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ORIGINAL ARTICLE

Contraction and closure

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In this paper, I consider the connection between consequence relations and closure operations. I argue that one familiar connection makes good sense of some usual applications of consequence relations, and that a largeish family of familiar noncontractive consequence relations cannot respect this familiar connection.

Keywords consequence; contraction; closure; multisets; axiomatisation; commitment

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This is a paper about *noncontractive* logical systems: systems not closed under contraction. (Contraction is what allows us to conclude that $\Gamma, A \vdash B$ from the claim that $\Gamma, A, A \vdash B$.) These systems form one family of *substructural* logical systems: systems that do not impose the full budget of what are called *structural rules*, or in which there are counterexamples to the *structural metainferences* associated with these rules. My main goal here is to consider the connection between consequence and *closure*, a connection recently emphasized in (Beall 2015), and point to a seeming difficulty faced by many familiar noncontractive consequence relations in upholding this connection. The trouble I will point out in what follows applies to approaches explored and defended in Barker (2010), Beall and Murzi (2013), Grišin (1982), Mares and Paoli (2014), Petersen (2000), Restall (1994), and Zardini (2011). It does not apply to every noncontractive approach — there are ways to avoid the trouble, which I will flag — but I think it applies to enough of the usual suspects to be of interest. I will not argue (and indeed I don't believe) that the difficulty I point to here is fatal for these approaches; my point is simply to call attention to the difficulty that they must face.

1 Noncontractive consequence

First, let me outline a few relevant features of the noncontractive consequence relations I will consider.[1] Whether or not consequence relations should allow for multiple *conclusions* is a matter of some controversy (see, e.g., Rumfitt 2008; Steinberger 2011), but whether or not they should allow for multiple *premises* is not: they should.

Contraction (really, *left contraction*) is the following principle about consequence: whenever $\Gamma, A, A \vdash B$, then $\Gamma, A \vdash B$.[2] (In multiple-conclusion settings, there is also *right contraction*: whenever $\Gamma \vdash A, A, \Delta$, then $\Gamma \vdash A, \Delta$. In these settings, left contraction should

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be reformulated as: whenever $\Gamma, A, A \vdash \Delta$, then $\Gamma, A \vdash \Delta$. I will stick to single conclusions here; this makes the connection to closure easier to formulate.) Noncontractive consequence relations are those that are not closed under contraction; they feature *counterexamples* to contraction: cases where $\Gamma, A, A \vdash B$ but $\Gamma, A \nvdash B$.

Such consequence relations have been advocated for a wide variety of purposes. As an example, one important philosophical application has to do with their potential for addressing paradoxes of truth and validity; for discussion, see, for example, Beall and Murzi (2013), Restall (1994), Ripley (2015), Shapiro (2011), Weber (2014), and Zardini (2011). However, I will not pursue any application here; my concern is rather to look at whether noncontractive systems can play the roles we want a consequence relation for at all, whether we are dealing with paradoxes or with something else.

1.1 Multisets and collecting premises

Rejecting contraction forces us to think about how premises are combined with each other. After all, if Γ, A, A is the same thing as Γ, A, then contraction follows straightforwardly. But if premises are combined by being gathered into a set — one very common approach — then Γ, A, A is indeed the same thing as Γ, A. So if we are to reject contraction, we cannot allow that premises are combined by being gathered into a set.

One usual approach is instead to take premises to be combined by collecting them into a *multiset*.[3] A multiset is just like a set, except that number of occurrences matters: while $\{A, A, B\}$ and $\{A, B\}$ are the same set, $[A, A, B]$ and $[A, B]$ are different multisets. Alternately, a multiset is just like a sequence, except that order doesn't matter: while A, A, B and A, B, A are different sequences, $[A, A, B]$ and $[A, B, A]$ are the same multiset. (There is no need here to decide whether or not infinite multisets are allowed; the points to follow are not affected either way.)

Many options besides multisets are possible. However, my arguments here are restricted to systems based on multisets. This is a substantial restriction, and I do not know to what extent these arguments can be extended to systems that are not multiset-based. See Bimbó (2015), Galatos et al. (2007), Paoli (2002), Read (1988), and Restall (1994, 2000) for discussion of other options; I set all other options aside here, without further qualification.

