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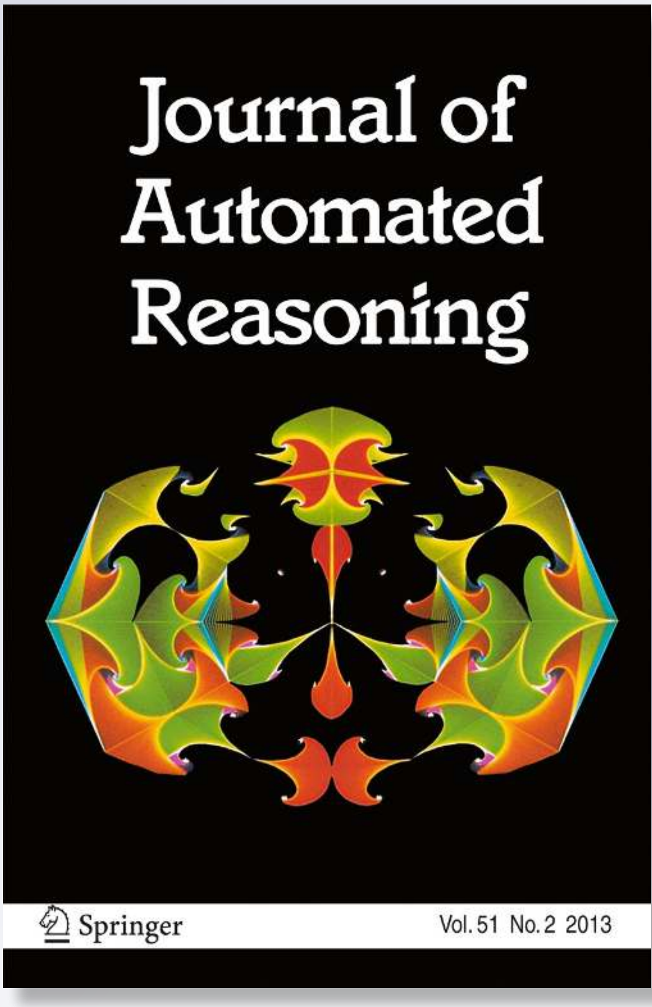
Journal of Automated Reasoning

ISSN 0168-7433
Volume 51
Number 2

J Autom Reasoning (2013) 51:129-149
DOI 10.1007/s10817-012-9252-7



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Contraction-Free Linear Depth Sequent Calculi for Intuitionistic Propositional Logic with the Subformula Property and Minimal Depth Counter-Models

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Received: 24 May 2011 / Accepted: 24 May 2012 / Published online: 10 June 2012

Abstract In this paper we present **LSJ**, a contraction-free sequent calculus for Intuitionistic propositional logic whose proofs are linearly bounded in the length of the formula to be proved and satisfy the subformula property. We also introduce a sequent calculus **RJ** for intuitionistic unprovability with the same properties of **LSJ**. We show that from a refutation of **RJ** of a sequent σ we can extract a Kripke counter-model for σ. Finally, we provide a procedure that given a sequent σ returns either a proof of σ in **LSJ** or a refutation in **RJ** such that the extracted counter-model is of minimal depth.

Keywords Intuitionistic propositional logic · Sequent calculi · Subformula property · Decision procedures · Counter-models generation

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1 Introduction

The research on the design of efficient decision procedures for Intuitionistic propositional logic has a long and articulated history. In the context of sequent calculi the main concern is the treatment of *left implicative formulas*, that is implicative formulas occurring in the left-hand side of a sequent (analogously **T**-signed implicative formulas in tableau calculi). These formulas are the main source of inefficiency in proof-search for known calculi and their treatment makes the problem of deciding Intuitionistic propositional logic PSPACE-complete [16].

Gentzen's early sequent calculi [6] for Intuitionistic logic were based on the re-use of left implicative formulas. The major drawback of this solution is that deductions may have infinite depth, hence some loop-checking mechanism is needed to guarantee termination. Vorob'ev [20] introduced rules to treat left-implicative formulas according to the main connective of the antecedent. See also [4, 11, 14], where calculi with analogous properties are given. In these cases, the re-use of formulas is avoided by replacing $A \to B$ on the left with "simpler" formulas. However, such simpler formulas are not subformulas of $A \to B$, thus the calculi do not have the subformula property: as an example, the formula $(A \vee B) \to C$ is replaced with $A \to C$ and $B \to C$. Giving a suitable measure on formulas one can guarantee that derivations in these calculi have bounded depth. Although the decision procedures for these calculi do not need loop-checking mechanisms, the rules to treat left implicative formulas of the kind $(A \vee B) \to C$ and $(A \to B) \to C$ still generate proofs whose depth is not linear in the size of the formula to be proved. This problem is overcome in [11], where proofs have linear depth and the related decision procedures require $O(n \log n)$-SPACE. A further refinement is given in [5] where, in the context of tableau calculi, extra rules are added to treat implications of the kind $(A \to B) \to C$ according with the main connective of B.

In spite of the efficiency of the related decision procedures, all the above mentioned calculi lack of a fundamental feature: the *subformula property*. The calculus **LSJ** we present in Section 3 meets the subformula property, is terminating and its proofs have linear depth. Following the ideas of a previous work of the authors on constructive description logics [1], **LSJ** handles sequents with a third set of formulas besides the usual sets of left and right formulas and it uses a three-premise rule to treat left implicative formulas.

We remark that, even if termination can be easily achieved also for the calculi in [4, 5, 11, 14, 20], the subformula property provides a more elegant termination argument. Moreover, our rules better capture the original goal of Gentzen [6] to justify connectives via introduction rules acting on their subformulas.

In Section 3 we also present the sequent calculus **RJ**, strongly related with **LSJ**, for asserting Intuitionistic unprovability. **RJ** is similar to the refutation calculi described in [15, 21]. In Section 4 we provide a decision procedure for Intuitionistic propositional logic which returns either a proof (a derivation in **LSJ**) or a refutation (a derivation in **RJ**) of the input sequent. Since, as discussed in Section 4, from a refutation of a sequent σ we can extract a Kripke counter-model for σ, the correctness of the decision procedure implies the completeness of **LSJ**. As discussed in Section 5 the above decision procedure can be modified so as to generate refutations giving rise to counter-models with minimal depth. In particular, in the case of classical non-valid formulas it generates Kripke counter-models consisting of a single world.

2 Preliminaries

We consider the propositional language $\mathcal{L}$ based on a denumerable set of propositional variables $\mathcal{PV}$, the logical connectives $\wedge$, $\vee$, $\rightarrow$ and the logical constant $\bot$. Writing formulas we assume that $\wedge$ and $\vee$ bind stronger than $\rightarrow$. We treat $\neg A$ as a shorthand for $A \rightarrow \bot$. A formula is *atomic* if it is a propositional variable or $\bot$.

A *(finite) Kripke model* for $\mathcal{L}$ is a structure $\mathcal{K} = \langle P, \leq, \rho, V \rangle$, where:

- $\langle P, \leq, \rho \rangle$ is a finite poset with minimum ρ;
- V is a function mapping every $\alpha \in P$ to a subset of $\mathcal{PV}$ such that $\alpha \leq \beta$ implies $V(\alpha) \subseteq V(\beta)$.

We write $\alpha < \beta$ to mean $\alpha \leq \beta$ and $\alpha \neq \beta$. The *forcing relation* $\Vdash \subseteq P \times \mathcal{L}$ of $\mathcal{K}$ is defined as follows:

- $\mathcal{K}, \alpha \nVdash \bot$ and, for every $p \in \mathcal{PV}$, $\mathcal{K}, \alpha \Vdash p$ iff $p \in V(\alpha)$;
- $\mathcal{K}, \alpha \Vdash A \wedge B$ iff $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \alpha \Vdash B$;
- $\mathcal{K}, \alpha \Vdash A \vee B$ iff $\mathcal{K}, \alpha \Vdash A$ or $\mathcal{K}, \alpha \Vdash B$;
- $\mathcal{K}, \alpha \Vdash A \rightarrow B$ iff, for every $\beta \in P$ such that $\alpha \leq \beta$, $\mathcal{K}, \beta \nVdash A$ or $\mathcal{K}, \beta \Vdash B$.

Monotonicity property holds for arbitrary formulas, i.e.: $\mathcal{K}, \alpha \Vdash A$ and $\alpha \leq \beta$ imply $\mathcal{K}, \beta \Vdash A$. A formula A *is valid in* $\mathcal{K}$ iff $\mathcal{K}, \rho \Vdash A$. It is well-known that Intuitionistic propositional logic **Int** coincides with the set of formulas valid in all (finite) Kripke models [2]. A *final world of* $\mathcal{K}$ is an element $\phi \in P$ such that, for every $\alpha \in P$, $\phi \leq \alpha$ implies $\phi = \alpha$. The *depth of* $\mathcal{K}$, denoted by $\text{depth}(\mathcal{K})$, is the length of the longest path from its root to a final world of $\mathcal{K}$.

3 Sequent Calculi

A *sequent* σ is an expression of the kind $\Theta ; \Gamma \Rightarrow \Delta$ where Θ, Γ and Δ are (possibly empty) sets of formulas. We semantically justify sequents as follows: given a Kripke model $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ and $\alpha \in P$, α *refutes* $\Theta ; \Gamma \Rightarrow \Delta$ in $\mathcal{K}$, written $\mathcal{K}, \alpha \triangleright \Theta ; \Gamma \Rightarrow \Delta$, iff the following hold:

(a) for every $H \in \Theta$ and for every $\beta \in P$ such that $\alpha < \beta$, $\mathcal{K}, \beta \Vdash H$;
(b) for every $H \in \Gamma$, $\mathcal{K}, \alpha \Vdash H$;
(c) for every $H \in \Delta$, $\mathcal{K}, \alpha \nVdash H$.

