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# Contraction of State Variables in Non-Equilibrium Open Systems. II

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A projector elimination and an adiabatic elimination of irrelevant degrees-of-freedom are developed for the contraction of state variables in stochastic equations of motion. For multiplicative stochastic equations, a master equation for the probability density of relevant variables $A(t)\equiv\{A_i(t)\}$ is derived by means of the projector method and is shown to reduce to a Fokker-Planck equation if the stochastic forces $S_i(a,t)$ are Gaussian processes with time correlations of the form $\langle S_i(a,t)S_j(a',t')\rangle=2[\xi_{ij}(a,a')\delta_+(t-t')+\xi_{ji}(a',a)\delta_+(t'-t)]$, where $\delta_+(t)$ is the right half of the $\delta$ function $\delta(t)$, nonvanishing only at $t=0+$. If $\xi_{ij}(a,a')=\xi_{ji}(a',a)$, then this reduces to the conventional form $2\xi_{ij}(a,a')\delta(t-t')$.

With the aid of stochastic processes of this new type, an adiabatic elimination from the Langevin equations is proposed for a stochastic Haken-Zwanzig model for non-equilibrium phase transitions. A projector elimination from the Langevin equations and an adiabatic elimination from the Fokker-Planck equation are also explored. Calculation is carried out up to second order in the slowness parameter. Three different methods are thus developed with consistent results and are applied to a laser model for illustration.

## § 1. Introduction

Macroscopic properties are described by a relevant subset of macrovariables of the system, and it becomes necessary to obtain closed equations of motion for the subset by eliminating the rest. In a previous paper[1] we have developed a projector elimination for such a contraction of state variables when the system is governed by deterministic equations of motion.

Many systems are, however, described by stochastic equations of motion; for example, in a one-variable case,

$$dA(t)/dt=V(A(t))+S(A(t),t), \tag{1·1}$$

where $S(a,t)$ is a stochastic force which depends on the value $a$ of $A(t)$ explicitly. Conventional theories assume the form[2)~4)]

$$S(a,t)=g_k(a)r_k(t), \tag{1·2}$$

where $r_k(t)$ are Gaussian white noises and repeated indices $k$ are to be summed up. Such a stochastic force appears in the magnetic resonance absorption when one observes the motion of a spin in a fluctuating magnetic field.[2] Multiplica-

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tive stochastic forces of this type also appear when one replaces external parameters in equations of motion by fluctuating ones in order to take into account fluctuations of the surroundings,[5] and also appear when one eliminates irrelevant variables in coupled Langevin equations by an adiabatic procedure.[6),7)]

As will be shown later, however, an improved adiabatic procedure leads to a stochastic force of the memory form

$$S(a,t)=h_j(a)\int_0^\infty ds[e^{-s\gamma(a)}]_{jk}r_k(t-s), \tag{1·3}$$

where $\gamma(a)$ is a relaxation-rate matrix which may depend on $a$. The conventional procedure replaces the time integral by $[\gamma^{-1}(a)]_{jk}r_k(t)$ in the coarse-graining limit $\tau_B\to 0$, where $\tau_B$ is the time scale of the relaxation $e^{-s\gamma(a)}$, thus leading to (1·2). This replacement, however, is incorrect since $\tau_B\gg\tau_m$ even in the limit, where $\tau_m$ is the microscopic time scale of $r_k(t)$. In fact, as will be shown in § 5, (1·3) leads, in the limit $\tau_B\to 0$, to a time correlation of the form

$$\langle S(a,t)S(a',t')\rangle=2[\xi(a,a')\delta_+(t-t')+\xi(a',a)\delta_+(t'-t)] \tag{1·4}$$

with asymmetric coefficients $\xi(a,a')\neq\xi(a',a)$, where $\delta_+(t)$ is the right half of the $\delta$ function $\delta(t)$. This asymmetry comes from the memory effect due to $\tau_B\gg\tau_m$, and would be important when the degrees-of-freedom of the time scale $\tau_B$ are far from equilibrium.

In § 2, we treat multiplicative stochastic processes, including the new type (1·4), and derive reduced equations of motion and a master equation with the aid of the projector method. In § 3, a projector elimination from the Langevin equations is also studied with the aid of Fujisaka and Mori's projector method.[8] In §§ 4 and 5, we develope an adiabatic elimination from the Fokker-Planck equation and the Langevin equations. In § 6, we treat a laser model. Section 7 is devoted to a summary and remarks.

## § 2. Projector elimination in multiplicative stochastic processes

Let us denote a relevant subset of macrovariables by $A(t)=\{A_i(t)\}$ and assume, as a generalization of (1·1)~(1·3), that they are governed by stochastic equations of motion

$$dA_i(t)/dt=V_i(A(t),t)+S_i(A(t),t), \tag{2·1}$$

where $V_i(a,t)$ are unique functions of $a=\{a_i\}$ and $t$, and $S_i(a,t)$ are stochastic processes whose statistical properties are supposed to be known. Let us introduce the generating functional

$$\Pi_a(t)=\delta(A(t)-a). \tag{2·2}$$

Its time evolution is governed by

$$\partial \Pi_a(t)/\partial t = L^+(a,t)\Pi_a(t), \tag{2·3}$$

where $L^+$ is the linear operator

$$L^+(a,t) \equiv -(\partial/\partial a_i)[V_i(a,t)+S_i(a,t)] \tag{2·4}$$

with repeated indices $i$ being to be summed up. Integrating (2·3) formally leads to

$$\Pi_a(t) = \exp_+\left(\int_0^t L^+(a,s)\,ds\right)\delta(A(0)-a), \tag{2·5}$$

where $\exp_+$ denotes the time-ordered exponential ordered from left to right in decreasing order. In order to utilize the projector elimination, let us introduce the adjoint operator of $L^+(a,t)$,

$$L(b,t) \equiv [V_i(b,t)+S_i(b,t)](\partial/\partial b_i). \tag{2·6}$$

Then, since $\Pi_a(t) = \int db \Pi_b(t)\delta(a-b)$, we can write (2·3) by partial integration as

$$\partial \Pi_a(t)/\partial t = -\frac{\partial}{\partial a_i}\int db\,\delta(A(0)-b)\,U(b,t) \times [V_i(b,t)+S_i(b,t)]\delta(a-b), \tag{2·7}$$

where

$$U(b,t) \equiv \exp_-\left(\int_0^t L(b,s)\,ds\right). \tag{2·8}$$

Our problem is now to eliminate the degrees-of-freedom $\Omega$ associated with the stochastic forces $S_i(b,t)$. Let us suppose that at $t=0$ the relevant set $A(0)$ is known to take a set of values $a_0 \equiv \{a_{0i}\}$. The average of a functional $G(b)$ *of* $\{S_i(b,s)\}$ over $\Omega$ with this initial condition is denoted by $\langle G(b);a_0\rangle$. The projection onto this conditional average is denoted by the projector $P$:

$$PG(b) = \langle G(b);b\rangle. \tag{2·9}$$

Let us assume that the mean value of $S_i(b,t)$ is zero:

$$\langle S_i(b,t);a_0\rangle = 0. \tag{2·10}$$

Then, as will be shown in Appendix A, the equations of motion (2·7) can be transformed into

$$\frac{\partial}{\partial t}\Pi_a(t) = -\frac{\partial}{\partial a_i}\Big[V_i(a,t)\Pi_a(t) + F_{ia}(t) + \int_0^t d\tau \int db\,\Pi_b(\tau)\left\langle S_j(b,\tau)\frac{\partial}{\partial b_j}F_{ia}(b,t,\tau);b\right\rangle\Big], \tag{2·11}$$

where

$$F_{ia}(t)\equiv\int db\delta(A(0)-b)F_{ia}(b,t,0), \tag{2·12}$$

$$F_{ia}(b,t,\tau)\equiv U_Q(b,t,\tau)S_i(b,t)\delta(a-b), \tag{2·13}$$

$$U_Q(b,t,\tau)\equiv\exp_-\Big[\int_\tau^t QL(b,s)ds\Big] \tag{2·14}$$

with $Q\equiv1-P$. Since $PQ=0$, we have

$$\langle F_{ia}(t);a_0\rangle=\langle F_{ia}(b,t,\tau);a_0\rangle=0. \tag{2·15}$$

Namely, the $F_{ia}$'s are statistically independent of $A(0)$. Equation (2·11) is the fundamental equation corresponding to (3·3) of I.