Many operations and relations familiar from set theory generalize nicely to multisets, which can make them pleasant to work with. For example, there is a natural and well-behaved notion of *submultiset*; $X \sqsubseteq Y$ iff everything that occurs in X occurs *at least as many times* in Y. There is also a natural notion of *intersection*: something occurs in $X \sqcap Y$ the *minimum* of the number of times it occurs in X and the number of times it occurs in Y.

Interestingly, however, there are *two* natural union-like notions for multisets, both of which will be important here. One of them is dual to $\sqcap$; something occurs in $X \sqcup Y$ the *maximum* of the number of times it occurs in X and the number of times it occurs in Y. (As it happens, $\sqsubseteq$ is a lattice order, with $\sqcap$ the meet and $\sqcup$ the join.) The other is sui generis: something occurs in $X \boxplus Y$ the *sum* of the number of times it occurs in X and the number of times it occurs in Y.

When it comes to combining premises, both $\sqcup$ and $\boxplus$ may seem like natural choices. After all, both are ways to combine multisets. But the noncontractivist cannot simply have their pick. If Γ, A, A (considered as a collection of premises) is to be distinct from Γ, A, the comma cannot be understood as $\sqcup$; $\Gamma \sqcup [A] \sqcup [A]$ is always the same multiset as $\Gamma \sqcup [A]$, and this again forces contraction.

So $\boxplus$ is the way for noncontractivists to collect their premises into multisets. $\Gamma \boxplus [A] \boxplus [A]$ really is distinct from $\Gamma \boxplus [A]$; the former has one more occurrence of A than the latter. When we see things like 'Γ, A, A' written as a collection of premises, we should interpret this as $\Gamma \boxplus [A] \boxplus [A]$; I'll abbreviate freely in this way.

All this is routine; I have belabored it only because $\sqcup$, which is not often a player in discussions of noncontractive approaches, turns out to matter for present purposes, and it is important to have the distinct roles of $\sqcup$ and $\boxplus$ firmly distinguished. I will also appeal to the following fact: if A does not occur in Γ, then $\Gamma \boxplus [A] = \Gamma \sqcup [A]$. (This is because the sum of 0 and 1 is the same as their maximum.)

1.2 Transitivity without contraction

A binary R relation on a set S is *transitive* iff: for any $x, y, z \in S$, if xRy and yRz, then xRz; call this *transitivity proper*. When we speak of a consequence relation being transitive, transitivity proper is almost never what is meant.[4] (A multiset-to-formula consequence relation like those considered here isn't a binary relation on a single set at all.)

So what do people mean, when they call a multiset-to-formula consequence relation *transitive*? One natural thing to mean is the following: for any formulas A, B, C, if $A \vdash B$ and $B \vdash C$, then $A \vdash C$.[5] Call this *simple transitivity*; it is the closest well-typed approximation possible to transitivity proper. But simple transitivity, too, is almost never what is meant. Much more usual is the following: for any multisets Γ, Δ, and any formulas A, B, if $\Gamma \vdash A$ and $\Delta, A \vdash B$, then $\Delta, \Gamma \vdash B$. This property is strictly stronger than simple transitivity. I will call it the *cut property*, after the structural rule of cut, which it is clearly intimately related to:

$$\text{Cut}: \ \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Delta, \Gamma \vdash B}$$

When a noncontractive system is said to be transitive, the cut property is usually what is meant.[6] All the above-cited noncontractive consequence relations exhibit the cut property; this is the usual sense of 'transitive' in play, when these systems are called 'transitive'. It turns out, however, that when we consider the relationship between consequence and closure, the cut property causes trouble.

2 Closure

It is often thought that there is an intimate connection between consequence and *closure*. To draw this out, I will consider two distinct ways in which this connection gets put to use; these ways provide running examples for the rest of the paper. First, we use consequence relations to give large (usually infinite) theories manageable presentations: we give only part of the theory, and let a consequence relation take care of the rest. For example, we can present an arithmetic theory by giving a selection of axioms and choosing a logic to close those axioms under. Here, the axioms give us something like a seed, from which we are able to grow an entire theory with the help of a consequence relation. Second, we use consequence relations to connect what someone has explicitly stated to the broader body of claims they are thereby committed to. Again, their explicit utterances give us something like a seed, from which we are able to grow the entire body of their commitments.

