

Contractions of the irreducible representations of the quantum algebras $su_q(2)$ and $so_q(3)$

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The contractions of the irreducible representations of the unitary quantum algebra $su_q(2)$ and the orthogonal quantum algebra $so_q(3)$ in the Gel'fand-Tsetlin basis are regarded in detail with the help of the dual numbers.

1 Introduction

The purpose of this paper is to give a constructive description of the contractions and the analytic continuations of the irreducible representations of the unitary and the orthogonal quantum algebras of low dimensions. The development of the quantum inverse problem method [1] gives rise to the notion of the quantum group and quantum algebra [2]–[4]. This field is now under active investigation [5]–[7]. In particular, Jimbo [4] and Ueno et al. [8] have shown the existence of a q analog of the Gel'fand-Tsetlin basis for the quantum group $GL_q(n+1)$. The representation operators of $GL_q(n+1)$ are obtained from those of $GL(n+1)$ by replacing simple Gel'fand-Tsetlin multipliers with their q analogs.

In this paper we generalize, in the case of a quantum groups, the unified description of the Wigner-Inonu contractions and the analytic continuations (or Weyl unitary tneck) of the classical groups [9]–[11].

The paper is organized as follows. In Sec. II. we recall the unified description of the unitary Cayley-Klein algebras $su(2; j_1)$. Section III contains the definition of the quantum algebra $su_q(2; j_1)$, and the contraction and the analytic continuation of their irreducible representations. The different contractions of the quantum algebra $so_q(3; j)$ are discussed in Sec. IV.

2 The unitary Cayley-Klein algebras $su(2; j_1)$

Let us define the map of the two-dimensional complex space \mathbf{C}_2 into the complex space $\mathbf{C}_2(j_1)$ as follows:

$$\begin{aligned}\psi : \mathbf{C}_2 &\rightarrow \mathbf{C}_2(j_1), \\ \psi z_0^* &= z_0, \quad \psi z_1^* = j_1 z_1,\end{aligned}\tag{1}$$

where $z_0^*, z_1^* \in \mathbf{C}_2$, $z_0, z_1 \in \mathbf{C}_2(j_1)$ are the complex Cartesian coordinates and parameter j_1 may be equal to the real unit or to the Clifford dual unit ι_1 or to the imaginary unit i .

The dual numbers are not often used and we briefly mentioned their algebraic properties. Each of the dual units is not equal to zero: $\iota_k \neq 0$; a different dual unit obeys the commutative

law of multiplication $\iota_k \iota_m = \iota_m \iota_k \neq 0$, $k \neq m$; the square of a dual unit is always equal to zero $\iota_k^2 = 0$. Division of a real or complex number by a dual unit is not defined, but division of a dual unit by itself is equal to the real unit $\iota_k / \iota_k = 1$. A function of a dual argument is defined by its Taylor expansion.

The quadratic form $|z_0^*|^2 + |z_1^*|^2$ of \mathbf{C}_2 is transformed under the map (1) into the following quadratic form of $\mathbf{C}_2(j_1)$:

$$(\mathbf{z}, \mathbf{z}) = |z_0|^2 + j_1^2 |z_1|^2. \quad (2)$$

The unitary Cayley-Klein group $SU(2; j_1)$ is defined [9]–[11] as the group of all transformations in the space $\mathbf{C}_2(j_1)$ that keep the form (2) invariant and have the unit determinant. The map (1) induces the transformation of the group $SU(2)$ into the group $SU(2; j_1)$ and, respectively, transformation of the algebra $su(2)$ into the algebra $su(2; j_1)$. The generators J_\pm^*, J_3^* of $su(2)$ are transformed as follows:

$$J_\pm = j_1 J_\pm^*(\rightarrow), \quad J_3 = J_3^*(\rightarrow), \quad (3)$$

where, by $J_\pm^*(\rightarrow), J_3^*(\rightarrow)$ are denoted the Wigner-Inonu [12] singular-transformed generators and j_1 play the role of the zero-tending parameter, when j_1 is equal to a dual unit. The generators $J_\pm^*(\rightarrow), J_3^*(\rightarrow)$ are defined in different ways for different representations of $su(2)$. In particular, for the Gel'fand-Tsetlin representation of $su(2)$, these generators are specified by the transformation law of the components of the Gel'fand-Tsetlin pattern $|l^*, m^*\rangle$, namely, $l = j_1 l^*, m = m^*$. Then, Eqs. (3) give the following generators of $su(2; j_1)$:

$$\begin{aligned} J_\pm |l, m\rangle &= j_1 \sqrt{(j_1^{-1} l \mp m)(j_1^{-1} l \pm m + 1)} |l, m \pm 1\rangle = \\ &= \sqrt{(l \mp j_1 m)(l \pm j_1 m + j_1)} |l, m \pm 1\rangle, \\ J_3 |l, m\rangle &= m |l, m\rangle, \end{aligned} \quad (4)$$

which satisfy the commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = j_1^2 2J_3. \quad (5)$$

