\frac{1}{6}$ when the value is nonnegative. Since the marginal cost of production is $c = 0.2 > \frac{1}{6}$, the only reason that the seller would offer a positive quantity of the product to the L-type in an optimal scheme is to induce the sender to talk.[16] Since the sender's cost of talking is 10, the L-type is offered a product only if $10 > \alpha r$ (assuming that doing so results in a nonnegative profit). Moreover, when the L-type is offered a product under the optimal scheme, q_L^* must be the lowest quantity in order for the L-type to use the product. Hence, we must have $\sqrt{q_L^*} - 3 = 0$, or $q_L^* = 9$.

If the seller offers a contract to the H-type only, then as in the standard model of screening, the price is set to extract the entire surplus from the H-type, and q_H^* is a solution of the first-order condition of the seller's problem, $\frac{10}{2\sqrt{q_H^*}} - 0.2 = 0$, i.e., $q_H^* = 625$. Since the optimal price for the H-type p_H^* satisfies the PC, we must have $p_H^* - (10\sqrt{625} - 3) = 0$, or $p_H^* = 247$. The sender's IC constraint is $\xi = 10 \leq \alpha r$.

If both types are offered a contract, then only the H-type's IC and L-type's PC are binding as

[16]This conclusion can be different if the L-type buyer generates other revenues such as advertising revenue. In our applications (Skype, Dropbox, Uber, etc.), however, advertising revenue seems not to play an important role.

in the standard screening models:

$$10\sqrt{q_H^*} - p_H^* = 10\sqrt{q_L^*} - p_L^* \quad \text{and} \quad \sqrt{q_L^*} - 3 - p_L^* = 0.$$

Since we know $q_L^* = 9$, we have $p_L^* = 0$. Also, $q_H^* = 625$ implies that $p_H^* = 220$. The profit is thus

$$\alpha(220 - 0.2 \cdot 625) + (1-\alpha)(0 - 0.2 \cdot 9) = 96.8\alpha - 1.8.$$

This is strictly positive if and only if $\alpha > \frac{1.8}{96.8}$. The sender's IC constraint is simply $\xi = 10 \leq r$.

Given this, the following theorem characterize the optimal scheme.

Proposition 1 (Characterization for the No Rewards Model)**.** *1.* **(Positive profits)** *There exists an optimal scheme generating a strictly positive profit if and only if*

$$\alpha > \frac{1.8}{96.8} \text{ and } r > 10. \tag{1}$$

If (1) *is satisfied, then* $\mathcal{S}^{NR} \subseteq \{((0,0),(247,625),0),((0,9),(220,625),0)\}$.

2. **(Free vs. no free contracts)** $((0,9),(220,625),0) \in \mathcal{S}^{NR}$ *if and only if* $r \leq \frac{10}{\alpha}$.

Figure 2 depicts the optimal scheme for each (α, r) pair. The figure labels each region with a description of an optimal scheme in that region (including the boundaries of the region) whenever its interior generates a strictly positive profit. It also shows a region in which the maximized profit is zero (including the boundaries of the region). In the interior of each region with a name of a scheme, Theorem 1 implies that the scheme achieves the unique optimum.[17] The theorem implies that a free contract is used in an optimal scheme if $10 \leq r \leq \frac{10}{\alpha}$ and $\alpha \geq \frac{1.8}{96.8}$. The reason is that, if the externality r is too low ($r < 10$), then the sender cannot be incentivized to talk even if a free contract is offered and if it is too high ($r > \frac{10}{\alpha}$), the sender talks anyway to receive externalities from the H-type even absent a free contract. If the probability α is too low ($\alpha < \frac{1.8}{96.8}$), the revenue from the H-type is not enough to cover the cost of a free contract (we will be more explicit about what this cost is in Section 3.3).

[17]The same remark applies to other figures in this paper, too.

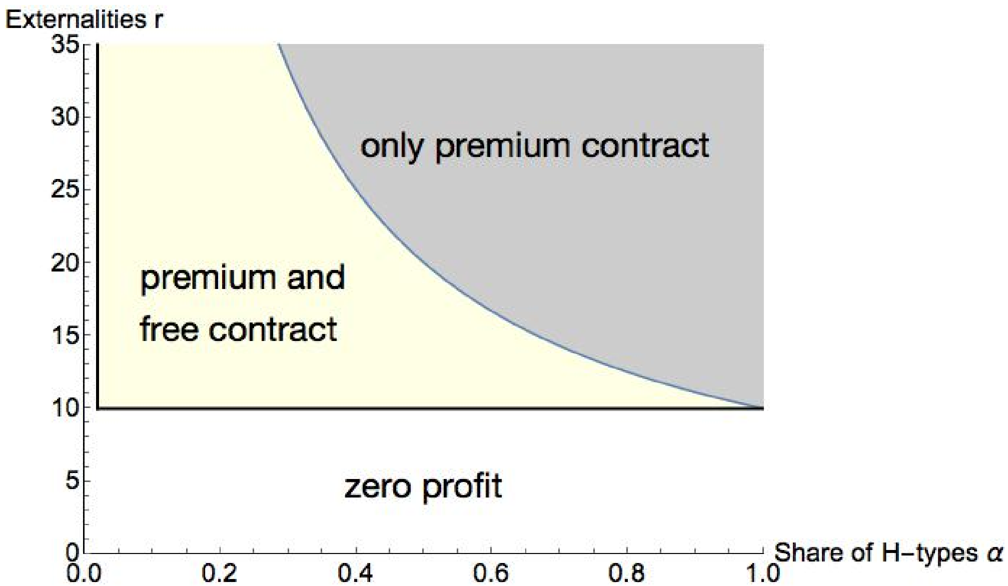


Figure 2: No Rewards Model

Remark 1. Note that a free contract arises endogenously in our model. If the firm serves the L-type customer to incentivize WoM, it is optimal to offer just enough to make her use the product, making zero the only possible price to the L-type in any optimal scheme. Although this might seem like an artifact of having only two types, we show in the Online Appendix that free contracts arise endogenously even with a continuous type space. Thus, in a sense, the L-type in the two-type model can be interpreted as the customer who the firm should not serve absent of the need to encourage WoM. In the extension with a continuous type space, we also show that only the marginal type who buys a free contract is made indifferent between using the product and not using the product while other "higher" low types enjoy some surplus from using the free product.

3.2 Benchmark without a Free Contract

Second, we consider a model in which the seller is restricted to offer only one contract to the receiver. We call this model the *no free-contract model*. The set of optimal schemes in the no free-contract model given parameters (α, r) is denoted by

$$\mathcal{S}^{NF}(\alpha, r) \subseteq (\{0\} \times \{0\}) \times (\mathbb{R}_+ \times \mathbb{R}_+) \times \mathbb{R}_+.$$

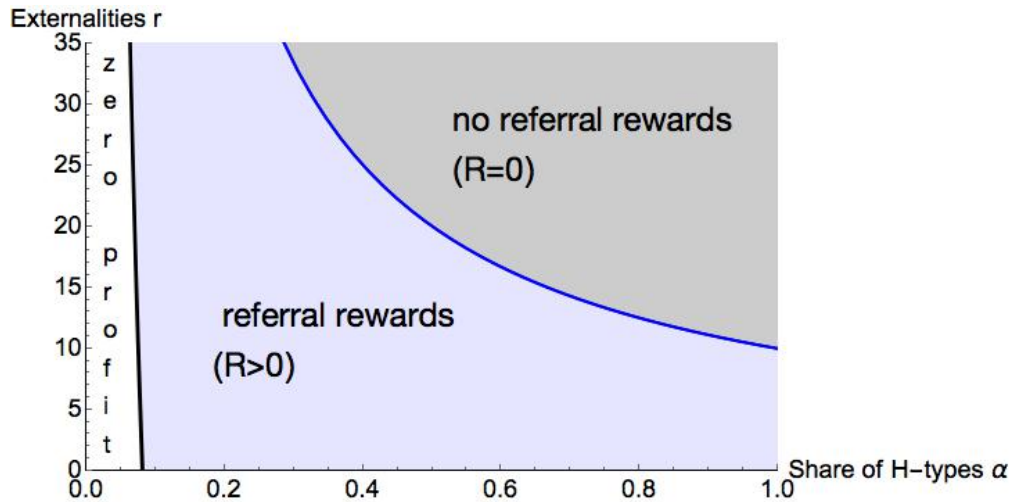


Figure 3: No Free-Contract Model

In this model, in a reasoning similar to the one in the no rewards model, the firm only offers one contract that only the H-type buys, which means that if $((0,0),(p^*_H,q^*_H),R^*) \in \mathcal{S}^{NF}(\alpha,r)$, then we must have $(p^*_H,q^*_H) = (247,625)$. The sender is incentivized to talk even with $R^* = 0$ if $\alpha r \geq 10$, while referral rewards of the amount $10 - \alpha r$ need to be paid to incentivize WoM otherwise. However, if the revenue from the H-type, which is 122α $(= \alpha(247 - 0.2 \times 625))$, is less than the reward payment $10 - \alpha r$, then WoM cannot be incentivized under the optimal scheme and the maximized profit is zero. This leads to the following characterization.

Proposition 2 (Characterization for the No Free-Contract Model)**.** *1.* **(Positive profits)** *There exists an optimal scheme generating a strictly positive profit if and only if*

$$10 < \alpha r + 122\alpha. \tag{2}$$

If (2) *is satisfied, then* $\mathcal{S}^{NF}(\alpha,r) = \{((0,0),(247,625),R)\}$ *for some* $R \geq 0$*.*

2. **(Rewards vs. no rewards)** $\mathcal{S}^{NF} = \{((0,0),(247,625),R)\}$ *with* $R > 0$ *if and only if* $r < \frac{10}{\alpha}$*.*

The result is illustrated in Figure 3. The intuition is simple: Referral rewards are useful if the size of the expected externalities is not enough to cover the cost of talking ($\alpha r < 10$), while covering the cost of talking by paying the referral rewards is not too expensive relative to the revenue from the receiver. Since the rewards payment and the revenue from the receiver (conditional on the receiver buying) are both decreasing in α, the region for which referral rewards are used in a optimal scheme requires α not to be too low.

3.3 The Full Model

Now we consider the full model. As in the no rewards model, the optimal menu of contracts is either $\{(0,9),(220,625)\}$ or $\{(247,625)\}$, depending on whether the L-type is served or not. One can completely characterize the optimal scheme, which we present below. To state the result formally, it is useful to define the following "cost of a free contract," denoted by CF^*:

$$CF^* = 27\alpha + 1.8(1-\alpha).$$

To understand this, note that there are two disadvantages of providing a free contract. The first is that the seller has to pay the cost of production when the buyer is of L-type L. The quantity provided to the L-type is 9, and the firm incurs the marginal cost $c = 0.2$ for each unit. Since there is a $1-\alpha$ probability of the buyer being the L-type, this part of the cost amounts to $0.2\times 9\times(1-\alpha)$, which is the second term of CF^*. Second, the fact that the L-type is offered a positive quantity implies that the H-type must be incentivized to choose the contract offered to her over the one offered to the L-type. For this purpose, the seller needs to reduce the price by the amount of information rent, which is the valuation difference between the two types for the quantity that the L-type is offered, which is given by $(10\sqrt{9}-3)-(\sqrt{9}-3) = 27$. Since the probability of the receiver being an H-type is α, this part of the cost amounts to $27\times\alpha$, which is the first term of CF^*.

Furthermore, it is useful to note that the profit for a hypothetical case in which, as in a classic screening model, the cost of talking is zero and hence the sender always informs the buyer of the existence of the product is given by $\alpha(10\sqrt{625} - 3 - 0.2\cdot 625) = 122\alpha$, where 625 is the quantity that we solved for in the previous section in analyzing the no rewards model.

Using these two values, we can now characterize the optimal scheme for the full model.

Proposition 3 (Characterization for the Full Model)**.** *1.* **(Positive profits)** *There exists an*

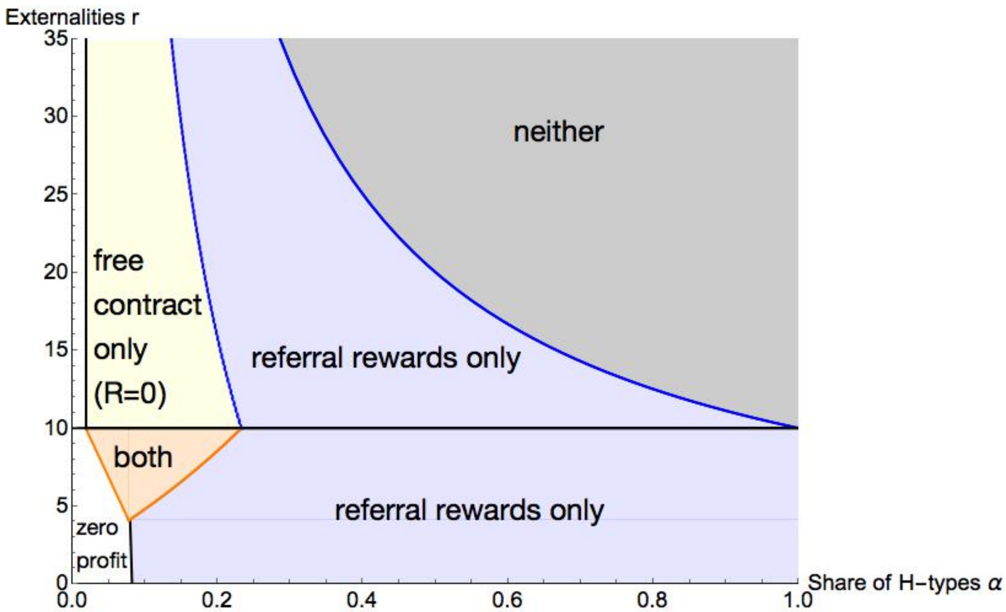


Figure 4: Full Model

optimal scheme generating a strictly positive profit if and only if[18]

$$10 < \max\left\{122\alpha - CF^* + \min\{r, 10\},\ 122\alpha + \alpha r\right\}. \tag{3}$$

If (3) *is satisfied, then* $\mathcal{S} \subseteq \{((0,0),(247,625),R)|R \in \mathbb{R}_+\} \cup \{((0,9),(220,625),R)|R \in \mathbb{R}_+\}$.

2. **(Free vs. no free contracts)** *There exists* $((0,9),(220,625),R) \in \mathcal{S}$ *for some* R *if and only if* $r \in \left[\frac{CF^*}{1-\alpha}, \frac{10-CF^*}{\alpha}\right]$.[19]

3. **(Rewards vs. no rewards)**

   (a) **(With free contracts)** *If* $r \in [\frac{CF^*}{1-\alpha}, \frac{\xi-CF^*}{\alpha}]$, *then* $((0,9),(220,625),R) \in \mathcal{S}$ *with* $R > 0$ *if and only if* $r < 10$, *and*

   (b) **(With no free contracts)** *If* $r \notin [\frac{CF^*}{1-\alpha}, \frac{\xi-CF^*}{\alpha}]$, *then* $((0,0),(247,625),R) \in \mathcal{S}$ *with* $R > 0$ *if and only if* $r < \frac{10}{\alpha}$.

The optimal scheme is illustrated in Figure 4. As one can see, the characterization of the optimal scheme in the full model entails a rich pattern. In particular, there are five different regions, a region in which the profit is zero, only a free contract is used, only referral rearwards are

[18] $10 < 122\alpha - CF^* + 10$ is equivalent to $\alpha > \frac{1.8}{96.8}$.

[19] If $\frac{CF^*}{1-\alpha} > \frac{10-CF^*}{\alpha}$, then $[\frac{CF^*}{1-\alpha}, \frac{10-CF^*}{\alpha}] = \emptyset$.

used, both are used, and none is used while the profit is strictly positive. The detailed intuition for this rich pattern will be investigated in the next section by comparing the full model with the no rewards and no free-contract models. We only note here that if the probability of the H-type is too small (i.e., $\alpha < \frac{1.8}{96.8}$), then profits generated become too small to make it worthwhile to encourage WoM (i.e., the maximized profit is zero).[20] With small externalities r, the sender has little innate benefit from WoM, so the lower bound of α above which the profit is positive is large.

4 Substitution and Complementation

In Section 3 we show that the optimal scheme can take many different forms depending on the relevant parameters in the model. Using the characterizations in the previous section, this section aims to shed light on the interaction of referral rewards and free contracts. There are two fundamentally different situations that a company might be in. First, a company might only use one of the tools and be successfully incentivizing WoM, but substituting the tool with the other one could be cost-saving. Second, the company might not be able to generate WoM with any one of the two tools alone, but if it used both tools together it could successfully encourage talking. For example, in the leading example of Dropbox that we discussed in the Introduction, it seems that it was important to use both tools together. To understand the profit-maximization problem faced by such companies, we address the following research questions:

1. When are referral rewards better substitutes for free contracts and vice versa?
2. When can referral rewards and free contracts complement each other?

To answer those questions, we first clarify what we mean by substitution and complementation. First, the introduction of the possibility of referral rewards can make it unnecessary to use a free contract in incentivizing the sender to talk in the optimal scheme. In such a case, we say that *referral rewards substitute a free contract*.[21] In contrast, the introduction of the possibility of referral rewards may make a free contract useful in incentivizing the sender to talk in the optimal

[20] This region disappears with heterogeneous priors as we show in the Online Appendix.

[21] Formally, it corresponds to the case where it is uniquely optimal for the seller to offer a free contract under the no rewards model, while it is uniquely optimal not to offer it while offering referral rewards in the full model.

scheme. In such a case, we say that *referral rewards complements a free contract.*[22] A free contract substituting and complementing referral rewards is defined analogously, by comparing the full model with the no free-contract model.

We will discuss for which parameters α and r substitution and complementation occur. We can interpret markets with high α as *mass markets* and markets with small α as *niche markets.* We can also interpret products with high degrees of positive externalities r as *social products*, and those with low degrees as *private products.* In the following, we identify for which α and r one of the two tools substitutes or complements the other and explain the intuition behind it (Sections 4.1 and 4.2). We show that there is a subtle difference between referral rewards substituting and complementing a free contract, and a free contract substituting and complementing referral rewards (Section 4.3).

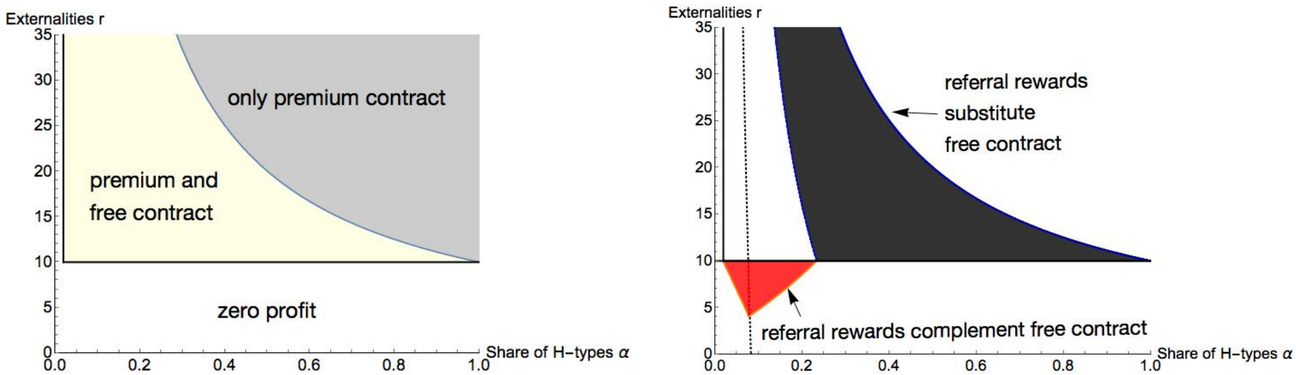
4.1 Referral Rewards Substituting and Complementing a Free Contract

We first compare the full model with the no rewards model. By doing so, we aim to understand when referral rewards can substitute as well as complement a free contract. To this end, the left panel of Figure 5 reproduces Figure 2, while the right panel shows the regions where substitution and complementation occur due to the introduction of referral rewards. Specifically, the interior of the black region in the right panel of Figure 5 corresponds to the parameter combinations under which referral rewards substitute a free contract, and the interior of the red region of the same panel shows the parameter combinations under which referral rewards complement a free contract. Substitution and complementation occur in the regions shown in Figure 5 for the following reasons.

- **Substitution:** Substituting a free contract with referral rewards is an effective strategy if it is cheap enough to do so. Notice that offering a free contract boosts up the benefit of talking by $(1-\alpha)r$. By using referral rewards instead of a free contract, the seller must pay referral rewards up to that amount. This payment is small if α is high (mass market). Hence, substitution occurs when α is high.

- **Complementation:** The reward payment required to induce the sender to talk can be kept low enough if a free contract alone could have already covered most of the cost of talking.

[22]Formally, it corresponds to the case where it is uniquely optimal for the seller not to offer a free contract under the no rewards model, while it is uniquely optimal to offer it with also offering referral rewards.



(a) No Rewards Model

(b) Regions for substitution and complementation

Figure 5: Referral Rewards Substituting and Complementing a Free Contract

This is the case when the externality level r is not too low (social product). Moreover, for a free contract to be offered, α cannot be too low as then the revenue from the receiver is too low, while it cannot be too high as then $(1-\alpha)r$ is too small so offering a free contract is not worth the cost and it is better to use only referral rewards.

