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Research Article

Contractive-Like Mapping Principles in Ordered Metric Spaces and Application to Ordinary Differential Equations

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The purpose of this paper is to present a fixed point theorem for generalized contractions in partially ordered complete metric spaces. We also present an application to first-order ordinary differential equations.

1. Introduction

Existence of fixed point in partially ordered sets has been considered recently in [1–17]. Tarski's theorem is used in [9] to show the existence of solutions for fuzzy equations and in [11] to prove existence theorems for fuzzy differential equations. In [2, 6, 7, 10, 13] some applications to ordinary differential equations and to matrix equations are presented. In [3–5, 17] some fixed point theorems are proved for a mixed monotone mapping in a metric space endowed with partial order and the authors apply their results to problems of existence and uniqueness of solutions for some boundary value problems.

In the context of ordered metric spaces, the usual contraction is weakened but at the expense that the operator is monotone. The main tool in the proof of the results in this context combines the ideas in the contraction principle with those in the monotone iterative technique [18].

Let S denote the class of the class of the functions $\beta : [0,\infty) \to [0,1)$ which satisfies the condition

$$\beta(t_n) \longrightarrow 1 \Longrightarrow t_n \longrightarrow 0. \tag{1.1}$$

In [19] the following generalization of Banach's contraction principle appears.

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \leq \beta(d(x, y)) \cdot d(x, y), \quad \text{for } x, y \in X, \tag{1.2}$$

where $\beta \in S$. Then T has a unique fixed point $z \in X$ and $\{T^n(x)\}$ converges to z for each $x \in X$.

Recently, in [2] the authors prove a version of Theorem 1.1 in the context of ordered complete metric spaces. More precisely, they prove the following result.

Theorem 1.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping such that*

$$d(Tx, Ty) \leq \beta(d(x, y)) \cdot d(x, y), \quad \text{for } x, y \in X \text{ with } x \leq y, \tag{1.3}$$

where $\beta \in S$. Assume that either T is continuous or X satisfies the following condition:

$$\text{if } \{x_n\} \text{ is a nondecreasing sequence in } X \text{ such that } \; x_n \longrightarrow x, \text{ then } \; x_n \leq x \;\; \forall n \in \mathbb{N}. \tag{1.4}$$

Besides, suppose that for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a unique fixed point.

The purpose of this paper is to generalize Theorem 1.2 with the help of the altering functions.

We recall the definition of such functions.

Definition 1.3. An altering function is a function $\psi : [0, \infty) \to [0, \infty)$ which satisfies the following.

(a) ψ is continuous and nondecreasing.

(b) $\psi(t) = 0$ if and only if $t = 0$.

Altering functions have been used in metric fixed point theory in recent papers [20–22].

In [7] the authors use these functions and they prove some fixed point theorems in ordered metric spaces.

2. Fixed Point Theorems

Definition 2.1. If $(X, \leq)$ is a partially ordered set and $T : X \to X$, we say that T is monotone nondecreasing if for $x, y \in X$,

$$x \leq y \Longrightarrow T(x) \leq T(y). \tag{2.1}$$

This definition coincides with the notion of a nondecreasing function in the case $X = \mathbb{R}$ and $\leq$ represents the usual total order in $\mathbb{R}$.

In the sequel, we prove the main result of the paper.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that*

$$\psi(d(T(x), T(y))) \leq \beta(d(x, y)) \cdot \psi(d(x, y)), \quad \text{for } x \geq y, \tag{2.2}$$

where ψ is an altering function and $\beta \in S$.

If there exist $x_0 \in X$ with $x_0 \leq T(x_0)$, then T has a fixed point.

Proof. If $T(x_0) = x_0$, then the proof is finished. Suppose that $x_0 < T(x_0)$. Since $x_0 < T(x_0)$ and T is a nondecreasing mapping, we obtain by induction that

$$x_0 < T(x_0) \leq T^2(x_0) \leq T^3(x_0) \leq \cdots \leq T^n(x_0) \leq T^{n+1}(x_0) \leq \cdots. \tag{2.3}$$

Put $x_{n+1} = T(x_n)$. Taking into account that $\beta \in S$ and since $x_n \leq x_{n+1}$ for each $n \in \mathbb{N}$, then, by (2.2), we get

$$\begin{aligned}\psi(d(x_{n+1}, x_n)) &= \psi(d(T(x_n), T(x_{n-1}))) \\ &\leq \beta(d(x_n, x_{n-1})) \cdot \psi(d(x_n, x_{n-1})) \\ &\leq \psi(d(x_n, x_{n-1})).\end{aligned} \tag{2.4}$$

