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Contributions to Balanced Fractional 2^m Factorial Designs Derived from Balanced Arrays of Strength 2l

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0. Introduction and summary

The theory of fractional factorial designs, first introduced by Finney [12], has found increasing use in agricultural, biological, industrial, and other various experimentations. One reason for the usefulness of fractional designs in preference to complete factorials is that they involve a lesser number of assemblies or treatment combinations, since higher order effects can be in general assumed negligible. In the beginning, the theory was developed for orthogonal fractional designs in which the estimates of various effects of interest are all uncorrelated. However, as is well known, they are available only for special values of N assemblies. Moreover they are in general uneconomic in that they require a large value of N in comparison with the number of unknown effects. As generalizations of orthogonal fractional designs, Chakravarti [5] first introduced the concept of balanced fractional designs. In these designs the covariance matrix of the estimates of effects has desirable features second to orthogonal fractional designs, although the estimates are not uncorrelated. Of course, balanced fractional designs are flexible in the number of N assemblies with the fact that more experimental situations can be handled. Such economic designs are very attractive and often practical.

After important work of Bose and Srivastava [2, 3], Srivastava and/or Chopra have developed balanced fractional 2^m factorial (briefly, 2^m-BFF) designs of resolution V (cf. [7–10, 28, 34, 35, 37]). It is known from their results that these designs have close relationships with balanced arrays (B-arrays) of strength 4, which make it possible to interpret the problems into those in combinatorial fields. For some work in these fields, see Chakravarti [6], Srivastava [29], Srivastava and Chopra [36], Rafter and Seiden [18]. The above investigations, however, have been restricted to the effects up to two-factor interactions only. Since three factor or higher order interactions can not always be neglected, it is desirable to study fractional designs of higher resolution.

Recently, Yamamoto, Shirakura and Kuwada [41] have established a general connection between a 2^m-BFF design of resolution $2l+1$ and a B-array of strength $2l$. In the above paper, the authors also have discussed some properties of a triangular type multidimensional partially balanced (TMDPB) association scheme, defined among the effects up to l-factor interactions, which are useful for clarifying the algebraic structures of 2^m-BFF designs of resolution $2l+1$. The concept of MDPB association schemes was first introduced by Bose and Srivastava [3] in relation to the analysis of fractional designs. Using the decomposition of the TMDPB association algebra $\mathfrak{A}$ into its two-sided ideals, Yamamoto, Shirakura and Kuwada [42] have obtained an explicit expression for the characteristic polynomial of the information matrix M_T of a 2^m-BFF design T of resolution $2l+1$. (This result includes that of a 2^m-BFF design of resolution V ($l=2$) given by Srivastava and Chopra [35].) It is used for comparing 2^m-BFF designs of higher odd resolution by popular criteria such as minimizing the trace, determinant or largest root of M_T^{-1}. Indeed, Shirakura [23] has presented optimal 2^m-BFF designs of resolution VII ($l=3$) with respect to the trace criterion for each $6 \leqq m \leqq 8$ and for the reasonable number of N assemblies. On the other hand, the study of balanced designs of even resolution is much more rare. For work on such designs, see Shirakura [24], Srivastava and Anderson [30, 33]. Particularly, by use of the properties of the TMDPB association algebra $\mathfrak{A}$, Shirakura

[24] has obtained a general result that some B-arrays of strength $2l$ yield 2^m-BFF designs of resolution $2l$.

This paper will make further deep investigations on 2^m-BFF designs of odd or even resolution on the basis of the above mentioned results. 2^m-BFF designs derived from B-arrays of strength $2l$ will be characterized. This paper thus consists of three parts. In Part I, the algebraic structures of 2^m-BFF designs are discussed. In Section 1, fractional 2^m factorial designs of resolution $2l$ or $2l+1$ are treated. In Section 2, 2^m-BFF designs of resolution $2l$ or $2l+1$ are defined. A relation between a 2^m-BFF design of resolution $2l+1$ and a B-array of strength $2l$, m constraints and index set $\{\mu_0, \mu_1, \dots, \mu_{2l}\}$ is also given. Section 3 gives definitions of an $l+1$ sets TMDPB association scheme and its relationship algebra $\mathfrak{A}$. Furthermore it is observed that $\mathfrak{A}$ called the $l+1$ sets TMDPB association algebra is decomposed into the direct sum of $l+1$ two-sided ideals $\mathfrak{A}_\beta$ ($\beta=0, 1, \dots, l$). Section 4 presents the irreducible representation K_β of the information matrix M_T for a B-array T of strength $2l$ with respect to each ideal $\mathfrak{A}_\beta$. For later use, explicit expressions for K_β are given for each case $l=2$ and 3. As will be seen, many of the results in this part have been already established by the authors [41, 42]. For clarification of this paper, however, we shall recall them.