If we put $j_1 = \iota_1$ then we obtain the infinite-dimensional representation of the contracted algebra $su(2; \iota_1)$ in the form

$$J_\pm |l, m\rangle = l |l, m \pm 1\rangle, \quad J_3 |l, m\rangle = m |l, m\rangle, \quad l \geq 0, l \in \mathbf{R}, m \in \mathbf{Z}. \quad (6)$$

The infinite-dimensional representation of the analytic continued algebra $su(2; i) \equiv su(1, 1)$ is described by Eqs (4), with $j_1 = i$ and $l = a + ib \in \mathbf{C}$. The Hermitic condition for generators $(J_-)^+ = J_+$ gives the following irreducible representations of the pseudo-unitary algebra $su(1, 1)$:

(i) $l = a - \frac{i}{2}$, $m \in \mathbf{Z}$ — the principal series:

$$J_\pm |l, m\rangle = \sqrt{\left(\frac{1}{2} \pm m + ia\right)\left(\frac{1}{2} \pm m - ia\right)} |l, m \pm 1\rangle; \quad (7)$$

(ii) $l = ib$, $-1 < b < 0$, $m \in \mathbf{Z}$ — the supplementary series:

$$J_\pm |l, m\rangle = i \sqrt{(b \mp m)(b \pm m - 1)} |l, m \pm 1\rangle; \quad (8)$$

(iii) $l = ib$, $b \in \mathbf{N}$, $\mathbf{N} + \frac{1}{2}$, $m \in (b, b + 1, \dots)$ or $m \in \{-b, -b - 1, \dots\}$ — the discrete series with the generators J_\pm s in (8)

3 The quantum Cayley-Klein algebras $su_q(2; j_1)$

In accordance with Refs. 4 and 8 we replace the multipliers in the final expressions (4) for the generators J_{\pm} with their q analogs. Then we obtain the generators of the irreducible representations of the quantum algebras $su_q(2; j_1)$ in the form

$$\begin{aligned} J_{\pm}|l, m\rangle &= \sqrt{[l \mp j_1 m][l \pm j_1 m + j_1]}|l, m \pm 1\rangle, \\ J_3|l, m\rangle &= m|l, m\rangle, \end{aligned} \quad (9)$$

where the q analog of the number x is defined by

$$[x] = \frac{\sinh(xh)}{\sinh(h)}, \quad q = e^{2h}, \quad h \in \mathbf{R}. \quad (10)$$

The generators (9) satisfy the commutation relations

$$[J_3, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = [j_1][2j_1 J_3], \quad (11)$$

which for $j_1 = 1$ coincide with the well-known commutation relations of $su_q(2)$.

The following formulas for co-product Δ , co-unit ϵ and antipode γ define on the quantum algebras $su_q(2; j_1)$ the structure of a Hopf algebra:

$$\begin{aligned} \Delta(J_{\pm}) &= J_{\pm} \otimes e^{-hJ_3} + e^{hJ_3} \otimes J_{\pm}, \\ \Delta(J_3) &= J_3 \otimes 1 + 1 \otimes J_3, \\ \epsilon(J_{\pm}) &= \epsilon(J_3) = 0, \\ \gamma(J_3) &= -J_3, \\ \gamma(J_{\pm}) &= -e^{hJ_3} J_{\pm} e^{-hJ_3} = -J_{\pm} e^{\pm h}. \end{aligned} \quad (12)$$

From definition (10), we find the q analog of the dual unit

$$[\iota_1] = \frac{\sinh(\iota_1 h)}{\sinh(h)} = \frac{\iota_1}{[1]_h}, \quad [1]_h = \frac{\sinh(h)}{h}, \quad (13)$$

then the commutator $[J_+, J_-] = 0$ and the generators (9) in the form

$$J_{\pm}|l, m\rangle = [l]|l, m \pm 1\rangle, \quad J_3|l, m\rangle = m|l, m\rangle, \quad (14)$$

$$l \in \mathbf{R}, \quad l \geq 0, \quad m \in \mathbf{Z}$$

realize the irreducible representation of the quantum-contracted algebra $su_q(2; \iota_1)$.