Before closing this subsection, we formalize our findings. Recall that each of $((p_L, q_L), (p_H, q_H), R) \in \mathcal{S}$ and $((p_L, q_L), (p_H, q_H), 0) \in \mathcal{S}^{\text{NR}}$ implies $p_L = 0$ and $q_H = 625$, and either (i) $q_L = 0$ and $p_H = 247$ or (ii) $q_L = 9$ and $p_H = 220$. Say that $(\alpha, r) \in \text{SUB}^{\text{rewards}}$ if $((0,9),(220,625),0) \in \mathcal{S}^{\text{NR}}(\alpha, r)$ while there is no $R \geq 0$ such that $((0,9),(220,625),R) \in \mathcal{S}(\alpha, r)$. Similarly, we say that $(\alpha, r) \in \text{COM}^{\text{rewards}}$ if $((0,9),(220,625),0) \notin \mathcal{S}^{\text{NR}}(\alpha, r)$ while there exists an $R \geq 0$ such that $((0,9),(220,625),R) \in \mathcal{S}(\alpha, r)$. That is, $\text{SUB}^{\text{rewards}}$ and $\text{COM}^{\text{rewards}}$ correspond to the parameter regions such that referral rewards substitute and complement, respectively, a free contract. Finally, let $\Pi^{NR}(\alpha, r)$ be the maximized profit under parameters (α, r) in the no rewards model.

Theorem 1 (The Effect of Referral Rewards)**.**

1. *(Substitution) Suppose that* $(\alpha, r) \in \text{SUB}^{\text{rewards}}$*. Then, for any* α' *such that* $((0,9),(220,625),0) \in \mathcal{S}^{NR}(\alpha', r)$*,* $(\alpha', r) \notin \text{SUB}^{\text{rewards}}$ *implies* $\alpha' < \alpha$*.*

2. *(Complementation)*

   (a) *Suppose that* $(\alpha, r) \in \text{COM}^{\text{rewards}}$*. For any* r' *such that* $\Pi^{NR}(\alpha, r') = 0$*,* $(\alpha, r') \notin \text{COM}^{\text{rewards}}$ *implies* $r' < r$*.*

(b) *Fix r. There are $\underline{\alpha} > 0$ and $\bar{\alpha} < \infty$ with $\underline{\alpha} \leq \bar{\alpha}$ such that the following holds. If $(\alpha, r) \in \mathrm{COM}^{\mathrm{rewards}}$, then, $\alpha \in (\underline{\alpha}, \bar{\alpha})$.*

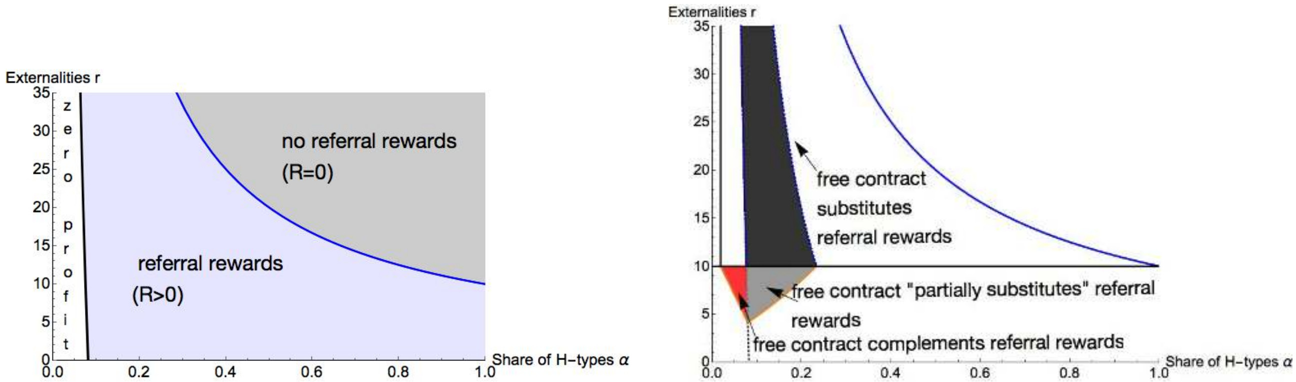
The proof immediately follows from Propositions 1 and 3. The first part of this theorem states that it is cost efficient to completely substitute referral rewards with a free contract in mass markets, but not in niche markets. The second part states that if a firm cannot incentivize WoM only with a free contract alone, referral rewards can complement a free contract and help to incentivize WoM in markets that are niche but not too niche to guarantee a positive profit, while having a sufficiently high level of externalities.

Remark 2. We note that a parameter combination (α, r) is in the region in which neither tool is needed to incentivize WoM in the no rewards model if and only if it is in such a region in the full model since the sender talks anyway without any additional incentives in either model in such a region. Also, the region in which the profit is zero under the full model is a subset of such a region in the no rewards model because the profit is always weakly greater in the full model than in the no rewards model. An analogous set of comments applies to the comparison between the full model and the no free-contract model.

4.2 A Free Contract Substituting and Complementing Referral Rewards

Now we compare the full model with the no free-contract model. Analogous to Section 4.1, we aim to understand when a free contract can substitute as well as complement referral rewards. The comparison is displayed in the right panel of Figure 6, together with a reproduction of Figure 3 in the left panel. It entails a region such that a free conract "partially substitutes" referral rewards. We explain this in Remark 3. Substitution and complementation occur in the respective parameter regions for the following reasons:

- **Substitution:** The interior of the black region in Figure 6 is such that it is uniquely optimal for the seller to offer referral rewards under the no free-contract model, while it is uniquely optimal not to pay referral rewards when offering a free contract. This region has the feature that r is not too low and α is not too high so that the size of the additional expected externalities, $(1-\alpha)r$, is high enough.



(a) No Free-Contract Model

(b) Regions for substitution and complementation

Figure 6: A Free Contract Substituting and Complementing Referral Rewards

- **Complementation:** The interior of the red region in the right panel of Figure 3 is such that it is uniquely optimal for the seller not to offer referral rewards under the no free-contract model, while it is uniquely optimal to offer both referral rewards and a free contract under the full model. In this region, r is not too high but not too low. On the one hand, r cannot be too high because high r implies high additional expected externalities $(1-\alpha)r$, so a free contract would rather substitute, not complement, the referral rewards. On the other hand, r cannot be too low because even in the presence of a free contract, the referral rewards of $10-r$ must be paid to incentivize WoM. As a result, the reduction in the referral rewards due to the introduction of a free contract is not enough to cover the cost of a free contract CF^*. In addition, α needs to be high enough in this region because otherwise the revenue from the H-type is too low and thus incentivizing WoM would not generate a positive profit.

As before, we say $(\alpha, r) \in \text{SUB}^{\text{free}}$ if there is $R > 0$ such that $((0,0),(247,625),R) \in \mathcal{S}^{\text{NF}}(\alpha, r)$ while there are no (q'_L, p'_H) and $R' > 0$ such that $((0,q'_L),(p'_H,625),R') \in \mathcal{S}(\alpha, r)$. Similarly, we say $(\alpha, r) \in \text{COM}^{\text{free}}$ if there is no $R > 0$ such that $((0,0),(247,625),R) \in \mathcal{S}^{\text{NF}}(\alpha, r)$ while there is $R' > 0$ such that $((0,0),(247,625),R') \in \mathcal{S}(\alpha, r)$.

Theorem 2 (The Effect of a Free Contract)**.**

1. *(Substitution) Suppose that* $(\alpha, r) \in \text{SUB}^{\text{free}}$.

   (a) *For any* α' *such that there is* $R' > 0$ *satisfying* $((0,0),(247,625),R') \in \mathcal{S}^{\text{NF}}(\alpha', r)$, $(\alpha', r) \notin \text{SUB}^{\text{free}}$ *implies* $\alpha < \alpha'$.

(b) $r \geq 10$.

2. *(Complementation)*

(a) Fix α. *There are* $\underline{r} > 0$ *and* $\bar{r} < \infty$ *with* $\underline{r} \leq \bar{r}$ *such that the following holds. If* $(\alpha, r) \in \text{COM}^{\text{free}}$*, then,* $r \in (\underline{r}, \bar{r})$*.*

(b) Suppose that $(\alpha, r) \in \text{COM}^{\text{free}}$*. Then, for any* α' *such that* $\Pi^{\text{NF}}(\alpha', r) = 0$, $(\alpha', r) \notin \text{COM}^{\text{free}}$ *implies* $\alpha' < \alpha$*.*

The proof immediately follows from Propositions 2 and 3. The first part of this theorem states that it is cost efficient to completely substitute a free contract with referral rewards in niche markets, but not in mass markets. It also shows that substitution can only occur if the product is "sufficiently social" compared to the cost of talking. The second part states that if a firm cannot incentivize WoM only with referral rewards alone, a free contract can complement referral rewards and help to incentivize WoM in markets that are "sufficiently mass" and with intermediate positive externalities.

Remark 3. One can think of the grey region as describing parameters under which a free contract "partially substitutes" referral rewards. In the interior of this region, the unique optimal scheme under the full model entails positive payment of referral rewards, but its amount is strictly less than under the no free-contract model. The substitution is only "partial" because the benefit from a free contract $(1-\alpha)r$ is not large enough to incentivize WoM due to small r. Note that an analogous region (referral rewards "partially substituting" a free contract) does not exist in Figure 5 because a free contract can either do or do not exit, being different from referral rewards that can take a value from $\mathbb{R}_+$.

4.3 Comparing the Two Directions of Substitution and Complementation

Having completed the analysis of substitution and complementation in both directions, we are now in a position to make a few remarks on the similarities and differences between referral rewards substituting/complementing a free contract and vice versa.

First, a company that can induce WoM with one of the tools alone should think about substituting it with the other tool based on whether the company expects a high level of externalities between customers and whether the market is niche or not. In general, substitution occurs when the externalities are expected to be high. This is because a strictly positive profit needs to be generated in the no reward model for substitution to occur, and for that, r must be high enough to cover the cost of talking. If the market is niche, the company should use a free contract, while if it is mass, referral rewards should be used.

A company that cannot incentivize WoM with one of the tools, can incentivize WoM by adding the other tool and offering both tools together when externalities are not too small but not too high either. Even if referral rewards complement a free contract, the latter may not complement the former under the same parameter values. The reason for the difference is that a free contract is useful in incentivizing WoM only in markets that are not too mass, so there is a region in which referral rewards are used in the no free-contract model while a free contract is not used in the no rewards model. Thus, it is not sufficient to test the effectiveness of the two tools separately, but it is essential to consider the combined effect. More precisely, the intersections of the red areas in panels (b) of Figures 5 and 6 (which is just the red region of panel (b) in Figure 6) is the region where both tools are absolutely necessary in order to incentivize WoM. Dropbox seems to be in this region given the history of how it grew. The grey region in panel (b) of Figure 6, in turn, is a region where a free contract only is not effective to incentivize WoM, but referral rewards alone are. However, combining the two is the most cost-effective. Thus, as long as the profit is positive, the more "niche" the market, the more important it becomes to combine both strategies.

We formalize the findings about the difference between the two-way substitution and complementation follows.

Theorem 3 (Difference between the Two-Way Substitution and Complementation)**.**

1. *(Substitution) Fix r. Suppose that $(\alpha, r) \in \mathrm{SUB}^{\mathrm{free}}$ and $(\alpha', r) \in \mathrm{SUB}^{\mathrm{rewards}}$. Then, $\alpha < \alpha'$ holds.*

2. *(Complementation) Fix r. Suppose that $(\alpha, r) \in \mathrm{COM}^{\mathrm{free}}$ and $(\alpha', r) \in \mathrm{COM}^{\mathrm{rewards}} \setminus \mathrm{COM}^{\mathrm{free}}$. Then, $\alpha < \alpha'$ holds.*

5 Comparative Statics for the Full Model

The characterization in Section 3.3 also allows us to conduct comparative statics of the full model. We first analyze the different implications for the optimal scheme as the market size α varies.

Proposition 4 (Market Structure and Free Contracts)**.**
(i) Consider two markets that are identical to each other except for the probability of the buyer being the H-type, denoted α_1 and α_2. Suppose that a free contract is offered under an optimal scheme in the market with α_1, the maximized profit is strictly positive in the market with α_2, and $\alpha_2 < \alpha_1$. Then, a free contract is offered under any optimal scheme in the market with α_2.
(ii) If $\alpha > \frac{r-1.8}{r+28.2}$ ($\Leftrightarrow r < \frac{CF^*}{1-\alpha}$), then a free contract is never offered under any optimal scheme.

This proposition shows that the monopolist should encourage WoM using a free contract in a niche market with a small fraction α of H-type buyers as long as the market is profitable enough. Intuitively, if the probability of the H-type is low, the seller is better off using a free contract because a free contract significantly increases the probability of purchase. The exact trade-off is determined by the comparison of the information rent and the per-low-type surplus $r - 1.8$ that the seller can extract. The cutoff for α is increasing in this surplus.

These findings are consistent with the observation that digital service providers with small production costs who successfully offer free contracts (e.g., Dropbox or Skype), have a large number of free users. Moreover, free contracts are combined with a reward program, if the externalities are not large (as in Dropbox: one may use it for oneself to store files and access them from multiple computers, or share files with others), while only free contracts are offered if the externalities are large (as in Skype: any usage generates externalities). In contrast, transportation services such as Amtrak or Uber that solely rely on referral rewards programs would correspond to monopolists facing high α and low r, as many customers would be willing to pay for such services and those services would not be subject to significant externalities.[23]

We next consider how the optimal scheme depends on r. One might think that the smaller the

[23]Note that the fraction of the consumers purchasing free contracts is an endogenous variable, and one might think that our association of observable fractions for these real products to the exogenous parameter α is not justifiable. However, such association is justified because the map from consumer types to the choices of contracts is one-to-one given that free contracts are used. That is, if a positive fraction of consumers purchases free contracts, then within our model, such a fraction is exactly equal to $1 - \alpha$. Yet, it may be hard to empirically test our predictions for firms that do not offer free contracts because we do not observe α when free contracts are absent.

Externalities	$r < \frac{CF^*}{1-\alpha}$	$\frac{CF^*}{1-\alpha} < r < 10$	$10 < r < \frac{10-CF^*}{\alpha}$	$\frac{10-CF^*}{\alpha} < r < \frac{10}{\alpha}$	$\frac{10}{\alpha} < r$
Referral rewards	Yes	Yes	No	Yes	No
Free contract	No	Yes	Yes	No	No
Profit	Positive or zero	Positive or zero	Positive	Positive	Positive

Table 1: Comparative Statics with respect to r when $10 < \frac{CF}{1-\alpha}$. The use of referral rewards and free contracts is conditional on the firm generating positive profits.

externalities are, the more likely rewards are used. Figure 4 illustrates that this type of comparative statics fails for externalities. For example, at $\alpha = 0.15$, referrals are used when $r = 30$ but not when $r = 15$. The reason is that (i) when r is high, only one of a free contract and referral rewards suffices to incentivize the sender, i.e., these two are substitutes, and (ii) the cost of offering a free product CF^* is constant across r's while the optimal reward monotonically decreases with r. Thus, conditional on offering a free contract being sufficient to encourage WoM (i.e., $r \geq 10$), offering a free contract is more cost-saving for smaller r while rewards are more cost-saving for larger r. Table 1 summarizes the different regions as functions of r for the case in which $10 < \frac{CF^*}{1-\alpha}$.[24]

6 Conclusion

In this paper we propose a model that shows how referral rewards and offering a free contract can be effective tools to incentivize WoM for new products. They can be used separately, but substitution can result in cost savings while the two tools can also complement each other and encourage WoM in markets in which one tool alone is not effective. The main take-aways can be summarized as follows:

1. In general, substitution and complementation may or may not occur depending on whether the market is niche and whether the product is social.

2. Substitution occurs when the product is social, while complementation occurs when it is not too social but not too private either.

3. For social products, it is better to substitute a free contract with referral rewards when the market is mass, while it is better to substitute referral rewards with a free contract in niche markets.

[24] If this condition is not satisfied, some regions cease existing.

4. For less social products, a free contract can complement referral rewards in niche markets. Referral rewards complement a free contracts for the same markets, but also for markets that have more premium customers (i.e., are "more mass"). Those markets, however, cannot be too niche as the revenue from the product needs to be sufficiently high to make it worthwhile to incentivize WoM.

5. The pattern of the optimal scheme is consistent with the strategies we observe for companies such as Dropbox, Skype, Uber, and Amtrak.

In the Appendix and the Online Appendix we employ several robustness checks in order to show that our insights are not an artifact of the assumptions we impose in the model and analyze a few extensions. First, we generalize the functional forms and also allow for the possibility that the seller makes the referral rewards conditional on the receiver's purchase behavior. We show that such conditioning does not increase the seller's profit. We also prove that introducing heterogeneity in the costs of WoM does not change the qualitative results. Moreover, for a continuous type space of receivers (rather than only allowing for low-valuation and high-valuation receivers), we show that free contracts correspond to bunching at the bottom, i.e., among the customers who purchase positive quantities, customers buying the free contract correspond to a positive mass at the bottom of the type distribution. Importantly, all receivers who buy the free contract (except for the very lowest type) receive positive surplus. We also consider a model in which a receiver can be reached by many senders, and illustrate qualitative robustness of our results. If, in contrast, a sender can reach many receivers, he can be thoguht of as simply solving many identical WoM problems. For posting on social media, the cost of talking per receiver is lower, while the main tradeoffs remain unchanged.

A few results change if we allow for externalities both on the sender and receiver side, as well as when externalities depend on the quantity consumed. In yet another extension, we let the senders be better informed than the firm, and conclude that in general the optimal reward must additionally depend on the type of receiver being acquired. We then discuss what the socially optimal contract scheme would look like if the social planner had control over the sender's actions. It turns out that free contracts are underutilized under the optimal scheme relative to the social optimum because the firm does not fully internalize the benefits from externalities and gains from trade with the receivers (corresponding to the information rent). Our final extension concerns with dynamics. We consider a steady-state overlapping generations model in which a receiver in one period considers a continuation value of becoming a sender, which complicates the problem. We show that the basic pattern of the characterization of optimal schemes is unchanged. In particular, there is an open set of parameter combinations such that providing both a free contract and referral rewards is optimal.

There are many direction of future research that are beyond the scope of this paper. For example, we have enumerated potential reasons for the use of free products in Section 1.1, and it would be interesting include those effects in the analysis. One possibility is to enrich our model by having the receiver take the role of the sender once she is informed. This can be done in either a diffusion-type model in which the penetration takes place over time, or in a stationary environment in which the population size is constant through time. Possible challenges in such models are that, when a customer decides whether to adopt the product, she not only considers the price and quantity (as in the receiver in our model), but also the future benefit from talking as a sender. In turn, the sender has to take into account this tradeoff of the receiver.

In another interesting extension, the receiver could be uncertain about the quality of the product, and the sender might have a higher incentive to talk when he knows the quality is higher. In such a model, if the receiver knows that the sender would receive referral rewards, then she may adjust their belief about the quality downwards. This requires a significant divergence from the Maskin-Riley model, but may be a worthwhile direction for future research.

References

Ajorlou, Amir, Ali Jadbabaie, and Ali Kakhbod, "Dynamic Pricing in Social Networks: The Word of Mouth Effect," Technical Report, Mimeo 2015.

Anderson, Eugene W, "Customer satisfaction and word of mouth," *Journal of service research*, 1998, *1* (1), 5–17.

Aral, Sinan and Dylan Walker, "Creating social contagion through viral product design: A randomized trial of peer influence in networks," *Management science*, 2011, *57* (9), 1623–1639.

Berger, Jonah, "Word of mouth and interpersonal communication: A review and directions for future research," *Journal of Consumer Psychology*, 2014, *24* (4), 586–607.

_ **and Eric M Schwartz**, "What drives immediate and ongoing word of mouth?," *Journal of Marketing Research*, 2011, *48* (5), 869–880.

Bimpikis, Kostas, Asuman Ozdaglar, and Ercan Yildiz, "Competitive targeted advertising over networks," *Operations Research*, 2016, *64* (3), 705–720.

Biyalogorsky, Eyal, Eitan Gerstner, and Barak Libai, "Customer referral management: Optimal reward programs," *Marketing Science*, 2001, *20* (1), 82–95.

Campbell, Arthur, "Word of mouth and percolation in social networks," *American Economic Review*, 2012.

_ **, Dina Mayzlin, and Jiwoong Shin**, "Managing Buzz," 2015.

Chen, Yubo and Jinhong Xie, "Online consumer review: Word-of-mouth as a new element of marketing communication mix," *Management Science*, 2008, *54* (3), 477–491.

Csorba, Gergely, "Screening contracts in the presence of positive network effects," *International Journal of Industrial Organization*, 2008, *26* (1), 213–226.

den Bulte, Christophe Van and Yogesh V Joshi, "New product diffusion with influentials and imitators," *Marketing Science*, 2007, *26* (3), 400–421.

Dichter, Ernest, "How word-of-mouth advertising works," *Harvard business review*, 1966, *44* (6), 147–160.

Eisenmann, Thomas R., "Skype," *Harvard Business School Publishing*, 2006, *9-806-165.*

Fainmesser, Itay P, Dominique Olié Lauga, and Elie Ofek, "Ratings, Reviews, and the Marketing of New Products," 2018.