Using the fact that ψ is nondecreasing, we have

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}). \tag{2.5}$$

If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0-1}) = 0$, then $x_{n_0} = T(x_{n_0-1}) = x_{n_0-1}$ and x_{n_0-1} is a fixed point and the proof is finished. In another case, suppose that $d(x_{n+1}, x_n) \neq 0$ for all $n \in \mathbb{N}$. Then, taking into account (2.5), the sequence $\{d(x_{n+1}, x_n)\}$ is decreasing and bounded below, so

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r \geq 0 \tag{2.6}$$

Assume that $r > 0$.

Then, from (2.4), we have

$$\frac{\psi(d(x_{n+1}, x_n))}{\psi(d(x_n, x_{n-1}))} \leq \beta(d(x_n, x_{n-1})) < 1. \tag{2.7}$$

Letting $n \to \infty$ in the last inequality and by the fact that ψ is an altering function, we get

$$1 \leq \lim_{n\to\infty} \beta(d(x_n, x_{n-1})) \leq 1 \tag{2.8}$$

and, consequently, $\lim_{n\to\infty}\beta(d(x_n, x_{n-1})) = 1$. Since $\beta \in S$ this implies that $\lim_{n\to\infty}(d(x_{n+1}, x_n)) = 0$ and this contradicts our assumption that $r > 0$. Hence,

$$\lim_{n\to\infty}(d(x_{n+1}, x_n)) = 0. \tag{2.9}$$

In what follows, we will show that $\{x_n\}$ is a Cauchy sequence.

Suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$d\left(x_{n(k)}, x_{m(k)}\right) \geq \epsilon. \tag{2.10}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.10), then

$$d\left(x_{n(k)-1}, x_{m(k)}\right) < \epsilon. \tag{2.11}$$

Using (2.10), (2.11), and the triangular inequality, we have

$$\begin{aligned}\epsilon &\leq d\left(x_{n(k)}, x_{m(k)}\right)\\ &\leq d\left(x_{n(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{m(k)}\right)\\ &< d\left(x_{n(k)}, x_{n(k)-1}\right) + \epsilon.\end{aligned} \tag{2.12}$$

Letting $k \to \infty$ and using (2.9), we get

$$\lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)}\right) = \epsilon. \tag{2.13}$$

Again, the triangular inequality gives us

$$\begin{aligned}&d\left(x_{n(k)}, x_{m(k)}\right) \leq d\left(x_{n(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{m(k)-1}\right) + d\left(x_{m(k)-1}, x_{m(k)}\right),\\ &d\left(x_{n(k)-1}, x_{m(k)-1}\right) \leq d\left(x_{n(k)-1}, x_{n(k)}\right) + d\left(x_{n(k)}, x_{m(k)}\right) + d\left(x_{m(k)}, x_{m(k)-1}\right).\end{aligned} \tag{2.14}$$

Letting $k \to \infty$ in the above two inequalities and using (2.9) and (2.13), we have

$$\lim_{k\to\infty} d\left(x_{n(k)-1}, x_{m(k)-1}\right) = \epsilon. \tag{2.15}$$

As $n(k) > m(k)$ and $x_{n(k)-1} \geq x_{m(k)-1}$, by (2.2), we obtain

$$\begin{aligned}\psi\left(d\left(x_{n(k)}, x_{m(k)}\right)\right) &= \psi\left(d\left(Tx_{n(k)-1}, Tx_{m(k)-1}\right)\right)\\ &\leq \beta\left(d\left(x_{n(k)-1}, x_{m(k)-1}\right)\right) \cdot \psi\left(d\left(x_{n(k)-1}, x_{m(k)-1}\right)\right)\\ &\leq \psi\left(d\left(x_{n(k)-1}, x_{m(k)-1}\right)\right).\end{aligned} \tag{2.16}$$

Taking into account (2.13) and (2.15) and the fact that ψ is continuous and letting $k \to \infty$ in (2.16), we get

$$\psi(\epsilon) \leq \lim_{k \to \infty} \beta\left(d\left(x_{n(k)-1}, x_{m(k)-1}\right)\right) \cdot \psi(\epsilon) \leq \psi(\epsilon). \tag{2.17}$$

As ψ is an altering function, $\psi(\epsilon) > 0$, the last inequality gives us

$$\lim_{k \to \infty} \beta\left(d\left(x_{n(k)-1}, x_{m(k)-1}\right)\right) = 1. \tag{2.18}$$

Since $\beta \in S$, this means that

$$\lim_{k \to \infty} d\left(x_{n(k)-1}, x_{m(k)-1}\right) = 0. \tag{2.19}$$

This fact and (2.15) give us $\epsilon = 0$ which is a contradiction.