As will be shown in Appendix A, (2·11) can be further transformed into

$$\begin{aligned}\frac{\partial}{\partial t}\Pi_a(t)=&-\frac{\partial}{\partial a_i}[V_i(a,t)\Pi_a(t)+F_{ia}(t)]\\&+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\partial}{\partial a_{i_1}}\cdots\frac{\partial}{\partial a_{i_n}}\int_0^t d\tau\Big[-C_{i_1\cdots i_n}(a,t,\tau)\\&+\frac{\partial}{\partial a_j}E_{i_1\cdots i_n;j}(a,t,\tau)\Big]\Pi_a(\tau),\end{aligned} \tag{2·16}$$

where we have defined

$$C_{i_1\cdots i_n}(a,t,\tau)\equiv\Big\langle S_j(a,\tau)\frac{\partial}{\partial a_j}\mathscr{S}(i_1\cdots i_n,a,t,\tau);a\Big\rangle, \tag{2·17}$$

$$E_{i_1\cdots i_n;j}(a,t,\tau)\equiv\langle S_j(a,\tau)\mathscr{S}(i_1\cdots i_n,a,t,\tau);a\rangle \tag{2·18}$$

in terms of the generalized fluctuating forces

$$\begin{aligned}\mathscr{S}(i_1\cdots i_n,a,t,\tau)\equiv&\int_\tau^t ds_2\int_{s_2}^t ds_3\cdots\int_{s_{n-1}}^t ds_n U_Q(a,s_2,\tau)\\&\times Q[V_{i_2}(a,s_2)+S_{i_2}(a,s_2)]\\&\times\cdots U_Q(a,s_n,s_{n-1})Q[V_{i_n}(a,s_n)+S_{i_n}(a,s_n)]R_{i_1}(a,t,\tau),\end{aligned} \tag{2·19}$$

$$R_i(a,t,\tau)\equiv U_Q(a,t,\tau)S_i(a,t). \tag{2·20}$$

Equation (2·16) is a generalization of the $\delta$ expansion (3·14) of I.

The probability distribution function that $A(t)$ takes a set of values $a$ at time $t$ is given by

$$P(a,t)=\langle\Pi_a(t);a_0\rangle, \tag{2·21}$$

where $P(a,0)=\delta(a-a_0)$. Therefore, (2·15) and (2·16) lead to

$$\frac{\partial}{\partial t}P(a,t)=-\frac{\partial}{\partial a_i}[V_i(a,t)P(a,t)]+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\partial}{\partial a_{i_1}}\cdots\frac{\partial}{\partial a_{i_n}}$$
$$\times\int_0^t d\tau\Big[-C_{i_1\cdots i_n}(a,t,\tau)+\frac{\partial}{\partial a_j}E_{i_1\cdots i_n;j}(a,t,\tau)\Big]P(a,\tau). \qquad (2\cdot22)$$

Since $A_i(t)=\int a_i\Pi_a(t)\,da$, (2·16) also leads to

$$dA_i(t)/dt=V_i(A(t),t)+\int_0^t d\tau C_i(A(\tau),t,\tau)+R_i(t), \qquad (2\cdot23)$$

where

$$R_i(t)\equiv\int F_{ia}(t)\,da=R_i(A(0),t,0). \qquad (2\cdot24)$$

Equation (2·23) is a stochastic equation whose features differ from those of (2·1), and is called the generalized *Langevin equation.* Equation (2·22) is called the generalized *master equation.* It should be noted that these equations are all exact under the assumption (2·10).

Let us now assume that the stochastic forces $S_i(a,t)$ are Gaussian processes with

$$\langle S_i(a,t)\,;a_0\rangle=0, \qquad (2\cdot25\text{a})$$
$$\langle S_i(a,t)S_j(a',t')\,;a_0\rangle=2[\xi_{ij}(a,a')\delta_+(t-t')+\xi_{ji}(a',a)\delta_+(t'-t)], \qquad (2\cdot25\text{b})$$

where $\delta_+(t)$ is the right half of $\delta(t)$ and is defined by

$$\delta_+(t)\equiv\begin{cases}\lim_{\tau\to0^+}(1/2\tau)e^{-t/\tau}, & \text{if } t\geq0,\\ 0, & \text{otherwise},\end{cases} \qquad (2\cdot26)$$

and $\int_0^\infty f(t)\delta_+(t)\,dt=f(0+)/2$. If $\xi_{ij}(a,a')=\xi_{ji}(a',a)$, then (2·25b) becomes the conventional form $2\xi_{ij}(a,a')\delta(t-t')$. One important assumption involved here is that $S_i(a,t)$ are independent of the initial value $A(0)$ of the relevant variables. Then, as will be shown in Appendix B, (2·11) reduces to

$$\frac{\partial}{\partial t}\Pi_a(t)=\Big[-\frac{\partial}{\partial a_i}H_i(a,t)+\frac{\partial^2}{\partial a_i\partial a_j}E_{ij}(a)\Big]\Pi_a(t)-\frac{\partial}{\partial a_i}F_{ia}(t), \qquad (2\cdot27)$$

where

$$H_i(a,t)=V_i(a,t)+\Big[\frac{\partial}{\partial b_j}\xi_{ij}(b,a)\Big]_{b=a}, \qquad (2\cdot28)$$

$$E_{ij}(a)=\xi_{ij}(a,a). \qquad (2\cdot29)$$

Therefore, the master equation (2·22) reduces to the Fokker-Planck equation

$$\frac{\partial}{\partial t}P(a,t)=\Big[-\frac{\partial}{\partial a_i}H_i(a,t)+\frac{\partial^2}{\partial a_i\partial a_j}E_{ij}(a)\Big]P(a,t), \qquad (2\cdot30)$$

and the generalized Langevin equation (2·23) reduces to the usual form

$$dA_i(t)/dt=H_i(A(t),t)+R_i(t), \tag{2·31}$$

where

$$\langle R_i(t);a_0\rangle=0, \tag{2·32}$$

$$\langle R_i(t)R_j(0);a_0\rangle=2E_{ij}(a_0)\delta(t). \quad (t\geqq 0) \tag{2·33}$$

Equations (2·30) and (2·31) describe the mean values and fluctuations of the relevant variables $A_i(t)$ in the conventional way.

## § 3. Projector elimination from the Langevin equations

Let us consider the following model equations:

$$dA_i(t)/dt=v_i(A(t))-\alpha_{ij}(A(t))B_j(t)+r_i^A(t), \tag{3·1}$$

$$dB_j(t)/dt=\beta_j(A(t))-\gamma_{jk}(A(t))B_k(t)+r_j^B(t), \tag{3·2}$$

where $r_i^A(t)$, $r_j^B(t)$ are Gaussian white noises with mean values zero and correlations

$$\langle r_i^A(t)r_l^A(t');a_0b_0\rangle=2L_{il}^{AA}\delta(t-t'),\quad L_{il}^{AA}=L_{li}^{AA}, \tag{3·3}$$

$$\langle r_i^A(t)r_j^B(t');a_0b_0\rangle=2L_{ij}^{AB}\delta(t-t'),\quad L_{ij}^{AB}=L_{ji}^{BA}, \tag{3·4}$$

$$\langle r_j^B(t)r_k^B(t');a_0b_0\rangle=2L_{jk}^{BB}\delta(t-t'),\quad L_{jk}^{BB}=L_{kj}^{BB}. \tag{3·5}$$

Here $\langle\cdots;a_0b_0\rangle$ denotes the conditional average with the initial values $A(0)=a_0$ and $B(0)=b_0$. This is a generalization of the Haken-Zwanzig model for non-equilibrium phase transitions[9] which has been treated in I. We assume that $A(t)$ and $B(t)$ are relevant and irrelevant variables, respectively, and the time scale $\tau_A$ of $A(t)$ is distinctly larger than the time scale $\tau_B$ of $B(t)$. Our problem is to eliminate $B(t)$ and to derive a reduced equation of motion for $A(t)$.