In either case, we see consequence playing an important *closure* role. Whether we are concerned with presenting a full theory via one of its subtheories, or with connecting explicit claims to commitments thereby taken up, we use consequence to tie a smaller body of information Γ to a larger one $C(\Gamma)$, its closure, in the following way: a claim A is in $C(\Gamma)$ iff it is entailed by Γ; that is, iff $\Gamma \vdash A$. This use has been recently emphasized in Beall (2015, notation tweaked): "Give to logic your theory Γ, and then sit back: logic 'freely' or 'automatically' expands your theory to $C(\Gamma)$, which contains all of Γ's (singleton) consequences."

2.1 Closure and multisets

It is natural to look at the operation C here as a closure operation in the usual algebraic sense. Given a partially ordered set P with order $\leq$, a *closure operation* on P is a unary operation C on P that meets the following three conditions, for all $x, y \in P$:

- Increasingness: $x \leq Cx$
- Monotonicity: If $x \leq y$, then $Cx \leq Cy$
- Idempotence: $Cx = CCx$

The closure-related uses of consequence considered above seem to motivate all three of these principles quite directly. First, consider a consequence relation used to present a theory via one of its parts. Call the part used a 'presentation' of the theory so determined; in an axiomatic presentation of an arithmetic theory, for example, the axioms are a presentation of the full arithmetic theory they axiomatise. In this use, increasingness tells us that every presentation is part of the theory it presents. Monotonicity tells us that to remove something from the theory presented, we must remove something from its presentation. Finally, idempotence tells us that a full theory, when treated as a presentation, presents itself.

Second, consider a consequence relation used to determine the range of commitments taken on by someone who has made certain explicit claims. In this use, increasingness tells us that they are committed to the claims they have explicitly made. Monotonicity tells us that to remove a commitment, some explicit claim must be removed. And finally, idempotence tells us that rendering our commitments explicit does not add still more to them. In either use, then, all three conditions on closure operations seem reasonable.

All this is perfectly neutral as to what kind of thing the bodies of information under consideration are. For example, if we want consequence to apply to multisets, it is easy to have such a closure operation on multisets.[7] (Here and throughout, when I consider multisets as partially ordered, I assume $\sqsubseteq$ as the relevant ordering.) Recall that what we want from the connection between closure and consequence is the following: that B occurs in $C(\Gamma)$ iff $\Gamma \vdash B$. Call $\vdash$ a *closure relation* iff it is connected to some closure operation on multisets in this way.

Here's my main point: no familiar noncontractive consequence relation is a closure relation, and this is not a coincidence. The expected connection between consequence and closure turns out to play havoc with almost any alleged failure of contraction, when it is combined with the cut property.

The trouble is created by the following fact:

FACT 1. *For any closure operation C on multisets, if A occurs in $C(\Gamma)$, and if B occurs in $C(\Delta \sqcup [A])$, then B also occurs in $C(\Delta \sqcup \Gamma)$.*

Proof. Take any A, B, Γ, Δ satisfying the antecedents of the claim. Since A occurs in $C(\Gamma)$, we have $[A] \sqsubseteq C(\Gamma)$. By increasingness, $\Delta \sqsubseteq C(\Delta)$. Now, $\Delta \sqcup [A]$ is the least upper bound of Δ and $[A]$; it is contained in anything that contains them both.[8] $C(\Delta) \sqcup C(\Gamma)$ contains them both; so $\Delta \sqcup [A] \sqsubseteq C(\Delta) \sqcup C(\Gamma)$. By monotonicity, $C(\Delta \sqcup \Gamma)$ contains both $C(\Delta)$ and $C(\Gamma)$, so we also have $C(\Delta) \sqcup C(\Gamma) \sqsubseteq C(\Delta \sqcup \Gamma)$. Chaining, $\Delta \sqcup [A] \sqsubseteq C(\Delta \sqcup \Gamma)$. By monotonicity again, $C(\Delta \sqcup [A]) \sqsubseteq C(C(\Delta \sqcup \Gamma))$; idempotence then gives $C(\Delta \sqcup [A]) \sqsubseteq C(\Delta \sqcup \Gamma)$. By assumption, B occurs in $C(\Delta \sqcup [A])$; it must then also occur in $C(\Delta \sqcup \Gamma)$. □