We say that σ is *refutable* if there exist a Kripke model $\mathcal{K}$ and an element $\alpha \in P$ such that $\mathcal{K}, \alpha \triangleright \sigma$; in this case we say that $\mathcal{K}$ is a *counter-model* for σ. The notion of refutability is related to the notions of intuitionistic and classical validity by the following proposition:

Theorem 1 *Let* Θ, Γ *and* Δ *be finite sets of formulas. Then:*

1. $\emptyset ; \Gamma \Rightarrow \Delta$ *is refutable iff* $\bigwedge \Gamma \rightarrow \bigvee \Delta$ *is not intuitionistically valid;*
2. *If* $\bot \in \Theta$*, then* $\Theta ; \Gamma \Rightarrow \Delta$ *is refutable iff* $\bigwedge \Gamma \rightarrow \bigvee \Delta$ *is not classically valid.*

Proof Point 1 easily follows by the definition of refutability. As for point 2, let $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ be a Kripke model and $\alpha \in P$ such that $\mathcal{K}, \alpha \triangleright \Theta ; \Gamma \Rightarrow \Delta$. Since $\bot \in \Theta$, point (a) above implies that every $\beta > \alpha$ obeys $\mathcal{K}, \beta \Vdash \bot$. By definition of forcing, $\mathcal{K}, \beta \nVdash \bot$, hence such a β cannot exist. That is, α must be a final world of $\mathcal{K}$. Let $\mathcal{I}$ be a classical interpretation such that, for every propositional variable p, we have $\mathcal{I} \models p$ iff $p \in V(\alpha)$. Then, for every formula A, it holds that $\mathcal{K}, \alpha \Vdash A$ iff $\mathcal{I} \models A$. Since $\mathcal{K}, \alpha \Vdash \bigwedge \Gamma$ and $\mathcal{K}, \alpha \nVdash \bigvee \Delta$, it follows that $\mathcal{I} \not\models \bigwedge \Gamma \rightarrow \bigvee \Delta$, hence $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is not classically valid. Conversely, let us assume that $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is not classically valid and let $\mathcal{I}$ be a classical interpretation such that $\mathcal{I} \models \bigwedge \Gamma$ and $\mathcal{I} \not\models \bigvee \Delta$. Let $\mathcal{K} = \langle P, \leq \rho, V \rangle$ be a Kripke model where P only contains ρ and, for every propositional variable p, it holds that $p \in V(\rho)$ iff $\mathcal{I} \models p$. It is easy to check that, for every formula A, we have $\mathcal{K}, \rho \Vdash A$ iff $\mathcal{I} \models A$. This implies that $\mathcal{K}, \rho \triangleright \Theta\,;\Gamma \Rightarrow \Delta$, hence $\Theta\,;\Gamma \Rightarrow \Delta$ is refutable. □

Note that, by point 1 of the above theorem, a sequent $\sigma = \Theta\,;\Gamma \Rightarrow \Delta$ with empty Θ can be represented by a formula. We do not know if it is possible to extend the above translation to a generic σ.

In this paper we introduce two sequent-based calculi, one for proving sequents and one for refuting them. To treat these calculi in a uniform way we introduce the following definitions. A *sequent rule* has the form:

$$\frac{\sigma_1 \quad \ldots \quad \sigma_n}{\sigma} \ \mathcal{R}$$

where σ is a sequent, $\sigma_1, \ldots, \sigma_n$ ($n \geq 0$) is a sequence of sequents and $\mathcal{R}$ is the name of the rule. The sequent σ is called the *conclusion* of the rule, while $\sigma_1, \ldots, \sigma_n$ are called the *premises* of the rule. If a rule has no premises, we call it an *axiom-rule*. A *sequent calculus* $\mathcal{C}$ is a finite set of sequent rules.

A *tree* is a directed graph where every node is reachable from some unique *root* node via a finite number of directed edges; every node except the root has one edge directed into it, and the root node has no edges directed into it. Given a tree T, we denote with $root(T)$ the root of T. Given a node $a \in T$, $children(a)$ denotes the set of the immediate successors of a in T. A *leaf* is any node a of T such that $children(a) = \emptyset$. $leaves(T)$ denotes the set of all the leaves of T. Given a sequent calculus $\mathcal{C}$, a $\mathcal{C}$-*tree* is a triple $\pi = \langle T, s, r \rangle$ where:

- T is a finite tree;
- s is a function associating a sequent with every node of T;
- r is a function associating a rule of $\mathcal{C}$ with every node of T.

A $\mathcal{C}$-*derivation* is a $\mathcal{C}$-*tree* $\pi = \langle T, s, r \rangle$ such that, for every node $a \in T$, if $children(a) = \{b_1, \ldots, b_n\}$ then

$$\frac{s(b_1) \ldots s(b_n)}{s(a)} \ r(a)$$

is an instance of a rule in $\mathcal{C}$. We say that π is a $\mathcal{C}$-derivation of the sequent $s(root(T))$. The *depth* of the derivation π, denoted by $\mathrm{depth}(\pi)$, is the depth of the tree T, that is the maximal length of a path from the root of T to a leaf.

3.1 The Calculus **LSJ**

The rules of the sequent calculus **LSJ** are given in Fig. 1. The calculus consists of *left* (L) and *right* (R) introduction rules for the logical constants plus the axiom-rules

$$\frac{}{\Theta\,;\bot,\Gamma\Rightarrow\Delta}\,\bot L \qquad \frac{}{\Theta\,;A,\Gamma\Rightarrow A,\Delta}\,\mathrm{Id}$$

$$\frac{\Theta\,;A,B,\Gamma\Rightarrow\Delta}{\Theta\,;A\wedge B,\Gamma\Rightarrow\Delta}\,\wedge L \qquad \frac{\Theta\,;\Gamma\Rightarrow A,\Delta \quad \Theta\,;\Gamma\Rightarrow B,\Delta}{\Theta\,;\Gamma\Rightarrow A\wedge B,\Delta}\,\wedge R$$

$$\frac{\Theta\,;A,\Gamma\Rightarrow\Delta \quad \Theta\,;B,\Gamma\Rightarrow\Delta}{\Theta\,;A\vee B,\Gamma\Rightarrow\Delta}\,\vee L \qquad \frac{\Theta\,;\Gamma\Rightarrow A,B,\Delta}{\Theta\,;\Gamma\Rightarrow A\vee B,\Delta}\,\vee R$$

$$\frac{\Theta\,;B,\Gamma\Rightarrow\Delta \quad B,\Theta\,;\Gamma\Rightarrow A,\Delta \quad B\,;\Theta,\Gamma\Rightarrow A}{\Theta\,;A\to B,\Gamma\Rightarrow\Delta}\,\to L$$

$$\frac{\Theta\,;A,\Gamma\Rightarrow B,\Delta \quad \emptyset\,;A,\Theta,\Gamma\Rightarrow B}{\Theta\,;\Gamma\Rightarrow A\to B,\Delta}\,\to R$$

Fig. 1 The **LSJ** calculus

$\bot L$ and Id. Given a sequent $\Theta\,;\Gamma\Rightarrow\Delta$, left rules act on formulas in Γ and right rules act on formulas in Δ; the set Θ is modified only by the rules $\to L$ and $\to R$. In the formulation of the rules we write H,Σ as a shorthand for $\{H\}\cup\Sigma$. In the conclusion of a rule, writing H,Σ we assume that $H\notin\Sigma$, thus the formula H (the *principal* formula) is not retained in the premises; e.g., in the rule $\to L$ we assume that $A\to B\notin\Gamma$. We call *initial sequent of* **LSJ** every sequent that can occur as a conclusion of an axiom-rule.

Differently from standard presentations, the rule $\to L$ has three premises and the rule $\to R$ has two premises; in some cases the premises of these rules can coincide. As an example, if Θ and Δ are empty, the two premises of the rule $\to R$ are equal and the two rightmost premises of the rule $\to L$ are equal. We also notice that, since sequents act on sets and the calculus is multi-succedent, we do not need structural rules.

We call *proof* an **LSJ**-derivation $\pi=\langle T,s,r\rangle$. We remark that, for every $a\in leaves(T)$, $r(a)$ is an axiom-rule. We say that π is a proof of the sequent $s(root(T))$. A formula H is provable in **LSJ** if there exists a proof of the sequent $\emptyset\,;\emptyset\Rightarrow H$.

As the reader can easily check inspecting the rules, **LSJ** is a contraction-free calculus and, differently from other well-known contraction-free proposals [4, 5, 11, 14, 20], it has the subformula property: every formula occurring in a derivation is a subformula of the root sequent.

Example 1 The following is a proof of the formula $((p\vee(p\to q))\to q)\to q$. We use the calculus applying the rules backward, hence we read proofs from the root to the leaves. We remark that the first rule applied in the proof is an instance of $\to R$ where the two premises coincide. Hence, for the sake of conciseness, we only draw one of the corresponding subproofs.

$$\dfrac{\dfrac{}{\emptyset\,;q\Rightarrow q}\,\mathrm{Id} \quad \dfrac{\dfrac{\dfrac{}{q\,;p\Rightarrow p,q}\,\mathrm{Id} \quad \dfrac{}{\emptyset\,;p,q\Rightarrow q}\,\mathrm{Id}}{q\,;\emptyset\Rightarrow p,\ p\to q,q}\,\to R}{q\,;\emptyset\Rightarrow p\vee(p\to q),q}\,\vee R \quad \dfrac{\dfrac{\dfrac{}{q\,;p\Rightarrow p,q}\,\mathrm{Id} \quad \dfrac{}{\emptyset\,;p,q\Rightarrow q}\,\mathrm{Id}}{q\,;\emptyset\Rightarrow p,\ p\to q}\,\to R}{q\,;\emptyset\Rightarrow p\vee(p\to q)}\,\vee R}{\dfrac{\emptyset\,;(p\vee(p\to q))\to q\Rightarrow q}{\emptyset\,;\emptyset\Rightarrow((p\vee(p\to q))\to q)\to q}\,\to R}\,\to L$$

Example 2 The following is a proof of the double negation of the *tertium-non-datur* principle.

$$\dfrac{\dfrac{}{\emptyset\,;\bot \Rightarrow \bot}\,\bot L \qquad \dfrac{\dfrac{\dfrac{}{\bot\,;p \Rightarrow p,\bot}\,\text{Id} \quad \dfrac{}{\emptyset\,;\bot,p \Rightarrow \bot}\,\bot L}{\bot\,;\emptyset \Rightarrow p,\neg p,\bot}\,\to R}{\bot\,;\emptyset \Rightarrow p \vee \neg p,\bot}\,\vee R \qquad \dfrac{\dfrac{\dfrac{}{\bot\,;p \Rightarrow \bot,p}\,\text{Id} \quad \dfrac{}{\emptyset\,;\bot,p \Rightarrow \bot}\,\bot L}{\bot\,;\emptyset \Rightarrow p,\neg p}\,\to R}{\bot\,;\emptyset \Rightarrow p \vee \neg p}\,\vee R}{\dfrac{\emptyset\,;\neg(p \vee \neg p) \Rightarrow \bot}{\emptyset\,;\emptyset \Rightarrow \neg\neg(p \vee \neg p)}\,\to R}\,\to L$$

We remark that the above proof contains redundancies since the two rightmost subproofs of the $\to L$ application essentially coincide. Indeed, applying the rules of the calculus one could always delete the occurrences of $\bot$ in the right-hand side of a sequent without affecting soundness and completeness.

Redundancies due to the implicit treatment of negation can be avoided by introducing the following rules:

$$\dfrac{\bot,\Theta\,;\Gamma \Rightarrow A,\Delta \quad \bot\,;\Theta,\Gamma \Rightarrow A}{\Theta\,;\neg A,\Gamma \Rightarrow \Delta}\,\neg L \qquad\qquad \dfrac{\Theta\,;A,\Gamma \Rightarrow \Delta \quad \emptyset\,;A,\Theta,\Gamma \Rightarrow \emptyset}{\Theta\,;\Gamma \Rightarrow \neg A,\Delta}\,\neg R$$

We remark that to prove the formulas of the above examples using the Gentzen calculus LJ or the analogous **G1i** of [19] one has to apply the contraction rule. In the case of intuitionistic unprovable sequents the use of contraction requires loop checking mechanisms to get termination, as in the case of deciding with LJ the sequent $p \to q \vdash q$.