If we put $j = i$, then we find

$$\begin{aligned} [i] &= \frac{\sinh(ih)}{\sinh(h)} = i \frac{\sin(h)}{\sinh(h)}, \\ [i]^2 &= -\frac{\sin^2(h)}{\sinh^2(h)} \neq -1, \end{aligned} \quad (15)$$

and the commutation relations (11) for $j_1 = i$ do not coincide with the following commutation relations:

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -[2J_3], \quad (16)$$

of the pseudo-unitary quantum algebra $su_q(1, 1)$. Therefore, the generators (9) for $j_1 = i$ do not describe the representation of $su_q(1, 1)$.

But we have the other possibility of replacing the multipliers with their q analogs in the intermediate expressions for the generators J_\pm (4), namely,

$$J_\pm |l, m\rangle = j_1 \sqrt{[j_1^{-1}l \mp m][j_1^{-1} \pm m + 1]} |l, m \pm 1\rangle, \quad (17)$$

$$J_3 |l, m\rangle = m |l, m\rangle.$$

Then, the generators (17) satisfy the commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = j_1^2 [2J_3]. \quad (18)$$

which, for $j_1 = i$, are exactly the commutation relations of $su_q(1, 1)$. In the case of $j_1 = i$, the component l is an arbitrary complex number $l \in \mathbf{C}$. The Hermitic requirement for generators $(J_-)^+ = J_+$ give the following series of the unitary irreducible representations of the quantum pseudo-unitary algebra $su_q(1, 1)$ (see also Refs. [13] and [14]):

(i) $l = a - \frac{i}{2}$, $0 \leq a \leq \pi/2h$, $m \in \mathbf{Z}$ — the principal series

$$J_\pm |l, m\rangle = \sqrt{[\frac{1}{2} \pm m + ia][\frac{1}{2} \pm m - ia]} |l, m \pm 1\rangle =$$

$$= \frac{1}{\sqrt{2} |\sinh(h)|} \sqrt{\cosh((1 \pm 2m)h) - \cos(2ah)} |l, m \pm 1\rangle; \quad (19)$$

(ii) $l = \pi/2h - i/2 + it$, $t > 0$, $m \in \mathbf{Z}, \mathbf{Z} + 1/2$ — the strange series, which do not have the classical analogs

$$J_\pm |l, m\rangle = \frac{1}{\sqrt{2} |\sinh(h)|} \sqrt{\cosh((1 \pm 2m)h) + \cosh(2th)} |l, m \pm 1\rangle; \quad (20)$$

(iii) $l = ib$, $-i < b < 0$, $m \in \mathbf{Z}$ — the supplementary series

$$J_\pm |l, m\rangle = \frac{1}{\sqrt{2} |\sinh(h)|} \sqrt{\cosh((1 \pm 2m)h) - \cosh((1 + 2b)h)} |l, m \pm 1\rangle; \quad (21)$$

(iv) $l = -ib$, $b \in \mathbf{N}, \mathbf{N} + 1/2$, $m \in \{b, b + 1, \dots\}$ or $m \in \{-b, -b - 1, \dots\}$ — the discrete series with the generators J_\pm as in (21).

It will be noted that formulas (17) are not applicable in the case of contraction since they are not defined for dual value of the parameter $j_1 = \iota_1$. Indeed, the generators (17) include the following undefined functions:

$$\iota_1 \left[\frac{l}{\iota_1} \right] = \iota_1 \frac{\sinh(lh/\iota_1)}{\sinh(h)} = \frac{\iota_1}{\sinh(h)} \left\{ \frac{lh}{\iota_1} + \frac{(lh)^3}{3!} \cdot \frac{1}{\iota_1^3} + \dots \right\}$$

$$= \frac{1}{\sinh(h)} \left\{ lh + \frac{(lh)^3}{3!} \cdot \frac{1}{\iota_1^2} + \dots \right\}. \quad (22)$$

As we have discussed previously, the generators given by Eqs. (9) must be used in the case of contractions (see also Refs. [15] and [16])

Therefore, the unified description of the contractions and analytic continuations of the classical groups (algebras) and their representations [9]–[11] is split on two different methods for the quantum groups (algebras) and their representations. The same is true in higher dimensions.