In Part II, optimal 2^9-BFF designs of resolution VII with respect to the trace and determinant criteria are presented for any given N assemblies with $130 \leqq N \leqq 150$. For this purpose, Section 5 gives explicit expressions for the trace and determinant of M_T^{-1} for a 2^m-BFF design T of resolution $2l+1$. These can be obtained from the characteristic polynomial of M_T, due to [42]. As a by-product, the existence conditions for 2^m-BFF designs of resolution $2l+1$ or B-arrays of strength $2l$ are also given in terms of the m and μ_i ($i=0, 1, \dots, 2l$). Sections 6 and 7 deal with constructions of B-arrays of strength t. Simple arrays in Section 7 have been introduced by Shirakura [22], as special cases of B-arrays. In Section 8, the required designs are given with the covariance matrices of the estimates and other useful informations.

In Part III, 2^m-BFF designs of even resolution derived from various B-arrays of strength $2l$ are investigated. Section 9 deals with 2^m-BFF designs of resolution $2l$ obtained from B-arrays of strength $2l$ with index $\mu_l=0$, which are called S_l type 2^m-BFF designs. For the case $l=3$, Section 10 presents optimal S_3 type 2^m-BFF designs with respect to the generalized trace (GT) criterion, due to [24], for $m=6$, 7, and for every value of N within a certain practical range. Note that the optimal S_3 type 2^8-BFF designs have been already presented by [24]. As in Section 8, the covariance matrices of the estimates and other useful informations are also given for such designs. In Section 11, alias structures of l-factor interactions in S_l type 2^m-BFF designs and their estimability derived from these structures are discussed. Section 12 shows that there exists a 2^m-BFF design of resolution IV with the minimum number of assemblies $N=2m$. It can be obtained from a B-array of strength 4 with $\mu_2=0$. Section 13 shows that some 2^m-BFF designs of resolution $2l$ can be also obtained from B-arrays of strength $2l$ with $\kappa_\beta^{l-\beta,l-\beta}=0$, where $\kappa_\beta^{l-\beta,l-\beta}$ ($\beta=0, 1,\dots, l$) are the last diagonal elements of K_β. Such designs are called $S_l(\beta_1, \beta_2,\dots, \beta_r)$ type 2^m-BFF designs if $\kappa_{\beta_1}^{l-\beta_1,l-\beta_1}=\kappa_{\beta_2}^{l-\beta_2,l-\beta_2}=\cdots=\kappa_{\beta_r}^{l-\beta_r,l-\beta_r}=0$ and $\kappa_\alpha^{l-\alpha,l-\alpha}\neq 0$ for $\alpha\neq\beta_i$. For given N assemblies, there are a large number of possible $S_l(\beta_1,\dots, \beta_r)$ type 2^m-BFF designs. A criterion for comparing these designs is also given which is called the partial generalized trace (PGT) criterion. In Section 14, for the case $l=3$, optimal $S_3(\beta_1,\dots, \beta_r)$ type 2^m-BFF designs with respect to the PGT criterion are presented for $m=6$, 7, 8, and for desirable values of N.