Given a formula H we denote with $dg(H)$ the number of logical connectives occurring in H. The degree $dg(\sigma)$ of a sequent σ is the sum of the degrees of the formulas occurring in σ. The reader can easily check that the rules of **LSJ** have the following property:

Lemma 1 *Let $\mathcal{R}$ be an instance of a rule of* **LSJ** *and let σ be the conclusion of $\mathcal{R}$. For every premise τ of $\mathcal{R}$, $dg(\tau) < dg(\sigma)$.*

By the above lemma we get:

Theorem 2 *Let π be an* **LSJ**-*derivation of σ, then* $\text{depth}(\pi) \leq dg(\sigma)$.

Let $\mathcal{R}$ be a rule of **LSJ**:

- $\mathcal{R}$ is *sound* if the refutability of the conclusion of $\mathcal{R}$ implies the refutability of at least one of its premises;
- a premise of $\mathcal{R}$ is *invertible* if its refutability implies the refutability of the conclusion;
- $\mathcal{R}$ is *invertible* if all its premises are invertible.

The following is the main lemma to prove the soundness of **LSJ**.

Lemma 2 *Let $\mathcal{R}$ be a rule of* **LSJ** *with conclusion σ and premises $\sigma_1, \ldots, \sigma_n$, let $\mathcal{K} = \langle P, \leq, \rho, V\rangle$ be a Kripke model and $\alpha \in P$ such that $\mathcal{K}, \alpha \rhd \sigma$. There exist a premise σ_i and $\beta \in P$ such that $\alpha \leq \beta$ and $\mathcal{K}, \beta \rhd \sigma_i$.*

Proof We only discuss the rules $\rightarrow L$ and $\rightarrow R$, the other cases being trivial. Let us consider the rule $\rightarrow L$ and let us suppose that $\mathcal{K}, \alpha \triangleright \Theta\,;\, A \rightarrow B, \Gamma \Rightarrow \Delta$. This means that $\mathcal{K}, \alpha \Vdash A \rightarrow B$. If $\mathcal{K}, \alpha \Vdash B$, then $\mathcal{K}, \alpha \triangleright \Theta\,;\, B, \Gamma \Rightarrow \Delta$ and the assertion holds. Otherwise, $\mathcal{K}, \alpha \nVdash B$ and hence $\mathcal{K}, \alpha \nVdash A$. Since $\mathcal{K}$ is finite, there exists $\beta \in P$ such that $\alpha \leq \beta$, $\mathcal{K}, \beta \nVdash A$ and, for every $\gamma \in P$ such that $\beta < \gamma$, it holds that $\mathcal{K}, \gamma \Vdash A$. If $\beta = \alpha$, then $\mathcal{K}, \beta \triangleright B, \Theta\,;\, \Gamma \Rightarrow A, \Delta$, otherwise $\alpha < \beta$ and $\mathcal{K}, \beta \triangleright B\,;\, \Theta, \Gamma \Rightarrow A$. As for the rule $\rightarrow R$, let us assume that $\mathcal{K}, \alpha \triangleright \Theta\,;\, \Gamma \Rightarrow A \rightarrow B, \Delta$. Then, there exists $\beta \geq \alpha$ such that $\mathcal{K}, \beta \Vdash A$ and $\mathcal{K}, \beta \nVdash B$. If $\beta = \alpha$, then $\mathcal{K}, \beta \triangleright \Theta\,;\, A, \Gamma \Rightarrow B, \Delta$, otherwise $\alpha < \beta$ and $\mathcal{K}, \beta \triangleright \emptyset\,;\, A, \Theta, \Gamma \Rightarrow B$. □

By the above lemma, all the rules of **LSJ** are sound, hence:

Theorem 3 (Soundness of **LSJ**) *If a sequent σ is provable in* **LSJ** *then it is not refutable.*

Proof Let σ be provable and let π be a proof of σ. If σ were refutable then, by Lemma 2, some of the sequents in the leaves of π would be refutable, a contradiction since initial sequents are not refutable. □

As for invertibility of the rules we note that:

- the rules $\wedge L$, $\wedge R$, $\vee L$, $\vee R$ are invertible;
- the two leftmost premises of the rule $\rightarrow L$ are invertible;
- the leftmost premise of the rule $\rightarrow R$ is invertible.

Instead, the rightmost premise of $\rightarrow L$ and the rightmost premise of $\rightarrow R$ are not invertible. Indeed, these premises do not retain the set Δ occurring in the conclusion and hence, in general, the refutability of these premises does not imply the refutability of the conclusion.

We remark that, as a consequence of the completeness we discuss in Section 4, the following cut-rule is admissible.

$$\frac{\Theta\,;\,\Gamma \Rightarrow A, \Delta \qquad \Theta\,;\,\Gamma, A \Rightarrow \Delta}{\Theta\,;\,\Gamma \Rightarrow \Delta}\ cut$$

On the other hand, it is easy to see that the other possible formulation of the cut-rule

$$\frac{\Theta\,;\,\Gamma \Rightarrow A, \Delta \qquad \Theta, A\,;\,\Gamma \Rightarrow \Delta}{\Theta\,;\,\Gamma \Rightarrow \Delta}$$

is not sound.

3.1.1 On Infinite Models

In the literature, the propositional logic **Int** has been semantically defined as the set of formulas valid in all finite and infinite Kripke models (that is, Kripke models $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ where P can be infinite). Subsequently it has been proved that **Int** has the *finite model property*, namely **Int** coincides with the set of formulas valid in all finite Kripke models, see [2] for details. The proof of soundness of **LSJ** discussed above crucially exploits the finite model property. The problematic rule is $\rightarrow L$ and one can easily check that in infinite models $\rightarrow L$ might not be sound.

Nevertheless, the soundness of **LSJ** can be proved without using the finite model property. We firstly state a property of infinite Kripke models (K1) which implies the soundness of $\rightarrow L$. Then we prove that, if the sequent σ is refutable in an infinite Kripke model, then it is refutable in a model satisfying (K1).

Let us consider possibly infinite Kripke models $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ having the following property:

(K1) For every formula A and every $\alpha \in P$, if $\mathcal{K}, \alpha \nVdash A$, there is β such that $\alpha \leq \beta$, $\mathcal{K}, \beta \nVdash A$ and, for every $\gamma \in P$, $\alpha < \gamma$ implies $\mathcal{K}, \gamma \Vdash A$.

An element β satisfying (K1) is also called *maximal world in $\mathcal{K}$ relative to A* [2]. Note that in Lemma 2 we use Property (K1) to prove the soundness of the rule $\rightarrow L$. Hence Lemma 2 holds in (finite or infinite) models satisfying (K1). To prove the soundness of the calculus **LSJ** in infinite models, we have to show that:

(K2) If a sequent σ is (finitely or infinitely) refutable, then σ is refutable in a model satisfying (K1).

For propositional Intuitionistic logic, (K2) is guaranteed by *canonical models*. Indeed, it is well-known [2] that canonical models satisfy (K1) (this can be proved using Zorn's Lemma). To prove (K2), let $\sigma = \Theta\,;\Gamma \Rightarrow \Delta$ and let $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ be a model such that $\mathcal{K}, \alpha \triangleright \sigma$, with $\alpha \in P$; we show that there is a world γ of the canonical model $\mathcal{C}$ such that $\mathcal{C}, \gamma \triangleright \sigma$. Let Γ^* be the set of all the formulas A such that $\mathcal{K}, \alpha \Vdash A$. In the canonical model $\mathcal{C}$ there exists a world γ such that $\mathcal{C}, \gamma \Vdash A$ iff $A \in \Gamma^*$. This implies that $\mathcal{C}, \gamma \Vdash A$ for every $A \in \Gamma$ and $\mathcal{C}, \gamma \nVdash B$ for every $B \in \Delta$. Now, suppose that there exists $C \in \Theta$ and γ' such that $\gamma < \gamma'$ in $\mathcal{C}$ and $\mathcal{C}, \gamma' \nVdash C$. By properties of canonical models, there exists a formula D such that $\mathcal{C}, \gamma' \Vdash D$ and $\mathcal{C}, \gamma \nVdash D$, which implies $\mathcal{C}, \gamma \nVdash D \rightarrow C$. It follows that $\mathcal{K}, \alpha \nVdash D$ and $\mathcal{K}, \alpha \nVdash D \rightarrow C$, hence, for some β such that $\alpha < \beta$ in $\mathcal{K}$, it holds that $\mathcal{K}, \beta \Vdash D$ and $\mathcal{K}, \beta \nVdash C$, in contradiction with the fact that that $\mathcal{K}, \alpha \triangleright \sigma$ and $C \in \Theta$. This proves that $\mathcal{C}, \gamma \triangleright \sigma$. To conclude, the soundness of **LSJ** can be proved using canonical models.

3.2 The Refutation Calculus **RJ**

In this section, following ideas from [15], we introduce a refutation calculus **RJ** for Intuitionistic propositional logic, that is a calculus to prove that a sequent σ is refutable. As we discuss later, from an **RJ**-derivation π of a sequent σ we can extract a counter-model $\mathrm{Mod}(\pi)$ for σ.

The rules of the calculus **RJ** are given in Fig. 2. As for **LSJ**, when H, Σ occurs in the conclusion of a rule, we assume $H \notin \Sigma$. In the formulation of the rules we denote with Γ_{At} and Δ_{At} sets of atomic formulas, and with $\Gamma^{\rightarrow}$ and $\Delta^{\rightarrow}$ sets of implicative formulas. A sequent $\Theta\,;\Gamma \Rightarrow \Delta$ is *simple* if Γ and Δ only contain atomic formulas. We call *initial sequent of* **RJ** every sequent that can occur as a conclusion of the rule Irr (the name stands for irreducible), that is all the simple sequents $\Theta\,;\Gamma_{At} \Rightarrow \Delta_{At}$ where Γ_{At} and Δ_{At} are disjoint and $\bot \notin \Gamma_{At}$.

$$\frac{}{\Theta\,;\Gamma_{At} \Rightarrow \Delta_{At}}\ \mathrm{Irr}$$

$$\frac{\Theta\,;A, B, \Gamma \Rightarrow \Delta}{\Theta\,;A \wedge B, \Gamma \Rightarrow \Delta}\ \wedge L \qquad \frac{\Theta\,;\Gamma \Rightarrow A_i, \Delta}{\Theta\,;\Gamma \Rightarrow A_1 \wedge A_2, \Delta}\ \wedge R_i$$

$$\frac{\Theta\,;A_i, \Gamma \Rightarrow \Delta}{\Theta\,;A_1 \vee A_2, \Gamma \Rightarrow \Delta}\ \vee L_i \qquad \frac{\Theta\,;\Gamma \Rightarrow A, B, \Delta}{\Theta\,;\Gamma \Rightarrow A \vee B, \Delta}\ \vee R$$

$$\frac{\Theta\,;B, \Gamma \Rightarrow \Delta}{\Theta\,;A \to B, \Gamma \Rightarrow \Delta} \to L_1 \qquad \frac{B, \Theta\,;\Gamma \Rightarrow A, \Delta}{\Theta\,;A \to B, \Gamma \Rightarrow \Delta} \to L_2$$

$$\frac{\Theta\,;A, \Gamma \Rightarrow B, \Delta}{\Theta\,;\Gamma \Rightarrow A \to B, \Delta} \to R$$

$$\frac{\{B\,;\Theta, \Gamma_{At}, \Gamma^{\to} \setminus \{A \to B\} \Rightarrow A\}_{A \to B \in \Gamma^{\to}} \quad \{\emptyset\,;C, \Theta, \Gamma_{At}, \Gamma^{\to} \Rightarrow D\}_{C \to D \in \Delta^{\to}}}{\Theta\,;\Gamma_{At}, \Gamma^{\to} \Rightarrow \Delta_{At}, \Delta^{\to}}\ \mathrm{Succ}$$

$-\Gamma_{At}$ and Δ_{At} are sets of atomic formulas such that $\bot \notin \Gamma_{At}$ and $\Gamma_{At} \cap \Delta_{At} = \emptyset$
$-\Gamma^{\to}$ and $\Delta^{\to}$ are sets of implicative formulas with $\Gamma^{\to} \cup \Delta^{\to} \neq \emptyset$.