In this section we use the projector method developed by Fujisaka and Mori.[8] Statistical properties of fluctuating forces $r_i^A(t)$, $r_j^B(t)$ do not depend on the initial values of $A(0)$, $B(0)$. Therefore, (2·27) leads to

$$\partial\Pi_{ab}(t)/\partial t=\Gamma^+(a,b)\Pi_{ab}(t)+F_{ab}(t), \tag{3·6}$$

where $\Pi_{ab}(t)\equiv\delta(A(t)-a)\delta(B(t)-b)$, and $\Gamma^+$ is the adjoint operator of $\Gamma(a,b)$,

$$\Gamma(a,b)\equiv[v_i(a)-\alpha_{ij}(a)b_j](\partial/\partial a_i)+[\beta_j(a)-\gamma_{jk}(a)b_k](\partial/\partial b_j)$$
$$+L_{il}^{AA}\frac{\partial^2}{\partial a_i\partial a_l}+2L_{ij}^{AB}\frac{\partial^2}{\partial a_i\partial b_j}+L_{jk}^{BB}\frac{\partial^2}{\partial b_j\partial b_k}, \tag{3·7}$$

and $F_{ab}(t)$ is the master fluctuating force related to $r_i^A(t)$, $r_j^B(t)$ by

$$\iint a_i F_{ab}(t)\, da\, db = r_i^A(t), \quad \iint b_j F_{ab}(t)\, da\, db = r_j^B(t). \tag{3·8}$$

Let $q(b;a_0)$ be the conditional probability density that at $t=0$ $B(0)$ takes a value $b$ when $A(0)$ takes a value $a_0$, and let us introduce the projector $P$ by

$$PG(a,b) = \langle G(a,b)\,;\, a\rangle \equiv \int db\, G(a,b)\, q(b;a) \tag{3·9}$$

for an arbitrary function $G(a,b)$. We have $P^2=P$ since $Pg(a)=g(a)$ for an arbitrary function $g(a)$. Then, as will be shown in Appendix C, we obtain the following reduced equation for $\Pi_a(t) \equiv \delta(A(t)-a)$:[8]

$$\frac{\partial}{\partial t}\Pi_{a'}(t) = \int da \langle \Gamma(a,b)\,\delta(a-a')\,;\, a\rangle \Pi_a(t)$$
$$+ \int_0^t d\tau \int da \langle \Gamma(a,b)\, f_{a'}(a,b,\tau)\,;\, a\rangle \Pi_a(t-\tau) + \tilde{F}_{a'}(t), \tag{3·10}$$

where

$$f_{a'}(a,b,t) \equiv e^{tQ\Gamma(a,b)} Q\Gamma(a,b)\,\delta(a-a'), \quad (Q \equiv 1-P) \tag{3·11}$$

$$\tilde{F}_{a'}(t) \equiv f_{a'}(A(0), B(0), t) + \int db\, F_{a'b}(t)$$
$$+ \int_0^t ds \iint da\, db\, F_{ab}(t-s)\, f_{a'}(a,b,s). \tag{3·12}$$

Since $A_i(t) = \int a_i' \Pi_{a'}(t)\, da'$, (3·10) leads to

$$dA_i(t)/dt = \langle \Gamma(a,b)\, a_i;\, A(t)\rangle$$
$$+ \int_0^t d\tau \langle \Gamma(a,b)\, q_i(a,b,\tau)\,;\, A(t-\tau)\rangle + \tilde{R}_i(t), \tag{3·13}$$

where

$$q_i(a,b,t) \equiv e^{tQ\Gamma(a,b)} Q\Gamma(a,b)\, a_i, \tag{3·14}$$

$$\tilde{R}_i(t) \equiv q_i(A(0), B(0), t) + r_i^A(t) + \int_0^t ds \iint da\, db\, F_{ab}(t-s)\, q_i(a,b,s). \tag{3·15}$$

Since $\Gamma(a,b)\, a_i = v_i(a) - \alpha_{ij}(a)\, b_j$, (3·13) is a transformation of (3·1) which takes into account the renormalization due to the mode couplings to the $B$ modes.

Since the time scale of $A_i(t)$ is distinctly larger than that of $B_j(t)$, we may expand the memory term of (3·10) in powers of the slowness parameter $\delta \equiv \tau_B/\tau_A \ll 1$ similarly to § 4 of I. Then, as will be shown in Appendix D, (3·10) is reduced, to order $\delta^2$, to

$$\frac{\partial}{\partial t}\Pi_a(t)=\left[-\frac{\partial}{\partial a_i}H_i(a)+\frac{\partial^2}{\partial a_i\partial a_l}E_{il}(a)\right]\Pi_a(t)+\tilde{F}_a(t), \qquad (3\cdot16)$$

where

$$\begin{aligned}H_i(a)&=v_i(a)-\alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}\beta_k(a)\\&\quad+\alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}\left\{\frac{\partial}{\partial a_l}[\gamma^{-1}(a)]_{kp}\beta_p(a)\right\}\\&\quad\times\{v_l(a)-\alpha_{lm}(a)[\gamma^{-1}(a)]_{mn}\beta_n(a)\}\\&\quad+\left\{\frac{\partial}{\partial a_l}\alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}\right\}\{\alpha_{lm}(a)\langle\hat{b}_k\hat{b}_m;a\rangle-2L_{lk}^{AB}\},\end{aligned} \qquad (3\cdot17)$$

$$E_{il}(a)=L_{il}^{AA}+\alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}\{\alpha_{lm}(a)\langle\hat{b}_k\hat{b}_m;a\rangle-2L_{lk}^{AB}\} \qquad (3\cdot18)$$

with

$$\hat{b}_j=b_j-\langle b_j;a\rangle, \qquad (3\cdot19)$$

$$\langle b_j;a\rangle=[\gamma^{-1}(a)]_{jk}\beta_k(a)+O(\delta). \qquad (3\cdot20)$$

Therefore, (3·13) reduces to the Langevin equation

$$dA_i(t)/dt=H_i(A(t))+\tilde{R}_i(t). \qquad (3\cdot21)$$

Equations (3·17) and (3·18) agree with the previous results (4·13) and (4·17) of I if $\gamma(a)$ is a constant diagonal matrix and $L_{lk}^{AB}$ are negligible.

The degrees-of-freedom involved in the fluctuating forces $\tilde{F}_a(t)$ and $\tilde{R}_i(t)$ are $A(0)$, $B(0)$ and external degrees-of-freedom $\Omega$ associated with $F_{ab}(t)$. Let $\langle\cdots;a_0\rangle$ also denote the conditional average over $A(0)$, $B(0)$ and $\Omega$ with $A(0)$ being fixed so as to be $a_0$. Then, as will be shown in Appendix C, we have

$$\langle\tilde{F}_a(t);a_0\rangle=\langle\tilde{R}_i(t);a_0\rangle=0. \qquad (3\cdot22)$$

Therefore (3·16) leads to the following Fokker-Planck equation:

$$\begin{aligned}(\partial/\partial t)P(a,t)&=-(\partial/\partial a_i)[I_i(a)P(a,t)]\\&\quad+(\partial/\partial a_i)\{\alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}(\partial/\partial a_l)[\alpha_{lm}(a)\langle\hat{b}_k\hat{b}_m;a\rangle\\&\quad-2L_{lk}^{AB}]P(a,t)\}+L_{il}^{AA}(\partial^2/\partial a_i\partial a_l)P(a,t),\end{aligned} \qquad (3\cdot23)$$

where $I_i(a)$ represents the first three terms of (3·17).

## § 4. Adiabatic reduction of the Fokker-Planck equation

The Fokker-Planck equation corresponding to the Langevin equations (3·1) and (3·2) is given by

$$\partial P(a,b,t)/\partial t=\Gamma^{+}(a,b)P(a,b,t). \qquad (4\cdot1)$$

The probability density $P(a, t)$ is given by

$$P(a, t) = \int db P(a, b, t). \tag{4·2}$$

Let us put

$$P(a, b, t) = P(a, t) q(b|a, t), \tag{4·3}$$

$$\int db q(b|a, t) = 1. \tag{4·4}$$

Then, integrating (4·1) over $b$ leads to

$$\frac{\partial}{\partial t} P(a, t) = \left[-\frac{\partial}{\partial a_i}[v_i(a) - \alpha_{ij}(a)\langle b_j|a, t\rangle] + \frac{\partial^2}{\partial a_i \partial a_l} L_{il}^{AA}\right] P(a, t), \tag{4·5}$$

where, for an arbitrary function $G(b)$,

$$\langle G(b)|a, t\rangle \equiv \int db G(b) q(b|a, t). \tag{4·6}$$

From (4·1), (4·3) and (4·5), we obtain

$$\begin{aligned}\frac{\partial}{\partial t} q(b|a, t) = &\left\{-\frac{\partial}{\partial b_j}[\beta_j(a) - \gamma_{jk}(a) b_k] + \frac{\partial^2}{\partial b_j \partial b_k} L_{jk}^{BB}\right\} q(b|a, t) \\ &+ \frac{1}{P(a, t)} \left\{\frac{\partial}{\partial a_i} \alpha_{ij}(a) \delta b_j(t) + 2 \frac{\partial^2}{\partial a_i \partial b_j} L_{ij}^{AB}\right\} P(a, t) q(b|a, t) \\ &- [v_i(a) - \alpha_{ij}(a)\langle b_j|a, t\rangle] \frac{\partial}{\partial a_i} q(b|a, t),\end{aligned} \tag{4·7}$$

where

$$\delta b_j(t) \equiv b_j - \langle b_j|a, t\rangle. \tag{4·8}$$

In (4·7), each term is of order $\delta^0$ or $\delta$, and, since $L_{il}^{AA} \sim O(\delta^2)$, we have neglected the terms with $L_{il}^{AA}$. Then (4·7) leads to

$$\begin{aligned}\partial\langle b_j|a, t\rangle/\partial t = &\beta_j(a) - \gamma_{jk}(a)\langle b_k|a, t\rangle \\ &- [v_l(a) - \alpha_{lm}(a)\langle b_m|a, t\rangle] \left\{\frac{\partial}{\partial a_l}\langle b_j|a, t\rangle\right\} \\ &+ \frac{1}{P(a, t)} \frac{\partial}{\partial a_l}\{\alpha_{lm}(a) \chi_{jm}(a, t) - 2L_{lj}^{AB}\} P(a, t),\end{aligned} \tag{4·9}$$

where

$$\chi_{jm}(a, t) \equiv \langle \delta b_j(t) \delta b_m(t)|a, t\rangle. \tag{4·10}$$