Galeotti, Andrea, "Talking, searching, and pricing*," *International Economic Review*, 2010, *51* (4), 1159–1174.

— **and Sanjeev Goyal**, "Influencing the influencers: a theory of strategic diffusion," *The RAND Journal of Economics*, 2009, *40* (3), 509–532.

Godes, David and Dina Mayzlin, "Firm-created word-of-mouth communication: Evidence from a field test," *Marketing Science*, 2009, *28* (4), 721–739.

— , — , **Yubo Chen, Sanjiv Das, Chrysanthos Dellarocas, Bruce Pfeiffer, Barak Libai, Subrata Sen, Mengze Shi, and Peeter Verlegh**, "The firm's management of social interactions," *Marketing letters*, 2005, *16* (3-4), 415–428.

Goldenberg, Jacob, Barak Libai, and Eitan Muller, "Talk of the network: A complex systems look at the underlying process of word-of-mouth," *Marketing letters*, 2001, *12* (3), 211–223.

Hahn, Jong-Hee, "Nonlinear pricing of telecommunications with call and network externalities," *International Journal of Industrial Organization*, 2003, *21* (7), 949–967.

Hollenbeck, Brett, Sridhar Moorthy, and Davide Proserpio, "Advertising Strategy in the Presence of Reviews: An Empirical Analysis," 2017.

Houston, Drew, 2010. `http://www.slideshare.net/gueste94e4c/dropbox-startup-lessons-learned-3836587`. Accessed: 2016-05-05.

Iyengar, Raghuram, Christophe Van den Bulte, and Thomas W Valente, "Opinion leadership and social contagion in new product diffusion," *Marketing Science*, 2011, *30* (2), 195–212.

Joshi, Yogesh and Andres Musalem, "Does Word of Mouth Reduce Advertising That Signals Quality?," 2017.

Katona, Zsolt, Peter Pal Zubcsek, and Miklos Sarvary, "Network effects and personal influences: The diffusion of an online social network," *Journal of marketing research*, 2011, *48* (3), 425–443.

Kornish, Laura J. and Qiuping Li, "Optimal Referral Bonuses with Asymmetric Information: Firm-Offered and Interpersonal Incentives," *Marketing Science*, 2010, *29* (1), 108–121.

Lambrecht, Anja and Kanishka Misra, "Fee or Free: When Should Firms Charge for Online Content?," *Management Science*, 2016, *63* (4), 1150–1165.

Leduc, Matt V, Matthew O Jackson, and Ramesh Johari, "Pricing and referrals in diffusion on networks," *Games and Economic Behavior*, 2017, *104*, 568–594.

Lee, Clarence, Vineet Kumar, and Sunil Gupta, "Designing freemium: a model of consumer usage, upgrade, and referral dynamics," Technical Report, Mimeo 2013.

_ , _ , and _ , "Designing Freemium: Balancing Growth and Monetization Strategies," 2015.

Manchanda, Puneet, Ying Xie, and Nara Youn, "The role of targeted communication and contagion in product adoption," *Marketing Science*, 2008, *27* (6), 961–976.

Maskin, Eric and John Riley, "Monopoly with incomplete information," *The RAND Journal of Economics*, 1984, *15* (2), 171–196.

Mayzlin, Dina, "Promotional chat on the Internet," *Marketing Science*, 2006, *25* (2), 155–163.

Schmitt, Philipp, Bernd Skiera, and Christophe Van den Bulte, "Referral Programs and Customer Value," *Journal of Marketing*, 2011, *75* (1), 46–59.

Segal, Ilya, "Coordination and discrimination in contracting with externalities: Divide and conquer?," *Journal of Economic Theory*, 2003, *113* (2), 147–181.

Shapiro, Carl and Hal R Varian, "Versioning: the smart way to," *Harvard Business Review*, 1998, *107* (6), 107.

Shi, Zijun, Kannan Srinivasan, and Kaifu Zhang, "Freemium as an Optimal Strategy under Network Externalities," Technical Report, Mimeo 2017.

Economist

Economist, 2012. `http://www.economist.com/blogs/babbage/2012/12/dropbox`. Accessed: 2016-06-27.

APPENDIX

This Appendix generalizes the "full model" provided in the main text. Section A presents such a generalized model and Section B analyzes the model. Section C provides proofs of the general results, which also prove the results presented in the main text.

A Generalized Model and Results

Basics. We consider a monopolist producing a single product at constant marginal cost $c > 0$. *Senders* (male) $\{1, \ldots, N\}$ can inform *receivers*, (female) $\{1, \ldots, N\}$ about the existence of the product. The monopolist's goal is to maximize the expected profit generated by receivers by offering them a menu of contracts and, in addition, offering a referral scheme to senders.

Receivers' preferences. Each receiver privately observes her type $\theta \in \{L, H\}$ that determines her valuation of the product. It is drawn independently such that a receiver is of type H with probability $\alpha \in (0, 1)$ and of type L otherwise. A type-θ receiver is associated with a valuation function $v_\theta : \mathbb{R}_+ \to \mathbb{R}$ that assigns to each quantity (or quality) q her valuation $v_\theta(q)$. Over the strictly positive domain, i.e., $q \in (0, \infty)$, we assume that v_θ is continuously differentiable, strictly increasing, strictly concave, $v_H(q) > v_L(q)$, $v'_H(q) > v'_L(q)$ for all q and $\lim_{q\to\infty} v'_H(q) < c$. We assume that $v_H(0) = v_L(0) = 0$, which can be interpreted as the utility of the outside option of not using the product at all. We make the following additional assumptions:

Assumptions.

1. **(Minimum quantity for low types)** $\exists\, \underline{q} > 0$ such that $v_L(\underline{q}) = 0$.
2. **(No gains from trade with low types)** $v'_L(q) < c$ for all $q \geq \underline{q}$.
3. **(Gains from trade with high types)** There exists a $q > 0$ such that $v_H(q) > q \cdot c$.

The first assumption can be interpreted as low types incurring some fixed installation cost of the product, and the low valuation buyer only wanting to start using the product if a minimum quantity of $\underline{q} > 0$ is consumed.[25] This makes a distinciton between purchase and consumption necessary. This distinction only becomes relevant if prices are negative. In that case, the receiver can buy the product in order to receive the negative price, but then refuse to use it if she receives

[25]Note that this does not preclude the possibility of positive fixed installation costs for high types.

negative utility from it (e.g., because the installation cost is too high). Note that $\underline{q}$ is uniquely defined because v_L is strictly increasing. This first assumption together with the normalization that $v_L(0) = 0$ and the assumption that v_L is strictly increasing in the strictly positive domain implies that the function v_L is necessarily discontinuous at $q = 0$ because v_L is strictly increasing on the strictly positive domain.[26]

The second assumption captures that there are some consumers who would never use the product if they were not needed to incentivize WoM. Without the third assumption, the monopolist would not be able to earn positive profits, so the problem becomes trivial.

Senders' preferences and WoM technology. First, each sender i observes the monopolist's choice of menu of contracts and referral scheme (specified below). Each sender i then decides whether to inform receiver i at a cost $\xi \geq 0$ or not. We denote sender i's action by $a_i \in \{\text{Refer}, \text{Not}\}$, where $a_i = \text{Refer}$ if sender i refers receiver i and $a_i = \text{Not}$ otherwise. If (and only if) receiver i learns about the product, she decides whether to purchase a contract or not, and whether to consume the product or not upon purchasing. If receiver i consumes a positive quantity, sender i receives *externalities* $r \geq 0$.

Monopolist's problem. As in Maskin and Riley (1984), the monopolist offers a menu of contracts given by $((p_L, q_L), (p_H, q_H)) \in (\mathbb{R} \times \mathbb{R}_+)^2$ to receivers, where q_θ is the quantity type θ is supposed to buy at a price p_θ. Furthermore, she offers a reward scheme $\mathbf{R} : \{L, H\} \to \mathbb{R}_+$ such that a sender receives $\mathbf{R}(\theta)$ if he has referred a receiver who purchases the θ-contract. Rewards are assumed to be nonnegative because otherwise senders would be able to secretly invite new customers. We assume that the monopolist only receives revenue from new customers who do not know about the product unless a sender talks to them. In order to exclusively focus on the senders' incentive to talk, we assume that the monopolist receives no revenue from senders. Thus, the monopolist solves

$$\max_{p_L,\, p_H \in \mathbb{R},\, q_L, q_H \geq 0,\, \mathbf{R} \in \mathbb{R}_+^{\{L,H\}}} \sum_{i=1}^{N} \mathbf{1}(a_i = \text{Refer}) \cdot \Big(\underbrace{\alpha \cdot (p_H - q_H \cdot c)) + (1-\alpha) \cdot (p_L - q_L \cdot c)}_{\text{total average profit per referred receiver}} - (\alpha \mathbf{R}(H) + (1-\alpha)\mathbf{R}(L)) \Big) \tag{4}$$

[26]Recall also that continuous differentiability of v_L is assumed only on the strictly positive domain.

subject to the incentive compatibility and participation constraints given by

$$\left.\begin{aligned}
\max\{v_H(q_H),0\}-p_H &\geq \max\{v_H(q_L),0\}-p_L && \text{(H-type's IC)}\\
\max\{v_L(q_L),0\}-p_L &\geq \max\{v_L(q_H),0\}-p_H && \text{(L-type's IC)}\\
\max\{v_H(q_H),0\}-p_H &\geq 0 && \text{(H-type's PC)}\\
\max\{v_L(q_L),0\}-p_L &\geq 0 && \text{(L-type's PC)}
\end{aligned}\right\} \quad (5)$$

and for all i, $a_i =$ Refer if and only if

$$\xi \leq r\left(\alpha + (1-\alpha)\cdot \mathbf{1}(q_L \geq \underline{q})\right) + (\alpha \mathbf{R}(H) + (1-\alpha)\mathbf{R}(L)) \quad \text{(Senders' IC)}$$

Let Π^* denote the value of this problem. The monopolist chooses contracts given by quantities and prices, while *managing WoM*. The management of WoM appears as the senders' incentive compatibility (IC) constraint. On the left hand side is the cost of talking, ξ, which we assume to be homogeneous across senders. This simple case allows us to illustrate the main trade-offs. As a robustness check, the Online Appendix analyzes the case of heterogeneous costs in detail. On the right hand side, the quantity sold to L-type receivers q_L affects WoM by controlling the *expected externalities* given by $r\left(\alpha + (1-\alpha)\cdot \mathbf{1}(q_L \geq \underline{q})\right)$. The senders' optimal decision determines the value of the indicator function in the objective function and thereby controls the number of informed receivers.

Let us explain a few assumptions implicit in this formulation. First, as standard in contract theory, we assume tie-breaking conditions for senders and receivers that are most favorable for the monopolist. Senders who are indifferent between referring and not will refer, and receivers that are indifferent between buying and not buying always buy. Second, we assume that if the buyer purchases a contract (p, q) such that $v_\theta(q) < 0$, then the monopolist cannot "force" the receiver to consume even if she pays the buyer a negative price. Thus, a type-θ receiver who purchases such a contract enjoys utility $\max\{v_\theta(q), 0\}$. There is no such max operation in the constraints in the main text. This is because, in the model in the main text, we assume for simplicity that prices are nonnegative. Under such an assumption it is straightforward to see that $v_\theta(q)$ is always the maximum under any optimal scheme.

A.1 Benchmark with free WoM

We first consider a benchmark case where $\bar{\xi} = 0$, i.e., WoM is costless and customers are automatically informed about the product. Then, the monopolist simply solves the classic problem as in Maskin and Riley (1984):

$$\Pi^{\text{classic}} \equiv \max_{p_H,\, p_L \in \mathbb{R}\; q_H, q_L \geq 0} \qquad \alpha \cdot (p_H - q_H \cdot c) + (1 - \alpha) \cdot (p_L - q_L \cdot c)$$

subject to the constraints (5). It is always optimal for the seller not to sell to L-type buyers such that $q_L^* = 0$ and the optimal quantity q_H^* sold to H-type buyers satisfies $v_H'(q_H^*) = c$. Assumption 3, strict concavity, continuous differentiability of v_H and $\lim_{q \to \infty} v_H'(q) < c$ ensure that there is a unique such q_H^*. The price for high types is given by $p_H^* = v_H(q_H^*)$ and the maximal static profit is $\Pi^{\text{classic}} = \alpha \cdot (p_H^* - q_H^* \cdot c)$. All in all, we can summarize our findings as follows:

$$v_H'(q_H^*) = c, \quad p_H^* = v_H(q_H^*), \quad \text{and } \Pi^{\text{classic}} = \alpha \cdot (p_H^* - q_H^* \cdot c).$$

A.2 Preliminaries

Before proceeding to the main analysis, we present several preliminary results. First, observe that $\mathbf{R}(\cdot)$ affects the monopolist's optimization problem only through the ex ante expected reward $R \equiv \alpha \mathbf{R}(H) + (1 - \alpha)\mathbf{R}(L)$. Thus, profits are identical for all reward schemes $\mathbf{R}(\cdot)$ that share the same expected value. Formally, this means:

Lemma 1 (Reward Reduction)**.** *If a menu of contracts* $((p_L, q_L), (p_H, q_H)) \in (\mathbb{R} \times \mathbb{R}_+)^2$ *and a reward scheme* $\mathbf{R}^{**} : \{L, H\} \to \mathbb{R}_+$ *solve* (4)*, then the same menu of contracts* $((p_L, q_L), (p_H, q_H))$ *and any reward scheme* $\mathbf{R} : \{L, H\} \to \mathbb{R}_+$ *with* $\mathbb{E}[\mathbf{R}] = \mathbb{E}[\mathbf{R}^{**}]$ *solve* (4)*.*

Despite being a simple observation, this result implies an important feature of the optimization problem faced by the firm. As long as the firm and the senders have the same expectation about the receivers' types, there is no reason for the firm to condition their payment on the purchased contract. Indeed, in the Online Appendix, we show that if the senders have more accurate information about the receivers' types than the firm, the conclusion of Lemma 1 no longer holds. Thus, the detail of the optimal reward scheme crucially depends on the senders' knowledge. We relegate the analysis of this detail to the Online Appendix, while here we consider senders who have the same information about the receiver's types as the firm does. Note also that Lemma 1 does not imply that the sender receives referral rewards when the receiver does not end up using the product, for example when the low types are offered zero quantity.[27]

Plugging the sender's IC constraint into the objective function and noting that all senders share the same IC constraint, Lemma 1 allows us to simplify the problem as follows:

$$\Pi^* = \max_{p_L, p_H \in \mathbb{R},\, q_L, q_H \geq 0,\, R \in \mathbb{R}_+} \quad \begin{aligned} & N \cdot \mathbf{1}(\xi \leq r\left(\alpha + (1-\alpha)\cdot \mathbf{1}(q_L \geq \underline{q})\right) + R\} \cdot \\ & \left[\alpha \cdot (p_H - q_H \cdot c) + (1-\alpha)\cdot(p_L - q_L \cdot c) - R\right] \end{aligned} \tag{6}$$

subject to the constraints (5). We prove the existence of a solution to this problem. It is not immediate as the objective function is not necessarily continuous, but right-continuity of those functions and the fact that the number of discontinuous points is finite suffices to establish existence.[28]

Proposition 5 (Existence)**.** *The maximization problem (6) subject to* (5) *has a solution.*

Given parameters (α, r), we denote the (non-empty) *set of solutions* to this problem by

$$\mathcal{S} \subseteq (\mathbb{R} \times \mathbb{R}_+)^2 \times \mathbb{R}_+.$$

Moreover, for any menu of contracts $((p_L, q_L), (p_H, q_H))$ satisfying (5), we denote the firm's expected profits obtained from a receiver conditional on being informed by

$$\pi((p_L, q_L), (p_H, q_H)) \quad = \quad \alpha(p_H - q_H \cdot c) + (1-\alpha)(p_L - q_L \cdot c).$$

The monopolist can always choose not to sell to anyone and attain zero profits, i.e., $\Pi^* \geq 0$. Furthermore, whenever $\Pi^* = 0$ the seller can attain the maximum by inducing no sender to talk. This can be done by offering unacceptable contracts to receivers and no rewards.[29] We, thus, focus

[27] We can set $\mathbf{R}(\mathbf{L}) = \mathbf{0}$ and $\mathbf{R}(H) = R/\alpha$, so that senders who refer low types receive zero referral rewards.

[28] The proof is done in a more general context, in which after each sender i sees the menu of contrast, he privately observes his cost of talking drawn from an independent and identical distribution with a cumulative distribution function that has at most finitely many jumps.

[29] Note that if there is a positive mass of senders with $\xi = 0$, then by Assumption 3 the seller can attain strictly positive profits by only selling to H-receivers and offering no reward.

the characterization of optimal menu of contracts and rewards programs on the case when $\Pi^* > 0$.[30] The following lemma summarizes some basic properties of optimal menus of contracts.

Lemma 2. *If* $\Pi^* > 0$ *and* $((p_L, q_L), (p_H, q_H), R) \in \mathcal{S}$*, then:*

(i) **Low types don't pay:** $q_L \in \{0, \underline{q}\}$ *and* $p_L = 0$.[31]

(ii) **No distortions at the top:** $q_H = q_H^*$.

(iii) **No free contracts:** *If* $q_L = 0$*, then* $p_H = p_H^*$.

(iv) **Free contracts:** *If* $q_L = \underline{q}$*, then* $p_H = p_H^* - \underbrace{v_H(\underline{q})}_{\text{information rent}} \equiv \tilde{p}_H^*$.

We have illustrated these results in the context of the no rewards model in the main text. Note that parts (iii) and (iv) follow because the incentive compatibility constraint of H-type receivers must be binding.

Lemma 2 restricts the set of possible optimal contracts significantly. In particular, it uniquely pins down the price offered to low types and the quantity offered to high types whenever $\Pi^* > 0$. At a price of zero for low types, the seller either chooses $q_L = 0$ (*no free contracts*) or $q_L = \underline{q}$ (*free contracts*). A full characterization of optimal contracts requires us to characterize the optimal reward scheme R and whether free contracts are optimal for the monopolist. These choices depend on the parameters that have not been used so far: the cost structure, the magnitude of externalities, and the composition of different types of buyers.

B Analysis of the Generalized Model

Here we aim to characterize the optimal schemes. Technically, the full characterization is involved for two reasons. First, there is a non-monotonicity of the use of rewards with respect to the size of externalities. That is, it is possible that the optimal reward changes from positive to zero and back to positive when externalities are increased because free contracts substitutes rewards. Second, the total cost of offering free contracts is determined by two factors, that is, the production cost (which is low for products such as Skype and Dropbox) of the free products and informational asymmetry, which forces the firm to pay an information rent to high-valuation buyers. This total cost of offering free contracts plays a key role in fully characterizing the optimal incentive scheme.

[30] In part 1 of Theorem 4, we give a necessary and sufficient condition for $\Pi^* > 0$ to hold.

[31] The proof shows that we do not need to restrict prices to be nonnegative in order to obtain this result.

B.1 Characterization of Optimal Scheme

We characterize the optimal contracts in steps. First, we characterize the optimal referral reward scheme given a menu of contracts satisfying (5) (Lemma 3). Then, we solve for the optimal menu of contracts (Lemma 4) and finally, use these optimal contracts to derive the optimal reward using Lemma 3 (Theorem 4).

With homogeneous costs of talking, if $r\left(\alpha + (1-\alpha)\cdot \mathbf{1}_{\{q_L>0, v_L(q_L)\geq 0\}}\right) + R \geq \xi$, then for any menu of contracts satisfying the constraints (5), profits are given by $\pi((p_L, q_L), (p_H, q_H)) - R$. Otherwise, profits are zero. Thus, if incentivizing WoM is not more expensive than the expected profits, the monopolist would like to pay senders just enough to make them talk. The following lemma formalizes this intuition. Let

$$R^{**}((p_L, q_L), (p_H, q_H)) \quad = \quad \max\left\{\xi - \underbrace{r\cdot\left[\alpha + (1-\alpha)\cdot \mathbf{1}(q_L \geq \underline{q})\right]}_{\text{expected externalities}}, 0\right\}. \tag{7}$$

Lemma 3 (Referral Program)**.** *Given contracts* (p_L, q_L) *and* (p_H, q_H) *satisfying* (5) *and* $v_H(q_H) \geq 0$*, the optimal referral reward is unique as long as* $R^{**}((p_L, q_L), (p_H, q_H)) < \pi((p_L, q_L), (p_H, q_H))$ *and is given by* $R^{**}((p_L, q_L), (p_H, q_H))$*.*

Using Lemma 2 and the formula of the optimal reward function R^{**} in Lemma 3, we can determine whether it is optimal to offer free contracts or not, which then pins down the full optimal menu of contracts. As in the main text, we define the cost of free contracts:

$$CF^{*} \quad \equiv \quad \alpha \quad \underbrace{\cdot v_H(\underline{q})}_{\text{information rent}} \quad +(1-\alpha)\cdot \quad \underbrace{c\cdot \underline{q}}_{\text{production cost of free product}} \quad . \tag{8}$$

Using this variable, let us first provide a heuristic argument: In order for free contracts to be optimal, this cost has to be outweighed by the benefit generated by providing the product to low types, i.e.,

$$CF^{*} \leq (1-\alpha)r, \tag{9}$$

or equivalently $\frac{CF^{*}}{1-\alpha} \leq r$. Notice that $\frac{CF^{*}}{1-\alpha}$ represents the "break-even externalities" necessary to compensate for the cost of free contracts. Moreover, $\frac{CF^*}{1-\alpha}$ is increasing in α. The average profit generated by a receiver if free contracts are offered can be written as

$$\pi((0, \underline{q}), (\tilde{p}_H^*, q_H^*)) = \Pi^{\text{classic}} - CF^*$$

The following result shows that, with additional boundary conditions, (9) is also sufficient to guarantee optimality of free contracts. We denote the set of optimal q_L by Q_L^{**}.