This gives, for any closure relation $\vdash$: if $\Gamma \vdash A$ and $\Delta \sqcup [A] \vdash B$, then $\Delta \sqcup \Gamma \vdash B$.[9] If you squint at this fact, it will start to look like the cut property—but *don't!* it *ain't!* The cut property involves $\boxplus$, not $\sqcup$—this is something distinct. We have, not the cut property itself, but a relative built on $\sqcup$ rather than $\boxplus$; call this the *maxi-cut property*. It is immediate from Fact 1 that every closure relation has the maxi-cut property. Now, consider the following:

FACT 2. *Suppose $\Gamma, P, P \vdash R$, and suppose there is some Q that does not occur in $\Gamma \boxplus [P]$ such that $P \vdash Q$ and $Q \vdash P$. Then if $\vdash$ is a closure relation that has the cut property, $\Gamma, P \vdash R$.*

Proof. Since $\Gamma, P, P \vdash R$ and $Q \vdash P$, the cut property gives us $\Gamma, P, Q \vdash R$; that is, $\Gamma \boxplus [P] \boxplus [Q] \vdash R$. But since Q does not occur in $\Gamma \boxplus [P]$, we have $\Gamma \boxplus [P] \boxplus [Q] = (\Gamma \boxplus [P]) \sqcup [Q]$, so $(\Gamma \boxplus [P]) \sqcup [Q] \vdash R$. We also have (by assumption) $P \vdash Q$. By the maxi-cut property, then, $(\Gamma \boxplus [P]) \sqcup [P] \vdash R$. But $(\Gamma \boxplus [P]) \sqcup [P]$ is just $\Gamma \boxplus [P]$, so $\Gamma \boxplus [P] \vdash R$. That is, $\Gamma, P \vdash R$. □

That is, if consequence is a closure relation obeying the cut property, we can *never* have a failure of contraction where the formula being contracted is equivalent to some distinct formula that is absent from the remaining premises. But noncontractive logics, like many others, typically have denumerably many formulas equivalent to any given formula. In such systems, then, there can't be *any* counterexamples to contraction with a finite collection of premises.

Interestingly, although the cut property is usually considered to be the sole form of transitivity a noncontractivist should want, we can see from the above that it is not only *not motivated by* the alleged connection between consequence and closure, it is in fact *incompatible with* such a connection, at least if contraction is to fail! Yet all of the multiset-based systems explored in Barker (2010), Beall and Murzi (2013), Grišin (1982), Mares and Paoli (2014), Petersen (2000), Restall (1994), and Zardini (2011) preserve the cut property rather than the connection to closure.

3 Discussion

The usually assumed connection between consequence and closure cannot be respected by any usual multiset-based noncontractive consequence relation. This connection brings the maxi-cut property with it, and this property and the cut property together bring contraction, or at least very many instances of it.

How should an advocate of a noncontractive approach to consequence respond? I see three broad options. First, they might advocate a noncontractive system not based around multisets. This would get around the above argument directly, as the argument turns on features of closure operations on multisets. However, one reason my argument has left systems involving more complicated structures to one side is precisely that there is often not such a clear ordering available for these structures — and yet an ordering is needed even to state what a closure operation on these structures should be. Still, there may well be some kind of closure that can fulfill the expected connection for such systems.

Second, such advocates might simply drop the cut property in favor of exploring noncontractive closure relations. These will exhibit the maxi-cut property *rather than* the cut property. This is perhaps the most interesting possibility, as it holds the potential to open up a new direction for noncontractive explorations, which have so far not extended to systems without the cut property. This kind of response would allow for maintaining the usual connection to closure, while still allowing for failures of contraction. This option has not yet been explored; it calls for further work.

Finally, noncontractivists might reject the supposed connection between consequence and closure. This too would block the trouble, and it is the only option for understanding the multiset-based noncontractive consequence operations already in the literature. There are at least two ways in which this might be pursued.