Integrating (4·9) leads to

$$\langle b_j|a,t\rangle=[e^{-t\gamma}]_{jk}\langle b_k|a,0\rangle+\int_0^t ds[e^{-s\gamma}]_{jk}$$
$$\times\Big[\beta_k(a)-\{v_l(a)-\alpha_{lm}(a)\langle b_m|a,t-s\rangle\}\left\{\frac{\partial}{\partial a_l}\langle b_k|a,t-s\rangle\right\}$$
$$+\frac{1}{P(a,t-s)}\frac{\partial}{\partial a_l}\{\alpha_{lm}(a)\chi_{km}(a,t-s)-2L_{lk}^{AB}\}P(a,t-s)\Big]. \tag{4·11}$$

For $t\gg\tau_B$, we thus obtain, to order $\delta$,

$$\langle b_j|a,t\rangle=[\gamma^{-1}(a)]_{jk}\Big[\beta_k(a)-\{v_l(a)-\alpha_{lm}(a)[\gamma^{-1}(a)]_{mn}\beta_n(a)\}$$
$$\times\left\{\frac{\partial}{\partial a_l}[\gamma^{-1}(a)]_{kp}\beta_p(a)\right\}$$
$$+\frac{1}{P(a,t)}\frac{\partial}{\partial a_l}\{\alpha_{lm}(a)\chi_{km}(a,\infty)-2L_{lk}^{AB}\}P(a,t)\Big]. \tag{4·12}$$

From (4·7) and (4·9) we also obtain, to order $\delta^0$,

$$\partial\langle b_kb_m|a,t\rangle/\partial t=\beta_k(a)\langle b_m|a,t\rangle+\beta_m(a)\langle b_k|a,t\rangle$$
$$-\gamma_{kl}(a)\langle b_lb_m|a,t\rangle-\gamma_{ml}(a)\langle b_kb_l|a,t\rangle+2L_{km}^{BB}, \tag{4·13}$$

$$\partial\chi_{km}(a,t)/\partial t=-\gamma_{kl}(a)\chi_{lm}-\gamma_{ml}(a)\chi_{kl}+2L_{km}^{BB}, \tag{4·14}$$

which lead to the fluctuation-dissipation relation[10)]

$$\chi_{km}(a,\infty)=2\int_0^\infty[e^{-s\gamma(a)}]_{ki}L_{ij}^{BB}[e^{-s\gamma(a)}]_{mj}ds. \tag{4·15}$$

Therefore, inserting (4·12) into (4·5), we obtain, to order $\delta^2$, the Fokker-Planck equation (3·23) in which the variances $\langle\hat{b}_k\hat{b}_m;a\rangle$ are given by (4·15).

## § 5. Adiabatic elimination from the Langevin equations

Let us consider the Langevin equations (3·1) and (3·2). The conventional elimination of $B_j(t)$ assumes $dB_j/dt=0$, which leads to

$$B_j(t)=[\gamma^{-1}(A(t))]_{jk}\{\beta_k(A(t))+r_k^B(t)\}. \tag{5·1}$$

Inserting this into (3·1) leads to

$$dA_i(t)/dt=v_i(A(t))-\alpha_{ij}(A(t))[\gamma^{-1}(A(t))]_{jk}\beta_k(A(t))$$
$$-\alpha_{ij}(A(t))[\gamma^{-1}(A(t))]_{jk}r_k^B(t)+r_i^A(t). \tag{5·2}$$

This approximation is unsatisfactory as follows. Since (5·2) has the form (2·1), we obtain the Fokker-Planck equation (2·30) with

$$H_i(a,t)=v_i(a)-\alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}\beta_k(a)$$
$$+\left\{\frac{\partial}{\partial a_l}\alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}\right\}\{L^{BB}_{kn}[\gamma^{-1}(a)]_{mn}\alpha_{lm}(a)-L^{BA}_{kl}\}, \quad (5\cdot3)$$

$$E_{il}(a)=L^{AA}_{il}+\alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}L^{BB}_{kn}[\gamma^{-1}(a)]_{mn}\alpha_{lm}(a)$$
$$-\alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}L^{BA}_{kl}-\alpha_{lk}(a)[\gamma^{-1}(a)]_{kj}L^{AB}_{ij}. \quad (5\cdot4)$$

This result differs from that of § 4, and implies that (5·1) does not treat the random motion of $B_j(t)$ due to the fluctuating forces properly. This will be improved in the following.

Integrating (3·2) formally leads to

$$B_j(t)=\left[\exp_+\left\{-\int_0^t\gamma(A(s))ds\right\}\right]_{jk}B_k(0)$$
$$+\int_0^t d\tau\left[\exp_-\left\{-\int_0^\tau\gamma(A(t-s))ds\right\}\right]_{jk}\left\{\beta_k(A(t-\tau))+r^B_k(t-\tau)\right\}. \quad (5\cdot5)$$

Since, up to order $\delta$,

$$\left[\exp_-\left\{-\int_0^\tau\gamma(A(t-s))ds\right\}\right]_{jk}$$
$$=\left[\exp_-\left\{-\int_0^\tau\left[\gamma(a)-\int_0^s\partial\gamma(a)/\partial a_l\dot{A}_l(t-s')ds'\right]ds+O(\delta^2)\right\}\right]_{jk}$$
$$=[e^{-\tau\gamma(a)}]_{jk}+\int_0^\tau ds\int_0^s ds'[e^{-s\gamma(a)}]_{jm}\partial\gamma_{mn}(a)/\partial a_l[e^{-(\tau-s)\gamma(a)}]_{nk}\dot{A}_l(t-s'),$$

$$\beta_k(A(t-\tau))=\beta_k(a)-\int_0^\tau ds\,\partial\beta_k(a)/\partial a_l\dot{A}_l(t-s)$$

with $a\equiv A(t)$, we have for $t\gg\tau_B$

$$B_j(t)=\int_0^\infty d\tau[e^{-\tau\gamma(a)}]_{jk}\left\{\beta_k(a)+r^B_k(t-\tau)-\partial\beta_k(a)/\partial a_l\int_0^\tau ds\dot{A}_l(t-s)\right\}$$
$$+\int_0^\infty d\tau\int_0^\tau ds\int_0^s ds'[e^{-s\gamma(a)}]_{jm}\partial\gamma_{mn}(a)/\partial a_l[e^{-(\tau-s)\gamma(a)}]_{nk}\dot{A}_l(t-s')$$
$$\times\{\beta_k(a)+r^B_k(t-\tau)\}+O(\delta^2). \quad (5\cdot6)$$

Then we also have, from (3·1),

$$\dot{A}_l(t-s)=v_l(a)-\alpha_{lm}(a)[\gamma^{-1}(a)]_{mn}\beta_n(a)$$
$$-\alpha_{lm}(a)\int_0^\infty d\sigma[e^{-\sigma\gamma(a)}]_{mn}r_n{}^B(t-s-\sigma)+r_l{}^A(t-s)+O(\delta^2). \quad (5\cdot7)$$

Inserting (5·6) with (5·7) into (3·1), we obtain an equation of the form (2·1).