Lemma 4 (Free Contract)**.** *Whenever $\Pi^* > 0$, an optimal contract to the type-L receiver must satisfy the following:*

(i) Let $r \in [\frac{\bar{\xi}}{\alpha}, \infty)$. Then, $Q_L^{**} = \{0\}$ (i.e., it is not optimal to provide free contracts).

(ii) Let $r \in [\bar{\xi}, \frac{\bar{\xi}}{\alpha})$.

1. **(Free contracts)** $\underline{q} \in Q_L^{**}$ *if and only if*

$$\underbrace{\xi - \alpha r}_{\text{reward w/o free contract}} \geq CF^*. \tag{10}$$

2. **(No free contracts)** $0 \in Q_L^{**}$ *if and only if* $\xi - \alpha r \leq CF^*$.

(iii) Let $r \in [0, \bar{\xi})$.

1. **(Free contracts)** $\underline{q} \in Q_L^{**}$ *if and only if* $r \geq \frac{CF^*}{1-\alpha}$.

2. **(No free contracts)** $0 \in Q_L^{**}$ *if and only if* $r \leq \frac{CF^*}{1-\alpha}$.

The intuition for this lemma is the following. First, there is no need for the seller to provide any incentives for WoM (i.e., $q_L = 0$) if the cost of talking ξ is smaller than the lowest expected externalities αr because in that case people talk anyway (Lemma 4 (i)). If $r \in [\bar{\xi}, \frac{\bar{\xi}}{\alpha})$ (Lemma 4 (ii)), then the cost of talking is larger than αr, but free contracts can boost the expected externalities to $r \geq \xi$. Then, free contracts are used whenever the referral reward that the seller had to pay without free contracts $\xi - \alpha r$ is larger than the cost of offering a free contract CF^* which is the sum of the information rent and cost of producing $\underline{q}$. Note that in this case, whenever free contracts are offered, the optimal reward is zero by Lemma 3. Finally, for high costs of talking $\xi > r$ (Lemma 4 (iii)), by Lemma 3 the seller pays a reward as long as the optimal reward does not exceed expected profits. If free contracts are offered, the expected externalities can be increased by $(1-\alpha)r$. Hence, free contracts are offered only if this benefit exceeds the cost of production and the information rent so that $r \geq \frac{CF^*}{1-\alpha}$ as explained above.

Lemmas 2, 3 and 4 pave the way for a full characterization of the optimal menu of contracts and reward scheme summarized in the following theorem. It shows that the optimal incentive scheme depends on the market structure given by parameters such as the cost of production c, the externalities r, the cost of talking ξ, and the fraction of H-type receivers α. Note that Proposition 3 is a special case of this theorem applied to the model in the main text.

Theorem 4 (Full Characterization). *1.* **(Positive profits)** $\Pi^* > 0$ *if and only if*

$$\xi < \max\left\{\Pi^{classic} - CF^* + \min\{r, \xi\},\ \Pi^{classic} + \alpha r\right\}. \tag{11}$$

For the following cases, assume that (11) *is satisfied:*

2. **(Free vs. no free contracts)** *There exists* $((0, \underline{q}), (\tilde{p}_H^*, q_H^*), R) \in \mathcal{S}$ *for some* R *if and only if* $r \in \left[\frac{CF^*}{1-\alpha}, \frac{\xi - CF^*}{\alpha}\right]$.[32]

3. **(Rewards vs. no rewards)**

   (a) **(With free contracts)** *If* $r \in [\frac{CF^*}{1-\alpha}, \frac{\xi-CF^*}{\alpha}]$*, then* $((0, \underline{q}), (\tilde{p}_H^*, q_H^*), R) \in \mathcal{S}$ *with* $R > 0$ *if and only if* $r < \xi$*, and*

   (b) **(With no free contracts)** *If* $r \notin [\frac{CF^*}{1-\alpha}, \frac{\xi-CF^*}{\alpha}]$*, then* $((0, 0), (p_H^*, q_H^*), R) \in \mathcal{S}$ *with* $R > 0$ *if and only if* $r < \frac{\xi}{\alpha}$.

First, it is straightforward that the monopolist should provide no incentives for WoM either if senders talk anyway because the cost of talking is small (i.e., $\xi < \alpha r$) or if it is too expensive because the cost of talking ξ is too large relative to its benefits given in (11). A necessary condition for free contracts to be optimal is that r is large enough (i.e., $r > \frac{CF^*}{1-\alpha}$). An immediate implication is that without any externalities, free contracts are of no value to the seller. At the same time, free contracts are more effective to encourage WoM than rewards only if the cost of talking ξ is sufficiently large relative to r (i.e., $\xi > CF^* + \alpha r$ which is derived from the upper bound of r in part 2 of Theorem 4). Otherwise, it is cheaper to pay a small reward for talking.

[32] If $\frac{CF^*}{1-\alpha} > \frac{\xi-CF^*}{\alpha}$, then $[\frac{CF^*}{1-\alpha}, \frac{\xi-CF^*}{\alpha}] = \emptyset$.

We can also generalize the comparative statics in Proposition 4 as follows.

Proposition 6 (Market Structure and Free Contracts)**.**
(i) Consider two markets that are identical to each other except for the share of H-types, denoted α_1 *and* α_2*. Suppose that free contracts are offered under an optimal scheme in the market with* α_1*,* $\Pi^* > 0$ *in the market with* α_2*, and* $\alpha_2 < \alpha_1$*. Then, free contracts are offered under any optimal scheme in the market with* α_2*.*
(ii) Suppose $v_H(\underline{q}) + r > c\underline{q}$*. Then,* $\alpha > \frac{r - c\underline{q}}{v_H(\underline{q}) + r - c\underline{q}}$ $(\Leftrightarrow r < \frac{CF^*}{1-\alpha})$ *implies that free contracts are never offered under any optimal scheme.*

In the interest of brevity, we do not generalize Theorems 1 and 2 here, but one can show that analogous results hold in the generalize model, too.

C Proofs of the General Results

Proof. **(Proposition 5)** As discussed in footnote 28, we prove the result for a general environment in which, after each sender i sees the menu of contrast, he privately observes his cost of talking ξ_i, drawn from an independent and identical distribution with a cumulative distribution function $G : \mathbb{R}_+ \to [0, 1]$ that has at most finitely many jumps. With this formulation, the present proof shows that the existence result is also valid for the general setup discussed in the Online Appendix. First, we show that it is without loss of generality to restrict attention to choice variables in a compact set. To see this, first note that, as we will show in the proof of Lemma 2, a scheme $((p_L, q_L), (p_H, q_H), R)$ with $q_L \in (0, \underline{q})$ generates a strictly lower profit than a scheme $((p_L, 0), (p_H, q_H), R)$. The same proof also shows that a scheme $((p_L, q_L), (p_H, q_H), R)$ with $q_L > \underline{q}$ generates a strictly lower profit than a scheme $((p_L, \underline{q}), (p_H, q_H), R)$. Thus it is without loss of generality to restrict attention to $\{0, \underline{q}\}$ as the space from which q_L is chosen. This and the participation constraint for low types imply that if a scheme $((p_L, q_L), (p_H, q_H), R)$ satisfies the constraints then $p_L \leq 0$. Also, the proof for Lemma 2 shows that for any scheme $((p_L, q_L), (p_H, q_H), R)$, $p_L < 0$ implies that the participation constraints for both types are non-binding, hence there exists $\epsilon > 0$ such that there exists a scheme $((p_L + \epsilon, q_L), (p_H + \epsilon, q_H), R)$ that satisfies the constraints and generates a higher profit than the original scheme. Consequently, it is without loss of generality to restrict attention to a scheme $((p_L, q_L), (p_H, q_H), R)$ with $p_L = 0$.

Also, since $\lim_{q\to\infty} v'_H(q) < c$, there exists q' such that any scheme $((p_L, q_L), (p_H, q_H), R)$ with $q_H > q'$ generates a strictly negative profit. Thus it is without loss of generality to restrict attention to $[0, q']$ for the space for q_H, where q' is any number satisfying $v'_H(q') < c$. Fix such $q' < \infty$ arbitrarily. Then, any scheme $((p_L, q_L), (p_H, q_H), R)$ with $R > v_H(q')$ generates a strictly negative profit, so again it is without loss to restrict attention to $[0, v_H(q')]$ as the space for R.

These bounds for q_H and q_L together with the PC constraints imply that it is without loss of generality to consider $p_H \leq v_H(q')$. The incentive compatibility condition for low types implies that $0 = \max\{v_L(q_L), 0\} - p_L \geq \max\{v_L(q_H), 0\} - p_H$, which implies $p_H \geq \max\{v_L(q_H), 0\} \geq 0$. Thus, it is without loss of generality to consider $p_H \in [0, v_H(q')]$.

These facts and the fact that all constraints are weak inequalities with continuous functions imply that the optimal scheme is chosen from a compact set. Now, note that the objective function is right-continuous in each choice variable because G is a cumulative distribution function, and all jumps are upwards.

These facts and the assumption that G has only finitely many discontinuities imply that there exists a partition of the compact space of the choice variables C with a finite number of cells $(P_1, \ldots, P_K)$ for some integer $K \in \mathbb{N}$, such that over each cell, the objective function is continuous.

Let $\hat{\pi}$ be the supremum of the objective function over C. Then there exists a sequence $(y^k)_{k=1,2,\ldots}$ with $y^k \in C$ for all k such that the value of the objective function under y^k converges to $\hat{\pi}$. Since $K < \infty$, this implies that there exists a cell of the partition, denoted P_{i^*} (choose one arbitrarily if there are multiples of such cells), and a subsequence $(z^k)_{k=1,2,\ldots}$ of $(y^k)_{k=1,2,\ldots}$ such that $z^k \in P_{i^*}$ for all k.

Since P_{i^*} is a bounded set, $(z^k)_{k=1,2,\ldots}$ has an accumulation point. Let an arbitrary choice of an accumulation point be z^*. If $z^* \in P_{i^*}$, then by continuity the objective function attains the value $\hat{\pi}$ at z^*. If $z^* \notin P_{i^*}$, then by the assumption of the upward jumps, the objective function attains the value strictly greater than $\hat{\pi}$ at z^*, which is a contradiction. This completes the proof. □

Proof. **(Lemma 2)** Let $((p_L, q_L), (p_H, q_H), R)$ be an optimal scheme.

(i) Given a menu of contracts with $q_L > \underline{q}$ that satisfy (5), continuity of v_L implies that the monopolist can decrease q_L and p_L slightly, such that $\max\{v_L(q_L), 0\} - p_L$ remains constant (by Assumption 1) without violating (5) because $v_H(q_L) - p_L$ decreases with such a change (as

$v_H' > v_L'$). This strictly increases profits by Assumption 2. Similarly, given a menu of contracts with $0 < q_L < \underline{q}$ that satisfy (5) and such that $\Pi^* > 0$, the monopolist can decrease q_L to zero and increase profits without violating (5).

The equation $p_L = 0$ can be shown by noting that type L's participation constraint must be binding: Assume $p_L < \max\{v_L(q_L), 0\} = 0$. First, note that then type H's participation constraint cannot be binding: If it was, then

$$0 = \max\{v_H(q_H), 0\} - p_H \geq \max\{v_H(q_L), 0\} - p_L \geq \max\{v_L(q_L), 0\} - p_L > 0$$

which is a contradiction. Thus, the monopolist can strictly increase profits by increasing p_L and p_H by the same small amount such that (5) remains to be satisfied. Consequently, $p_L = \max\{v_L(q_L), 0\} = 0$.

(ii) Given a R, $p_L = 0$ and fixing $q_L \in \{0, \underline{q}\}$, H-type's contract (p_H, q_H) must solve $\max_{p_H, q_H} \alpha(p_H - q_H c)$ subject to $\max\{v_H(q_H), 0\} - p_H \geq \max\{v_H(q_L), 0\}$ and $\max\{v_H(q_H), 0\} - p_H \geq 0$. If we ignored the participation constraint, and solved a relaxed problem, the incentive compatibility constraint must be binding and it follows that $q_H = q_H^*$ and $p_H = \max\{v_H(q_H^*), 0\} - \max\{v_H(q_L), 0\}$. This automatically satisfies the participation constraint:

$$\max\{v_H(q_H^*), 0\} - [\max\{v_H(q_H^*), 0\} - \max\{v_H(q_L), 0\}] = \max\{v_H(q_L), 0\} > \max\{v_L(q_L), 0\} = 0.$$

The above proof shows that IC constraint of the H-type is binding. Using this fact, parts (iii) and (iv) follow by plugging q_L into type-H's incentive compatibility constraint. □

Proof. **(Lemma 3, Referral Program)** A sender talks if and only if

$$\xi \leq r\left(\alpha + (1-\alpha) \cdot \mathbf{1}(q_L \geq \underline{q})\right) + R.$$

As a result, the monopolist must pay at least (7) in order to assure that senders talk and thus, the monopolist pays exactly this as long as it is profitable to inform receivers, i.e., as long as $R^{**}((p_L, q_L), (p_H, q_H)) < \pi((p_L, q_L), (p_H, q_H))$ holds. □

Proof. **(Lemma 4, Free Contracts)** (i) If $\xi \leq \alpha r$, then the senders' IC constraint is always satisfied, so that the seller's problem collapses to

$$\max_{p_L,p_H\in\mathbb{R},\, q_L,q_H\geq 0} N\cdot\left[\alpha\cdot(p_H-q_H\cdot c)+(1-\alpha)\cdot(p_L-q_L\cdot c)-R\right]$$

which is equivalent to the maximization problem in the benchmark case with free WoM. Thus, no free contracts are offered under any optimal scheme.

(ii) First, note that if $\Pi^*>0$, it suffices to show when profits with free contracts (and the optimal reward scheme given by Lemma 3) are greater than profits without free contracts.

Let $\alpha r<\xi\leq r$. First, if $\xi-\alpha r>\Pi^{\text{classic}}$, then by Lemma 3, not offering free contracts yields negative profits and cannot be optimal. If $\xi-\alpha r\leq\Pi^{\text{classic}}$, then by Lemma 3, the optimal reward is $R=0$ whenever $q_L=\underline{q}$ and is $R=\xi-\alpha r$ whenever $q_L=0$. With $p_L=0$ and (p_H,q_H) as in Lemma 2, it follows immediately that offering free contracts generates weakly higher profits than offering $q_L=0$ if and only if $\Pi^{\text{classic}}-\alpha v_H(\underline{q})-(1-\alpha)\cdot\underline{q}\cdot c\geq\Pi^{\text{classic}}-(\xi-\alpha r)$, which is equivalent to (10).

(iii) Let $\xi>r$. Then, by Lemma 3 if the monopolist chooses $q_L=\underline{q}$, then profits are given by $\Pi^{\text{classic}}-CF^*-(\xi-r)$ and if $q_L=0$, then profits are given by $\Pi^{\text{classic}}-(\xi-\alpha r)$. Thus, offering free contracts generates a weakly higher profit than offering no free contracts if and only if $\Pi^{\text{classic}}-CF^*-(\xi-r)\geq\Pi^{\text{classic}}-(\xi-\alpha r)$, which is equivalent to $CF^*\leq(1-\alpha)r$. □

Proof. **(Proposition 3 and Theorem 4, Full Characterization)** Since Proposition 3 is a corollary of Theorem 4, we only prove the latter.

1. By Lemmas 2 and 3, $\Pi^*>0$ if and only if $\Pi^{\text{classic}}-CF^*-\max\{\xi-r,0\}>0$ or $\Pi^{\text{classic}}-\max\{\xi-\alpha r,0\}>0$. Since $\Pi^{\text{classic}}>0$, this can be rewritten as $\Pi^{\text{classic}}-CF^*-\max\{\xi-r,0\}>0$ or $\Pi^{\text{classic}}-(\xi-\alpha r)>0$.

2. This follows immediately from Lemma 4.

3. (a) By Lemma 3, in the presence of free contracts, a reward must only be paid if $r>\xi$.

(b) Similarly, if no free contracts are offered, positive rewards are only being paid if $\alpha r<\xi$. □

Proof. **(Propositions 4 and 6)**

Since Proposition 4 is a corollary of Proposition 6, we only prove the latter.

(i) Denote the maximal expected profit without free contracts (i.e., $q_L=0$ is offered to low types) under α by $\Pi^{\text{not free}}(\alpha)$. Similarly, denote the maximal expected profit with free contracts under α by $\Pi^{\text{free}}(\alpha)$.[33] The function $\Pi^{\text{not free}}(\alpha)$ is concave as long as $\Pi^{\text{not free}}(\alpha) > 0$, and $\Pi^{\text{free}}(\alpha)$ is linear in α as long as $\Pi^{\text{free}}(\alpha) > 0$. Moreover, we have that

$$\begin{aligned}\lim_{\alpha\to 1}\Pi^{\text{free}}(\alpha) &= \lim_{\alpha\to 1}\alpha(p_H^* - q_H^* c - v_H(\underline{q})) - (1-\alpha)\underline{q}c - \max\{\xi - r, 0\} \\ &< \lim_{\alpha\to 1}\alpha(p_H^* - q_H^* c) - \max\{\xi - \alpha r, 0\} = \Pi^{\text{not free}}(\alpha).\end{aligned}$$

This implies that $\Pi^{\text{not free}}(\alpha)$ and $\Pi^{\text{free}}(\alpha)$ intersect at most once. Hence, if $\Pi^{\text{free}}(\alpha_1) \geq \Pi^{\text{not free}}(\alpha_1)$, then $\Pi^{\text{free}}(\alpha_2) > \Pi^{\text{not free}}(\alpha_2)$ for all $\alpha_2 < \alpha_1$. This concludes the proof.

(ii) This part follows directly from part 2 of Theorem 4. □

Proof. **(Proposition 6)** (i) Denote the maximal expected profit without free contracts (i.e., $q_L = 0$ is offered to low types) under α by $\Pi^{\text{not free}}(\alpha)$. Similarly, denote the maximal expected profit with free contracts under α by $\Pi^{\text{free}}(\alpha)$.[34] The function $\Pi^{\text{not free}}(\alpha)$ is concave as long as $\Pi^{\text{not free}}(\alpha) > 0$, and $\Pi^{\text{free}}(\alpha)$ is linear in α as long as $\Pi^{\text{free}}(\alpha) > 0$. Moreover, we have that

$$\begin{aligned}\lim_{\alpha\to 1}\Pi^{\text{free}}(\alpha) &= \lim_{\alpha\to 1}\alpha(p_H^* - q_H^* c - v_H(\underline{q})) - (1-\alpha)\underline{q}c - \max\{\xi - r, 0\} \\ &< \lim_{\alpha\to 1}\alpha(p_H^* - q_H^* c) - \max\{\xi - \alpha r, 0\} = \Pi^{\text{not free}}(\alpha).\end{aligned}$$

This implies that $\Pi^{\text{not free}}(\alpha)$ and $\Pi^{\text{free}}(\alpha)$ intersect at most once. Hence, if $\Pi^{\text{free}}(\alpha_1) \geq \Pi^{\text{not free}}(\alpha_1)$, then $\Pi^{\text{free}}(\alpha_2) > \Pi^{\text{not free}}(\alpha_2)$ for all $\alpha_2 < \alpha_1$. This concludes the proof.

(ii) This part follows directly from part 2 of Theorem 4. □

[33] Existence of these maxima follows from an analogous proof to the one for Proposition 5.
[34] Existence of these maxima follows from an analogous proof to the one for Proposition 5.

Supplemental Material for
Contracting with Word-of-Mouth Management

By

Yuichiro Kamada and Aniko Öry

August 2018

COWLES FOUNDATION DISCUSSION PAPER NO. 2048SR2

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.yale.edu/

Online Supplementary Appendix to: Contracting with Word-of-Mouth Management

Yuichiro Kamada and Aniko Öry*

August 29, 2018

D Discussion

In this section we discuss various extensions and their implications, as well as the social planner's problem. For generality, we base the discussion on the generalized model introduced in the Appendix.

D.1 Heterogeneous WoM Cost

In Section B, we have entirely focused on homogeneous costs of talking, in order to emphasize the core trade-off faced by a firm when encouraging senders to engage in WoM. In the Online Appendix, we consider an extension in which different senders have different costs of talking. With heterogeneous costs of talking, the optimal reward scheme is more complicated as it can be used to *fine-tune the amount of WoM*, while with homogeneous costs either everyone or no one talks. We analyze the optimal scheme for a fairly general class of cost distribution G, and discuss how our results from Section B change. Here, we summarize the main findings of that section.