4 The orthogonal Cayley-Klein algebras $so_q(3; j)$

The $SU(2)$ group is the covering group of $SO(3)$ and the generators of the $so(3)$ algebra are connected with the generators of $su(2)$ by the equations

$$J_1^* = \frac{i}{2}(J_+^* - J_-^*), \quad J_2^* = \frac{1}{2}(J_+^* + J_-^*), \quad J_3^* = J_3^*. \quad (23)$$

The map

$$\begin{aligned} \psi : \mathbf{R}_3 &\rightarrow \mathbf{R}_3(\mathbf{j}), \\ \psi x_0^* &= x_0, \quad \psi x_1^* = j_1 x_1, \quad \psi x_2^* = j_1 j_2 x_2, \end{aligned} \quad (24)$$

where $x_k^* \in \mathbf{R}_3$, $x_k \in \mathbf{R}_3(\mathbf{j})$ are the Cartesian coordinates, and $\mathbf{j} = (j_1 j_2)$, $j_1 = 1, \iota_1, i, j_2 = 1, \iota_2, i$ transform the rotation group $SO(3)$ in \mathbf{R}_3 into the group $SO(3; \mathbf{j})$, i. e. the rotation group in $\mathbf{R}_3(\mathbf{j})$ keeps invariant the quadratic form $(\mathbf{x}, \mathbf{x}) = x_0^2 + j_1^2 x_1^2 + j_1^2 j_2^2 x_2^2$. The generators (23) of $so(3)$ are transformed under the map (24) as follows:

$$J_1 = j_1 J_1^*(\rightarrow), \quad J_2 = j_1 j_2 J_2^*(\rightarrow), \quad J_3 = j_2 J_3^*(\rightarrow) \quad (25)$$

and satisfy the commutation relations of the Cayley-Klein algebra $so(3; \mathbf{j})$:

$$[J_3, J_1] = iJ_2, \quad [J_2, J_3] = j_2^2 iJ_1, \quad [J_1, J_2] = j_1^2 iJ_3. \quad (26)$$

In accordance with the results of the previous section, the generators of the representations of the quantum Cayley-Klein algebras $so(3; \mathbf{j})$ in the case of contractions are obtained from the generators of $so(3)$ if we transform the components of Gel'fand-Tsetlin patterns as follows: $l = j_1 j_2 l^*$, $m = j_2 m^*$, then transform the generators according to Eqs. (25), and replace the simple multipliers with their q analogs. As a result, we have the generators

$$\begin{aligned} J_1 |l, m\rangle &= \frac{i}{2j_2} (\alpha^+ |l, m + j_2\rangle - \alpha^- |l, m - j_2\rangle), \\ J_2 |l, m\rangle &= \frac{1}{2} (\alpha^+ |l, m + j_2\rangle + \alpha^- |l, m - j_2\rangle), \\ J_3 |l, m\rangle &= m |l, m\rangle, \\ \alpha^\pm &= \{[l \mp j_1 m][l \pm j_1 m + j_1 j_2]\}^{1/2} \end{aligned} \quad (27)$$

with the commutation relations

$$\begin{aligned} [J_3, J_1] &= iJ_2, \quad [J_2, J_3] = j_2^2 iJ_1, \\ [J_1, J_2] &= \frac{i}{2j_2} [j_1 j_2][2j_1 J_3] \end{aligned} \quad (28)$$

of the quantum algebra $so_q(3; \mathbf{j})$.

The structure of a Hopf algebra on the quantum algebras $so_q(3; \mathbf{j})$ is defined by the following formulas for co-products Δ , co-unit ϵ , and antipode γ :

$$\begin{aligned}
\Delta(J_{1,2}) &= J_{1,2} \otimes e^{-hJ_3} + e^{hJ_3} \otimes J_{1,2}, \\
\Delta(J_3) &= J_3 \otimes 1 + 1 \otimes J_3, \\
\epsilon(J_1) &= \epsilon(J_2) = \epsilon(J_3) = 0, \\
\gamma(J_3) &= -J_3, \\
\gamma(J_1) &= -e^{hJ_3} J_1 e^{-hJ_3} = -\{J_1 \cosh(j_2 h) + i J_2 j_2^{-1} \sinh(j_2 h)\}, \\
\gamma(J_2) &= -e^{hJ_3} J_2 e^{-hJ_3} = -\{J_2 \cosh(j_2 h) - i J_1 j_2 \sinh(j_2 h)\}.
\end{aligned} \tag{29}$$