Taking its conditional average and using (3·4) and (3·5), we have, to order $\delta^2$,

$$V_i(a,t)=v_i(a)-\alpha_{ij}(a)\int_0^\infty d\tau[e^{-\tau\gamma(a)}]_{jk}[\beta_k(a)-\tau\partial\beta_k(a)/\partial a_l$$
$$\times\{v_l(a)-\alpha_{lm}(a)[\gamma^{-1}(a)]_{mn}\beta_n(a)\}]$$
$$+\alpha_{ij}(a)\int_0^\infty d\tau\int_0^\tau ds\int_0^s ds'[e^{-s\gamma(a)}]_{jm}\partial\gamma_{mn}(a)/\partial a_l[e^{-(\tau-s)\gamma(a)}]_{nk}$$
$$\times\Big[\beta_k(a)\{v_l(a)-\alpha_{lp}(a)[\gamma^{-1}(a)]_{pq}\beta_q(a)\}$$
$$-2\alpha_{lp}(a)\int_0^\infty d\sigma[e^{-\sigma\gamma(a)}]_{pq}L^{BB}_{qk}\delta(\tau-s'-\sigma)+2L^{BA}_{kl}\delta(\tau-s')\Big].$$

This turns out to be

$$V_i(a,t)=v_i(a)-\alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}$$
$$\times\Big[\beta_k(a)-\{v_l(a)-\alpha_{lm}(a)[\gamma^{-1}(a)]_{mn}\beta_n(a)\}$$
$$\times\Big\{\frac{\partial}{\partial a_l}[\gamma^{-1}(a)]_{kp}\beta_p(a)\Big\}\Big]-\alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}\alpha_{lm}(a)$$
$$\times\Big[\frac{\partial}{\partial a_l}\tilde{\chi}_{km}(a,a')\Big]_{a'=a}+O(\delta^3), \tag{5·8}$$

where

$$\tilde{\chi}_{km}(a,a')\equiv 2\int_0^\infty d\tau[e^{-\tau\gamma(a)}]_{ki}L^{BB}_{ij}[e^{-\tau\gamma(a')}]_{mj}, \tag{5·9}$$

$$\Big[\frac{\partial}{\partial a_l}\tilde{\chi}_{km}(a,a')\Big]_{a'=a}=-\int_0^\infty du[e^{-u\gamma(a)}]_{ki}\frac{\partial\gamma_{ij}{}^{(a)}}{\partial a_l}\tilde{\chi}_{jn}(a,a)[e^{-u\gamma(a)}]_{mn}. \tag{5·10}$$

We may retain terms up to order $\delta$ in the fluctuating forces because they already give the second-order contribution to the Fokker-Planck equation for $A(t)$. Then we obtain

$$S_i(a,t)=r_i{}^A(t)-\alpha_{ij}(a)\int_0^\infty d\tau[e^{-\tau\gamma(a)}]_{jk}r_k{}^B(t-\tau). \tag{5·11}$$

Since $S_i(a, t)$ are the linear transformation of Gaussian processes $r_i{}^A(t)$ and $r_k{}^B(t)$, $S_i(a,t)$ are also Gaussian processes. Their time correlation functions are given, from (5·11) and (3·3)$\sim$(3·5), by

$$\langle S_i(a,t)S_l(a',t');a_0\rangle=2L^{AA}_{il}\delta(t-t')-2\alpha_{ij}(a)[e^{-(t-t')\gamma(a)}]_{jk}L^{BA}_{kl}$$
$$+\alpha_{ij}(a)[e^{-(t-t')\gamma(a)}]_{jk}\tilde{\chi}_{km}(a,a')\alpha_{lm}(a') \tag{5·12}$$

for $t>t'$. On the time scale of order $\tau_A(\gg\tau_B)$,

$$e^{-(t-t')\gamma(a)} \doteqdot 2\gamma^{-1}(a)\delta_+(t-t'). \tag{5·13}$$

Then (5·12) leads to (2·25) with

$$\xi_{il}(a,a') = L_{il}^{AA} - 2\alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}L_{kl}^{BA} + \alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}\tilde{\chi}_{km}(a,a')\alpha_{lm}(a'). \tag{5·14}$$

Although $\tilde{\chi}_{km}(a,a') = \tilde{\chi}_{mk}(a',a)$, the third term of (5·14) as well as the second term gives an asymmetric part of $\xi_{il}(a,a')$. Therefore, from (2·28)~(2·30), we obtain the Fokker-Planck equation (2·30) with

$$H_i(a,t) = \left[\delta_{i,l} + \alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}\left\{\frac{\partial}{\partial a_l}[\gamma^{-1}(a)]_{kp}\beta_p(a)\right\}\right] \times \{v_l(a) - \alpha_{lm}(a)[\gamma^{-1}(a)]_{mn}\beta_n(a)\} + \left\{\frac{\partial}{\partial a_l}\alpha_{ij}(a)[\gamma^{-1}(a)]_{jk}\right\}\{\tilde{\chi}_{km}(a,a)\alpha_{lm}(a) - 2L_{kl}^{BA}\} \tag{5·15}$$

and $E_{il}(a) = \xi_{il}(a,a)$. Since $\tilde{\chi}(a,a) = \chi(a,\infty)$, this result agrees with that of § 4. It is worth noting that, since (4·15) satisfies the matrix equation

$$\gamma^{-1}\cdot\chi + \chi\cdot(\gamma^T)^{-1} = 2\gamma^{-1}\cdot L^{BB}\cdot(\gamma^T)^{-1} \tag{5·16}$$

with $\gamma^T$ denoting the transpose of $\gamma$, the symmetric part of $\xi_{il}(a,a)$ agrees with (5·4) so that the conventional procedure gives the correct diffusion term though it gives an incorrect drift term.

## § 6. A single-mode laser model

In this section we consider a single mode laser interacting with two-level atoms, and discuss its stochastic equations by applying the foregoing results. If we assume exact resonance, then we have the following five equations for the complex slowly varying amplitude of the electromagnetic field $b$, the total atomic dipole moment $R$ and the total inversion $Z$:[6),7),9)]

$$db/dt = -\kappa b - igR + F(t), \tag{6·1a}$$

$$dR/dt = -\gamma_\perp R + 2igbZ + \Gamma(t), \tag{6·1b}$$

$$dZ/dt = -\gamma_\parallel(Z - Z_0) + ig(b^*R - bR^*) + \Gamma_z(t), \tag{6·1c}$$

and the conjugates of (6·1a) and (6·1b), where $\kappa$, $\gamma_\perp$, $\gamma_\parallel$ are relaxation rates, $g$ is a coupling constant, $Z_0$ is a pumping parameter and asterisks indicate the complex conjugate. Fluctuating forces $F$, $\Gamma$ and $\Gamma_z$ are assumed to be Gaussian white noises with $\langle F(t)\rangle = \langle\Gamma(t)\rangle = \langle\Gamma_z(t)\rangle = 0$ and

$$\langle F(t)F^*(t')\rangle = 2\kappa\bar{n}\delta(t-t'), \tag{6·2a}$$

$$\langle \Gamma(t)\Gamma^*(t')\rangle = 2\gamma_\perp \overline{M}\delta(t-t'), \tag{6·2b}$$

$$\langle \Gamma_z(t)\Gamma_z(t')\rangle = \gamma_\| \overline{M}\delta(t-t') \tag{6·2c}$$

and all other correlations vanishing, where

$$\bar{n} = \langle b^* b\rangle_{eq}, \qquad \overline{M} = \langle R^* R\rangle_{eq} \tag{6·3}$$

with $\langle\cdots\rangle_{eq}$ denoting the equilibrium average.

Then (6·1) has the form of (3·1) and (3·2). If we set $A=\mathrm{Col}(b, b^*)$, $B=\mathrm{Col}(R, R^*, Z)$, then we have

$$v_i = \begin{pmatrix} -\kappa b \\ -\kappa b^* \end{pmatrix}, \qquad \alpha_{ij} = \begin{pmatrix} ig & 0 & 0 \\ 0 & -ig & 0 \end{pmatrix}, \tag{6·4a}$$

$$\beta_j = \begin{pmatrix} 0 \\ 0 \\ \gamma_\| Z_0 \end{pmatrix}, \qquad \gamma_{jk} = \begin{pmatrix} \gamma_\perp & 0 & -2igb \\ 0 & \gamma_\perp & 2igb^* \\ -igb^* & igb & \gamma_\| \end{pmatrix}, \tag{6·4b}$$

$$L_{il}^{AA} = \begin{pmatrix} 0 & \kappa\bar{n} \\ \kappa\bar{n} & 0 \end{pmatrix}, \qquad L_{jk}^{BB} = \begin{pmatrix} 0 & \gamma_\perp \overline{M} & 0 \\ \gamma_\perp \overline{M} & 0 & 0 \\ 0 & 0 & \gamma_\| \overline{M}/2 \end{pmatrix}, \tag{6·5}$$

and $L^{AB}=O$. Then we have

$$\alpha_{ij}[\gamma^{-1}(a)]_{jk}\beta_k = -\frac{\gamma_\| s Z_0}{2(1+sn)}\begin{pmatrix} b \\ b^* \end{pmatrix}, \tag{6·6}$$

where

$$s = 4g^2/\gamma_\perp\gamma_\|, \qquad n = b^* b\,. \tag{6·7}$$

Equation (6·6) gives the main drift term of $db/dt$ due to the mode couplings to $R$. Therefore, assuming that

$$\kappa/\gamma_\perp, \quad \kappa/\gamma_\|, \quad |\gamma_\| s Z_0/\gamma_\perp|, \quad |sZ_0| \ll 1, \tag{6·8}$$

we eliminate $(R, R^*, Z)$ and derive reduced equations for $(b, b^*)$ to order $\delta^2$, $\delta$ denoting the order of magnitude of the small parameters (6·8).