Part I. 2^m-BFF designs and their algebraic structures

1. Fractional 2^m factorial designs

Consider a factorial experiment with m factors $f_1, f_2,\dots, f_m$, each at two levels (i.e., a 2^m factorial design). An assembly (or treatment combination) will be represented by $(j_1, j_2,\dots, j_m)$ where j_t, the level of the factor f_t, equals 0 or 1. There are 2^m assemblies in all. Consider the observations $y(j_1, j_2,\dots, j_m)$ corresponding to assemblies $(j_1, j_2,\dots, j_m)$ and their expectations $\eta(j_1, j_2,\dots, j_m)=\mathrm{Exp}\,[y(j_1, j_2,\dots, j_m)]$. It is well known (cf. [41]) that the various factorial effects can be expressed as linear combinations of all expectations $\eta(j_1, j_2,\dots, j_m)$, i.e.,

$$(1.1)\qquad \theta(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_m)=\frac{1}{2^m}\sum_{j_1,j_2,\dots,j_m} d_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m}^{j_1,j_2,\dots,j_m}\,\eta(j_1, j_2,\dots, j_m)$$

$$\text{for}\quad \varepsilon_r=0,\ 1;\quad r=1, 2,\dots, m,$$

where

$$d_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m}^{j_1,j_2,\dots,j_m}=d_{j_1}(\varepsilon_1)d_{j_2}(\varepsilon_2)\cdots d_{j_m}(\varepsilon_m).$$

Here $d_0(0)=d_1(0)=d_1(1)=1$ and $d_0(1)=-1$. In particular the general mean is represented by $\theta(0, 0,\dots, 0)$ and the main effect of the factor f_{t_1} is represented by $\theta(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_m)$, where $\varepsilon_{t_1}=1$ and $\varepsilon_r=0$ for $r\neq t_1$. The two-factor interaction of the factors f_{t_1} and f_{t_2} is represented by $\theta(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_m)$, where $\varepsilon_{t_1}=\varepsilon_{t_2}=1$ and $\varepsilon_r=0$ for $r\neq t_1$, t_2. In general the k-factor interaction of the factors $f_{t_1}, f_{t_2},\dots, f_{t_k}$ is represented by $\theta(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_m)$, where $\varepsilon_{t_1}=\varepsilon_{t_2}=\cdots=\varepsilon_{t_k}=1$ and the remaining ε_r are all zero.

Let

$$Y = \begin{bmatrix} y(0,\ldots,0,0) \\ y(0,\ldots,0,1) \\ \vdots \\ y(1,\ldots,1,1) \end{bmatrix} \quad \text{and} \quad \boldsymbol{\theta} = \begin{bmatrix} \theta(0,\ldots,0,0) \\ \theta(0,\ldots,0,1) \\ \vdots \\ \theta(1,\ldots,1,1) \end{bmatrix}$$

be respectively the $2^m \times 1$ vectors of all observations and effects in the binary order. From (1.1), $\boldsymbol{\theta}$ can be expressed in the following form:

$$\boldsymbol{\theta} = \frac{1}{2^m} D_{(m)} \mathrm{Exp}\,[Y], \tag{1.2}$$

where

$$D_{(m)} = D \otimes D \otimes \cdots \otimes D \quad (m \text{ times Kronecker products of } D).$$

Here

$$D = \begin{bmatrix} d_0(0) & d_1(0) \\ d_0(1) & d_1(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Note that $D_{(m)}$ is an Hadamard matrix of order 2^m. Thus $D_{(m)}D'_{(m)} = 2^m I_{2^m}$, where I_p denotes usually the identity matrix of order p. From (1.2), we thus have

$$\mathrm{Exp}\,[Y] = D'_{(m)}\boldsymbol{\theta} \tag{1.3}$$

or

$$\eta(j_1, j_2, \ldots, j_m) = \sum_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m} d^{j_1, j_2, \ldots, j_m}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m}\, \theta(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m). \tag{1.4}$$

For simplicity we shall write $\theta_\phi = \theta(0, 0, \ldots, 0)$ and $\theta_{t_1 t_2 \cdots t_k} = \theta(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ if $\varepsilon_{t_1} = \varepsilon_{t_2} = \cdots = \varepsilon_{t_k} = 1$ and $\varepsilon_r = 0$ for $r \neq t_1, t_2, \ldots, t_k$. Then (1.4) reduces to the following:

$$\begin{aligned} \eta(j_1, j_2, \ldots, j_m) &= \sum_{k=0}^{m} \sum_{\{t_1, \ldots, t_k\} \in \mathfrak{m}_k} d_{j_{t_1}} \cdots d_{j_{t_k}} \theta_{t_1 \cdots t_k} \\ &= \theta_\phi + \sum_{\{t_1\} \in \mathfrak{m}_1} d_{j_{t_1}} \theta_{t_1} + \sum_{\{t_1, t_2\} \in \mathfrak{m}_2} d_{j_{t_1}} d_{j_{t_2}} \theta_{t_1 t_2} \\ &\quad + \cdots + d_{j_1} d_{j_2} \cdots d_{j_m} \theta_{12 \cdots m}, \end{aligned} \tag{1.5}$$

where $\mathfrak{m}_k$ denotes the class of all subsets of $\{1, 2, \ldots, m\}$ with cardinality k and $d_j = 1$ or -1 according as $j = 1$ or 0.