Fig. 2 The calculus **RJ**

There is a tight correspondence between the rules of **LSJ** and those of **RJ**.

– The initial sequents of **RJ** are the simple sequents which are not initial sequents of **LSJ**.
– Let $\mathcal{R}$ be a rule of **LSJ** with premises $\sigma_1, \ldots, \sigma_n$ ($n \geq 1$) and conclusion σ and let σ_i be an invertible premise of $\mathcal{R}$. Then there exists a rule of **RJ** having σ_i as only premise and σ as conclusion. Rules of this kind are: $\wedge L$, $\wedge R_i$, $\vee L_i$, $\vee R$, $\to L_1$, $\to L_2$ and $\to R$.
– The non-invertible premises of the rules $\to R$ and $\to L$ of **LSJ** are collected in the rule Succ of **RJ**. We notice that the rule Succ can be applied only when Γ and Δ are composed exclusively of atomic or implicative formulas.

We call *refutation* an **RJ**-derivation $\pi = \langle T, s, r\rangle$ and we say that π is a refutation of the sequent $s(root(T))$.

The notion of soundness for **RJ** refutation rules is dual to the one given for **LSJ** rules. A rule $\mathcal{R}$ of **RJ** is a sound refutation rule if the refutability of all its premises implies the refutability of its conclusion. Accordingly, if there exists a refutation of σ, then σ is refutable. We prove the soundness in a stronger sense, showing how to extract from a refutation π of σ a counter-model $\mathrm{Mod}(\pi)$ for σ.

Let π be a refutation of $\sigma = \Theta\,;\Gamma \Rightarrow \Delta$; we define the Kripke model $\mathrm{Mod}(\pi) = \langle P, \leq, \rho, V\rangle$ by induction on the structure of π. Let $\mathcal{R}$ be the rule applied at the root of π:

– If $\mathcal{R}$ is Irr, then $\Gamma \subseteq \mathcal{PV}$. We set $\mathrm{Mod}(\pi) = \langle\{\rho\}, \{(\rho, \rho)\}, \rho, V\rangle$ with $V(\rho) = \Gamma$.

- If $\mathcal{R}$ is one of the rules $\wedge L$, $\wedge R_i$, $\vee L_i$, $\vee R$, $\rightarrow L_i$, $\rightarrow R$, let π' be the immediate subderivation of π: then $\mathrm{Mod}(\pi) = \mathrm{Mod}(\pi')$.
- If $\mathcal{R}$ is Succ, let $\pi_1, \ldots, \pi_n$ be the immediate subderivations of π and, for every $i \in \{1, \ldots, n\}$, let $\mathrm{Mod}(\pi_i) = \langle P_i, \leq_i, \rho_i, V_i \rangle$. Without loss of generality, we can assume that the P_i's are pairwise disjoint. Let $\rho \notin \bigcup_{i\in\{1,\ldots,n\}} P_i$, we set:
  - $P = \{\rho\} \cup \bigcup_{i\in\{1,\ldots,n\}} P_i$;
  - $\leq = \{(\rho, \alpha) \mid \alpha \in P\} \cup \bigcup_{i\in\{1,\ldots,n\}} \leq_i$;
  - $V(\rho) = \Gamma \cap \mathcal{PV}$;
  - for every $i \in \{1, \ldots, n\}$ and $\alpha \in P_i$, $V(\alpha) = V_i(\alpha)$.

It is easy to check that $\mathrm{Mod}(\pi)$ is a Kripke model. In particular, since passing from the consequence of a rule to one of its premises the set of propositional variables in the left-hand side of a sequent does not decrease, we get the monotonicity property on propositional variables. We also note that in the model obtained by an application of the rule Succ, for every formula H, every $i \in \{1, \ldots, n\}$ and every $\alpha \in P_i$, we have that $\mathrm{Mod}(\pi), \alpha \Vdash H$ iff $\mathrm{Mod}(\pi_i), \alpha \Vdash H$. The only rule of **RJ** generating new worlds in the counter-model is Succ. Thus, given a refutation π of σ, if k is the number of implications occurring in σ we have:

- the depth of $\mathrm{Mod}(\pi)$ is at most k;
- given a world α in $\mathrm{Mod}(\pi)$ the number of immediate successors of α is at most k.

We prove the main property of $\mathrm{Mod}(\pi)$.

Theorem 4 *Let π be a refutation of σ. Then,* $\mathrm{Mod}(\pi), \rho \rhd \sigma$*, where ρ is the root of* $\mathrm{Mod}(\pi)$.

Proof The proof goes by induction on the structure of π. We only discuss the case where the rule applied at the root of π is Succ, the other cases being easy. Let $\sigma = \Theta\,;\Gamma_{At}, \Gamma^{\rightarrow} \Rightarrow \Delta_{At}, \Delta^{\rightarrow}$ be defined as in Fig. 2 and let $\mathrm{Mod}(\pi) = \langle P, \leq, \rho, V \rangle$. Since $V(\rho) = \Gamma_{At}$ and $\Gamma_{At} \cap \Delta_{At} = \emptyset$, we immediately have that $\mathrm{Mod}(\pi), \rho \Vdash p$ for every $p \in \Gamma_{At}$ and $\mathrm{Mod}(\pi), \rho \nVdash p$ for every $p \in \Delta_{At}$. Let $\pi_1, \ldots, \pi_n$ be the immediate subderivations of π. Each π_i is a refutation of a sequent $\sigma_i = \Theta_i\,; \Gamma_i \Rightarrow \Delta_i$ occurring in the premise of Succ. By the induction hypothesis, denoting by ρ_i the root of $\mathrm{Mod}(\pi_i)$, it holds that $\mathrm{Mod}(\pi_i), \rho_i \rhd \sigma_i$, which implies $\mathrm{Mod}(\pi), \rho_i \rhd \sigma_i$. Let $A \rightarrow B \in \Gamma^{\rightarrow}$ and let $k \in \{1, \ldots, n\}$ such that $\sigma_k = B\,; \Theta, \Gamma_{At}, \Gamma^{\rightarrow} \setminus \{A \rightarrow B\} \Rightarrow A$. We show that:

(1) for every $j \in \{1, \ldots, n\}$ such that $j \neq k$, $\mathrm{Mod}(\pi), \rho_j \Vdash A \rightarrow B$;
(2) for every $\alpha \in P$ such that $\rho_k < \alpha$, $\mathrm{Mod}(\pi), \alpha \Vdash B$;
(3) $\mathrm{Mod}(\pi), \rho_k \nVdash A$.

If $j \neq k$, then $A \rightarrow B$ belongs to Γ_j; by the induction hypothesis, $\mathrm{Mod}(\pi_j), \rho_j \Vdash A \rightarrow B$, and this implies point (1). Let us consider the model $\mathrm{Mod}(\pi_k)$. Since $\mathrm{Mod}(\pi_k), \rho_k \rhd \sigma_k$, it holds that $\mathrm{Mod}(\pi_k), \rho_k \nVdash A$ and, for every $\alpha \in P_k$ such that $\rho_k < \alpha$, $\mathrm{Mod}(\pi_k), \alpha \Vdash B$; hence points (2) and (3) follow. Now, let $\alpha \in P$ such that $\mathrm{Mod}(\pi), \alpha \Vdash A$. By point (3) and monotonicity, it holds that $\alpha \neq \rho_k$ and $\alpha \neq \rho$, hence either $\rho_j \leq \alpha$, for some $j \neq k$, or $\rho_k < \alpha$. In both cases, by points (1) and (2), it follows that $\mathrm{Mod}(\pi), \alpha \Vdash B$; this proves that $\mathrm{Mod}(\pi), \rho \Vdash A \to B$. Let $C \to D \in \Delta^{\to}$. By definition of the rule Succ there exists $l \in \{1, \ldots, n\}$ such that $\sigma_l = \emptyset\,; C, \Theta, \Gamma_{At}, \Gamma^{\to} \Rightarrow D$. By the induction hypothesis, it follows that $\mathrm{Mod}(\pi_l), \rho_l \Vdash C$ and $\mathrm{Mod}(\pi_l), \rho_l \nVdash D$, thus $\mathrm{Mod}(\pi), \rho \nVdash C \to D$. It remains to prove that for every $H \in \Theta$ and for every $\alpha \in P$ such that $\rho < \alpha$, $\mathrm{Mod}(\pi), \alpha \Vdash H$. This follows by the fact that, for every $i \in \{1, \ldots, n\}$, $\Theta \subseteq \Gamma_i$ hence, by the induction hypothesis, $\mathrm{Mod}(\pi_i), \rho_i \Vdash H$. □

In the following examples, to emphasize the relation between the nodes of a refutation π and worlds of the extracted counter-model $\mathrm{Mod}(\pi)$, we label sequents occurring in π with an integer value denoting a world of $\mathrm{Mod}(\pi)$. The sequent at the root of the refutation is labelled with 0, which represents the root of the counter-model. The only rule which affects labels is Succ. When such a rule is applied its premises have new distinct labels. A sequent in π with label n is refuted in the world n of $\mathrm{Mod}(\pi)$. Models are represented as trees with the convention that $\alpha < \beta$ if the world β is drawn above node α. Every world is represented by its label followed by the list of forced propositional variables.