According to § 4, (4·15) and (6·5) lead to

$$\langle \hat{b}_k \hat{b}_m; a\rangle = \chi(a, \infty) = \begin{pmatrix} 0 & \overline{M} & 0 \\ \overline{M} & 0 & 0 \\ 0 & 0 & \overline{M}/2 \end{pmatrix}. \tag{6·9}$$

Therefore (3·23) leads to

$$\frac{\partial}{\partial t}P(b, b^*, t) = -\frac{\partial}{\partial b}\left\{-\kappa + \frac{\gamma_\| s Z_0}{2(1+sn)} + \frac{J(n)}{2n}\right\} bP$$

$$-\frac{1}{8}\gamma_{\|}s\overline{M}\frac{\partial}{\partial b}\left\{\frac{sb^2}{1+sn}\frac{\partial}{\partial b}-\frac{2+sn}{1+sn}\frac{\partial}{\partial b^*}\right\}P$$

$$+\kappa\bar{n}\frac{\partial^2 P}{\partial b\partial b^*}+\text{c.c.}, \tag{6·10}$$

where

$$J(n)=\frac{\gamma_{\|}sZ_0}{\gamma_\perp}\left[\kappa-\frac{\gamma_{\|}sZ_0}{2(1+sn)}\right]\frac{n[1-\{1+2(\gamma_\perp/\gamma_{\|})\}sn]}{(1+sn)^3}. \tag{6·11}$$

The $J(n)$ term comes from the third term of (3·17), and is of order $\delta^2$. The Langevin equation for $b$ is given by (3·21):

$$\frac{d}{dt}b=\left\{-\kappa+\frac{\gamma_{\|}sZ_0}{2(1+sn)}+\frac{J(n)}{2n}\right\}b$$

$$-\frac{1}{8}\gamma_{\|}s^2\overline{M}\frac{3+sn}{(1+sn)^2}b+\tilde{R}_b(t), \tag{6·12}$$

where $\tilde{R}_b(t)$ is a fluctuating force of the type (3·15).

Using

$$b=\sqrt{n}\,e^{i\phi},\qquad b^*=\sqrt{n}\,e^{-i\phi}, \tag{6·13}$$

we can transform (6·10) and (6·12) into the equations for the photon number $n=b^*b$ and the phase $\phi=(1/2i)\log(b/b^*)$. Since $\partial(n,\phi)/\partial(b,b^*)=i$, we have $P(n,\phi)\,dnd\phi=P(b,b^*)\,d^2b$. Thus we obtain

$$\frac{\partial}{\partial t}P(n,\phi,t)=-\frac{\partial}{\partial n}\left\{-2\kappa n+\frac{\gamma_{\|}sZ_0}{1+sn}n+J(n)\right\}P$$

$$+\frac{1}{2}\gamma_{\|}s\overline{M}\frac{\partial}{\partial n}\frac{n}{1+sn}\frac{\partial P}{\partial n}+2\kappa\bar{n}\frac{\partial}{\partial n}n\frac{\partial P}{\partial n}$$

$$+(\gamma_{\|}s\overline{M}+4\kappa\bar{n})\frac{1}{8n}\frac{\partial^2 P}{\partial\phi^2}. \tag{6·14}$$

The Langevin equation for $n$ takes the form

$$\frac{d}{dt}n=-2\kappa[n-\bar{n}]+\frac{\gamma_{\|}s}{1+sn}\left[Z_0 n+\frac{1}{2}\frac{\overline{M}}{1+sn}\right]+J(n)+\tilde{R}_n(t), \tag{6·15}$$

where $\tilde{R}_n(t)$ is a fluctuating force of the type (3·15).

These reduced equations differ from those which are obtained by the conventional adiabatic elimination (5·3) and (5·4). For example, in the Fokker-Planck equation for $P(b,b^*,t)$, though the diffusion terms are identical, the drift term in the conventional one does not have the $J(n)$ term in (6·10) but has an additional term $(\gamma_{\|}s\overline{M}/8)[sb/(1+sn)^2]$. In the Langevin equation for $n$, the conventional one does not produce the $J(n)$ term in (6·15) but produces an additional term $(\gamma_{\parallel}s\overline{M}/4)[sn/(1+sn)^2]$.

## § 7. Summary and remarks

We have formulated a projector elimination of irrelevant degrees-of-freedom in multiplicative stochastic process (2·1), and derived reduced equations of motion (2·23) and the corresponding master equation (2·22). Multiplicative stochastic processes of a new type are given by (5·11) which takes the memory form

$$S_i(a,t)=r_i^A(t)-\alpha_{ij}(a)\int_{-\infty}^{t}ds[e^{-(t-s)\gamma(a)}]_{jk}r_k^B(s) \tag{7·1}$$

with $r_i^A(t)$ and $r_k^B(t)$ being Gaussian white noises. On the time scale of order $\tau_A(\gg\tau_B)$, this leads to the form (2·25) with (5·14). Then (2·23) and (2·22) reduce to the Langevin equation (2·31) and the Fokker-Planck equation (2·30).

The asymmetric form (2·25) is a generalization of the conventional form $2\xi_{ij}(a,a')\delta(t-t')$. This generalization is indispensable since $\xi_{ij}(a,a')\neq\xi_{ji}(a',a)$. The asymmetry of the spectral density matrix arises from the memory effect due to $\tau_B\gg\tau_m$, $\tau_m$ being the time scale of $r_i^A(t)$ and $r_k^B(t)$. Consider (7·1). Then, in $\langle S_i(a,t)S_j(a',t');a_0\rangle$ with $t>t'$, contribution comes from $r_k^B(s)$ with $s\leqq t'$, whereas, if $t'>t$, contribution comes from $r_k^B(s')$ of $S_j(a',t')$ with $s'\leqq t$. The difference between the two contributions leads to the asymmetry. Here it is essential to distinguish *three* different time scales $\tau_A$, $\tau_B$ and $\tau_m$ with $\tau_A\gg\tau_B\gg\tau_m$, and then to take the limit $\tau_B\to 0$ with $\tau_B\gg\tau_m$ being kept. The conventional adiabatic elimination simply replaces the time integral of (7·1) by $[\gamma^{-1}(a)]_{jk}r_k^B(t)$ without distinguishing $\tau_B$ and $\tau_m$. In § 5, we have proposed its improvement.

In § 3 we have developed a projector elimination from coupled Langevin equations and obtained the drift term (3·17) and the spectral density (3·18). This is a generalization of the previous results derived in I with the aid of the projector elimination in dissipative dynamical systems.

The adiabatic elimination in deterministic equations[9] retains the first-order terms in the slowness parameter $\delta$. In §§ 4 and 5 we have retained up to order $\delta^2$ in order to take fluctuations into account. One of the most important features of the adiabatic elimination is to give $H_i(a)$ and $E_{il}(a)$ in terms of $L_{jk}^{BB}$ completely through the variance equations (4·15). This is different from the projector elimination which gives $H_i(a)$ and $E_{il}(a)$ in terms of the initial variances $\langle\hat{b}_k\hat{b}_m;a\rangle$. Since the variance equations (4·15) are the fluctuation-dissipation relations characteristic of the steady Gaussian distribution generated by the noise sources $L_{jk}^{BB}$, this means that the adiabatic elimination assumes the local equilibrium for the initial distribution of irrelevant variables.

The multiplicative stochastic equations with white noises have a mathematical ambiguity. The Itô interpretation and the Stratonovich interpretation are often used.[6),10)] Our results in § 2 agree with the Stratonovich interpretation. This is due to the fact that we have replaced the time correlation functions of the stochastic forces by those of white noises after transforming the stochastic equations into the master equation by means of the ordinary calculus.[10)] Thus (2·1) can be interpretated as the Stratonovich type stochastic differential equation, whereas the resulting Langevin equation (2·31) should be interpretated as the Itô type.