We show that the results from Section B are robust in the following sense. Free contracts are not optimal for large α because in that case the benefit of free contracts given by $(1-\alpha)r$ is small compared to the cost CF^*. Referrals and free contracts remain strategic substitutes. We also show how the homogeneous cost case can be thought of as the limit of models with heterogeneous costs.

New insights can be derived in the heterogeneous cost model with respect to the reward scheme. The optimal reward scheme is not constant in α when a free contact is offered (as it is when the cost of talking is homogeneous), but is increasing in α. The reason is that expected profits are higher with higher α and hence, the seller has a stronger incentive to increase WoM. If no free

*Kamada: Haas School of Business, University of California, Berkeley, Berkeley, CA 94720, e-mail: y.cam.24@gmail.com; Öry: School of Management, Yale University, New Haven, CT 06511, e-mail: aniko.oery@yale.edu.

contracts are offered, in addition to the aforementioned effect, there is an opposing effect (that is present also with homogeneous costs), as the seller only needs to pay less to senders if the expected externalities are large in order to induce the same number of senders to talk. Thus, if no free contracts are offered the effect of α on rewards is ambiguous, where rewards are decreasing in α if costs are sufficiently homogeneous.

D.2 Continuous Type Space

In the model with two receiver types, the optimal scheme results in the low-type customers experiencing zero value from the product, a feature that may not be realistic. Our intention in the main section was to provide the simplest model that highlights the role of free contracts as a way to incentivize WoM, and the unrealistic feature is an artifact of the simplification, not an implication of the effect we want to highlight. The aim of this section is to make this claim formal.

To this end, we provide an alternative model with a continuous type space and characterize the optimal scheme. In particular, the characterization shows that under an open set of parameter values, conditional on a customer purchasing a free contract (which happens with positive probability), with probability one the customer receives a strictly positive value from the product.

Formally, let us consider the same model as in the main section with a continuum of receiver types. The receivers' types θ are uniformly distributed on $[0, 1]$ and type θ's valuation for quantity q is given by

$$v_\theta(q) = \begin{cases} 0 & \text{if } q = 0 \\ \theta \ln(q+1) - K & \text{if } q > 0 \end{cases}$$

for some constant $K > 0$ that is independent of θ and q. Since $\lim_{q \searrow 0} v_\theta(q) = -K$, one can think of K as the fixed cost of starting to use the product. To simplify the exposition, let us assume $c < K < -\ln(c) - 1 + c$ and $c < 1$. Moreover, let us define, for $\theta \in (0, 1]$, $\underline{q}(\theta) = e^{\frac{K}{\theta}} - 1$ which is the smallest quantity that must be offered to a type-θ receiver to make her indifferent between using the product and not. Note that $\underline{q}(\theta) > 0$ for all $\theta \in (0, 1]$ and the receivers with $\theta = 0$ would not like to use the product for any $q \geq 0$. For simplicity, let us also assume $N = 1$.

The seller solves

$$\Pi^*(\xi) = \max_{p_\xi(\cdot),\, q_\xi(\cdot),\, \underline{\underline{\theta}}_\xi,\, R_\xi} \quad \mathbf{1}_{\{a_1=\text{Refer}\}} \cdot \left(\int_{\underline{\underline{\theta}}_\xi}^1 (p_\xi(\theta) - q_\xi(\theta)c)\, d\theta - R_\xi \right) \tag{12}$$

where $p_\xi \in \mathbb{R}^{[0,1]}$ and $q_\xi \in \mathbb{R}_+^{[0,1]}$ are integrable functions, $\underline{\underline{\theta}}_\xi \in [0,1]$, and $R_\xi \in \mathbb{R}$ subject to the receiver's incentive compatibility and participation constraints which are given by[1]

$$\begin{aligned} &\max\{v_\theta(q_\xi(\theta)), 0\} - p_\xi(\theta) \geq \max\{v_\theta(q_\xi(\theta')), 0\} - p_\xi(\theta') \qquad \forall \theta, \theta' \qquad (\theta\text{-type's IC}) \\ &\max\{v_\theta(q_\xi(\theta)), 0\} - p_\xi(\theta) \geq 0 \qquad \forall \theta \geq \underline{\underline{\theta}}_\xi \qquad (\theta\text{-type's PC}) \end{aligned} \tag{13}$$

and the sender's incentive compatibility (IC) constraint which is given by

$$a_1 = \text{Refer} \quad \text{if and only if} \quad \xi \leq r(1 - \underline{\underline{\theta}}_\xi) + R_\xi. \tag{14}$$

Define a strengthening of the constraint (14) by imposing a condition that the sender must talk, i.e.,

$$a_1 = \text{Refer holds and } \; \xi \leq r(1 - \underline{\underline{\theta}}_\xi) + R_\xi. \tag{14'}$$

We denote by $\tilde{\Pi}(\xi)$ the optimal profit of the problem (12) subject to (13) and (14').

In order to characterize the optimal scheme, we first define several notations. First, for $\xi = 0$, there exists a unique (up to measure-zero set of θ) solution to (12) subject to (13) and (14'), which satisfies

$$q_0^*(\theta) := \begin{cases} q^{**}(\theta) & \text{if } \theta \geq \underline{\underline{\theta}}_0^* \\ 0 & \text{if } \theta < \underline{\underline{\theta}}_0^* \end{cases} \tag{15}$$

where

$$q^{**}(\theta) := \frac{2\theta - 1}{c} - 1$$

and a $\underline{\underline{\theta}}_0^*$ which is the unique solution to $(2\theta - 1)\left[\ln\left(\frac{2\theta-1}{c}\right) - 1\right] - K + c = 0$ (we will prove this below).

[1] Note that an analogous result to Lemma 1 holds in this setup.

Second, let us denote by θ' the unique solution to $q^{**}(\theta') = \underline{q}(\theta')$. Finally, if

$$(2\theta' - 1)\left[\ln\left(\frac{2\theta' - 1}{c}\right) - 1\right] - K + c + r \leq 0, \tag{16}$$

let θ'' denote the unique value of θ that solves $(2\theta - 1)\left[\ln(\frac{2\theta-1}{c}) - 1\right] - K + c + r = 0$, which always exists.

Proposition 7. *Let $\xi > 0$.*

(i) Whenever $\tilde{\Pi}(\xi) > 0$, there exists a unique solution (up to measure-zero set of types)[2] *to the problem (12) subject to (13) and (14), and it is a solution to (12) subject to (13) and (14').*

(ii) There is a unique solution (up to measure-zero set of types) to the problem (12) subject to (13) and (14') given by $(p^*_\xi(\cdot), q^*_\xi(\cdot), \underline{\underline{\theta}}^*_\xi, R^*_\xi)$. It has the following properties:

1. *If $\xi < r(1 - \underline{\underline{\theta}}^*_0)$, then neither a free contract nor reward is offered, i.e., $p^*_\xi(\theta) > 0$ if and only if $q^*_\xi(\theta) > 0$, and $R^*_\xi = 0$. Moreover, $q^*_\xi(\cdot) = q^*_0(\cdot)$ for $\theta \in [0, 1]$ and $\underline{\underline{\theta}}^*_\xi = \underline{\underline{\theta}}^*_0$.*

2. *Suppose $r(1 - \underline{\underline{\theta}}^*_0) \leq \xi$.*

   (a) If (16) is satisfied, then the following hold.

      i. No free contract is offered, i.e., $p^*_\xi(\theta) > 0$, if and only if $q^*_\xi(\theta) > 0$.

      ii. $q^*_\xi(\theta) = q^{**}(\theta)$ for $\theta \geq \underline{\underline{\theta}}^*_\xi$ and $q^*_\xi(\theta) = 0$ otherwise.

      iii. $\underline{\underline{\theta}}^*_\xi = \theta''$

      iv. A positive reward is offered, i.e., $R^*_\xi = \xi - r(1 - \underline{\underline{\theta}}^*_\xi) > 0$, if and only if $\xi > r(1 - \theta'')$.

   (b) If (16) is not satisfied, then there exists a $\underline{\theta}_\xi > \theta'$ such that the following hold.[3]

      i. For $\theta > \underline{\theta}_\xi$, no free contract is offered, i.e., $p^*_\xi(\theta) > 0$. For $\theta \in [\underline{\underline{\theta}}^*_\xi, \underline{\theta}_\xi]$, a free contract is offered, i.e., $p^*_\xi(\theta) = 0$. Otherwise, $p^*_\xi(\theta) = 0$.

      ii. $q^*_\xi(\theta) = q^{**}(\theta)$ for $\theta > \underline{\theta}_\xi$, $q^*_\xi(\theta) = \underline{q}(\underline{\underline{\theta}}^*_\xi)$ for $\theta \in [\underline{\underline{\theta}}^*_\xi, \underline{\theta}_\xi]$, and $q^*_\xi(\theta) = 0$ otherwise.

      iii. $\underline{\underline{\theta}}^*_\xi < \theta'$.

      iv. A positive reward is offered, i.e., $R^*_\xi = \xi - r(1 - \underline{\underline{\theta}}^*_\xi) > 0$, if and only if $\xi > r(1 - \underline{\underline{\theta}}^*_\xi)$.

[2] It is not payoff-relevant for the firm if for a zero-mass of types a different contract satisfying the constraints is offered.

[3] The type $\underline{\theta}_\xi$ is determined such that $\frac{2\underline{\theta}_\xi - 1}{c} = \underline{q}(\underline{\underline{\theta}}^*_\xi)$.

The proof is presented at the end of this section. The proposition highlights that, as in the two-type case that we consider in the main analysis, the optimal scheme exhibits a rich pattern of the use of free contracts and referral rewards. In particular, it allows for the parameter regions such that both are used, only free contracts are used, only referral rewards are used, and none are used. To see our main point about the size of the surplus the receiver purchasing a free contract experiences, first note that a free contract is offered under an open set of parameter values because it is offered whenever $r(1-\underline{\underline{\theta}}^*) \le \xi$ holds and (16) is not satisfied, and those conditions hold (case 2b of Proposition 7) for an open set of parameter values. Second, whenever a free contract $(\underline{q}(\underline{\underline{\theta}}^*_\xi), 0)$ is offered, it is purchased with a positive probability as all types $[\underline{\underline{\theta}}_\xi, \underline{\theta}_\xi]$ purchase that contract and $\underline{\underline{\theta}}_\xi < \underline{\theta}_\xi$, but everyone but $\underline{\underline{\theta}}_\xi$ receives strictly positive surplus $v_\theta(\underline{q}(\underline{\underline{\theta}}_\xi))$ from it.

Proof. **(Proposition 7)** Part (i) is straightforward, so we prove part (ii). Fix a solution to the problem (12) subject to (13) and (14') and denote it by $(p^*_\xi(\cdot), q^*_\xi(\cdot), \underline{\underline{\theta}}^*_\xi, R^*_\xi)$. We first rewrite the firm's problem. To this end, let us denote the utility received by type θ under the contract $(p_\xi(\theta), q_\xi(\theta))$ by $U(\theta) = v_\theta(q_\xi(\theta)) - p_\xi(\theta)$. Then, by a standard argument in mechanism design, the receivers' IC constraints can be rewritten as

$$U(\theta) = \int_{\underline{\underline{\theta}}_\xi}^{\theta} \ln(q_\xi(\tilde{\theta}) + 1) d\tilde{\theta} + U(\underline{\underline{\theta}}_\xi)$$

for $\theta \ge \underline{\underline{\theta}}_\xi$, $q_\xi(\cdot)$ being non-decreasing and $q_\xi(\theta) \ge \underline{q}(\theta)$ for $\theta \ge \underline{\underline{\theta}}_\xi$. The PC constraint and optimality then imply $U(\underline{\underline{\theta}}^*_\xi) = 0$. Then, the seller's objective function can be rewritten by substituting $U(\theta) = v_\theta(q_\xi(\theta)) - p_\xi(\theta)$ into $\int_{\underline{\underline{\theta}}_\xi}^1 (p_\xi(\theta) - q_\xi(\theta)c)\, d\theta$:

$$\begin{aligned} \int_{\underline{\underline{\theta}}_\xi}^{1} (\theta \ln(q_\xi(\theta)+1) - K - q_\xi(\theta)c)\, d\theta &= \\ \int_{\underline{\underline{\theta}}_\xi}^{1} (\theta \ln(q_\xi(\theta)+1) - K - U(\underline{\theta}) - q_\xi(\theta)c)\, d\theta - \int_{\underline{\underline{\theta}}_\xi}^{1} \int_{\underline{\underline{\theta}}_\xi}^{1} \mathbf{1}_{\{\tilde{\theta} \le \theta\}} \cdot \ln(q_\xi(\tilde{\theta})+1) d\tilde{\theta}\, d\theta &= \\ \int_{\underline{\underline{\theta}}_\xi}^{1} ((2\theta - 1)\ln(q_\xi(\theta)+1) - K - q_\xi(\theta)c)\, d\theta & \end{aligned}$$

yielding

$$\tilde{\Pi}(\xi) = \max_{\underline{\underline{\theta}}_\xi} \max_{q_\xi(\cdot),\, R_\xi} \quad \int_{\underline{\underline{\theta}}_\xi}^1 ((2\theta - 1)\ln(q_\xi(\theta) + 1) - K - q_\xi(\theta)c)\, d\theta - R_\xi \tag{17}$$

subject to

$$q_\xi(\cdot) \text{ being non-decreasing and } q_\xi(\theta) \geq \underline{q}(\theta) \text{ for } \theta \geq \underline{\underline{\theta}}_\xi, \tag{13'}$$

(14') and $p^*(\theta) = v(q_\xi(\theta)) - \int_{\underline{\underline{\theta}}_\xi}^{\theta} \ln(q_\xi(\tilde{\theta}) + 1)d\tilde{\theta}$.

Next, we solve this maximization problem for $\xi = 0$. Point-wise maximization of the integral with respect to $q_0(\theta)$ for a fixed θ results in the first-order condition given by $\frac{2\theta-1}{q_0(\theta)+1} - c = 0$, i.e., $q_0(\theta) = \frac{2\theta-1}{c} - 1$ and a second-order condition given by $-\frac{2\theta-1}{(q_0(\theta)+1)^2} < 0$. Thus, the solution of the first-order condition gives a maximum if $\theta > \frac{1}{2}$ and otherwise the unique solution of the maximization problem is $q_0(\theta) = 0$.

If we plug this into $(2\theta - 1)\ln(q_0(\theta) + 1) - K - q_0(\theta)c$, we get for $\theta > \frac{1}{2}$,

$$(2\theta - 1)(\ln((2\theta - 1)/c) - 1) - K + 1$$

which is strictly greater than zero for $\theta = 1$ if $-\ln(c) - 1 - K + c > 0$ which we assumed. It is exactly zero at some $\underline{\underline{\theta}}_0^*$ as long as $K > c$. Thus, (15) is a solution to the maximization problem as it is increasing. Also, note that θ' given by $q_0(\theta') = \underline{q}(\theta')$ is well defined as the equation has a unique solution no more than 1 as long as $K < -\ln(c)$ which is implied by the parameter restriction $K < -\ln(c) - 1 + c$ and $c < 1$. Then, $q_0^*(\theta) > \underline{q}(\theta)$ if and only if $\theta > \theta'$.

Part 1: If $\xi < r(1 - \underline{\underline{\theta}}_0^*)$, then the unconstrained solution (the solution to (12) subject to (13)) is also achievable with the constraint (the solution to (12) subject to (13) and (14')), so it is the unique optimum and no free contracts or rewards are provided under the optimal scheme.

Part 2: If $\xi \geq r(1 - \underline{\underline{\theta}}_0^*)$, then profits are zero unless some reward is paid or the good is sold to more buyers. It is immediate that the sender's IC (14') must be binding. To find the optimal scheme, we can, hence, substitute $\xi - r(1 - \underline{\underline{\theta}}_\xi)$ for R_ξ in the optimization problem, yielding

$$\tilde{\Pi}(\xi) = \max_{\underline{\underline{\theta}}_\xi} \max_{q_\xi(\cdot)} \quad \int_{\underline{\underline{\theta}}_\xi}^1 ((2\theta - 1)\ln(q_\xi(\theta) + 1) - K - q_\xi(\theta)c + r)\, d\theta - \xi$$

subject to (13'), $R^*_\xi = \xi - r(1 - \underline{\underline{\theta}}_\xi)$ and $p^*(\theta) = v(q_\xi(\theta)) - \int_{\underline{\underline{\theta}}_\xi}^{\theta} \ln(q_\xi(\tilde{\theta}) + 1) d\tilde{\theta}$. Point-wise maximization of $((2\theta - 1)\ln(q_\xi(\theta) + 1) - K - q_\xi(\theta)c + r)$ with respect to $q_\xi(\theta)$ yields $q^*_\xi(\theta) = q^{**}(\theta)$ for $\theta \geq \underline{\underline{\theta}}^*_\xi$ where $\underline{\underline{\theta}}^*_\xi$ solves

$$(2\theta - 1)\ln\left(\frac{2\theta - 1}{c}\right) - K - 2\theta + 1 - c + r = 0$$

as long as the solution satisfies $q^{**}(\underline{\underline{\theta}}^*_\xi) \geq \underline{q}(\underline{\underline{\theta}}^*_\xi)$ (i.e., $\underline{\underline{\theta}}^*_\xi \geq \theta'$), which is equivalent to (16).

Otherwise, since $\underline{q}(\cdot)$ is strictly decreasing, we need to apply bunching and offer a free contract at the bottom because the pointwise solution $\max\{q^{**}(\theta), \underline{q}(\theta)\}$ is decreasing for $\theta \in (0, \theta')$. More precisely, there exist $\underline{\theta}_\xi$ and $\underline{\underline{\theta}}^*_\xi$ such that for $\theta \in [\underline{\underline{\theta}}^*_\xi, \underline{\theta}_\xi]$, a free contract is offered, i.e., $p^*_\xi(\theta) = 0$ and $q^*_\xi(\theta) = \underline{q}(\underline{\underline{\theta}}^*_\xi)$ for $\theta \in [\underline{\underline{\theta}}^*_\xi, \underline{\theta}_\xi]$ under the optimal scheme.

A strictly positive reward is paid if and only if ξ is strictly higher than the induced externalities $r(1 - \underline{\underline{\theta}}^*_\xi)$. This concludes the proof of (ii). □

D.3 Two-Sided Externalities

In the main analysis we assumed that only the senders receive externalities, and claimed that even if we assumed the receivers would receive externalities as well, the essence of the analysis would not change. The goal of this subsection is to make this formal. Consider a model as in Section 2, with an additional feature that if receiver i uses the product, she receives externalities r. In this model, for each $\theta \in \{H, L\}$, if a type-θ receiver uses quantity q, she experiences utility $v_\theta(q) + r$.

Note that this is a change that shifts the valuation functions by a constant, i.e., they change from $v_\theta(q)$ to $v_\theta(q) + r$ for each $\theta = H, L$. Hence, it does not alter the nature of the optimal contract scheme *under each fixed* r, assuming that our restrictions are met for the new valuation functions. This implies that all comparative statics with respect to parameters that are not r (e.g., Proposition 6) are robust. Below we show that our main comparative statics with respect to r (provided in Theorem 4) goes through as well.[4]

Note that Theorem 4 states that the use of free contracts is optimal if and only if the condition $r \in \left[\frac{CF^*}{1-\alpha}, \frac{\xi - CF^*}{\alpha}\right]$ is met. Then, the use of rewards is determined by conditions given by the bounds independent of the size of r (the conditions are $r < \xi$ in the presence of free contracts and

[4]We keep assuming that our restrictions are satisfied after the shifts of the valuation functions.

$r < \frac{\xi}{\alpha}$ otherwise, and ξ and $\frac{\xi}{\alpha}$ do not depend on r). It is immediate that the same characterization goes through in our modified model, but now the size of CF^* depends on r. If we show that CF^* is nonincreasing and $CF^* + \alpha r$ is nondecreasing in r, then the region of r such that free contracts are used is still given by a convex interval, guaranteeing that the essence of the comparative statics does not change. We first show that CF^* is strictly decreasing in r. To show this, let us write down the modified CF^* as follows:

$$CF^*(r) = \alpha(v_H(\underline{q}(r)) + r) + (1-\alpha)c\underline{q}(r),$$

where $CF^*(r)$ and $\underline{q}(r)$ denote the cost of free contracts under r and the break-even quantity for low-types under r (i.e., $v_L(\underline{q}(r)) + r = 0$), respectively. It is immediate that the second term is strictly decreasing in r because $v_L'(q)$ is strictly increasing in q and thus $\underline{q}(r)$ is strictly decreasing in r. The first term is strictly decreasing in r for the following reason: Take r and r' with $r < r'$. Then, by the assumption that $v_H'(q) > v_L'(q)$ and the definition of the $\underline{q}(\cdot)$ function, it must be the case that:

$$(v_H(\underline{q}(r')) + r') - (v_H(\underline{q}(r)) + r) < \big(v_L(\underline{q}(r')) - v_L(\underline{q}(r))\big) + (r' - r) = \big((-r') - (-r)\big) + (r' - r) = 0$$

Overall, $CF^*(r)$ is strictly decreasing in r. We next show that $CF^*(r) + \alpha r$ is strictly increasing in r under an additional assumption about the valuation functions. Specifically, suppose that $2v_L'(q) > v_H'(q) + \frac{1-\alpha}{\alpha}c$ for all $q > 0$. That is, the marginal values of the two types are not too different from each other, which ensures that the information rent $v_H(\underline{q}(r))$ does not vary too much with r. Then, taking the first-order condition of CF^* with respect to r and by noting $\underline{q}'(r) = -\frac{1}{v_L'(\underline{q}(r))}$ (by the Implicit Function Theorem), one can show that $CF^*(r) + \alpha r$ is strictly increasing in r. All in all, free contracts are used if and only if r is in a convex interval.