Example 3 The following is a refutation π of the *Scott principle* [2]

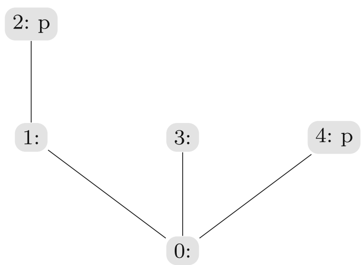
$$((\neg\neg p \to p) \to (\neg p \vee p)) \to (\neg\neg p \vee \neg p)$$

where $H = (\neg\neg p \to p) \to (\neg p \vee p)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{2:\bot\,;p \Rightarrow \bot}\,\mathrm{Irr}}{2:\bot\,;p \Rightarrow \neg p}\,{\to}R}{2:\bot\,;\neg p \vee p \Rightarrow \neg p}\,{\vee}L_2}{1:\neg p \vee p\,;\neg\neg p \Rightarrow p}\,\mathrm{Succ}}{1:\neg p \vee p\,;\emptyset \Rightarrow \neg\neg p \to p}\,{\to}R \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{3:\bot\,;\emptyset \Rightarrow p,\bot}\,\mathrm{Irr}}{3:\bot\,;\neg p \Rightarrow p,\bot}\,{\to}L_2}{3:\bot\,;\neg p \vee p \Rightarrow p,\bot}\,{\vee}L_1}{3:\bot\,;H \Rightarrow p,\bot}\,{\to}L_1}{3:\emptyset\,;H,\neg p \Rightarrow \bot}\,{\to}L_2 \quad \dfrac{\dfrac{\dfrac{}{4:\emptyset\,;p \Rightarrow \bot}\,\mathrm{Irr}}{4:\emptyset\,;\neg p \vee p, p \Rightarrow \bot}\,{\vee}L_2}{4:\emptyset\,;H, p \Rightarrow \bot}\,{\to}L_1}{0:\emptyset\,;H \Rightarrow \neg\neg p, \neg p}\,\mathrm{Succ}}{0:\emptyset\,;H \Rightarrow \neg\neg p \vee \neg p}\,{\vee}R}{0:\emptyset\,;\emptyset \Rightarrow H \to (\neg\neg p \vee \neg p)}\,{\to}R$$

As the reader can easily check, $\mathrm{Mod}(\pi)$ is a counter-model for the Scott principle.

$\mathrm{Mod}(\pi) =$



Example 4 The following is a refutation π of the formula

$$F = (p \to (q \to r)) \vee (((x \vee p) \to ((s \to t) \vee (w \to (z \to x)))) \vee (u \to (v \wedge u)))\,.$$

We note that this formula is not classically valid.

$$
\begin{array}{cl}
\hline
0 : \emptyset ; q, p, u, z, w, s \Rightarrow r, v, x, t & \mathrm{Irr} \\
\hline
0 : \emptyset ; p, u, z, w, s \Rightarrow q \to r, v, x, t & \to R \\
\hline
0 : \emptyset ; u, z, w, s, p \Rightarrow p \to (q \to r), v, x, t & \to R \\
\hline
0 : \emptyset ; u, z, w, s, p \Rightarrow v \wedge u, x, t, p \to (q \to r) & \wedge R_1 \\
\hline
0 : \emptyset ; z, w, s, p \Rightarrow u \to (v \wedge u), x, t, p \to (q \to r) & \to R \\
\hline
0 : \emptyset ; w, s, p \Rightarrow z \to x, t, u \to (v \wedge u), p \to (q \to r) & \to R \\
\hline
0 : \emptyset ; s, p \Rightarrow w \to (z \to x), t, u \to (v \wedge u), p \to (q \to r) & \to R \\
\hline
0 : \emptyset ; p \Rightarrow s \to t, w \to (z \to x), u \to (v \wedge u), p \to (q \to r) & \to R \\
\hline
0 : \emptyset ; x \vee p \Rightarrow s \to t, w \to (z \to x), u \to (v \wedge u), p \to (q \to r) & \vee L_2 \\
\hline
0 : \emptyset ; x \vee p \Rightarrow (s \to t) \vee (w \to (z \to x)), u \to (v \wedge u), p \to (q \to r) & \vee R \\
\hline
0 : \emptyset ; \emptyset \Rightarrow (x \vee p) \to ((s \to t) \vee (w \to (z \to x))), u \to (v \wedge u), p \to (q \to r) & \to R \\
\hline
0 : \emptyset ; \emptyset \Rightarrow ((x \vee p) \to ((s \to t) \vee (w \to (z \to x)))) \vee (u \to (v \wedge u)), p \to (q \to r) & \vee R \\
\hline
0 : \emptyset ; \emptyset \Rightarrow (p \to (q \to r)) \vee (((x \vee p) \to ((s \to t) \vee (w \to (z \to x)))) \vee (u \to (v \wedge u))) & \vee R
\end{array}
$$

The following is the counter-model for F extracted from the above refutation.

$$\mathrm{Mod}(\pi) = \boxed{0\!: \mathrm{q, p, u, z, w, s}}$$

We remark that our construction generates a Kripke counter-model consisting of a single world, that is a classical counter-model for F. We note that in [3] the generated counter-model for F has depth 3 and consists of 8 worlds.

4 Completeness

In this section we provide a function F that takes as input a sequent σ and returns either a proof or a refutation of σ and we prove its correctness. As a consequence we get the completeness of **LSJ** and **RJ**.

First of all, we define the following *gluing* constructor on $\mathcal{C}$-trees. Let us consider a list $[\pi_1, \ldots, \pi_n]$ of $\mathcal{C}$-trees, where $\pi_i = \langle T_i, s_i, r_i \rangle$; we assume without loss of generality that the T_i's are pairwise disjoint. Let σ be a sequent and let $\mathcal{R}$ be a rule of $\mathcal{C}$, we denote with $\mathrm{Glue}(\mathcal{C}, [\pi_1, \ldots, \pi_n], \sigma, \mathcal{R})$ the $\mathcal{C}$-tree $\pi = \langle T, s, r \rangle$ done as follows:

- let t be a node not occurring in $T_1, \ldots, T_n$; T is the tree having $T_1, \ldots, T_n$ as subtrees, t as root and $children(t) = \{root(T_1), \ldots, root(T_n)\}$;
- $s(t) = \sigma$ and $r(t) = \mathcal{R}$;
- for every $i \in \{1, \ldots, n\}$ and for every $a \in T_i$, $s(a) = s_i(a)$ and $r(a) = r_i(a)$.

Let us consider an instance of a rule $\mathcal{R}$ of **LSJ** having σ as conclusion and H as principal formula; we denote with $prem_i(\mathcal{R}, \sigma, H)$ the i-th premise of this rule application. If the rule has only one premise, we simply write $prem(\mathcal{R}, \sigma, H)$ instead of $prem_1(\mathcal{R}, \sigma, H)$.

The function F described in Fig. 3 takes as input a sequent σ and returns either an **LSJ**-tree or an **RJ**-tree. Essentially F searches for a proof or a refutation of σ by applying backward the rules of **LSJ** and **RJ**. Informally our algorithm works as follows:

- If σ is an initial sequent of **LSJ** (lines 2 and 3) an **LSJ**-tree is returned;
- If σ is an initial sequent of **RJ** (line 4) an **RJ**-tree is returned;

```
function F(Θ ; Γ ⇒ Δ)
let σ = Θ ; Γ ⇒ Δ
if (⊥ ∈ Γ) then return Glue(LSJ, [], σ, ⊥L)
if (Γ ∩ Δ ≠ ∅) then return Glue(LSJ, [], σ, Id)
if (σ is simple) then return Glue(RJ, [], σ, Irr)
if (one of the rules ∧L and ∨R is applicable to σ) then {
   if (there exists A ∧ B ∈ Γ) then select H = A ∧ B from Γ and let R = ∧L
   else select H = A ∨ B from Δ and let R = ∨R
   let σ′ = prem(R, σ, H) and let π = F(σ′)
   if (π is an LSJ-tree) then return Glue(LSJ, [π], σ, R)
   else return Glue(RJ, [π], σ, R)
}
if (one of the rules ∨L and ∧R is applicable to σ) then {
   if (there exists A ∨ B ∈ Γ) then select H = A ∨ B from Γ and let R = ∨L and R_i = ∨L_i
   else select H = A ∧ B from Δ and let R = ∧R and R_i = ∧R_i
   let π1 = F(prem_1(R, σ, H))
   if (π1 is an RJ-tree) then return Glue(RJ, [π1], σ, R_1)
   let π2 = F(prem_2(R, σ, H))
   if (π2 is an RJ-tree) then return Glue(RJ, [π2], σ, R_2)
   return Glue(LSJ, [π1, π2], σ, R)
}
let P = ∅ //set of RJ-trees
foreach(A → B in Γ) {
   let π1 = F(Θ ; B, Γ \ {A → B} ⇒ Δ)
   if (π1 is an RJ-tree) then return Glue(RJ, [π1], σ, → L_1)
   let π2 = F(B, Θ ; Γ \ {A → B} ⇒ A, Δ)
   if (π2 is an RJ-tree) then return Glue(RJ, [π2], σ, → L_2)
   let π3 = F(B ; Θ, Γ \ {A → B} ⇒ A)
   if (π3 is an LSJ-tree) then return Glue(LSJ, [π1, π2, π3], σ, → L)
   else P = P ∪ {π3}
}
foreach(C → D in Δ) {
   let π1 = F(Θ ; C, Γ ⇒ D, Δ \ {C → D})
   if (π1 is an RJ-tree) then return Glue(RJ, [π1], σ, → R)
   let π2 = F(∅ ; C, Θ, Γ ⇒ D)
   if (π2 is an LSJ-tree) then return Glue(LSJ, [π1, π2], σ, → R)
   else P = P ∪ {π2}
}
let P = {π1, ..., πn} //n ≥ 1
return Glue(RJ, [π1, ..., πn], σ, Succ)
```

Fig. 3 The function F

– If the previous cases do not hold, F tries to apply an invertible rule of **LSJ** (first trying the rules with one premise and then the branching ones). If this is not possible, it applies a non-invertible rule. In any case, if the recursive invocations of F return **LSJ**-trees, an **LSJ**-tree is returned, otherwise an **RJ**-tree is returned.

Now we prove that every execution of F(σ) terminates returning either a proof or a refutation of σ.

Theorem 5 *Let σ be a sequent:*

1. F(σ) *terminates and requires $dg(\sigma)$ nested recursive invocations at most;*
2. F(σ) *returns either a proof or a refutation of σ.*

Proof Point (1) immediately follows by the fact that in F(σ) every recursive invocation acts on a sequent σ' with $dg(\sigma') < dg(\sigma)$.

Now let $\pi = \text{Glue}(\mathcal{C}, L, \sigma, \mathcal{R})$ be the output of $\text{F}(\sigma)$. The proof of point (2) goes by induction on the number N of nested recursive invocations of F. If $N = 0$ then one of the `return` instructions at lines 2, 3 and 4 has been executed; in this case the assertion immediately follows.

Let us suppose that $\text{F}(\sigma)$ performs $N + 1$ nested recursive invocations. The proof goes by cases on the last executed `return` instruction. The assertion in the various cases easily follows by the induction hypothesis. We only discuss the cases where the last executed `return` instruction is one of those occurring in lines 21–39. We remark that if we are executing one of these instructions, no invertible rule of **LSJ** can be applied to σ. Hence, we can write σ as $\Theta\,;\Gamma_{At}, \Gamma^{\to} \Rightarrow \Delta_{At}, \Delta^{\to}$ where Γ_{At} and Δ_{At} are sets of atomic formulas such that $\bot \notin \Gamma_{At}$, $\Gamma_{At} \cap \Delta_{At} = \emptyset$, $\Gamma^{\to}$ and $\Delta^{\to}$ are sets of implicative formulas with $\Gamma^{\to} \cup \Delta^{\to} \neq \emptyset$.