This shows that $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is a complete metric space, there exists $z \in X$ such that $\lim_{n \to \infty} x_n = z$. Moreover, the continuity of T implies that

$$z = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = Tz, \tag{2.20}$$

and this proves that z is a fixed point. □

In what follows, we prove that Theorem 2.2 is still valid for T not necessarily continuous, assuming the following hypothesis in X (which appears in [10, Theorem 1]):

$$\text{if } (x_n) \text{ is a nondecreasing sequence in } X \text{ such that } x_n \longrightarrow x, \text{then } x_n \leq x \quad \forall n \in \mathbb{N}. \tag{2.21}$$

Theorem 2.3. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X satisfies* (2.21)*. Let $T : X \to X$ be a nondecreasing mapping such that*

$$\psi(d(T(x), T(y))) \leq \beta(d(x, y)) \cdot \psi(d(x, y)), \quad \text{for } x \geq y, \tag{2.22}$$

where ψ is an altering function and $\beta \in S$. If there exists $x_0 \in X$ with $x_0 \leq T(x_0)$, then T has a fixed point.

Proof. Following the proof of Theorem 2.2, we only have to check that $T(z) = z$. As (x_n) is a nondecreasing sequence in X and $\lim_{n \to \infty} x_n = z$ then, by (2.21), we have $x_n \leq z$ for all $n \in \mathbb{N}$, and, consequently,

$$\psi\left(d\left(x_{n+1}, f(z)\right)\right) = \psi(d(T(x_n), T(z))) \leq \beta(d(x_n, z)) \cdot \psi(d(x_n, z)) \leq \psi(d(x_n, z)). \tag{2.23}$$

Letting $n \to \infty$ and using the continuity of ψ, we have

$$0 \le \psi(d(z, T(z))) \le \psi(0) = 0, \tag{2.24}$$

or, equivalently,

$$\psi(d(z, T(z))) = 0. \tag{2.25}$$

As ψ is an altering function, this gives us $d(z, T(z)) = 0$ and, thus, $T(z) = z$. □

Now, we present an example where it can be appreciated that the hypotheses in Theorems 2.2 and 2.3 do not guarantee uniqueness of the fixed point. This example appears in [10].

Let $X = \{(1,0), (0,1)\} \subset \mathbb{R}^2$ and consider the usual order

$$(x, y) \le (z, t) \Longleftrightarrow x \le z, \quad y \le t. \tag{2.26}$$

$(X, \le)$ is a partially ordered set whose different elements are not comparable. Besides, (X, d_2) is a complete metric space considering d_2 as the Euclidean distance. The identity map $T(x, y) = (x, y)$ is trivially continuous and nondecreasing and condition (2.2) of Theorem 2.2 is satisfied since elements in X are only comparable to themselves. Moreover, $(1,0) \le T(1,0) = (1,0)$ and T has two fixed points in X.

In what follows, we give a sufficient condition for the uniqueness of the fixed point in Theorems 2.2 and 2.3. This condition appears in [16] and says that

$$\text{for } x, y \in X, \text{ there exists a lower bound or an upper bound.} \tag{2.27}$$

In [10] it is proved that condition (2.27) is equivalent to

$$\text{for } x, y \in X, \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{2.28}$$

Theorem 2.4. *Adding condition* (2.28) *to the hypotheses of Theorem 2.2 (resp., Theorem 2.3), we obtain uniqueness of the fixed point of f.*

Proof. Suppose that there exist $y, z \in X$ which are fixed points of T and $y \ne z$. We distinguish two cases.