Note that this analysis provides an interesting observation that the cost of free contracts decreases in the size of externalities because both the production cost and the information rent decrease. The reason is that if low types receive externalities it becomes easier for the firm to make them willing to use the product (implying low production cost) and high types have less incentives to switch to the low-type contract at such a level of quantity provided to low types (implying lower information rent).

To sum up, the model of two-sided externalities provides qualitatively equivalent comparative statics as our main model with one-sided externalities.

D.4 Quantity-Dependent Externalities

The main analysis is based on a model in which the magnitude of externalities is captured by a single parameter r. As Theorem 4 shows, this is the key parameter that determines the optimal scheme. However, one can imagine that a Dropbox user who wants to refer his co-author receives higher positive externalities from joint usage if the co-author uses Dropbox more. The objective of this section is to formalize the idea of quantity-dependent externalities and discuss how such dependencies affect our predictions.

To this end, consider a function $\bar{r} : \mathbb{R}_+ \to \mathbb{R}_+$ that assigns to each quantity level consumed the value of externalities generated. We employ the normalization that $\bar{r}(0) = 0$. Note that our main model corresponds to the case in which $\bar{r}(q) = r$ for all $q > 0$. In this section we assume that $\bar{r}$ is differentiable, strictly concave, $\bar{r}'(q) > 0$ for all $q \geq 0$ and $\lim_{q\to\infty} \bar{r}'(q) = 0$.

Fix an optimal scheme $((\bar{p}^*_L, \bar{q}^*_L), (\bar{p}^*_H, \bar{q}^*_H), \overline{R}^*)$. Then, the L-type's PC constraint and the H-type's IC constraint must be binding. First, consider the case when the sender's IC constraint is binding. In that case, (generically) positive rewards are being paid. Then, if a contract is offered to the low types ($\bar{q}^*_L > 0$), then the optimal scheme must satisfy the following first-order conditions:

$$\alpha(v'_H(\bar{q}^*_H) - c + \bar{r}'(\bar{q}^*_H)) \quad = \quad 0$$

and $\overline{q}^*_L \in \{0, \underline{q}\}$ (as in the main model) if

$$(1-\alpha)(v'_L(q_L) - c + \bar{r}'(q_L)) + \alpha(v'_L(q_L) - v'_H(q_L)) \quad < \quad 0 \tag{18}$$

holds for $q_L = \underline{q}$, and $\bar{q}^*_L$ satisfies the above inequality with equality otherwise.[5] For simplicity, we focus the discussion on the case when the inequality in (18) is satisfied for $q_L = \underline{q}$.

Otherwise, the contract has a positive price. If low types are not served under the optimal

[5] The solution exists and is unique as we assume $\bar{r}$ is strictly concave and the limit of its slope is zero.

contract scheme, then only the first-order condition for $\bar{q}_H^*$ need be satisfied. Thus, as in the main model, there are only three possible levels of realized externalities corresponding to the three contracts that the firm optimally chooses conditional on rewards being paid, $r_H := \bar{r}(\bar{q}_H^*)$, $r_L := \bar{r}(\underline{q})$ and $\bar{r}(0) = 0$. Note that in this case, $q_H^* \leq \bar{q}_H^*$ holds because $\bar{r}'(\bar{q}_H^*) > 0$ and v_H' is decreasing.

If the sender's IC constraint is not binding, then the sender's IC can be ignored and thus, the optimal contract is the same as in the main model, and in particular, $\bar{q}_H^* = q_H^*$. Let us denote the externalities received if the high type's contract is purchased by $r_h := \bar{r}(q_H^*)$.

Here we consider how the conditions for offering free contracts change. In the absence of free contracts, expected externalities are given by αr_H, while in the presence of free contracts, expected externalities are given by $\alpha r_H + (1-\alpha) r_L$. Now, consider part 2 of Theorem 4. It says that, for free contracts to be used in the optimal scheme, two conditions have to be met: $r(1-\alpha) \geq CF^*$ and $\xi - \alpha r \geq CF^*$. The first inequality says that the cost of free contracts has to be no more than the increment of the expected externalities. The second says that it has to be no more than the rewards necessary to be paid to compensate for the difference between the cost of talking and the externalities that are generated anyway by high types, in the absence of free contracts. Since the first inequality automatically holds when the sender's IC constraint does not bind, and the second inequality automatically holds when the sender's IC constraint binds, these conditions can be rewritten as:

$$r_L(1-\alpha) \geq CF^* \text{ and } \xi - \alpha r_h \geq CF^*.$$

Since CF^* is unchanged, these conditions imply that low externalities for low types and high externalities for high types both reduce the set of parameters for which free contracts are optimally offered. Thus, free contracts can be optimal only if the dependence of the magnitude of externalities does not vary too much with the quantity consumed by the receivers. Our main analysis corresponds to the (extreme) case with constant $\bar{r}$ functions, and hence best captures the role of free contracts.

D.5 Informed Senders

To simplify the analysis, in the main analysis we assume that each sender has the same information about the type of his receiver as the firm. However, in some markets one can imagine that senders have better information about their friends' willingness to pay than the firm. The objective of this section is to consider a model that accommodates this possibility, and to discuss robustness of and difference from the results of the main analysis. Specifically, let us assume that each sender independently observes a signal $s \in \{s_L, s_H\}$ about his receiver. If the receiver's type is $\theta = H$, the sender sees a signal $s = s_H$ with probability $\beta \in \left(\frac{1}{2}, 1\right)$, and if the receiver's type is $\theta = L$, the sender sees a signal $s = s_H$ with probability $1 - \beta$.[6] Thus, by Bayes rule, a sender who has received a signal s_H believes that the probability of facing a H-type receiver is $\alpha_H = \frac{\alpha\beta}{\alpha\beta + (1-\alpha)(1-\beta)} (> \alpha)$, while a sender who has received a signal s_L instead believes that the probability of facing a H-type receiver is $\alpha_L = \frac{\alpha(1-\beta)}{\alpha(1-\beta)+(1-\alpha)\beta} (< \alpha)$.

How does the firm's optimization problem change? The firm's objective function is a weighted sum of the profit generated by WoM of senders who have received a high signal and the profit generated by WoM of senders who have received a low signal. The two profit functions are as in (4) with the fraction of high valuation receivers being α_H and α_L, respectively. More precisely, a fraction $\alpha\beta + (1-\alpha)(1-\beta)$ of senders have received a high signal s_H and the expected profits generated by those senders is just (4) with the fraction of H-type receivers being α_H. A fraction $\alpha(1-\beta) + (1-\alpha)\beta$ of senders has received a low signal and the profit generated by those senders is (4) with the fraction of H-type receivers being α_L. Note that the receivers' constraints remain unchanged. However, the firm now faces two IC constraints for the senders - one for the senders who observed s_H and one for the senders who observed s_L.

An important difference to the model we consider in the main part is that Lemma 1 is not valid anymore as the firm can utilize the informational differences with the reward scheme.

Proposition 8 (Rewards with informed senders). *1. Suppose that all senders choose "Refer" under the optimal scheme.*

(a) If the firm does not offer free contracts, then the optimal reward scheme $\mathbf{R}$ *satisfies* $\mathbf{R}(H) \leq \mathbf{R}(L)$ *with the inequality being strict if* $r \in (0, \frac{\xi}{\alpha_L})$.[7]

(b) If the firm offers free contracts, then the optimal reward scheme $\mathbf{R}$ *satisfies* $\mathbf{R}(H) = \mathbf{R}(L) = \max\{\xi - r, 0\}$.

[6] If $\beta = \frac{1}{2}$ was the case, then senders and the firm would have exactly the same information about receivers. Our main model corresponds to this case.

[7] $\mathbf{R}(H) = \xi - r < \mathbf{R}(L) = \xi$ for $\xi \geq r$ and $\mathbf{R}(H) = 0 \leq \mathbf{R}(L) = \max\left\{\frac{\xi - \alpha_L r}{1-\alpha_L}, 0\right\}$ for $\xi < r$.

2. *Suppose that senders who received s_H choose "Refer" but other senders choose "Not" under the optimal scheme.*

   (a) *If the firm does not offer free contracts, then there exists an optimal reward scheme* $\mathbf{R}$ *such that* $\mathbf{R}(H) > \mathbf{R}(L) = 0$. *Moreover, any optimal reward scheme* $\mathbf{R}$ *satisfies* $\mathbf{R}(H) > \mathbf{R}(L) - r$.

   (b) *If the firm offers free contracts, then there exists an optimal reward scheme* $\mathbf{R}$ *such that* $\mathbf{R}(H) > \mathbf{R}(L) = 0$. *Moreover, any optimal reward scheme* $\mathbf{R}$ *satisfies* $\mathbf{R}(H) > \mathbf{R}(L)$.

Each of the four cases arises given a nonempty parameter region that we compute in the proof of Proposition 9 presented at the end of this section.[8] An important implication of this proposition is that, if the firm wants to incentivize all senders to talk, then she must pay *more* for referrals of L-type receivers than for H-type receivers because L-type senders' expected externalities are low. In contrast, if the firm is better off excluding senders who received signal s_L, then one optimal scheme only rewards referrals of premium users. Note that if the firm wants to induce s_L-senders to talk, it should also induce s_H-senders to talk because it is cheaper to provide incentives to s_H-senders and they talk to a better pool of receivers.

Solving the full problem is a daunting task because there are multiple cases to analyze depending on which type of senders are encouraged to talk. If the monopolist decides to encourage every sender to talk, the choice between free contracts and referral rewards can be tricky: offering free contracts can be very attractive in a market with fraction α_L of high types but not attractive in a market with fraction α_H of high types. As the firm cannot differentiate between buyers who have generated a high signal versus a low signal, it needs to trade off the benefits in both markets when deciding whether to offer free contracts. One can, however, easily derive the following results for the extreme cases:

Proposition 9 (Signal strength). 1. *If* $\xi - r < \alpha(p_H^* - cq_H^*)$, *then there exists* $\bar{\beta} < 1$ *such that for all* $\beta > \bar{\beta}$, *the unique optimal menu of contracts is given by* $((0,0),(p_H^*, q_H^*))$, *and there exists an optimal reward scheme* $\mathbf{R}$, *which satisfies* $\mathbf{R}(L) = 0$. *If* $\xi - r \geq \alpha(p_H^* - cq_H^*)$, *then for any* $\beta \in (\frac{1}{2}, 1)$, *the firm cannot make positive profits.*

[8]The proof for Proposition 8 is presented at the end of this section, too.

2. *Suppose that there exists a unique optimal menu of contracts* $((p_L, q_L), (p_H, q_H))$ *in the model without signals. Then, for all* $r \notin \left\{\frac{\xi}{\alpha}, \frac{CF^*}{1-\alpha}, \frac{\xi - CF^*}{\alpha}\right\}$*, there exists* $\bar{\beta} > \frac{1}{2}$ *such that for all* $\beta \in (\frac{1}{2}, \bar{\beta})$*, there exists a unique optimal menu of contracts and it is* $((p_L, q_L), (p_H, q_H))$.

Part 1 shows that, if the signal strength β is too large, free contracts are not used by the seller. Part 2 then shows that the model we analyze in the main section without signals is reasonable when we think of the introduction of a new product category because in such a case β would be close to $\frac{1}{2}$.

Proof. **(Proposition 8)** 1. If all senders choose Refer, the IC constraints for all senders— those who see s_H and those who see s_L— must be satisfied. (a) Without free contracts, the senders' IC constraints are given by:

$$\xi \leq \alpha_H r + (\alpha_H \mathbf{R}(H) + (1 - \alpha_H)\mathbf{R}(L)) \quad \text{and} \quad \xi \leq \alpha_L r + (\alpha_L \mathbf{R}(H) + (1 - \alpha_L)\mathbf{R}(L)).$$

The optimal reward conditional on these constraints minimizes referral reward payments by making both senders' IC constraints binding whenever possible. The firm is able to do this if and only if $r \leq \xi$ and in that case the optimal reward scheme is given by $\mathbf{R}(H) = \xi - r$ and $\mathbf{R}(L) = \xi$. If $r > \xi$, it is optimal to set $\mathbf{R}(H) = 0$ and $\mathbf{R}(L) = \max\left\{\frac{\xi - \alpha_L r}{1 - \alpha_L}, 0\right\}$.

(b) With free contracts, the senders' IC constraints are given by:

$$\xi \leq r + (\alpha_H \mathbf{R}(H) + (1 - \alpha_H)\mathbf{R}(L)) \quad \text{and} \quad \xi \leq r + (\alpha_L \mathbf{R}(H) + (1 - \alpha_L)\mathbf{R}(L)).$$

Thus, it is optimal to set $\mathbf{R}(H) = \mathbf{R}(L) = \max\{\xi - r, 0\}$.

2. If senders who saw s_L do not talk, then only the IC constraint of a sender who sees s_H must be satisfied and the IC constraint of the sender who sees s_L must be violated.

(a) Without free contracts, the firm minimizes reward payments subject to these constraints by minimizing $\alpha_H \mathbf{R}(H) + (1 - \alpha_H)\mathbf{R}(L)$ (i.e., making the IC for the sender with s_H binding whenever possible) such that

$$\alpha_L r + (\alpha_L \mathbf{R}(H) + (1 - \alpha_L)\mathbf{R}(L)) < \xi \leq \alpha_H r + (\alpha_H \mathbf{R}(H) + (1 - \alpha_H)\mathbf{R}(L)).$$

First, note that these inequalities imply $\mathbf{R}(H) > \mathbf{R}(L) - r$. Second, if a referral scheme with $\mathbf{R}(H), \mathbf{R}(L) \geq 0$ that satisfies these inequalities exists (this is the case whenever $\frac{\xi}{\alpha_L} - r \geq 0$), then the referral scheme given by $\mathbf{R}(L) = 0$, $\mathbf{R}(H) = \max\{\frac{\xi}{\alpha_H} - r, 0\}$ must maximize the seller's profits: The seller cannot increase profits by decreasing $\alpha_H \mathbf{R}(H) + (1 - \alpha_H)\mathbf{R}(L)$.

(b) With free contracts, the constraints become

$$r + (\alpha_L \mathbf{R}(H) + (1 - \alpha_L)\mathbf{R}(L)) < \xi \leq r + (\alpha_H \mathbf{R}(H) + (1 - \alpha_H)\mathbf{R}(L)),$$

which imply $\mathbf{R}(H) > \mathbf{R}(L)$. By an analogous argument as in (a), a reward scheme satisfying these constraints exists if and only if $\xi - r \geq 0$ and in that case the scheme given by $\mathbf{R}(H) = \frac{\xi - r}{\alpha_H}$, $\mathbf{R}(L) = 0$ maximizes profits. □

Proof. **(Proposition 9)** 1. First, note that any optimal scheme results in one of the following three types of behaviors by the senders: Either (i) no senders talks, or (ii) all senders talk, or (iii) only senders who have received a s_H signal talk.[9]

If $\xi - r \geq \alpha(p_H^* - cq_H^*)$, then for all $\beta \in (\frac{1}{2}, 1)$ the firm cannot make positive profits. We assume from now on $\xi - r < \alpha(p_H^* - cq_H^*)$. We will show that for sufficiently large β, the firm can make positive profits, i.e., that we are in case (ii) or (iii).

Fix $\beta \in (\frac{1}{2}, 1)$. If $\xi - r\alpha_L \leq 0$, then all senders talk even without any reward payments as long as H-type receivers consume a positive quantity. Thus, we are in case (ii), and so for any optimal scheme $((p_H, q_H), (p_L, q_L), \mathbf{R})$, $\mathbf{R}(L) = 0$ and $q_L = 0$ hold.

We assume from now on that $r\alpha_L < \xi < \alpha(p_H^* - cq_H^*) + r$. Under a reward scheme $\mathbf{R}$ with $\mathbf{R}(L) = 0$ (as specified in Proposition 8) and $\mathbf{R}(H) = \frac{\max\{\xi - \alpha_H r, 0\}}{\alpha_H}$, the senders who have seen s_H talk, while senders who have seen s_L do not talk.

Next we show that, there exists $\bar{\beta} < 1$ such that for all $\beta > \bar{\beta}$, it is not optimal to offer free contracts and the firm always chooses to be in case (iii). For this purpose, we compute the profits from cases (ii) and (iii).

- **Case (iii):** Since $\alpha_H \to 1$ as $\beta \to 1$, there exists $\bar{\beta} < 1$ such that for all $\beta > \bar{\beta}$, it

[9] Note that there is no optimal scheme in which s_L-senders talk while s_H-senders do not talk. This is because $\alpha_H > \alpha_L$ and thus, given a scheme $((p_H, q_H), (p_L, q_L), \mathbf{R})$ where only s_L-senders talk, the seller can strictly increase profits by choosing a reward scheme $\mathbf{R}'$ with $\mathbf{R}'(H) = \mathbf{R}'(L) = \alpha_L \mathbf{R}(H) + (1 - \alpha_L)\mathbf{R}(L)$ while holding the menu of contracts fixed. Under this scheme, both sender types talk.

is not optimal to offer free contracts by the analysis in Section B. Thus, the profits are given by $\alpha\beta(p_H^* - cq_H^*) - (\alpha\beta + (1-\alpha)(1-\beta))\max\{\xi - \alpha_H r, 0\}$, which is greater than zero for sufficiently large β because it converges to $\overline{\Pi}_H^* \equiv \alpha(p_H^* - cq_H^*) - \alpha\max\{\xi - r, 0\} \geq \max\{\alpha(p_H^* - cq_H^*) - (\xi - r), \alpha(p_H^* - cq_H^*)\} > 0$ as $\beta \to 1$.

- **Case (ii):** We consider two cases: $\xi \geq r$ and $\xi < r$.
  - $\xi \geq r$**:** By Proposition 8, without free contracts, profits are given by $\alpha(p_H^* - cq_H^*) - (\xi - \alpha r)$ and with free contracts they are given by $\alpha(p_H^* - cq_H^*) - CF^* - (\xi - r)$. Both profits are strictly smaller than $\overline{\Pi}_H^*$.
  - $\xi < r$**:** Without free contracts, profits are given by $\alpha(p_H^* - cq_H^*) - (1-\alpha)\max\left\{\frac{\xi - \alpha_L r}{1-\alpha_L}, 0\right\}$ and with free contracts, they are $\alpha(p_H^* - cq_H^*) - CF^*$. Both profits converge to numbers that are smaller than $\overline{\Pi}_H^*$ as $\beta \to 1$.

Hence, there exists $\bar{\beta} < 1$ such that for all $\beta > \bar{\beta}$, it is not optimal to offer free contracts and the firm always chooses to be in case (iii). This concludes the proof.

2. If $\beta = \frac{1}{2}$, then one can immediately see from the expressions above that profits coincide with the ones in the main section. Thus, by continuity, for any $r < \frac{\xi}{\alpha}$, there exists a $\bar{\beta} > \frac{1}{2}$ such that for all $\beta \in (\frac{1}{2}, \bar{\beta})$, $r < \frac{\xi}{\alpha_L}$ and $r < \frac{\xi}{\alpha_H}$. Similarly, for any $r \in \left(\frac{\xi}{\alpha_L}, \frac{CF^*}{1-\alpha_L}\right)$, there exists a $\bar{\beta} > \frac{1}{2}$ such that for all $\beta \in (\frac{1}{2}, \bar{\beta})$, $r \in \left(\frac{\xi}{\alpha_L}, \frac{CF^*}{1-\alpha_L}\right)$ and $r \in \left(\frac{\xi}{\alpha_H}, \frac{CF^*}{1-\alpha_H}\right)$. Analogous conclusions hold for intervals $\left(\frac{CF^*}{1-\alpha}, \frac{\xi - CF^*}{\alpha}\right)$ and $\left(\frac{\xi - CF^*}{\alpha}, \infty\right)$. Thus, there exists a $\bar{\beta} > \frac{1}{2}$ such that for all $\beta \in (\frac{1}{2}, \bar{\beta})$, the same analysis as in the main section applies for β. □

D.6 Multiple Senders per Receiver

In the main model, we consider a stylized network structure between senders and receivers, i.e., receiver i is connected only to sender i, and *vice versa*. In reality, however, it is possible that a receiver is connected to multiple potential senders of the same information. Similarly to the discussion in the Online Appendix where the receiver can learn from an advertisement, a receiver has multiple sources of information if there are multiple senders. Such a situation can arise when senders and receivers are connected through a general network structure.

In this section we discuss how the predictions change when there are multiple senders per receiver. To make our point as clear as possible, let us assume that once a receiver adopts a product, each sender who talked to the receiver experiences the same externalities of r. That is, if there are m senders for a given receiver, then the total externalities generated by the receiver are mr. The reward can be conditioned on the set of senders who talked.

Let $m > 1$ be the number of senders connected to a given receiver. Suppose that, when there is only one sender, R is the optimal expected referral reward. The conclusion in Lemma 1 (or the analysis in the Online Appendix on advertising) entails that, by paying R in expectation to each sender, the firm can give the same incentive of talking to the senders. However, such an adjustment changes the firm's total payment. This is because the expected payment of referral reward is no longer R, but mR.