If the last executed `return` instruction is one of those at lines 24, 26 and 33, then, by the induction hypothesis, the returned structure is a refutation of σ. If the last executed `return` instruction is one of those at lines 28 and 35, then, by the induction hypothesis, the returned structure is a proof of σ. Let us assume that the last executed `return` instruction is that at line 39 and $\mathcal{P} = \{\pi_1, \ldots, \pi_n\}$ with $n \geq 1$. Since, for every $A \to B \in \Gamma^{\to}$, the instruction at line 29 has been executed and, for every $C \to D \in \Delta^{\to}$, the instruction at line 36 has been executed, by induction hypothesis we get:

– for every $A \to B \in \Gamma^{\to}$, $\mathcal{P}$ contains a refutation of $B\,;\Theta, \Gamma_{At}, \Gamma^{\to} \setminus \{A \to B\} \Rightarrow A$;
– for every $C \to D \in \Delta^{\to}$, $\mathcal{P}$ contains a refutation of $\emptyset\,; C, \Theta, \Gamma_{At}, \Gamma^{\to} \Rightarrow D$.

Moreover, Γ_{At} and Δ_{At} satisfy the side conditions in Fig. 2. Hence the **RJ**-tree $\text{Glue}(\textbf{RJ}, [\pi_1, \ldots, \pi_n], \sigma, \text{Succ})$ is a refutation of σ. □

Theorem 6 (Completeness) *If the sequent σ is not refutable then it is provable.*

Proof By the above theorem, $\text{F}(\sigma)$ always terminates returning either a proof of σ or a refutation of σ. If σ is not provable, then $\text{F}(\sigma)$ returns a refutation π of σ. By Theorem 4, $\text{Mod}(\pi)$ refutes σ, hence σ is refutable. It follows that, if σ is not refutable, then it is provable. □

If we rewrite the function F of Fig. 3 as a decision procedure, that is ignoring proofs and refutations construction, we get a $O(n \log n)$-SPACE algorithm.

5 Properties of Counter-Models

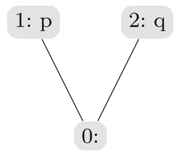
Given a refutable sequent $\sigma = \Theta\,;\Gamma \Rightarrow \Delta$, the *minimum depth* of σ is the minimum among the depths of all the counter-models for σ. Formally, the function md (*minimum depth*) assigns to a sequent σ an element of $\mathbf{N} \cup \{\infty\}$:

$$\text{md}(\sigma) = \begin{cases} \min\{\,\text{depth}(\mathcal{K}) \mid \mathcal{K} = \langle P, \leq, \rho, V\rangle \text{ and } \mathcal{K}, \rho \rhd \sigma\,\} & \text{if } \sigma \text{ is refutable} \\ \infty & \text{otherwise} \end{cases}$$

Clearly, if $\mathcal{K}$ is a counter-model for σ, then $\text{depth}(\mathcal{K}) \geq \text{md}(\sigma)$. In general, when σ is not provable in **LSJ**, the model $\text{Mod}(\text{F}(\sigma))$ has not the minimal depth $\text{md}(\sigma)$, because $\text{F}(\sigma)$ stops when the first refutation for σ is found. Since it is possible that F disregards some refutations, it is not guaranteed that the returned refutation describes a model with minimal depth. Here we provide two examples where F fails to return a counter-model of minimal depth.

Example 5 Let σ be the sequent $\emptyset\,;\emptyset \Rightarrow ((p \to q) \vee (q \to p)) \wedge r$. Clearly, $\mathrm{md}(\sigma) = 0$, since any model $\mathcal{K} = \langle \{\rho\}, \leq, \rho, V \rangle$ such that $\mathcal{K}, \rho \nVdash r$ is a counter-model for σ. On the other hand, $\mathrm{F}(\sigma)$ returns the refutation π_1 whose associated counter-model has depth 1.

$$\pi_1 = \dfrac{\dfrac{\dfrac{\dfrac{}{1:\emptyset;p \Rightarrow q}\,\text{Irr} \qquad \dfrac{}{2:\emptyset;q \Rightarrow p}\,\text{Irr}}{0:\emptyset;\emptyset \Rightarrow p \to q,\, q \to p}\,\text{Succ}}{0:\emptyset;\emptyset \Rightarrow (p \to q) \vee (q \to p)}\,\vee R}{0:\emptyset;\emptyset \Rightarrow ((p \to q) \vee (q \to p)) \wedge r}\,\wedge R_1 \qquad \mathrm{Mod}(\pi_1) =$$

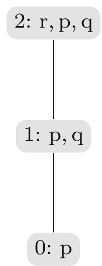


To get a counter-model of minimal depth, we have to apply $\wedge R_2$ (instead of $\wedge R_1$) to choose the subformula r. The related refutation π_2 generates a counter-model of depth 0.

$$\pi_2 = \dfrac{\dfrac{}{0:\emptyset;\emptyset \Rightarrow r}\,\text{Irr}}{0:\emptyset;\emptyset \Rightarrow ((p \to q) \vee (q \to p)) \wedge r}\,\wedge R_2 \qquad \mathrm{Mod}(\pi_2) =$$

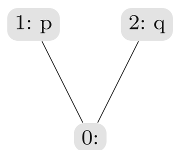
Example 6 Let σ be the sequent $\emptyset\,;\emptyset \Rightarrow p \to q, q \to p \wedge (p \to r \vee \neg r)$. $\mathrm{F}(\sigma)$ returns the refutation π_1 whose associated counter-model has depth 2.

$$\pi_1 = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{2:\emptyset;r,\, p,\, q \Rightarrow \emptyset}\,\text{Irr}}{1:\emptyset;p,\, q \Rightarrow r,\, \neg r}\,\text{Succ}}{1:\emptyset;p,\, q \Rightarrow r \vee \neg r}\,\vee R}{1:\emptyset;q,\, p \Rightarrow p \to r \vee \neg r}\,\to R}{1:\emptyset;q,p \Rightarrow p \wedge (p \to r \vee \neg r)}\,\wedge R_2}{0:\emptyset;p \Rightarrow q,\, q \to p \wedge (p \to r \vee \neg r)}\,\text{Succ}}{0:\emptyset;\emptyset \Rightarrow p \to q,\, q \to p \wedge (p \to r \vee \neg r)}\,\to R\ (a) \qquad \mathrm{Mod}(\pi_1) =$$



The function F applies in (a) the rule $\to R$, and this forces the variable p to be true in the root 0 of the counter-model. With this choice, the only way to falsify the formula $q \to p \wedge (p \to r \vee \neg r)$ in 0 is the generation of worlds 1 and 2, giving rise to a counter-model of depth 2. To build a model of depth 1 (the minimal depth of σ), we have to apply in (a) the rule Succ instead of $\to R$:

$$\pi_2 = \dfrac{\dfrac{}{1:\emptyset;p \Rightarrow q}\,\text{Irr} \qquad \dfrac{\dfrac{}{2:\emptyset;q \Rightarrow p}\,\text{Irr}}{2:\emptyset;q \Rightarrow p \wedge (p \to r \vee \neg r)}\,\wedge R_1}{0:\emptyset;\emptyset \Rightarrow p \to q,\, q \to p \wedge (p \to r \vee \neg r)}\,\text{Succ} \qquad \mathrm{Mod}(\pi_2) =$$



To avoid the above situations and to get counter-models with minimal depth we have to refine the algorithm F of Fig. 3 in such a way that all the possible refutations for σ are built and the one corresponding to a counter-model of minimal depth is returned. The new algorithm FMIN is given in Fig. 4.

The termination and correctness of FMIN can be proved along the lines of Theorem 5. The proof of minimality of the returned counter-models rests on the following properties of $\mathrm{md}(\sigma)$.

```
function FMIN(Θ ; Γ ⇒ Δ)
let σ = Θ ; Γ ⇒ Δ
if (⊥ ∈ Γ) then return Glue(LSJ, [], σ, ⊥L)
if (Γ ∩ Δ ≠ ∅) then return Glue(LSJ, [], σ, Id)
if (σ is simple) then return Glue(RJ, [], σ, Irr)
if (one of the rules ∧L and ∨R is applicable to σ) then {
   if (there exists A ∧ B ∈ Γ) then select H = A ∧ B from Γ and let 𝓡 = ∧L
   else select H = A ∨ B from Δ and let 𝓡 = ∨R
   let σ′ = prem(𝓡, σ, H) and let π = FMIN(σ′)
   if (π is an LSJ-tree) then return Glue(LSJ, [π], σ, 𝓡)
   else return Glue(RJ, [π], σ, 𝓡)
}
if (one of the rules ∨L and ∧R is applicable to σ) then {
   if (there exists A ∨ B ∈ Γ) then select H = A ∨ B from Γ and let 𝓡 = ∨L and 𝓡_i = ∨L_i
   else select H = A ∧ B from Δ and let 𝓡 = ∧R and 𝓡_i = ∧R_i
   let 𝒬 = ∅ //set of RJ-trees
   let π_1 = FMIN(prem_1(𝓡, σ, H)); if (π_1 is an RJ-tree) let 𝒬 = 𝒬 ∪ {π_1}
   let π_2 = FMIN(prem_2(𝓡, σ, H)); if (π_2 is an RJ-tree) let 𝒬 = 𝒬 ∪ {π_2}
   if (𝒬 is empty) then return Glue(LSJ, [π_1, π_2], σ, 𝓡)
   else {
      let π_i be an element in 𝒬 such that depth(Mod(π_i)) ≤ depth(Mod(π′)) for every π′ ∈ 𝒬
      return Glue(RJ, [π_i], σ, 𝓡_i)
   }
}
let 𝒫_succ = ∅ //candidates for Succ application
let 𝒬 = ∅ //set of RJ-trees
is_succ_applicable = true
foreach(A → B in Γ) {
   let π_1 = FMIN(Θ ; B, Γ \ {A → B} ⇒ Δ)
   let π_2 = FMIN(B, Θ ; Γ \ {A → B} ⇒ A, Δ)
   let π_3 = FMIN(B ; Θ, Γ \ {A → B} ⇒ A)
   if (π_1, π_2 and π_3 are LSJ-trees) then return Glue(LSJ, [π_1, π_2, π_3], σ, → L)
   else {
      if (π_1 is an RJ-tree) then 𝒬 = 𝒬 ∪ {Glue(RJ, [π_1], σ, → L_1)}
      if (π_2 is an RJ-tree) then 𝒬 = 𝒬 ∪ {Glue(RJ, [π_2], σ, → L_2)}
      if (π_3 is an RJ-tree) then 𝒫_succ = 𝒫_succ ∪ {π_3} else is_succ_applicable = false
   }
}
foreach(C → D in Δ) {
   let π_1 = FMIN(Θ ; C, Γ ⇒ D, Δ \ {C → D})
   let π_2 = FMIN(∅ ; C, Θ, Γ ⇒ D)
   if (π_1 and π_2 are LSJ-tree) then return Glue(LSJ, [π_1, π_2], σ, → R)
   else {
      if (π_1 is an RJ-tree) then 𝒬 = 𝒬 ∪ {Glue(RJ, [π_1], σ, → R)}
      if (π_2 is an RJ-tree) then 𝒫_succ = 𝒫_succ ∪ {π_2} else is_succ_applicable = false
   }
}
if (is_succ_applicable) then {
   let 𝒫_succ = {π_1, ..., π_n}
   let 𝒬 = 𝒬 ∪ {Glue(RJ, [π_1, ..., π_n], σ, Succ)}
}
let π̂ be an element in 𝒬 such that depth(Mod(π̂)) ≤ depth(Mod(π′)) for every π′ ∈ 𝒬
return π̂
```