One could attempt to retain the connection between noncontractive consequence on the one hand and axiomatisation or development of commitments on the other, and deny that this connection needs to be realized via a closure relation. In Cintula and Paoli (2015), for example, Cintula and Paoli seem to take something like this approach, responding to the difficulty in question by invoking operations that take multisets to *sets of* multisets, rather than closure operations. (It is not yet clear to me whether their operations can indeed play the roles in axiomatisation and commitment that I have indicated here as key reasons for the connection to closure; at the very least, they will not do so in the usual ways.)

Alternately, one could simply reject the connection I've indicated here between noncontractive consequence and axiomatisation or development of commitments. There might be no connection at all, or the connection might be more indirect. In particular, the connection might happen by way of a distinct consequence relation entirely, one that *does* obey contraction. For example, Priest (2015, §. 5.2), French and Ripley (2014, §. 3.4), and Mares and Paoli (2014, §. 3.3) all offer different ways in which this might be pursued.[10]

To sum up, then: usual multiset-based noncontractive consequence relations, of the kind explored and defended in Barker (2010), Beall and Murzi (2013), Grišin (1982), Mares and Paoli (2014), Petersen (2000), Restall (1994), and Zardini (2011), cannot sustain the connection to closure that we often expect of consequence relations. This is because they exhibit the cut property, and tying them to a closure operation on multisets would require them to exhibit the maxi-cut property; but the maxi-cut property and the cut property together come very close to ruling out failures of contraction altogether. Advocates of such consequence relations, then, must find some way to reject the usual connection between consequence and closure.[11]

Notes

1 Exactly which sort of consequence is at issue is something that differs from one theorist to another; I will try to address the issue at a general-enough level that I can legitimately avoid worrying about the differences.

2 Notational throat-clearing: I use $\vdash$ for consequence, with premises on the left and conclusion(s) on the right; capital Roman letters for formulas; and capital Greek letters (the ones that are not also capital Roman letters) for *collections* of formulas — more on these collections in a moment.

3 I just give what basics are relevant here. For more details, see, for example, Blizard (1988) and Singh et al. (2008).

4 As far as I know, exceptions are limited to Blok and Pigozzi (1989) and related work.

5 This might *look* like transitivity proper, but it is not; the formulas on the *right* sides of these turnstiles are indeed formulas, while the formulas on the *left* abbreviate singleton multisets. More explicitly, this property is: for any formulas A, B, C, if $[A] \vdash B$ and $[B] \vdash C$, then $[A] \vdash C$. Note that the system of Weir (2005), often called 'nontransitive', *does* exhibit (the set-based version of) this property.

6 As with so many things, this requires slight reformulation — in particular, a different cut property — in a multiple-conclusion setting. The reformulation wouldn't affect anything in what follows here, as the needed property for multiple conclusions implies the single-conclusion version. See French and Ripley (2014), Restall (2000, ch.6), and (Bimbó 2015, ch. 7) for discussion of multiple-conclusion cut properties in substructural logics.

7 For the idea that multisets are well-suited to represent bodies of information, see Mares and Paoli (2014).

8 To be clear: when I say that a multiset Λ 'contains both' Π and Θ, I mean that $\Pi \sqsubseteq \Lambda$ and $\Theta \sqsubseteq \Lambda$. I do *not* mean $\Pi \boxplus \Theta \sqsubseteq \Lambda$, which is a stronger condition.

9 As Cintula and Paoli (2015) points out, full idempotence is not necessary; all that is needed is *root set idempotence*: that closing multiple times can't add any new formulas. Adding new occurrences of formulas that are already there is incompatible with idempotence, but would not block the above proof.

10 Priest (2015) does not work with multisets, but it is clear how the ideas there might be adapted to a multiset-based setting.

11 For discussion of these ideas, thanks to Jc Beall, Petr Cintula, Rohan French, Francesco Paoli, the Pukeko Logic Group, the Melbourne Logic Group, the audience at the 2014 SILFS Satellite Workshop, and two anonymous referees. This research has been partially supported by the grant "Non-Transitive Logics", number FFI2013-46451-P, from the Ministerio de Economía y Competitividad, Government of Spain.

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