Case 1. If y and z are comparable, then $T^n(y) = y$ and $T^n(z) = z$ are comparable for $n = 0, 1, 2, \ldots$. Using the contractive condition appearing in Theorem 2.2 (or Theorem 2.3) and the fact that $\beta \in S$, we get

$$\begin{aligned} \psi(d(y, z)) &= \psi(d(T^n(y), T^n(z))) \\ &\le \beta\left(d\left(T^{n-1}(y), T^{n-1}(z)\right)\right) \cdot \psi\left(d\left(T^{n-1}(y), T^{n-1}(z)\right)\right) \\ &\le \beta(d(y, z)) \cdot \psi(d(y, z)) \\ &< \psi(d(y, z)), \end{aligned} \tag{2.29}$$

which is a contradiction.

Case 2. Using condition (2.28), there exists $x \in X$ comparable to y and z. Monotonicity of T implies that $T^n(x)$ is comparable to $T^n(y) = y$ and to $T^n(z) = z$, for $n = 0, 1, 2, \ldots$. Moreover, as $\beta \in S$, we get

$$\begin{aligned}\psi(d(z, T^n(x))) &= \psi(d(T^n(z), T^n(x))) \\ &\leq \beta\Big(d\Big(T^{n-1}(z), T^{n-1}(x)\Big)\Big) \cdot \psi\Big(d\Big(T^{n-1}(z), T^{n-1}(x)\Big)\Big) \\ &= \beta\Big(d\Big(z, T^{n-1}(x)\Big)\Big) \cdot \psi\Big(d\Big(z, T^{n-1}(x)\Big)\Big) \\ &\leq \psi\Big(d\Big(z, T^{n-1}(x)\Big)\Big).\end{aligned} \tag{2.30}$$

Since ψ is nondecreasing the above inequality gives us

$$d(z, T^n(x)) \leq d\Big(z, T^{n-1}(x)\Big). \tag{2.31}$$

Thus, $\lim_{n\to\infty} d(z, T^n(x)) = \gamma \geq 0$.

Assume that $\gamma > 0$.

Taking into account that ψ is an altering function and letting $n \to \infty$ in (2.30), we obtain

$$\psi(\gamma) \leq \lim_{n\to\infty} \beta\Big(d\Big(z, T^{n-1}(x)\Big)\Big) \cdot \psi(\gamma) \leq \psi(\gamma), \tag{2.32}$$

and this implies that $\lim_{n\to\infty} \beta(d(z, T^{n-1}(x))) = 1$.

Since $\beta \in S$ then we get

$$\lim_{n\to\infty} d\Big(z, T^{n-1}(x)\Big) = 0, \tag{2.33}$$

and, consequently, $\gamma = 0$, which is a contradiction.

Hence, $\lim_{n\to\infty} d(z, T^n(x)) = 0$.

Analogously, it can be proved that

$$\lim_{n\to\infty} d(y, T^n(x)) = 0. \tag{2.34}$$

Finally, as

$$d(z, y) \leq d(z, T^n(x)) + d(T^n(x), y) \tag{2.35}$$

and taking limit, we obtain $d(z, y) = 0$.

This finishes the proof. □

Remark 2.5. Under the assumptions of Theorem 2.4, it can be proved that for every $x \in X$, $\lim_{n\to\infty} T^n(x) = z$, where z is the fixed point (i.e., the operator T is Picard).

In fact, for $x \in X$ and x comparable to z then using the same argument that is in Case 1 of Theorem 2.4 can prove that $\lim_{n\to\infty} d(z, T^n(x)) = 0$ and, hence, $\lim_{n\to\infty} T^n(x) = z$.

If x is not comparable to z, we take that $y \in X$ is comparable to x and z. Using a similar argument that is in Case 2 of Theorem 2.4, we obtain

$$\lim_{n\to\infty} d\left(z, T^n(y)\right) = 0, \qquad \lim_{n\to\infty} d\left(T^n(x), T^n(y)\right) = 0. \tag{2.36}$$

Finally,

$$d(z, T^n(x)) \leq d\left(z, T^n(y)\right) + d\left(T^n(y), T^n(x)\right), \tag{2.37}$$

and taking limit as $n \to \infty$, we obtain $\lim_{n\to\infty} d(z, T^n(x)) = 0$ or, equivalently, $\lim_{n\to\infty} T^n(x) = z$.

Remark 2.6. Notice that if $(X, \leq)$ is totally ordered, condition (2.28) is obviously satisfied.

Remark 2.7. Considering ψ the identity mapping in Theorem 2.4, we obtain Theorem 1.2, being the main result of [2].

3. Application to Ordinary Differential Equations

In this section we present an example where our results can be applied.