Fig. 4 The function FMIN

Lemma 3 *Let Θ, Γ and Δ be sets of formulas.*

1. $\mathrm{md}(\Theta\,;\, A \wedge B, \Gamma \Rightarrow \Delta) = \mathrm{md}(\Theta\,;\, A, B, \Gamma \Rightarrow \Delta)$.
2. $\mathrm{md}(\Theta\,;\, \Gamma \Rightarrow A \wedge B, \Delta) = \min\{\mathrm{md}(\Theta\,;\, \Gamma \Rightarrow A, \Delta),\ \mathrm{md}(\Theta\,;\, \Gamma \Rightarrow B, \Delta)\}$.
3. $\mathrm{md}(\Theta\,;\, A \vee B, \Gamma \Rightarrow \Delta) = \min\{\mathrm{md}(\Theta\,;\, A, \Gamma \Rightarrow \Delta),\ \mathrm{md}(\Theta\,;\, \Gamma, B \Rightarrow \Delta)\}$.
4. $\mathrm{md}(\Theta\,;\, \Gamma \Rightarrow A \vee B, \Delta) = \mathrm{md}(\Theta\,;\, \Gamma \Rightarrow A, B, \Delta)$.

Proof The proof of point 1 is immediate, since a counter-model for $\Theta\,;\, A \wedge B, \Gamma \Rightarrow \Delta$ is a counter-model for $\Theta\,;\, A, B, \Gamma \Rightarrow \Delta$ and vice versa; point 4 has an analogous immediate proof. We prove point 2. Let us define:

$$\begin{aligned} \sigma &= \Theta\,;\, \Gamma \Rightarrow A \wedge B, \Delta \\ \sigma_A &= \Theta\,;\, \Gamma \Rightarrow A, \Delta \;=\; \mathit{prem}_1(\wedge R, \sigma, A \wedge B) \\ \sigma_B &= \Theta\,;\, \Gamma \Rightarrow B, \Delta \;=\; \mathit{prem}_2(\wedge R, \sigma, A \wedge B) \end{aligned}$$

and let $\delta = \mathrm{md}(\sigma)$, $\delta_A = \mathrm{md}(\sigma_A)$, and $\delta_B = \mathrm{md}(\sigma_B)$. Since a counter-model for σ_A is a counter-model for σ, it holds that $\delta \leq \delta_A$; similarly, $\delta \leq \delta_B$, hence $\delta \leq \min\{\delta_A, \delta_B\}$. Moreover, a counter-model for σ is either a counter-model for σ_A or a counter-model for σ_B, hence either $\delta_A \leq \delta$ or $\delta_B \leq \delta$, which implies $\min\{\delta_A, \delta_B\} \leq \delta$. We conclude $\delta = \min\{\delta_A, \delta_B\}$. The proof of point 3 is similar. □

An analogous property for implicative formulas requires a deeper case analysis. Let σ be the sequent $\Theta\,;\, \Gamma \Rightarrow \Delta$. For $A \to B \in \Gamma$ and for $1 \leq i \leq 3$, we denote with $\sigma_{Li}^{A\to B}$ the i-th premise of the rule $\to L$ applied to σ with principal formula $A \to B$, that is:

$$\begin{aligned} \sigma_{L1}^{A\to B} &= \mathit{prem}_1(\to L, \sigma, A \to B) &&= \Theta\,;\, B, \Gamma \setminus \{A \to B\} \Rightarrow \Delta \\ \sigma_{L2}^{A\to B} &= \mathit{prem}_2(\to L, \sigma, A \to B) &&= B, \Theta\,;\, \Gamma \setminus \{A \to B\} \Rightarrow A, \Delta \\ \sigma_{L3}^{A\to B} &= \mathit{prem}_3(\to L, \sigma, A \to B) &&= B\,;\, \Theta, \Gamma \setminus \{A \to B\} \Rightarrow A \end{aligned}$$

Similarly, given $C \to D \in \Delta$ we define:

$$\begin{aligned} \sigma_{R1}^{C\to D} &= \mathit{prem}_1(\to R, \sigma, C \to D) &&= \Theta\,;\, C, \Gamma \Rightarrow D, \Delta \setminus \{C \to D\} \\ \sigma_{R2}^{C\to D} &= \mathit{prem}_2(\to R, \sigma, C \to D) &&= \emptyset\,;\, C, \Theta, \Gamma \Rightarrow D \end{aligned}$$

Moreover, we set:

$$\begin{aligned} \delta_{Li}^{A\to B} &= \mathrm{md}(\sigma_{Li}^{A\to B}) \qquad \text{for } 1 \leq i \leq 3 \\ \delta_{Rj}^{C\to D} &= \mathrm{md}(\sigma_{Rj}^{C\to D}) \qquad \text{for } 1 \leq j \leq 2 \end{aligned}$$

Lemma 4 *Let $\sigma = \Theta\,;\, \Gamma_{At}, \Gamma^{\to} \Rightarrow \Delta_{At}, \Delta^{\to}$, where Γ_{At} and Δ_{At} are sets of atomic formulas such that $\bot \notin \Gamma_{At}$, $\Gamma_{At} \cap \Delta_{At} = \emptyset$, $\Gamma^{\to}$ and $\Delta^{\to}$ are sets of implicative formulas with $\Gamma^{\to} \cup \Delta^{\to} \neq \emptyset$. Let:*

$$\begin{aligned} \mathcal{D}_L &= \{\, \delta_{Lk}^{A\to B} \mid k \in \{1,2\} \textit{ and } A \to B \in \Gamma^{\to} \,\} \\ \mathcal{D}_R &= \{\, \delta_{R1}^{C\to D} \mid C \to D \in \Delta^{\to} \,\} \\ \delta_m &= \max(\{\, \delta_{L3}^{A\to B} \mid A \to B \in \Gamma^{\to} \,\} \,\cup\, \{\, \delta_{R2}^{C\to D} \mid C \to D \in \Delta^{\to} \})\end{aligned}$$

Then, $\mathrm{md}(\sigma) \;=\; \min(\mathcal{D}_L \cup \mathcal{D}_R \cup \{\, \delta_m + 1 \,\})$.[1]

[1] If $\delta_m = \infty$ we set $\delta_m + 1 = \infty$.

Proof For $A \to B \in \Gamma^{\to}$ and $k \in \{1, 2\}$, a counter-model for $\sigma_{Lk}^{A\to B}$ is a counter-model for σ, hence $\mathrm{md}(\sigma) \leq \delta_{Lk}^{A\to B}$. This implies that:

(P1) for every $\delta \in \mathcal{D}_L$, $\mathrm{md}(\sigma) \leq \delta$.

Similarly, since for every $C \to D \in \Delta^{\to}$ a counter-model for $\sigma_{R1}^{C\to D}$ is a counter-model for σ, we get:

(P2) for every $\delta \in \mathcal{D}_R$, $\mathrm{md}(\sigma) \leq \delta$.

We show:

(P3) $\mathrm{md}(\sigma) \leq \delta_m + 1$.

If, for some $A \to B \in \Gamma^{\to}$, $\sigma_{L3}^{A\to B}$ is not refutable, then $\delta_m = \infty$ and (P3) trivially holds. Similarly, if $\sigma_{R2}^{C\to D}$ is not refutable for some $C \to D \in \Gamma^{\to}$, $\delta_m = \infty$ and (P3) holds. Now, let us assume that, for every $A \to B \in \Gamma^{\to}$ and every $C \to D \in \Delta^{\to}$, all the sequents $\sigma_{L3}^{A\to B}$ and $\sigma_{R2}^{C\to D}$ are refutable. For every $A \to B \in \Gamma^{\to}$, let $\mathcal{K}_{L3}^{A\to B}$ be a counter-model for $\sigma_{L3}^{A\to B}$ such that $\mathrm{depth}(\mathcal{K}_{L3}^{A\to B}) = \delta_{L3}^{A\to B}$ and for every $C \to D \in \Delta^{\to}$ let $\mathcal{K}_{R2}^{C\to D}$ be a counter-model for $\sigma_{R2}^{C\to D}$ such that $\mathrm{depth}(\mathcal{K}_{R2}^{C\to D}) = \delta_{R2}^{C\to D}$. We can build a counter-model $\mathcal{K}$ for σ by gluing all the models $\mathcal{K}_{L3}^{A\to B}$ and $\mathcal{K}_{R2}^{C\to D}$ as described in the definition of $\mathrm{Mod}(\pi)$ (see Section 3.2). It follows that $\mathrm{depth}(\mathcal{K}) = \delta_m + 1$, which implies (P3). By (P1)–(P3) we conclude:

(P4) $\mathrm{md}(\sigma) \ \leq\ \min(\mathcal{D}_L \cup \mathcal{D}_R \cup \{\delta_m + 1\})$.

Now we prove the converse of (P4), that is:

(P5) $\mathrm{md}(\sigma) \ \geq\ \min(\mathcal{D}_L \cup \mathcal{D}_R \cup \{\delta_m + 1\})$.

If σ is not refutable, $\mathrm{md}(\sigma) = \infty$ and (P5) holds. Otherwise, let $\mathcal{K} = \langle P, \leq, \rho, V\rangle$ be a Kripke model such that $\mathcal{K}, \rho \triangleright \sigma$ and $\mathrm{depth}(\mathcal{K}) = \mathrm{md}(\sigma)$. Firstly, we show that one of the following properties (i)–(iii) holds:

(i) $\mathcal{K}, \rho \triangleright \sigma_{Lk}^{A\to B}$, for some $k \in \{1, 2\}$ and $A \to B \in \Gamma^{\to}$;
(ii) $\mathcal{K}, \rho \triangleright \sigma_{R1}^{C\to D}$, for some $C \to D \in \Delta^{\to}$;
(iii) For every $A \to B \in \Gamma^{\to}$ there exists $\alpha \neq \rho$ such that $\mathcal{K}, \alpha \triangleright \sigma_{L3}^{A\to B}$ and for every $C \to D \in \Delta^{\to}$ there exists $\beta \neq \rho$ such that $\mathcal{K}, \beta \triangleright \sigma_{R2}^{C\to D}$.