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## Appendix A

*——Derivation of (2·11) and (2·16)——*

Using the operator identity

$$U(b,t)=U_Q(b,t,0)+\int_0^t d\tau U(b,\tau)PL(b,\tau)U_Q(b,t,\tau), \tag{A·1}$$

we can rewrite (2·7) as

$$\begin{aligned}\frac{\partial}{\partial t}\Pi_a(t)=-\frac{\partial}{\partial a_i}\int db\,\delta(A(0)-b)\Big[U(b,t)P\\ +U_Q(b,t,0)Q+\int_0^t d\tau\,U(b,\tau)PL(b,\tau)U_Q(b,t,\tau)Q\Big]\\ \times[V_i(b,t)+S_i(b,t)]\delta(a-b).\end{aligned} \tag{A·2}$$

Since $PS_i(b,t)=QV_i(b,t)=0$, $QS_i(b,t)=S_i(b,t)$,

$$\int db\delta(A(0)-b)U(b,t)PF(b,t,\tau)=\int db\Pi_b(t)\langle F(b,t,\tau);b\rangle, \tag{A·3}$$

$PL(b,\tau)Q=PS_j(b,\tau)(\partial/\partial b_j)Q$, (A·2) leads to (2·11).

In order to derive (2·16), let us rewrite (2·13) as

$$F_{ia}(b,t,\tau)=\delta(a-b)R_i(b,t,\tau)+Y_{ia}(b,t,\tau), \tag{A·4}$$

where (2·20) has been used. Differentiating (A·4) with respect to $\tau$ leads to

$$\begin{aligned}\partial_\tau Y_{ia}(b,t,\tau)&=-QL(b,\tau)F_{ia}(b,t,\tau)+\delta(a-b)QL(b,\tau)R_i(b,t,\tau)\\ &=-QL_a(b,\tau)F_{ia}(b,t,\tau)-QM_a(b,\tau)Y_{ia}(b,t,\tau),\end{aligned} \tag{A·5}$$

where

$$L_a(b, \tau) \equiv -\frac{\partial}{\partial a_j}[V_j(b, \tau) + S_j(b, \tau)], \tag{A·6}$$

$$M_a(b, \tau) \equiv L(b, \tau) - L_a(b, \tau). \tag{A·7}$$

Since $Y_{ia}(b, t, t) = 0$, integrating (A·5) and inserting the $Y_{ia}(b, t, \tau)$ thus obtained into (A·4), we obtain

$$F_{ia}(b, t, \tau) = \delta(a-b) R_i(b, t, \tau) + \int_\tau^t ds \exp_-\left[\int_\tau^s QM_a(b, u) du\right] \times QL_a(b, s) F_{ia}(b, t, s). \tag{A·8}$$

Iterating this, we obtain

$$F_{ia}(b, t, \tau) = \sum_{n=0}^\infty \int_\tau^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \exp_-\left[\int_\tau^{s_1} QM_a(b, u)\right] QL_a(b, s_1) \times \cdots \exp_-\left[\int_{s_{n-1}}^{s_n} QM_a(b, u) du\right] QL_a(b, s_n) \delta(a-b) R_i(b, t, \tau). \tag{A·9}$$

Here each $QL_a(b, s)$ gives $\partial/\partial a_k$, and we have

$$QM_a(b, s)\left[f \frac{\partial}{\partial a_{k_1}} \cdots \frac{\partial}{\partial a_{k_n}} \delta(a-b)\right] = \left[\frac{\partial}{\partial a_{k_1}} \cdots \frac{\partial}{\partial a_{k_n}} \delta(a-b)\right] QL(b, s) f, \tag{A·10}$$

where $f$ is an arbitrary quantity which does not depend on $a$. Hence we obtain

$$F_{ia}(b, t, \tau) = \sum_{n=0}^\infty (-1)^n \frac{\partial}{\partial a_{k_1}} \cdots \frac{\partial}{\partial a_{k_n}} \delta(a-b) \mathscr{S}(i, k_1, \cdots, k_n, b, t, \tau). \tag{A·11}$$

Inserting this into (2·11), we obtain (2·16).

## Appendix B

*——Derivation of (2·27)——*

Using the interaction picture, we have

$$U_Q(b, t, \tau) Q = U_0^{-1}(b, \tau) \tilde{U}_Q(b, t, \tau) U_0(b, t) Q, \tag{B·1}$$

where

$$U_0(b, t) \equiv \exp_-\left[\int_0^t V_j(b, s) \frac{\partial}{\partial b_j} ds\right], \tag{B·2}$$

$$\tilde{U}_Q(b, t, \tau) \equiv \exp_-\left[\int_\tau^t Q\mathcal{L}(b, s) ds\right], \tag{B·3}$$

$$\mathcal{L}(b,s) \equiv U_0(b,s) S_j(b,s) \frac{\partial}{\partial b_j} U_0^{-1}(b,s), \tag{B·4}$$

and we have used the fact that

$$V_j(b,s)\,(\partial/\partial b_j) Q = Q V_j(b,s)\,(\partial/\partial b_j). \tag{B·5}$$

Then the integrand of the second term of (2·11) becomes

$$\begin{aligned}
&\left\langle S_j(b,\tau) \frac{\partial}{\partial b_j} F_{ia}(b,t,\tau)\,;\, b\right\rangle \\
&= \left\langle S_j(b,\tau) \frac{\partial}{\partial b_j} U_0^{-1}(b,\tau) \tilde{U}_Q(b,t,\tau) U_0(b,t) S_i(a,t)\,;\, b\right\rangle \delta(a-b) \\
&= \sum_{n=0}^{\infty} \int_\tau^t ds_1 \int_\tau^{s_1} ds_2 \cdots \int_\tau^{s_{n-1}} ds_n \left\langle S_j(b,\tau) \frac{\partial}{\partial b_j} U_0^{-1}(b,\tau) Q \mathcal{L}(b,s_n)\right. \\
&\quad \times \cdots Q\mathcal{L}(b,s_1) U_0(b,t) S_i(a,t)\,;\, b > \delta(a-b) \\
&= \sum_{m=1}^{\infty} \int_\tau^t ds_1 \int_\tau^{s_1} ds_2 \cdots \int_\tau^{s_{2m-1}} ds_{2m} \left\langle S_j(b,\tau) \frac{\partial}{\partial b_j} U_0^{-1}(b,\tau) \mathcal{L}(b,s_{2m})\right. \\
&\quad \times Q\mathcal{L}(b,s_{2m-1}) \mathcal{L}(b,s_{2m-2}) Q\mathcal{L}(b,s_{2m-3}) \cdots \\
&\quad \times \mathcal{L}(b,s_2) Q \mathcal{L}(b,s_1) U_0(b,t) S_i(a,t)\,;\, \left. b\right\rangle \delta(a-b),
\end{aligned} \tag{B·6}$$

where we have used the fact that the odd order correlations of $S_i(b,t)$ vanish. The integrands of (B·6) are written in terms of products of the double correlations of $S_i(b,t)$ and are proportional to products of $(m+1)$ delta functions, but do not include the terms proportional to

$$\delta(t-s_1)\,\delta(s_2-s_3) \cdots \delta(s_{2m}-\tau). \tag{B·7}$$

This can be shown by the mathematical induction. It can be shown, however, that products of $(m+1)$ delta functions except (B·7) vanish by the integration over time.[11)]

Hence the only non-vanishing term of (B·6) is the term with $m=0$. Then we obtain

$$\begin{aligned}
&\int_0^t d\tau \int db\, \Pi_b(\tau) \left\langle S_j(b,\tau) \frac{\partial}{\partial b_j} F_{ia}(b,t,\tau)\,;\, b\right\rangle \\
&\quad = \int_0^t d\tau \int db\, \Pi_b(\tau) \left\langle S_j(b,\tau) \frac{\partial}{\partial b_j} S_i(a,t)\,;\, b\right\rangle \\
&\quad \times U_0^{-1}(b,\tau) U_0(b,t)\, \delta(a-b) \\
&\quad = \int db\, \Pi_b(t)\, \xi_{ij}(a,b) \frac{\partial}{\partial b_j} \delta(a-b)
\end{aligned}$$

$$= -\frac{\partial}{\partial a_j}\xi_{ij}(a, a)\Pi_a(t) + \Pi_a(t)\left[\frac{\partial \xi_{ij}(a, b)}{\partial a_j}\right]_{b=a}. \tag{B·8}$$

Inserting this into (2·11), we obtain (2·27). Similarly we can show (2·33).