This example is inspired by [10].

We study the existence of solution for the following first-order periodic problem

$$\begin{aligned} u'(t) &= f(t, u(t)), \quad t \in [0, T], \\ u(0) &= u(T), \end{aligned} \tag{3.1}$$

where $T > 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Previously, we considered the space $C(I)$ ($I = [0, T]$) of continuous functions defined on I. Obviously, this space with the metric given by

$$d(x, y) = \sup\left\{\left|x(t) - y(t)\right| : t \in I\right\}, \quad \text{for } x, y \in C(I), \tag{3.2}$$

is a complete metric space. $C(I)$ can also be equipped with a partial order given by

$$x, y \in C(I), \quad x \leq y \Longleftrightarrow x(t) \leq y(t), \quad \text{for } t \in I. \tag{3.3}$$

Clearly, $(C(I), \leq)$ satisfies condition (2.28), since for $x, y \in C(I)$ the function $\max\{x, y\} \in C(I)$.

Moreover, in [10] it is proved that $(C(I), \leq)$ with the above-mentioned metric satisfies condition (2.21).

Now, let $\mathcal{A}$ denote the class of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following.

(i) ϕ is nondecreasing.

(ii) $\phi(x) < x$, for $x > 0$.

(iii) $\beta(x) = \phi(x)/x \in S$,

where S is the class of functions defined in Section 1.

Examples of such functions are $\phi(t) = \mu \cdot t$, with $0 \leq \mu < 1$, $\phi(t) = t/(1+t)$, and $\phi(t) = \ln(1+t)$.

Recall now the following definition

Definition 3.1. A lower solution for (3.1) is a function $\alpha \in C^1(I)$ such that

$$\begin{gathered} \alpha'(t) \leq f(t,\alpha(t)) \quad \text{for } t \in I, \\ \alpha(0) \leq \alpha(T). \end{gathered} \tag{3.4}$$

Now, we present the following theorem about the existence of solution for problem (3.1) in presence of a lower solution.

Theorem 3.2. *Consider problem* (3.1) *with $f : I \times \mathbb{R} \to \mathbb{R}$ continuous and suppose that there exist $\lambda, \alpha > 0$ with*

$$\alpha \leq \left(\frac{2\lambda(e^{\lambda T} - 1)}{T(e^{\lambda T} + 1)} \right)^{1/2}, \tag{3.5}$$

such that for $x, y \in \mathbb{R}$ with $x \leq y$

$$0 \leq f(t,y) + \lambda y - [f(t,x) + \lambda x] \leq \alpha \sqrt{(y-x)\phi(y-x)}, \tag{3.6}$$

where $\phi \in \mathcal{A}$. Then the existence of a lower solution for (3.1) *provides the existence of a unique solution of* (3.1).

Proof. Problem (3.1) can be written as

$$\begin{gathered} u'(t) + \lambda u(t) = f(t,u(t)) + \lambda u(t) \quad \text{for } t \in I = [0,T], \\ u(0) = u(T). \end{gathered} \tag{3.7}$$

This problem is equivalent to the integral equation

$$u(t) = \int_0^T G(t,s)[f(s,u(s)) + \lambda u(s)]\,\mathrm{d}s, \tag{3.8}$$

where $G(t,s)$ is the Green function given by

$$G(t,s)=\begin{cases}\dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1} & 0\le s<t\le T,\\[2mm] \dfrac{e^{\lambda(s-t)}}{e^{\lambda T}-1} & 0\le t<s\le T.\end{cases}\tag{3.9}$$

Define $F:\mathcal{C}(I)\to\mathcal{C}(I)$ by

$$(Fu)(t)=\int_0^T G(t,s)\left[f(s,u(s))+\lambda u(s)\right]\mathrm{d}s.\tag{3.10}$$

Notice that if $u\in\mathcal{C}(I)$ is a fixed point of F, then $u\in\mathcal{C}^1(I)$ is a solution of (3.1).

In the sequel, we check that hypotheses in Theorem 2.4 are satisfied.