Indeed, let us assume that (i) does not hold. By Lemma 2 applied to the rule $\to L$, for every $A \to B \in \Gamma^{\to}$ there exists α such that $\mathcal{K}, \alpha \triangleright \sigma_{L3}^{A\to B}$. We cannot have $\alpha = \rho$; indeed, if $\mathcal{K}, \alpha \triangleright \sigma_{L3}^{A\to B}$ and $\alpha = \rho$, by the fact that $\mathcal{K}, \rho \triangleright \sigma$ we would conclude $\mathcal{K}, \rho \triangleright \sigma_{L2}^{A\to B}$, against the assumption that (i) does not hold. Similarly, if (ii) does not hold, for every $C \to D \in \Delta^{\to}$ there exists $\beta \neq \rho$ such that $\mathcal{K}, \beta \triangleright \sigma_{R2}^{C\to D}$. Thus, one of the properties (i)–(iii) holds. We show that, according to the case, one of the following properties (iv)–(vi) holds:

(iv) $\mathrm{md}(\sigma) \geq \delta_{Lk}^{A\to B}$, for some $k \in \{1, 2\}$ and $A \to B \in \Gamma^{\to}$;
(v) $\mathrm{md}(\sigma) \geq \delta_{R1}^{C\to D}$, for some $C \to D \in \Delta^{\to}$;
(vi) $\mathrm{md}(\sigma) \geq \delta_{L3}^{A\to B} + 1$ for every $A \to B \in \Gamma^{\to}$ and $\mathrm{md}(\sigma) \geq \delta_{R2}^{C\to D} + 1$ for every $C \to D \in \Delta^{\to}$.

If (i) holds, we have $\mathrm{depth}(\mathcal{K}) \geq \delta_{Lk}^{A\to B}$ and, being $\mathrm{depth}(\mathcal{K}) = \mathrm{md}(\sigma)$, (iv) follows. Similarly, if (ii) holds, we get (v). Suppose now that (iii) holds. Let $A \to B \in \Gamma^{\to}$ and let $\alpha \in P$ such that $\alpha \neq \rho$ and $\mathcal{K}, \alpha \rhd \sigma_{L3}^{A\to B}$. Let $\mathcal{K}_\alpha$ be the submodel of $\mathcal{K}$ generated by α (namely, $\mathcal{K}_\alpha$ is the restriction of $\mathcal{K}$ to the worlds β such that $\alpha \leq \beta$). It is easy to check that $\mathcal{K}_\alpha$ is a counter-model for $\sigma_{L3}^{A\to B}$, hence $\text{depth}(\mathcal{K}_\alpha) \geq \delta_{L3}^{A\to B}$. By the fact that $\rho < \alpha$, we get $\text{depth}(\mathcal{K}) \geq \delta_{L3}^{A\to B} + 1$, namely $\text{md}(\sigma) \geq \delta_{L3}^{A\to B} + 1$. In a similar way, we prove that, for every $C \to D \in \Delta^{\to}$, $\text{md}(\sigma) \geq \delta_{R2}^{C\to D} + 1$, and this concludes the proof of (vi). Note that (vi) implies:

(vii) $\text{md}(\sigma) \geq \delta_m + 1$

Since one of the cases (iv), (v), (vii) holds, (P5) follows. By (P4) and (P5) we conclude $\text{md}(\sigma) = \min(\mathcal{D}_L \cup \mathcal{D}_R \cup \{\delta_m + 1\})$. □

From the above discussion, the main result of this section follows:

Theorem 7 *Let σ be a sequent:*

1. FMIN(σ) *terminates and requires $dg(\sigma)$ nested recursive invocations at most;*
2. FMIN(σ) *returns either a proof or a refutation of σ;*
3. *If* FMIN(σ) *returns a refutation π, then* $\text{Mod}(\pi)$ *has depth* $\text{md}(\sigma)$.

Proof The proof goes by induction on the number of nested recursive invocations of FMIN along the lines of the proof of Theorem 5. To prove point (3) we note that, if the refutation π is returned at line 4, the corresponding counter-model has depth 0. In all the other cases, point (3) follows by the induction hypothesis and Lemmas 3 and 4. We treat the more tricky case where π is returned at line 52. We note that, if lines 24–52 are executed, the input sequent σ can be written as $\Theta\,;\Gamma_{At}, \Gamma^{\to} \Rightarrow \Delta_{At}, \Delta^{\to}$ so that the hypothesis of Lemma 4 are satisfied. When line 51 is reached, $\mathcal{Q}$ and $\mathcal{P}_{succ}$ satisfy the following properties:

- For every $A \to B \in \Gamma^{\to}$ and $k \in \{1, 2\}$ such that $\sigma_{Lk}^{A\to B}$ is refutable, $\mathcal{Q}$ contains a refutation $\pi_{Lk}^{A\to B}$ of $\sigma_{Lk}^{A\to B}$ ($\pi_{Lk}^{A\to B}$ is the refutation added to $\mathcal{Q}$ at line 33 or 34). By the induction hypothesis $\text{depth}(\text{Mod}(\pi_{Lk}^{A\to B})) = \delta_{Lk}^{A\to B}$.
- For every $A \to B \in \Gamma^{\to}$ such that $\sigma_{L3}^{A\to B}$ is refutable, $\mathcal{P}_{succ}$ contains a refutation $\pi_{L3}^{A\to B}$ of $\sigma_{L3}^{A\to B}$. By the induction hypothesis $\text{depth}(\text{Mod}(\pi_{L3}^{A\to B})) = \delta_{L3}^{A\to B}$.
- For every $C \to D \in \Delta^{\to}$ such that $\sigma_{R1}^{C\to D}$ is refutable, $\mathcal{Q}$ contains a refutation $\pi_{R1}^{C\to D}$ of $\sigma_{R1}^{C\to D}$. By the induction hypothesis $\text{depth}(\text{Mod}(\pi_{R1}^{C\to D})) = \delta_{R1}^{C\to D}$.
- For every $C \to D \in \Delta^{\to}$ such that $\sigma_{R2}^{C\to D}$ is refutable, $\mathcal{P}_{succ}$ contains a refutation $\pi_{R2}^{C\to D}$ of $\sigma_{R2}^{C\to D}$. By the induction hypothesis $\text{depth}(\text{Mod}(\pi_{R2}^{C\to D})) = \delta_{R2}^{C\to D}$.
- Suppose that, for every $A \to B \in \Gamma^{\to}$ and $C \to D \in \Delta^{\to}$, all the sequents $\sigma_{L3}^{A\to B}$ and $\sigma_{R2}^{C\to D}$ are refutable. Then, the condition at line 47 is true (the variable `is_succ_applicable` is set to `false` only when one of the sequents $\sigma_{L3}^{A\to B}$ or $\sigma_{R2}^{C\to D}$ is provable), hence $\mathcal{Q}$ contains the refutation $\pi^* = \text{Glue}(\mathbf{RJ}, [\pi_1, \ldots, \pi_n], \sigma, \text{Succ})$, where $\pi_1, \ldots, \pi_n$ are all the refutations in $\mathcal{P}_{succ}$. Note that $\text{depth}(\text{Mod}(\pi^*)) = \delta_m + 1$, where δ_m is defined as in Lemma 4.

Let π be the refutation returned at line 52. Then, π is chosen in $\mathcal{Q}$ so that $\text{depth}(\text{Mod}(\pi)) \leq \text{depth}(\text{Mod}(\pi'))$, for every $\pi' \in \mathcal{Q}$. It follows that $\text{depth}(\text{Mod}(\pi))$ is the minimum of $\mathcal{D}_L \cup \mathcal{D}_R \cup \{\delta_m + 1\}$; by Lemma 4 we conclude that $\text{Mod}(\pi)$ has depth $\text{md}(\sigma)$. □

6 Related Works

The main difference among **LSJ** and the terminating calculi in [4, 5, 12, 13, 20] is that **LSJ** meets the subformula property. Paper [3] presents a sequent calculus with the subformula property whose termination is based on the rule *a-fortiori*; differently from **LSJ** the depth of its proofs is not linearly bounded. As for [12], we remark that STRIP implements some form of subformula property through the use of suitable data-structures.

The calculus **RJ** can be compared with CRIP [15], a sequent calculus which formalizes unprovability in Intuitionistic propositional logic. CRIP is based on LJT* [4], a multi-succedent variant of the well-known calculus LJT described in the same paper. CRIP does not meet the subformula property and the depth of its proofs is not linearly bounded.

In [7] it is provided a calculus which mixes together derivations and refutations for Bi-intuitionistic logic. An analogous calculus for Intuitionistic logic can be obtained disregarding the dual-intuitionistic connectives. Also this calculus does not require loop-checking. In this case, differently from our approach, the proof-search algorithm outputs derivations or refutations building a single proof-tree. However, the calculus of [7] does not obey to a linear bound on the depth of deductions.

To compare our approach with those based on histories, see e.g. [8, 10], we remark that histories and **LSJ** sequents are quite different mechanisms. Histories store goals already considered in proof-search and prevent rule applications which might lead to loops. For instance, the rule $\rightarrow R$ of [10] can be applied to the sequent $\Gamma \xrightarrow{\neg A} C; \mathcal{H}$ ($\mathcal{H}$ is the history) only if $C \notin \mathcal{H}$; if $C \in \mathcal{H}$ the proof-search fails and one has to backtrack. In our approach, the formulas in Θ of a sequent $\Theta\,;\Gamma\,\Rightarrow\Delta$ are never used to prevent the application of a rule; loop-checking is avoided by the fact that, when a rule is applied, at least a formula of the sequent is decomposed. Note that formulas stored in $\mathcal{H}$ are passive (no rule acts on them); in **LSJ**, the formulas in Θ can be added to Γ and become active (see the rules $\rightarrow L$ and $\rightarrow R$). Finally, we point out that history formulas are not part of the logical meaning of a sequent; for instance, the sequent $\Gamma \xrightarrow{\neg A} C; \mathcal{H}$ corresponds to the formula $(\bigwedge \Gamma \wedge \neg A) \rightarrow C$, regardless of the formulas in $\mathcal{H}$. In **LSJ**, the logical meaning of $\Theta\,;\Gamma\,\Rightarrow\Delta$ is expressed by a semantical condition involving all the components; as noticed in Section 3, we do not know if $\Theta\,;\Gamma\,\Rightarrow\Delta$ can be represented by a formula.

As for the procedures for counter-models generation, we quote [3, 9, 12, 17, 18]. As we noticed in Example 4 the counter-models extracted from the procedure described in [3] are not of minimal depth.

The decision procedure of [9] searches for *long normal form proofs* and relies on a non-terminating calculus requiring loop-checking. Also in this case the main difference with our proposal is that the generated counter-models are not of minimal depth. As an example, the counter-model for the non-classically valid formula described in [9] has 5 worlds and depth 3, while our procedure generates a counter-model consisting of a single (classical) world.

Papers [12, 17] describe tools inspired by the LJT calculus of [4]. In both cases the generated counter-models are not of minimal depth.

In [18] is presented a decision procedure which allows one to extract a counter-model from a failed attempt to find a proof. The procedure relies on a calculus whose proofs have depth $O(n^2)$. The author provides an upper bound on the depth and out-degree of generated counter-models. In both cases such a bound is the number of negative occurrences of implications in the sequent to be proved. By our proof of minimality we get that also our procedure obeys the bound on the depth of counter-models.

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