For $j_1 = \iota_1$, we obtain from Eqs. (27) the generators of $so_q(3; \iota_1, j_2)$ in the form

$$\begin{aligned}
J_1 |l, m\rangle &= \frac{i}{2} [l] (|l, m + j_2\rangle - |l, m - j_2\rangle), \\
J_3 |l, m\rangle &= m |l, m\rangle, \\
J_2 |l, m\rangle &= \frac{1}{2} [l] (|l, m + j_2\rangle + |l, m - j_2\rangle), \\
l &\geq 0, \quad l \in \mathbf{R} \quad m \in \mathbf{Z}
\end{aligned} \tag{30}$$

or the generators of the Euclidean quantum algebra on the plane if we put $j_2 = 1$. The Hopf algebra structure of $so_q(3; \iota_1, 1)$ is in full coincidence with those of the quantum motion group of the plane [17].

For $j_2 = \iota_2$, Eqs. (27) give the generators of $so_q(3; j_1, \iota_2)$:

$$\begin{aligned}
J_1 |l, m\rangle &= i\alpha |l, m\rangle' - i \frac{[2m]}{4\alpha [1]_h} |l, m\rangle, \\
J_3 |l, m\rangle &= m |l, m\rangle, \\
J_2 |l, m\rangle &= \alpha |l, m\rangle, \quad \alpha = \{[l]^2 - [j_1 m]^2\}^{1/2}, \\
l &\geq |m|, \quad l, m \in \mathbf{R},
\end{aligned} \tag{31}$$

where $|l, m\rangle' = \partial/\partial m |l, m\rangle$, with the commutation relations

$$[J_3, J_1] = iJ_2, \quad [J_2, J_3] = 0, \quad [J_1, J_2] = \frac{i}{2[1]_h} j_1 [2j_1 J_3]. \tag{32}$$

For $j_1 = 1$ the generators (31) form the semi-elliptic (or co-Euclidean or Newton or Hook or cylindrical) quantum algebra on the plane. The antipode γ in this case is obtained from Eqs. (29) as follows:

$$\gamma(J_1) = -\{J_1 + ihJ_2\}, \quad \gamma(J_2) = -J_2. \tag{33}$$

It must be emphasized that different contractions of the quantum algebra $so_q(3)$ give the different (nonisomorphic) algebras $so_q(3; \iota_1, 1)$ and $so_q(3; 1, \iota_2)$ as a result in contrast to the classical case, where the contracted algebras $so(3; \iota_1, 1)$ and $so(3; 1, \iota_2)$ are isomorphic [9]. This fact

reflects the nonequivalence of the generators of the quantum algebras. The equivalent generators of the classical orthogonal algebra $so(3)$ are nonequivalent after the quantum deformation to the algebra $so_q(3)$.

The two-dimensional contraction of $so_q(3)$ may be obtained in three different ways. We may put $j_1 = \iota_1, j_2 = \iota_2$ in Eqs. (27) or may put $j_2 = \iota_2$ in Eqs. (30) or put $j_1 = \iota_1$ in Eqs. (31), but we always get the same result, namely, the generators of the Galilei quantum algebra $so_q(3; \iota_1, \iota_2)$:

$$\begin{aligned} J_1|l, m\rangle &= i[l]|l, m\rangle', & J_2|l, m\rangle &= [l]|l, m\rangle, \\ J_3|l, m\rangle &= m|l, m\rangle, & l, m &\in \mathbf{R}, \end{aligned} \quad (34)$$

with the commutation relations

$$[J_3, J_1] = iJ_2, \quad [J_2, J_3] = 0, \quad [J_1, J_2] = 0, \quad (35)$$

and the antipode γ as in Eqs. (33).

There is another approach regarding the contractions of the quantum groups (algebras) [18]. The main feature of this approach is the transformation not only of the generators but also of the quantum (or deformation) parameter h . In our notation this transformation is $h = j_2 h^*$. Then the contraction $j_2 = \iota_2$ gives the Euclidean quantum group (algebra) with the commutation relations

$$[J_3, J_1] = iJ_2, \quad [J_2, J_3] = 0, \quad [J_1, J_2] = \frac{i}{2}[2J_3]_h, \quad (36)$$

where the h analog of x is defined by $[x]_h = h^{-1} \sinh(xh) = [x][1]_h$.

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