The mapping F is nondecreasing since, by hypothesis, for $u\ge v$

$$f(t,u)+\lambda u\ge f(t,v)+\lambda v,\tag{3.11}$$

and this implies, taking into account that $G(t,s)>0$ for $(t,s)\in I\times I$, that

$$(Fu)(t)=\int_0^T G(t,s)\left[f(s,u(s))+\lambda u(s)\right]\mathrm{d}s\ge\int_0^T G(t,s)\left[f(s,v(s))+\lambda v(s)\right]\mathrm{d}s=(Fv)(t).\tag{3.12}$$

Besides, for $u\ge v$, we have

$$\begin{aligned}d(Fu,Fv)&=\sup_{t\in I}|(Fu)(t)-(Fv)(t)|\\&=\sup_{t\in I}((Fu)(t)-(Fv)(t))\\&=\sup_{t\in I}\int_0^T G(t,s)\left[f(s,u(s))+\lambda u(s)-f(s,v(s))-\lambda v(s)\right]\mathrm{d}s\\&\le\sup_{t\in I}\int_0^T G(t,s)\alpha\sqrt{(u(s)-v(s))\phi(u(s)-v(s))}\mathrm{d}s\\&=\alpha\sup_{t\in I}\int_0^T G(t,s)\sqrt{(u(s)-v(s))\phi(u(s)-v(s))}\mathrm{d}s.\end{aligned}\tag{3.13}$$

Using the Cauchy-Schwarz inequality in the last integral, we get

$$\begin{aligned}&\int_0^T G(t,s)\alpha\sqrt{(u(s)-v(s))\phi(u(s)-v(s))}\mathrm{d}s\\&\quad\le\left(\int_0^T G(t,s)^2\mathrm{d}s\right)^{1/2}\left(\int_0^T(u(s)-v(s))\phi(u(s)-v(s))\mathrm{d}s\right)^{1/2}.\end{aligned}\tag{3.14}$$

The first integral gives us

$$\begin{aligned}\int_0^T G(t,s)^2 \mathrm{d}s &= \int_0^t G(t,s)^2 \mathrm{d}s + \int_t^T G(t,s)^2 \mathrm{d}s \\ &= \int_0^t \frac{e^{2\lambda(T+s-t)}}{\left(e^{\lambda T}-1\right)^2} \mathrm{d}s + \int_t^T \frac{e^{2\lambda(s-t)}}{\left(e^{\lambda T}-1\right)^2} \mathrm{d}s \\ &= \frac{1}{2\lambda\left(e^{\lambda T}-1\right)^2}\left(e^{2\lambda T}-1\right) \\ &= \frac{e^{\lambda T}+1}{2\lambda\left(e^{\lambda T}-1\right)}.\end{aligned} \tag{3.15}$$

As ϕ is nondecreasing, the second integral in (3.14) can be estimated by

$$\int_0^T (u(s)-v(s))\phi(u(s)-v(s))\mathrm{d}s \le T \cdot \|u-v\| \cdot \phi(\|u-v\|) = d(u,v) \cdot \phi(d(u,v)) \cdot T. \tag{3.16}$$

Taking into account (3.14), (3.15), and (3.16), from (3.13) we get

$$\begin{aligned}d(Fu,Fv) &\le \alpha \cdot \left(\frac{e^{\lambda T}+1}{2\lambda\left(e^{\lambda T}-1\right)}\right)^{1/2} \cdot d(u,v)^{1/2} \cdot \phi(d(u,v))^{1/2} \cdot T^{1/2} \\ &= \alpha \cdot \left(\frac{T \cdot \left(e^{\lambda T}+1\right)}{2\lambda\left(e^{\lambda T}-1\right)}\right)^{1/2} \cdot d(u,v)^{1/2} \cdot \phi(d(u,v))^{1/2}.\end{aligned} \tag{3.17}$$

Since $\alpha \le (2\lambda(e^{\lambda T}-1)/T \cdot (e^{\lambda T}+1))^{1/2}$, the last inequality gives us

$$d(Fu,Fv) \le d(u,v)^{1/2} \cdot \phi(d(u,v))^{1/2} \tag{3.18}$$

or, equivalently,

$$d(Fu,Fv) \le d(u,v) \cdot \left(\frac{\phi(d(u,v))}{d(u,v)}\right)^{1/2}. \tag{3.19}$$

This implies that

$$d(Fu,Fv)^2 \le \frac{\phi(d(u,v))}{d(u,v)} \cdot d(u,v)^2. \tag{3.20}$$

Putting $\psi(x) = x^2$, which is an altering function, and $\beta = \phi(x)/x \in S$ because $\phi \in \mathcal{A}$, we have

$$\psi(d(Fu,Fv)) \le \beta(d(u,v)) \cdot \psi(d(u,v)) \quad \text{for } u \ge v. \tag{3.21}$$

This proves that the operator F satisfies condition (2.2) of Theorem 2.2.