## Appendix C

*——Derivation of (3·10) and (3·22)——*

From (3·6) we obtain

$$\Pi_{ab}(t) = e^{t\Gamma^+(a,b)}\Pi_{ab}(0) + \int_0^t e^{s\Gamma^+(a,b)}F_{ab}(t-s)\,ds\,. \tag{C·1}$$

Then, for any functional $X(A(t), B(t))$, we have

$$\begin{aligned} X(A(t), B(t)) &= \int da \int db\, X(a, b)\Pi_{ab}(t) \\ &= \int da \int db \left\{\Pi_{ab}(0)\, e^{t\Gamma(a,b)} + \int_0^t ds\, F_{ab}(t-s)\, e^{s\Gamma(a,b)}\right\} \\ &\quad \times X(a, b), \end{aligned} \tag{C·2}$$

$$\begin{aligned} \frac{d}{dt}X(A(t), B(t)) = \int da \int db &\left\{\Pi_{ab}(0)\, e^{t\Gamma(a,b)} + \int_0^t ds F_{ab}(t-s)\, e^{s\Gamma(a,b)}\right\} \\ &\times \Gamma(a, b) X(a, b) + \int da \int db\, X(a, b) F_{ab}(t). \end{aligned} \tag{C·3}$$

Using the operator identity

$$e^{t\Gamma(a,b)} = e^{tQ\Gamma(a,b)} + \int_0^t d\tau\, e^{(t-\tau)\Gamma(a,b)} P\Gamma(a, b)\, e^{\tau Q\Gamma(a,b)} \tag{C·4}$$

and inserting $1 = P + Q$ in front of $\Gamma X$ in (C·3), we obtain[8]

$$\begin{aligned} \frac{d}{dt}X(A(t), B(t)) &= \int da \int db \Pi_{ab}(t) P\Gamma(a, b) X(a, b) \\ &+ \int da \int db\, \Pi_{ab}(0)\, e^{tQ\Gamma(a,b)} Q\Gamma(a, b) X(a, b) \\ &+ \int da \int db \int_0^t ds\, F_{ab}(t-s)\, e^{sQ\Gamma(a,b)} Q\Gamma(a, b) X(a, b) \\ &+ \int_0^t d\tau \int da \int db\, \Pi_{ab}(t-\tau) P\Gamma(a, b)\, e^{\tau Q\Gamma(a,b)} Q\Gamma(a, b) X(a, b) \\ &+ \int da \int db\, X(a, b) F_{ab}(t). \end{aligned} \tag{C·5}$$

Putting $X(a, b) = \delta(a - a')$, we obtain (3·10).

In order to derive (3·22), let $F(t)=F(A(0), B(0), \Omega, t)$ be an arbitrary function of $A(0)$, $B(0)$ and $\Omega$. Then we have

$$\langle F(t) ; a_0\rangle = \int da \int db \langle \Pi_{ab}(0) ; a_0\rangle f(a, b, t), \tag{C·6}$$

where $f(a, b, t)$ is the average of $F(a, b, \Omega, t)$ over $\Omega$ and have been assumed to be independent of $a_0$. Since

$$\begin{aligned}\langle \Pi_{ab}(0) ; a_0\rangle &= \int db_0 \langle \Pi_{ab}(0) ; a_0 b_0\rangle q(b_0; a_0) \\ &= \delta(a-a_0) q(b; a),\end{aligned} \tag{C·7}$$

(C·6) leads to

$$\begin{aligned}\langle F(t) ; a\rangle &= \int db f(a, b, t) q(b; a), \\ &= Pf(a, b, t).\end{aligned} \tag{C·8}$$

$\tilde{F}_a(t)$ and $\tilde{R}_i(t)$ have the form $QF(t)$ so that we have $Qf$ instead of $f$ in (C·8). Therefore $PQ=0$ leads to (3·22).

## Appendix D

*——Derivation of (3·16)——*

Since (3·7) leads to

$$\begin{aligned}\Gamma(a, b)\delta(a-a') &= -\frac{\partial}{\partial a_i'}[v_i(a') - \alpha_{ij}(a') b_j]\delta(a-a') \\ &\quad + \frac{\partial^2}{\partial a_i' \partial a_j'} L_{ij}^{AA}\delta(a-a'),\end{aligned} \tag{D·1}$$

$$\begin{aligned}\langle \Gamma(a, b)\delta(a-a') ; a\rangle &= -\frac{\partial}{\partial a_i'}[v_i(a') - \alpha_{ij}(a')\langle b_j; a\rangle]\delta(a-a') \\ &\quad + \frac{\partial^2}{\partial a_i' \partial a_j'} L_{ij}^{AA}\delta(a-a'),\end{aligned} \tag{D·2}$$

we obtain

$$f_{a'}(a, b, 0) = Q\Gamma(a, b)\delta(a-a') = \frac{\partial}{\partial a_i'}\alpha_{ij}(a')\hat{b}_j\delta(a-a'). \tag{D·3}$$

In the propagator of the memory term of (3·10), we treat $Q\Gamma$ similarly to (4·10) of I:

$$Q\Gamma \cong Q\hat{\Gamma} = Q\left\{[\beta_j(a) - \gamma_{jk}(a) b_k]\frac{\partial}{\partial b_j} + 2L_{ij}^{AB}\frac{\partial^2}{\partial a_i \partial b_j} + L_{ij}^{BB}\frac{\partial^2}{\partial b_i \partial b_j}\right\}. \tag{D·4}$$

Then we have

$$\int_0^\infty ds\langle \Gamma(a,b)\, e^{sQ\hat{\Gamma}(a,b)} f_{a'}(a,b,0)\,;\,a\rangle$$

$$= \frac{\partial}{\partial a_i'} \alpha_{ij}(a')\,[\gamma^{-1}(a')]_{jk}[\beta_k(a') - \gamma_{kl}(a')\langle b_l;\,a\rangle]\,\delta(a-a')$$

$$+ \frac{\partial}{\partial a_i'} \alpha_{ij}(a')\,[\gamma^{-1}(a')]_{jk} \frac{\partial}{\partial a_l'} \{\alpha_{lm}(a')\langle \hat{b}_m \hat{b}_k;\,a\rangle - 2L_{lk}^{AB}\}\,\delta(a-a')$$

$$- \frac{\partial}{\partial a_i'} \alpha_{ij}(a')\,[\gamma^{-1}(a')]_{jk} \frac{\partial\langle b_k;\,a'\rangle}{\partial a_l'} [v_l(a') - \alpha_{lm}(a')\langle b_m;\,a'\rangle]$$

$$\times\,\delta(a-a') + \frac{\partial}{\partial a_i'} \alpha_{ij}(a')\,[\gamma^{-1}(a')]_{jk} L_{lm}^{AA}$$

$$\times \left[2\frac{\partial\langle b_k;\,a'\rangle}{\partial a_l'} \frac{\partial}{\partial a_m'} + \frac{\partial^2\langle b_k;\,a'\rangle}{\partial a_l'\partial a_m'}\right]\delta(a-a'). \qquad (\mathrm{D}\cdot 5)$$

Since $L_{lm}^{AA} \sim O(\delta^2)$, the last term of (D·5) is of order $\delta^3$, but other terms are of order $\delta^2$. Hence we neglect the last term. Using the Markov approximation and inserting (D·2) and (D·5) into (3·10), we obtain (3·16).

### References


1) H. Mori, T. Morita and K. T. Mashiyama, Prog. Theor. Phys. **63** (1980), 1865. This will be referred to as I.
2) R. Kubo, in *Fluctuations, Relaxations and Resonance in Magnetic Systems*, ed. D. ter Haar (Oliver & Boyd, Edinburgh, 1962).
3) R. L. Stratonovich, *Topics in the Theory of Random Noise*, vol. 1 (Gordon and Breach, New York, 1963), pp. 100~103.
4) N. G. van Kampen, Phys. Reports **C24** (1976), 171.
5) K. Kitahara, W. Horsthemke and R. Lefever, Phys. Letters **A70** (1979), 377, and references cited therein.
6) H. Hasegawa and T. Nakagomi, J. Stat. Phys. **21** (1979), 191, and references cited therein.
7) A. Schenzle and H. Brand, Phys. Rev. **A20** (1979), 1628.
8) H. Fujisaka and H. Mori, Prog. Theor. Phys. **56** (1976), 754.
9) H. Haken, Rev. Mod. Phys. **47** (1975), 67.
10) R. F. Fox, Phys. Reports **48** (1978), 179.
11) R. F. Fox, J. Math. Phys. **13** (1972), 1196.