This implies that the firm becomes reluctant to use referral rewards. More precisely, if the optimal reward level is zero in the model with one sender per receiver, then it is still zero in the model with multiple senders per receiver. At the same time, free contracts become relatively more attractive as it incentivizes senders in the same way as with only one sender. Thus, when there are multiple senders per receiver, the range of parameter values such that only free contracts are used becomes wider because free contracts can substitute referral rewards.

D.7 Social Optimum

In order to understand the monopolist's strategy better, we consider the social planner's solution and compare it with the solution obtained in the main section. Specifically, we consider a social planner who has control over the senders' actions $a_i \in \{\text{Refer}, \text{Not}\}$ and the quantities q_L and q_H offered to receivers, while she does not have control over receivers' choice of whether to actually use the product after it is allocated.[10] Rewards and prices do not show up in the social planner's problem because they are only transfers between agents.

We start with two basic observations. First, whenever WoM takes place under the monopolist's solution, there is a surplus from WoM. Hence, it is also in the social planner's interest to encourage WoM. Second, under the monopolist's optimal scheme, free contracts always make senders weakly better off by increasing the probability of receiving externalities, high-type receivers better off

[10] In the classic setup of Maskin and Riley (1984), all buyers get positive utility from using the product, and thus, they always use the product after purchase. If we were to allow the social planner to have control over the use of the product and $v_L'(q) < c$ for all $q > 0$, then she would have low types use just a little bit of the quantity and generate the externalities r, which we view as implausible.

by reducing the price due to the information rent, and low-type receivers indifferent because their participation constraint is always binding. This implies that, if the monopolist firm optimally offers free contracts, then it is also socially optimal to offer it. We summarize these two observations in the following proposition:

Proposition 10. *1. If there exists a monopolist's solution under which $a_i = Refer$ for all i, then there exists a social planner's solution that entails $a_i = Refer$ for all i.*

2. If there exists $((0, \underline{q}), (\tilde{p}^*_H, q^*_H), R) \in \mathcal{S}$ for some R under the monopolist's solution, then there exists a social planner's solution that entails $q_L = \underline{q}$.

The converse of each part of the above proposition is not necessarily true, i.e., the monopolist may be less willing to encourage WoM than the social planner or not offer free contracts despite it being socially optimal. To see this clearly, we further investigate the social planner's problem in what follows.

Conditional on free contracts being offered, the welfare-maximizing menu of quantities (q_H, q_L) is exactly the same as the menu offered by the monopolist in the main section. To see why, first note that, as in the classic screening problem in Maskin and Riley (1984), the monopolist's solution results in no distortions at the top, i.e., $v'(q_H) = c$. Conditional on selling to the low types, the low-type quantity q_L under the second best in Maskin and Riley (1984) is distorted to deter high types to switch to the contract offered to low types. This means that the social planner's solution dictates that low types receive more quantity in the first best than in the second best. In our problem, however, the welfare-maximizing quantity cannot be strictly higher than $\underline{q}$ because the marginal cost c is higher than the marginal benefit $v'_L(q)$ for all $q \geq \underline{q}$ (Assumption 2), and the incentive-compatible quantity cannot be strictly lower than $\underline{q}$ because the low types would not use the product for $q_L < \underline{q}$.

Finally, whether or not the sender talks under the social planner's solution depends on the comparison between the total benefit from talking and the cost of talking, ξ: In total, WoM is efficient if and only if

$$\alpha(v_H(q^*_H) - cq^*_H + r) + (1-\alpha)\max\{r - c\underline{q}, 0\} \geq \xi. \tag{19}$$

Note that there are two social benefits of WoM. First, WoM creates network externalities because the senders and receivers become aware of each other using the product. Second, it creates gains from trade because some high-valuation buyers learn about the product.

In the monopolist's solution, free contracts are not used if $r < \frac{CF^*}{1-\alpha}$. Substituting the definition of CF^* shows that this is equivalent to $r - c\underline{q} < \frac{\alpha}{1-\alpha} v_H(\underline{q})$. Since the social planner uses free contracts if $0 < r - c\underline{q}$, the monopolist uses free contracts too little from the social planner's point of view conditional on it being socially optimal to encourage WoM if r is high, and α or $v_H(\underline{q})$ is high. The reason is as follows. On the one hand, high externalities r imply a high additional benefit r from having a receiver using the product, so that the social planner wants all receivers to use the product. However, such r pertains to the senders and the monopolist cannot extract the entire corresponding surplus. On the other hand, the monopolist is reluctant to use free contracts if the information rent necessary to induce high types to purchase a premium contracts is high relative to the number of low types who choose the free contracts. The "per low-type" information rent $\frac{\alpha}{1-\alpha} v_H(\underline{q})$ is high if α or $v_H(\underline{q})$ is high.

D.8 Effect of Advertising

In this section, we investigate how the optimal incentive scheme changes if the firm can also engage in classic advertising. Formally, consider the situation in which the firm has an option to conduct costly advertising before WoM takes place. The firm spends $a \in \mathbb{R}_+$ for advertising and this is observed by all senders but not by any receivers. Then, each receiver independently becomes aware of the product prior to the communication stage with probability $p(a)$, where $p(0) = 0$ and $p(a) > 0$ for $a > 0$. The firm simultaneously chooses a menu of contracts, a reward scheme, and advertising spending. We assume that the sender does not observe whether the receiver is already aware of the product and only enjoys externalities if the receiver starts using the product*and* she engages in WoM (independently of whether the receiver learns through advertising and/or WoM) since otherwise she cannot know whether the receiver uses the product or not. The reward scheme is now a function $\mathbf{R} : \{L, H\} \times \{A, N\} \to \mathbb{R}_+$. Here, $\mathbf{R}(\theta, A)$ denotes the reward paid to the sender whose receiver purchases the contract offered to θ-types and becomes aware of the product through advertising. Similarly, $\mathbf{R}(\theta, N)$ denotes the reward paid to the sender whose receiver purchases the contract offered to θ-types and does not become aware of the product through advertising.[11]

Having completely specified the model with advertising, let us now analyze it. Note first that Lemma 2 again holds without any modification. Suppose now that the reward scheme $\mathbf{R}$ and the advertising level a is part of the optimal scheme, and all senders choose Refer under such an optimal scheme. We assume $a > 0$ and derive a contradiction. To show this, consider the following modification of the scheme. First, let $R \equiv \alpha\left(p(a)\mathbf{R}(H,A) + (1-p(a))\mathbf{R}(H,N)\right) + (1-\alpha)\left(p(a)\mathbf{R}(L,A) + (1-p(a))\mathbf{R}(L,N)\right)$ be the expected reward, and construct a new reward scheme $\mathbf{R}'$ such that $\mathbf{R}'(\theta, x) = R$ for all $\theta = H, L$ and $x = A, N$. As in Lemma 1, this new scheme also satisfies the constraints and gives rise to the same expected profit, so it is optimal, too. Now, consider changing $a > 0$ to a new advertising level $a' = 0$. With the new scheme $(\mathbf{R}', a')$, the constraints are still satisfied; in particular all the senders choose Refer. Also, the expected profit to the monopolist increases by $a > 0$. This contradicts the assumption that the original scheme with $(\mathbf{R}, a)$ is optimal. All in all, this argument implies that either (i) the firm chooses a positive advertising level and no WoM takes place or (ii) WoM takes place and $a = 0$. Note that, in case (i), compared to the model in Section 2, advertising either substitutes WoM or allows the firm to inform some receivers if encouragement of WoM was too expensive.

D.9 Dynamic Extension

Our base model assumes a static environment, in which the receiver does not become a sender. A full analysis of a dynamic extension of the model is beyond the scope of this paper, but here we offer a simple dynamic model in a stationary environment to demonstrate the robustness of our results to dynamic extensions. Specifically, our objective is to show that coexistence of a free contract and referral rewards in the optimal scheme.

Specifically, suppose that time is discrete and double infinite, $t = \ldots, -2, -1, 0, 1, 2, \ldots$. Before the entire dynamic process starts, the seller offers a scheme $((p_L, q_L), (p_H, q_H), R) \in \mathbb{R}^5_+$. At each time t, there are a continuum of customers who know the product and consume a positive amount, and their measure is denoted by $\mu_t > 0$. Each of them talks to another new customer, so that measure μ_t of new customers are informed. Among the customers who are informed, a fraction ρ

[11] We assume that the externalities r do not depend on a. Such dependence may arise if WoM is conducted with self-enhancement motive as in Campbell et al. (2015). In such a model, r would be decreasing in a, and advertising becomes an even less attractive option for the firm than in the current model.

of them drop out from the market for some exogenous reason. Then, depending on the menu of contracts offered by the seller, each new customer makes a purchase decision. Finally, there is also an inflow of customers of size S, who know the product and make purchase decisions depending on the menu of the contract. The sum of the measures of the customers knowing the product and consuming a positive amount is then denoted μ_{t+1}. In total, each customer lives for two periods unless the customer drops out of the market with probability ρ.

We are interested in the steady state of this process, i.e. $\mu_{t+1} = \mu_t$. The seller's objective is to maximize the per-period profit, which we define to be the per-customer profit times stationary μ_t.

There are two differences from the static (full) model. First, the total population size is larger if a contract is offered to the L-type compared to if the L-type does not purchase in equilibrium. To see this, suppose first that the menu of contracts is such that the L-type customers do not make a purchase. Let $\mu^H := \mu_t$ for each t in this case. Then,

$$(1-\rho)\alpha\mu^H + \alpha S = \mu^H, \text{ or } \mu^H = \frac{S}{\rho + (\frac{1}{\alpha} - 1)}.$$

Second, let $\mu^{HL} := \mu_t$ for each t be the population size at each period when the menu of contracts is such that the both types make a purchase. Then,

$$(1-\rho)\mu^{HL} + S = \mu^{HL}, \text{ or } \mu^{HL} = \frac{S}{\rho},$$

hence $\mu^H < \mu^{HL}$.

The second difference is the participation constraint of the receiver. When deciding between purchasing and not, the receiver has to take into account the surplus from talking in the next period. Especially, if a free contract is used, this surplus may be strictly positive.

One can solve this model analytically for each parameter combination. In particular, for parameter combinations specifying a niche market and a not-too private product, one can show that both a free contract and referral rewards are used in an optimal scheme (e.g., any parameter combinations around $\alpha = 0.05$, $r = 58$, and $\rho = 0.8$). Although we do not present the full characterization for the entire parameter space as it is beyond the scope of this paper, this demonstrates that the key insights and tradeoffs are also present in a dynamic environment, showing the robustness of our results in a generalization to a dynamic model.

E Heterogeneous Costs of WoM

In this Online Appendix, we consider the case with heterogeneous costs of talking. Specifically, we assume that, after each sender i sees the menu of contrast, he privately observes his cost of talking ξ_i, drawn from an independent and identical distribution with a cumulative distribution function $G : \mathbb{R}_+ \to [0, 1]$. The firm maximizes the expected profits where the expectation is taken with respect to G. We restrict attention to twice differentiable G with $G' = g$ satisfying $g(\xi) > 0$ for all $\xi \in \mathbb{R}_+$ and

Assumption 4. *G is strictly log-concave, i.e., $\frac{g}{G}$ is strictly decreasing.*

This condition is satisfied by a wide range of distributions such as exponential distributions, a class of gamma, Weibull, and chi-square distributions, among others. Note that those restrictions are sufficient to imply the conditions for the existence result (Proposition 5) which are stated in the proof of Proposition 5.

Section E.1 characterizes the optimal scheme. Section E.2 conducts comparative statics of the optimal scheme. Section E.3 contains all the proofs for these results. Section E.4 discusses how the main model with homogeneous costs can be viewed as a limit of models with heterogeneous costs.

E.1 Properties of Optimal Contracts

First, we characterize the optimal reward. If free contracts are offered, it acts as a substitute for reward payments, which results in higher optimal rewards absent free contracts. The following proposition provides conditions under which a positive reward is optimally offered.

Lemma 5 (Optimal Reward)**.** *In the model with heterogeneous costs, there exists r^{free} and $r^{not\ free}$ with $r^{not\ free} > r^{free}$ such that the following are true:*

1. *If $r < r^{free}$, then $((p_L, q_L), (p_H, q_H), R) \in \mathcal{S}$ implies $R > 0$.*
2. *If $r^{free} \leq r < r^{not\ free}$, then $((p_L, q_L), (p_H, q_H), R) \in \mathcal{S}$ implies either $R > 0$ and $q_L = 0$, or $R = 0$ and $q_L = \underline{q}$.*

3. *If* $r^{\text{not free}} \leq r$, *then* $((p_L, q_L), (p_H, q_H), R) \in \mathcal{S}$ *implies* $R = 0$.

In order to prove this, we fix a menu of contracts with and without free contracts satisfying the conditions in Lemma 2 and solve for the optimal reward scheme. That is, conditional on offering free contracts ($q_L = \underline{q}$), we define the maximal profit under (r, α) by

$$\Pi^{\text{free}}(r, \alpha) = \max_{R \geq 0} \left(\left[\pi((0, \underline{q}), (\tilde{p}_H^*, q_H^*)) - R \right] \cdot G(r + R) \right)$$

and conditional on offering no free contracts ($q_L = 0$), define the maximal profit under (r, α) by

$$\Pi^{\text{not free}}(r, \alpha) = \max_{R \geq 0} \left([\pi((0, 0), (p_H^*, q_H^*)) - R] \cdot G(\alpha r + R) \right).$$

Let us also define the unique optimal reward given that free contracts are offered and that no free contracts are offered by $R^{\text{free}}(r, \alpha)$ and $R^{\text{not free}}(r, \alpha)$, respectively.

There are three reasons why $r^{\text{not free}} > r^{\text{free}}$ holds. As opposed to a situation without free contracts, with free contracts, (i) positive quantity is offered to low types, (ii) information rent is provided to high types, and (iii) the sender receives full externalities conditional on talking. All these effects reduce the incentive to provide referral rewards. Note that $r^{\text{not free}}$ corresponds to $\frac{\xi}{\alpha}$ in the homogeneous model, while r^{free} corresponds to ξ. In the homogeneous-cost setting, only reason (iii) affected the comparison of r^{free} and $r^{\text{not free}}$. The effects (i) and (ii) were present, but they only determined whether offering free contracts generates nonnegative profits.

The following theorem summarizes some general properties of optimal contracts. Unlike Theorem 4, it is not a full characterization, but it shows that many features of the optimal scheme with homogeneous cost carries over to the ones for heterogeneous costs.

Theorem 5 (Optimal Contracts)**.** *The following claims hold in the model with heterogeneous costs:*

1. **(Positive profits)** $\Pi^{\text{not free}}(r, \alpha) > 0$ *for all* $r \in [0, \infty)$ *and* $\alpha \in (0, 1)$.

2. **(Using both rewards and free contracts)** *There exists* $((0, \underline{q}), (\tilde{p}_H^*, q_H^*), R) \in \mathcal{S}$ *such that* $R > 0$ *(i.e., it is optimal to provide both free contracts and rewards) if and only if*

$$r^{\text{free}} > r \geq \frac{CF^*}{1 - \alpha}. \tag{20}$$

3. *Suppose that* $\frac{G(\xi)}{g(\xi)}$ *is convex.*

   (a) **(Free vs. no free contracts)** *There exist* $\underline{r}, \overline{r} \in [\frac{CF^*}{1-\alpha}, \infty)$ *such that there exists* $((0, \underline{q}), (\tilde{p}_H^*, q_H^*), R) \in \mathcal{S}$ *for some* $R \geq 0$ *(i.e., it is optimal to provide free contracts) if and only if* $r \in [\underline{r}, \overline{r}]$.

   (b) **(Never free contracts)** *If* $\frac{CF^*}{1-\alpha} > r^{not\ free}$, *then* $[\underline{r}, \overline{r}] = \emptyset$.

First, unlike in the homogeneous-cost model, profits without offering free contracts are always positive: With homogeneous costs, profits without free contracts are negative when the share of high types are low, so the expected externalities are low. This is because low expected externalities imply that a sufficient size of reward is necessary to encourage WoM, but such a cost cannot be compensated by the profits generated by only a small fraction of high types. With heterogenous costs, there always exists some fraction of customers with sufficiently small WoM costs, who do not need to be rewarded to initiate referrals.

Part 2 of the proposition shows that even with heterogeneous costs we can derive necessary and sufficient conditions for a combination of free contracts and rewards programs to be offered. As with homogeneous cost, free contracts are only optimal for sufficiently large externalities r and rewards are only offered for sufficiently small externalities.

For a full characterization of the optimal menu of contracts, it is useful to impose the additional assumption that $\frac{G}{g}$ is convex. This condition is, for example, satisfied by the exponential distribution. Given this assumption, free contracts are only offered for an intermediate connected range of externalities r. We can extend these results qualitatively as follows.

Remark 4. If we do not impose $\frac{G}{g}$ to be convex, one can still show that $\lim_{r\to 0} \Pi^{\text{not free}}(r, \alpha) > \lim_{r\to 0} \Pi^{\text{free}}(r, \alpha)$ and $\lim_{r\to\infty} \Pi^{\text{not free}}(r, \alpha) > \lim_{r\to\infty} \Pi^{\text{free}}(r, \alpha)$, i.e., free contracts can only be optimal if r is not too large and not too small.

Remark 5. With homogeneous cost $\xi > 0$, $\underline{r}$, r^{free}, $\overline{r}$ and $r^{\text{not free}}$ correspond to $\frac{CF^*}{1-\alpha}$, ξ, $\frac{\xi - CF^*}{\alpha}$, and $\frac{\bar{\xi}}{\alpha}$, respectively. In Section E.4, we formalize this correspondence by considering a limit of models with heterogeneous costs converging to the one with the homogeneous cost.

Table 2 summarizes the results of Lemma 5 and Theorem 5 for the case when $\frac{G(\xi)}{g(\xi)}$ is convex.

Externalities	$r < r^{\text{free}}$	$r^{\text{free}} < r < r^{\text{not free}}$			$r^{\text{not free}} < r$
Referral rewards	Yes	No	or	Yes	No
Free contracts	No $\Leftrightarrow r < \frac{CF^*}{1-\alpha}$	Yes		No	Yes $\Leftrightarrow r$ is small

Table 2: Comparative Statics with respect to r with heterogeneous WoM costs

E.2 Comparative Statics

Deriving precise comparative statics in the heterogeneous setup is daunting. While it is straightforward to show that $\Pi^{\text{not free}}(r, \alpha)$ and $\Pi^{\text{free}}(r, \alpha)$ are increasing in the size of externalities (r) and the fraction of the high types (α), it is hard to pin down how the comparison between these two values are affected as we change parameters (r and α). Nevertheless, using the partial characterization of the optimal contracts we can make comparative statics to understand robustness and changes of our results with the introduction of heterogeneity of WoM costs.

Proposition 11 (Market Structure and Free Contracts)**.** *The following claims hold in the model with heterogeneous costs for any fixed* $r \in [0, \infty)$*.* $\lim_{\alpha \to 0} \Pi^{\textit{not free}}(r, \alpha) > \lim_{\alpha \to 0} \Pi^{\textit{free}}(r, \alpha)$ *and* $\lim_{\alpha \to 1} \Pi^{\textit{not free}}(r, \alpha) > \lim_{\alpha \to 1} \Pi^{\textit{free}}(r, \alpha)$.[12]

The intuition for Proposition 11 is as follows. The only reason to offer free contracts is to boost up the expected externalities by $(1 - \alpha)r$, and such boosting is not significant if α is high, hence offering free contracts is suboptimal in those cases. With homogeneous costs, we showed in Section B that free contracts are optimal only when α is small. Similarly, with heterogeneous costs, a free contract cannot be optimal for high α. Moreover, if α is too small, $\Pi^{\text{free}}(r, \alpha) < 0$ holds because there are too few high types to compensate for the high cost of free contracts, and $\Pi^{\text{not free}}(r, \alpha) > 0$ holds because a strictly positive share of senders with very small WoM cost talk by part 1 of Theorem 5. This effect was not present with homogeneous costs, where the seller does not incentivize WoM at all, resulting in $\Pi^* = 0$.

The previous arguments imply that if there exists a set of parameters such that free contracts are optimal, then the choice of free versus non-free contracts is non-monotonic with respect to both r and α.

The comparative statics of the optimal reward scheme is more intricate with heterogeneous costs of WoM as the sender can fine-tune the number of senders that she wants to incentivize to

[12] These limits exist because of the monotonicity in α.

engage in WoM.

Proposition 12 (Optimal Reward Scheme)**.** *Let $r < r^{free}$. Then, the following hold in the model with heterogeneous costs:*

(i) $R^{free}(r,\alpha)$ is increasing in α. $R^{not\ free}(r,\alpha)$ is increasing in α if and only if $\alpha r \hat{G}'(\alpha r + R^{not\ free}(r,\alpha)) < \Pi^{classic}$, where we define $\hat{G}(\xi) \equiv \frac{G(\xi)}{g(\xi)}$ for all $\xi \in \mathbb{R}_+$.

(ii) $R^{free}(r,\alpha)$ and $R^{not\ free}(r,\alpha)$ are decreasing in r.