Finally, letting $\alpha(t)$ be a lower solution for (3.1), we claim that $\alpha \le F(\alpha)$.
In fact,

$$\alpha'(t) + \lambda\alpha(t) \le f(t,\alpha(t)) + \lambda\alpha(t), \quad \text{for } t \in I. \tag{3.22}$$

Multiplying by $e^{\lambda t}$,

$$\left(\alpha(t)e^{\lambda t}\right)' \le \left[f(t,\alpha(t)) + \lambda\alpha(t)\right]e^{\lambda t}, \quad \text{for } t \in I, \tag{3.23}$$

and this gives us

$$\alpha(t)e^{\lambda t} \le \alpha(0) + \int_0^t \left[f(s,\alpha(s)) + \lambda\alpha(s)\right]e^{\lambda s}\mathrm{d}s, \quad \text{for } t \in I. \tag{3.24}$$

As $\alpha(0) \le \alpha(T)$, the last inequality implies that

$$\alpha(0)e^{\lambda T} \le \alpha(T)e^{\lambda T} \le \alpha(0) + \int_0^T \left[f(s,\alpha(s)) + \lambda\alpha(s)\right]e^{\lambda s}\mathrm{d}s \tag{3.25}$$

and so

$$\alpha(0) \le \int_0^T \frac{e^{\lambda s}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s)\right]\mathrm{d}s. \tag{3.26}$$

This and (3.24) give us

$$\alpha(t)e^{\lambda t} \le \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s)\right]\mathrm{d}s + \int_t^T \frac{e^{\lambda s}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s)\right]\mathrm{d}s, \tag{3.27}$$

and, consequently,

$$\begin{aligned}\alpha(t) &\le \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s)\right]\mathrm{d}s + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}\left[f(s,\alpha(s)) + \lambda\alpha(s)\right]\mathrm{d}s \\ &= \int_0^T G(t,s)\left[f(s,\alpha(s)) + \lambda\alpha(s)\right]\mathrm{d}s \\ &= (F\alpha)(t), \quad \text{for } t \in I.\end{aligned} \tag{3.28}$$

Finally, Theorem 2.4 gives that F has a unique fixed point. □

Remark 3.3. Notice that if $\phi \in \mathcal{A}$, then $\varphi(x) = \sqrt{x\phi(x)} \in \mathcal{A}$. In fact, as $\phi \in \mathcal{A}$, then ϕ is nondecreasing and, consequently, φ is also nondecreasing.

Moreover, as $\phi(x) < x$, then $x\phi(x) < x^2$, and, thus, $\sqrt{x\phi(x)} < x$.

Finally, as $\varphi(x)/x = \sqrt{x\phi(x)}/x = \sqrt{\phi(x)/x}$, and as $\beta(x) = \phi(x)/x \in S$, then it is easily seen that $\varphi(x)/x \in S$.

Example 3.4. Consider $\phi_0 : [0,\infty) \to [0,\infty)$ given by

$$\phi_0(t) = \begin{cases} 0, & 0 \le t \le 2, \\ \dfrac{1}{2}t - 1, & 2 < t \le 4, \\ \dfrac{1}{4}t, & 4 < t. \end{cases} \tag{3.29}$$

It is easily seen that $\phi_0 \in \mathcal{A}$. Taking into account Remark 3.3, $\phi(x) = \sqrt{x\phi_0(x)} \in \mathcal{A}$.

Now, we consider problem (3.1) with $f : I \times \mathbb{R} \to \mathbb{R}$ continuous and suppose that there exist $\lambda, \alpha > 0$ with

$$\alpha \le \left(\frac{2\lambda(e^{\lambda T} - 1)}{T(e^{\lambda T} + 1)} \right)^{1/2} \tag{3.30}$$

such that for $x, y \in \mathbb{R}$ with $y \ge x$

$$0 \le f(t,y) + \lambda y - \left[f(t,x) + \lambda x\right] \le \alpha\sqrt{(y-x)\phi_0(y-x)} = \alpha\phi(y-x), \tag{3.31}$$

where ϕ_0 is the function above mentioned.

This example can be treated by our Theorem 3.2 but it cannot be covered by the results of [6] because $\psi(x) = x - \phi(x)$ is not increasing.

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