(iii) Referrals and free contracts are strategic substitutes, i.e. $R^{free}(r,\alpha) < R^{not\ free}(r,\alpha)$ for all $r \in (0, r^{not\ free})$ and $\alpha \in (0,1)$.

Although part (ii) has the same prediction as in the case with homogeneous WoM costs, the prediction in part (i) is different. We first explain the comparative statics regarding $R^{\text{free}}(r,\alpha)$. Under homogeneous costs, every sender talks and every receiver buys anyway under the usage of free contracts, so α does not affect the optimal reward level. With heterogeneous costs, however, the firm needs to tradeoff the gain and loss of increasing the rewards. The gain is the additional receivers who hear from the senders who start talking due to the increase of the rewards. The loss is the additional payments. The gain is increasing in α, so the firm has more incentive to raise the rewards.

The relationship of the optimal reward and α conditional on no free contracts being offered is ambiguous because two forces are present. First, higher α means more benefit from the receivers, and this contributes to the incentive to raise the rewards. On the other hand, higher α means more expected externalities, so there is less need to bribe a given sender. This contributes to lowering the rewards. Naturally, the second effect dominates when senders are relatively homogeneous, and indeed the optimal reward is strictly decreasing when G is completely homogeneous as in the main analysis. To formalize this idea, define

$$HMG \equiv \sup_x \left(\frac{G}{g}\right)'(x)$$

which can be interpreted as a measure of homogeneity of costs. If HMG is large, it means that there is a small range of costs of WoM that are held by many senders and HMG goes to infinity in the limit as G converges to the completely homogeneous one. An implication of the condition in part (i) of Proposition 12 is that there exists $\overline{HMG} > 0$ such that if $HMG < \overline{HMG}$, then

$R^{\text{not free}}(r, \alpha)$ is increasing in α.

Recall that both free contracts and positive rewards are used if and only if $r \in [\frac{CF^*}{1-\alpha}, r^{\text{free}})$.

Proposition 13 (Market Structure and Using Both Rewards and Free Contracts)**.** *The following claims hold in the model with heterogeneous costs:*

1. $\frac{CF^*}{1-\alpha}$ *and* $r^{\textit{free}}$ *are strictly increasing in* α.

2. $\frac{CF^*}{1-\alpha}$ *is strictly increasing and* $r^{\textit{free}}$ *is strictly decreasing in* c.

As in the homogeneous-cost model, free contracts can only be optimal if the size of externalities r is larger than $\frac{CF^*}{1-\alpha}$. Since this number is increasing in α, free contracts are optimal for small r in niche markets with small α. Thus, free contracts and referral rewards should be jointly used in niche markets (small α) if externalities are rather small, while they should be used in mass (larger α) markets if externalities are comparably larger.

With homogeneous costs, all receivers use the product under free contracts. Thus, what corresponds to r^{free} (which is ξ) does not vary with α or c. With heterogenous costs, however, it varies with these parameters. This is because the increase in α or decrease in c contributes to an increase of the expected profit per receiver, which increases the firm's incentive to offer referral rewards.

E.3 Proofs

Proof. **(Lemma 5)** First, we show the existence of unique cutoffs r^{free} and $r^{\text{not free}}$. The first-order condition of $\Pi^{\text{free}}(r, \alpha)$ with respect to R is that (i) $R = 0$ or (ii) $R > 0$ and

$$g(r+R) \cdot \left[\pi((0, \underline{q}), (\tilde{p}_H^*, q_H^*)) - R - \frac{G(r+R)}{g(r+R)}\right] = 0.$$

Note that the expression in the bracket on the left-hand side is strictly decreasing given Assumption 4 and varies continuously from ∞ to $-\infty$ as R varies from $-\infty$ to ∞. Hence, the optimal reward is always unique in $\mathbb{R}$. Also, the same argument implies that there exists a unique r such that $\pi((0, \underline{q}), (\tilde{p}_H^*, q_H^*)) - \frac{G(r)}{g(r)} = 0$. Let this unique r be r^{free}. That is, the left-hand side of the first-order condition is nonpositive and thus $R^{\text{free}}(r, \alpha) = 0$ if and only if $r \geq r^{\text{free}}$.

Analogously, conditional on offering no free contracts ($q_L = 0$), the optimal reward is unique in $\mathbb{R}$ and there exists a unique r such that $\pi((0, 0), (p_H^*, q_H^*)) - \frac{G(\alpha r)}{g(\alpha r)} = 0$. We denote this r by

$r^{\text{not free}}$. As before, we have that $R^{\text{not free}}(r,\alpha) = 0$ if and only if $r \geq r^{\text{not free}}$.

Finally, we show that $r^{\text{free}} < r^{\text{not free}}$. To see this, note that Assumption 4 implies $\frac{G(\alpha r)}{\alpha r} < \frac{G(r)}{r}$ for $r > 0$ and $\alpha \in (0,1)$. Together with $\pi((0,0),(p_H^*, q_H^*)) > \pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*))$, $r^{\text{free}} < \alpha r^{\text{not free}}$ follows by Assumption 4 and the definitions of r^{free} and $r^{\text{not free}}$. Since $\alpha < 1$, this implies $r^{\text{free}} < r^{\text{not free}}$. □

Proof. **(Theorem 5)**

1. By Assumption 3, $\pi((0,0),(p_H^*, q_H^*)) > 0$ holds. Also, since $g(\xi) > 0$ for all $\xi \in \mathbb{R}_+$, $G(\xi) > 0$ for all $\xi > 0$. Hence, for any $r \in [0,\infty)$ and $\alpha \in (0,1)$, $[\pi((0,0),(p_H^*, q_H^*)) - R] \cdot G(\alpha r + R) > 0$ holds if $R \in (0, \pi((0,0),(p_H^*, q_H^*)))$. Thus, $\Pi^{\text{not free}}(r,\alpha) > 0$.

2. Note that the use of both, free contracts and positive rewards, is optimal only if $r < r^{\text{free}}$. Also, $r < r^{\text{free}}$ implies that rewards are positive. Furthermore, in that case the maximization problems defining $\Pi^{\text{free}}(r,\alpha)$ and $\Pi^{\text{not free}}(r,\alpha)$ both have inner solutions, so the two maximization problems can be rewritten as:

$$\begin{aligned} \Pi^{\text{free}}(r,\alpha) &= \max\nolimits_{x\in\mathbb{R}}(A^{\text{free}} - x) \cdot G(x) \\ \Pi^{\text{not free}}(r,\alpha) &= \max\nolimits_{x\in\mathbb{R}}(A^{\text{not free}} - x) \cdot G(x) \end{aligned} \tag{21}$$

   where $A^{\text{free}} = \pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*)) + r$ and $A^{\text{not free}} = \pi((0,0),(p_H^*, q_H^*)) + \alpha r$. Thus, $\Pi^{\text{free}}(r,\alpha) \geq \Pi^{\text{not free}}(r,\alpha)$ if and only if

$$\pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*)) + r \geq \pi((0,0),(p_H^*, q_H^*)) + \alpha r.$$

   This is equivalent to $r \geq \frac{CF^*}{1-\alpha}$. Also, by part 1 of the current theorem, $\Pi^{\text{free}}(r,\alpha) \geq \Pi^{\text{not free}}(r,\alpha)$ implies $\Pi^{\text{free}}(r,\alpha) > 0$. Overall, there exists an optimal scheme such that both free contracts and positive rewards are used if and only if $r \in [\frac{CF^*}{1-\alpha}, r^{\text{free}})$.

3. Consider a variable

$$\frac{\Pi^{\text{free}}(r,\alpha)}{\Pi^{\text{not free}}(r,\alpha)}. \tag{22}$$

   This variable is well-defined because the denominator is always strictly positive by part 1 of the current theorem.

Step 1: Note that for $r \geq r^{\text{not free}}$, Lemma 5 shows that the rewards are zero in any optimal scheme. Hence, $\Pi^{\text{free}} = \pi((0, \underline{q}), (\tilde{p}_H^*, q_H^*)) \cdot G(r)$ and $\Pi^{\text{not free}} = \pi((0,0), (p_H^*, q_H^*)) \cdot G(\alpha r)$ hold, and thus (22) is differentiable with respect to r. If $\frac{G}{g}$ is convex, then

$$\frac{\partial}{\partial r}\frac{\Pi^{\text{free}}(r,\alpha)}{\Pi^{\text{not free}}(r,\alpha)} = \frac{\partial}{\partial r}\frac{\pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*)) \cdot G(r)}{\pi((0,0),(p_H^*, q_H^*)) \cdot G(\alpha r)} =$$

$$\frac{\left(\frac{G(\alpha r)}{g(\alpha r)} - \alpha\frac{G(r)}{g(r)}\right) \cdot \pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*)) \cdot g(r) \cdot g(\alpha r)}{[\pi((0,0),(p_H^*, q_H^*))] \cdot G(\alpha r)^2} < 0.$$

Thus, when $r \geq r^{\text{not free}}$, either (i) free contracts are not optimal for any $r \in [r^{\text{not free}}, \infty)$, or (ii) there exists a $\overline{r}' \geq r^{\text{not free}}$ such that there exists an optimal scheme in which free contracts are offered for $r \in [r^{\text{not free}}, \overline{r}']$, and no free contracts are offered under any optimal scheme for $r > \overline{r}'$. It must be the case that $\overline{r}' < \infty$ because

$$\lim_{r\to\infty}\frac{\pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*)) \cdot G(r)}{\pi((0,0),(p_H^*, q_H^*)) \cdot G(\alpha r)} = \frac{\pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*))}{\pi((0,0),(p_H^*, q_H^*))} < 1.$$

We let $\bar{r} = \bar{r}'$ in case (ii).

Step 2: Next, we consider the following three different cases: $\frac{CF^*}{1-\alpha} < r^{\text{free}}$, $\frac{CF^*}{1-\alpha} \in [r^{\text{free}}, r^{\text{not free}}]$, and $\frac{CF^*}{1-\alpha} > r^{\text{not free}}$.

- Let $\frac{CF^*}{1-\alpha} < r^{\text{free}}$. Then, it follows from part 2 of the current theorem that no free contracts are offered for $r < \frac{CF^*}{1-\alpha}$ and free contracts are offered for $r \in [\frac{CF^*}{1-\alpha}, r^{\text{free}}]$. For $r \in [r^{\text{free}}, r^{\text{not free}}]$,

$$\frac{\partial}{\partial r}\frac{\Pi^{\text{free}}(r,\alpha)}{\Pi^{\text{not free}}(r,\alpha)} = \frac{\partial}{\partial r}\frac{[\pi((0,\underline{q}),(p_H^*, q_H^*))] \cdot G(r)}{\max_{R\in\mathbb{R}}[\pi((0,0),(p_H^*, q_H^*)) - R] \cdot G(\alpha r + R)} =$$

$$\frac{\left(\frac{G(\alpha r + R^{\text{not free}}(r,\alpha))}{g(\alpha r + R^{\text{not free}}(r,\alpha))} - \alpha\frac{G(r)}{g(r)}\right) \cdot \pi((0,\underline{q}),(p_H^*, q_H^*)) \cdot g(r) \cdot g(\alpha r + R^{\text{not free}}(r,\alpha))}{[\pi((0,0),(p_H^*, q_H^*)) - R^{\text{not free}}(r,\alpha)] \cdot G(\alpha r + R^{\text{not free}}(r,\alpha))^2}.$$

  Note that $\Pi^{\text{not free}}(r,\alpha)$ is differentiable in r by the Envelope Theorem. Moreover, if $\frac{G(\alpha r+R)}{g(\alpha r+R)} - \alpha\frac{G(r)}{g(r)} < 0$, then $\alpha r + R < r$ because $\frac{G(\xi)}{g(\xi)}$ is increasing in ξ by Assumption 4. Moreover, $R^{\text{not free}}(r,\alpha)$ is differentiable in r by the implicit function theorem applied to the first-order condition of $\Pi^{\text{not free}}$ and, letting $\hat{G}(\xi) := \frac{G(\xi)}{g(\xi)}$ for all ξ,

$$\frac{\partial}{\partial r}R^{\text{not free}}(r,\alpha) = -\frac{\alpha\hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha))}{1 + \hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha))} < 0.$$

Thus,

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{G(\alpha r+R^{\text{not free}}(r,\alpha))}{g(\alpha r+R^{\text{not free}}(r,\alpha))}-\alpha\frac{G(r)}{g(r)}\right) &= \\
\alpha\left(\hat{G}'(\alpha r+R^{\text{not free}}(r,\alpha))-\hat{G}'(r)\right)+\hat{G}'(\alpha r+R^{\text{not free}}(r,\alpha))\frac{\partial}{\partial r}R^{\text{not free}}(r,\alpha) &= \\
\alpha\left(\hat{G}'(\alpha r+R^{\text{not free}}(r,\alpha))-\hat{G}'(r)-\frac{\hat{G}'(\alpha r+R^{\text{not free}})^2}{1+\hat{G}'(\alpha r+R^{\text{not free}}(r,\alpha))}\right) &< 0.
\end{aligned}$$

Thus, if the derivative of (22) is negative at $r' \in [r^{\text{free}}, r^{\text{not free}}]$ then (22) is decreasing for all $r \in [r', r^{\text{not free}}]$. Together with Step 1, this implies the following. In case (i), there exists $\overline{r} \in [r^{\text{free}}, r^{\text{not free}})$ such that free contracts are offered in an optimal scheme if and only if $r \in [\frac{CF^*}{1-\alpha}, \overline{r}]$. In case (ii), the current analysis shows that it is optimal to offer free contracts for all $r \in [r^{\text{free}}, r^{\text{not free}}]$, so free contracts are offered if and only if $r \in [\frac{CF^*}{1-\alpha}, \overline{r}]$, where $\overline{r}$ is the variable that we defined in Step 1.

- Let $\frac{CF^*}{1-\alpha} \in \left[r^{\text{free}}, r^{\text{not free}}\right]$. In that case, offering free contracts is not optimal for any $r < r^{\text{not free}}$. Then, either free contracts are not optimal for any r or by the same argument as above, if free contracts are not used in an optimal scheme for $r = r'$ then they are not used in any optimal scheme for any $r > r'$. This proves the desired claim for this case.

- If $\frac{CF^*}{1-\alpha} > r^{\text{not free}}$, then offering free contracts is not optimal for any $r < r^{\text{free}}$. For $r \in \left[r^{\text{free}}, r^{\text{not free}}\right]$ free contracts are also not optimal because

$$1 > \frac{\max_{R\in\mathbb{R}}[\pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))-R]\cdot G(r+R)}{\max_{R\in\mathbb{R}}[\pi((0,0),(p_H^*,q_H^*))-R]\cdot G(\alpha r+R)} \geq \frac{[\pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))]\cdot G(r)}{\max_{R\in\mathbb{R}}[\pi((0,0),(p_H^*,q_H^*))-R]\cdot G(\alpha r+R)}.$$

  The first inequality follows from the proof of part 2 of the current theorem. For $r \geq r^{\text{not free}}$, offering free contracts is never optimal by Step 1.

This concludes the proof.

□

Proof. **(Proposition 11)** First, note that we can write the limiting profits as

$$\begin{aligned}\lim_{\alpha\to 1}\Pi^{\text{free}}(r,\alpha) &= \max_{x\geq r}(\tilde{p}_H^* - cq_H^* + r - x)G(x) < \\ \lim_{\alpha\to 1}\Pi^{\text{not free}}(r,\alpha) &= \max_{x\geq r}(p_H^* - cq_H^* + r - x)G(x).\end{aligned}$$

It follows immediately from part 1 of Theorem 5 that there exist $\alpha' > 0$ and $\epsilon > 0$ such that $\Pi^{\text{free}}(r,\alpha) + \epsilon < \Pi^{\text{not free}}(r,\alpha)$ for any $\alpha \in (0,\alpha')$, hence the limit result as $\alpha \to 0$ holds. □

Proof. **(Proposition 12)** Applying the implicit function theorem to the first-order conditions of Π^{free} and $\Pi^{\text{not free}}$ gives us:

(i) $\frac{R^{\text{free}}(r,\alpha)}{\partial\alpha} = -\frac{p_H^* - q_H^* c - v_H(\underline{q}) + c\underline{q}}{-1-\hat{G}'(r+R)} > 0$ and

$$\frac{\partial R^{\text{not free}}(r,\alpha)}{\partial\alpha} = -\frac{p_H^* - q_H^* c - r\hat{G}'(\alpha r + R)}{-1 - \hat{G}'(r+R)}$$

which is strictly greater than zero if and only if $r\hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha)) < p_H^* - q_H^* c$, or $\alpha r\hat{G}'(\alpha r + R^{\text{not free}}(r,\alpha)) < \Pi^{\text{classic}}$.

(ii) $\frac{R^{\text{free}}(r,\alpha)}{\partial r} = -\frac{-\hat{G}'(R+r)}{-1-\hat{G}'(R+r)} < 0$ and $\frac{R^{\text{not free}}(r,\alpha)}{\partial r} = -\frac{-\alpha\hat{G}'(R+\alpha r)}{-1-\hat{G}'(R+\alpha r)} < 0$ because $\hat{G}'(x) > 0$ for all $x > 0$, so $-1 - \hat{G}'(x) < 0$.

(iii) First, note that for $r > r^{\text{free}}$, referral rewards are always zero when free contracts are offered, i.e., the statement is trivially true. If $r \leq r^{\text{free}}$, then the optimal reward with free contracts $R^{\text{free}}(r,\alpha)$ satisfies the first-order condition:

$$R^{\text{free}}(r,\alpha) = \pi((0,\underline{q}),(\tilde{p}_H^*, q_H^*)) - \frac{G(r + R^{\text{free}}(r,\alpha))}{g(r + R^{\text{free}}(r,\alpha))}. \tag{23}$$

By the first-order condition for the maximization problem for the case with no free contracts with respect to the reward, the solution $R^{\text{not free}}(r,\alpha)$ must satisfy:

$$g(\alpha r + R^{\text{not free}}(r,\alpha)) \cdot \left(\pi((0,0),(p_H^*, q_H^*)) - R^{\text{not free}}(r,\alpha) - \frac{G(\alpha r + R^{\text{not free}}(r,\alpha))}{g(\alpha r + R^{\text{not free}}(r,\alpha))}\right) = 0.$$

Since $g(\cdot) > 0$, this implies that

$$\pi((0,0),(p_H^*,q_H^*)) - R^{\text{not free}}(r,\alpha) - \frac{G(\alpha r + R^{\text{not free}}(r,\alpha))}{g(\alpha r + R^{\text{not free}}(r,\alpha))} = 0. \tag{24}$$

Now, substitute $R^{\text{not free}}(r,\alpha)$ by the expression for $R^{\text{free}}(r,\alpha)$ given by (23) on the left hand side of (24), to obtain:

$$\pi((0,0),(p_H^*,q_H^*)) - \pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*)) + \frac{G(r + R^{\text{free}}(r,\alpha))}{g(r + R^{\text{free}}(r,\alpha))} - \frac{G(\alpha r + R^{\text{free}}(r,\alpha))}{g(\alpha r + R^{\text{free}}(r,\alpha))}.$$

This is strictly positive by log-concavity of G (Assumption 4) and because $\pi((0,0),(p_H^*,q_H^*)) > \pi((0,\underline{q}),(\tilde{p}_H^*,q_H^*))$. Noting that the left hand side of (24) is strictly decreasing in referral rewards, the optimal reward without free contracts $R^{\text{not free}}(r,\alpha)$ is strictly greater than $R^{\text{free}}(r,\alpha)$. □

Proof. **(Proposition 13)** The comparative statics with respect to $\frac{CF^*}{1-\alpha}$ are straightforward from the formula of CF^*. The ones for r^{free} follow from the first-order condition with respect to rewards that appears in the proof of Lemma 5 and Assumption 4. □

E.4 Homogeneous Costs as the Limit of Heterogeneous Costs

Consider a sequence $\{G^n\}_1^\infty$ that converges pointwise to the G defined by $G = \mathbf{1}_{\{\bar{\xi} \leq \xi\}}$ such that for each n, G^n is twice differentiable with $(G^n)'(\xi) = g^n(\xi) > 0$ for all ξ, and Assumption 4 holds. Let the set of all such sequences be $\mathcal{G}$. The set $\mathcal{G}$ is nonempty. For example, consider $\{G^n\}_1^\infty$ such that for each $n \in \mathbb{N}$, G^n is a normal distribution with mean $\bar{\xi} \geq 0$ and variance $\frac{1}{n}$ truncated at $\xi = 0$. By inspection one can check that $\{G^n\}_1^\infty \in \mathcal{G}$. For any given G^n, we can define $\underline{r}^n$, $r^{\text{free},n}$, $\bar{r}^n$, and $r^{\text{not free},n}$. Then, the following statement can be shown: For any $\{G^n\}_1^\infty \in \mathcal{G}$,

$$\lim_{n\to\infty} \underline{r}^n = \frac{CF^*}{1-\alpha}, \quad \lim_{n\to\infty} r^{\text{free},n} = \bar{\xi}, \quad \lim_{n\to\infty} \bar{r}^n = \frac{\bar{\xi} - CF^*}{\alpha}, \quad \text{and} \quad \lim_{n\to\infty} r^{\text{not free},n} = \frac{\bar{\xi}}{\alpha}.$$