The formula (1.3), (1.4) or (1.5) is used as a statistical linear model in a 2^m factorial design. For any fixed integer l ($1 \leqq l \leqq m/2$), we shall assume a general situation where $(l+1)$-factor and higher order interactions are negligible (i.e., $\theta_{t_1 t_2 \cdots t_k} = 0$ for $k \geqq l+1$). (Throughout this paper, note that we are considering such a situation.) The number of unknown effects, therefore, is $v_l = 1 + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{l}$ and the vector of these effects is written as

$$(1.6) \quad \boldsymbol{\theta}' = (\theta_\phi;\ \theta_1, \theta_2, \ldots, \theta_m;\ \theta_{12}, \theta_{13}, \ldots, \theta_{m-1m};\ \cdots;\ \theta_{12\cdots l}, \ldots, \theta_{m-l+1\cdots m})$$
$$= (\theta_\phi;\ \{\theta_{t_1}\};\ \{\theta_{t_1t_2}\};\ \cdots;\ \{\theta_{t_1t_2\cdots t_l}\}).$$

For later use, we shall provide the following vectors:

$$(1.7) \quad \begin{aligned} \boldsymbol{\theta}'_0 &= (\{\theta_{t_1}\};\ \{\theta_{t_1t_2}\};\ \cdots;\ \{\theta_{t_1t_2\cdots t_{l-1}}\}), && (1 \times (v_{l-1} - 1)), \\ \boldsymbol{\theta}'_1 &= (\theta_\phi : \boldsymbol{\theta}'_0), && (1 \times v_{l-1}), \\ \boldsymbol{\theta}'_2 &= (\{\theta_{t_1t_2\cdots t_l}\}), && \left(1 \times \binom{m}{l}\right), \end{aligned}$$

i.e., $\boldsymbol{\theta}' = (\boldsymbol{\theta}'_1 : \boldsymbol{\theta}'_2) = (\theta_\phi : \boldsymbol{\theta}'_0 : \boldsymbol{\theta}'_2)$. From (1.5), we can obtain the following model for the expectation of the observation corresponding to an assembly $(j_1, j_2, \ldots, j_m)$:

$$(1.8) \quad \eta(j_1, j_2, \ldots, j_m) = \theta_\phi + \sum_{k=1}^{l} \sum_{\{t_1, t_2, \ldots, t_k\} \in \mathfrak{m}_k} d_{j_{t_1}} d_{j_{t_2}} \cdots d_{j_{t_k}} \theta_{t_1t_2\cdots t_k}.$$

Let T be a suitable set of N assemblies (called a fraction) in which any given assembly may not occur or occur once or more times. Then T can be considered as a (0, 1) matrix of size $m \times N$ whose α-th column $(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)})'$ denotes the α-th assembly for $\alpha = 1, 2, \ldots, N$. Let $\boldsymbol{y}_T$ be the $N \times 1$ observation vector whose α-th element is $y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)})$ and further consider the N observations in $\boldsymbol{y}_T$ as independent random variables with common variance σ^2 (>0). From (1.8) $\boldsymbol{y}_T$ can be expressed as

$$(1.9) \quad \begin{aligned} \mathrm{Exp}[\boldsymbol{y}_T] &= E_T \boldsymbol{\theta}, \\ \mathrm{Var}[\boldsymbol{y}_T] &= \sigma^2 I_N, \end{aligned}$$

where E_T is the $N \times v_l$ design matrix of T whose elements of the first column corresponding to the general mean θ_ϕ are all 1, and whose elements of α-th rows corresponding to an effects $\theta_{t_1t_2\cdots t_k}$ are $d_{j_{t_1}^{(\alpha)}} d_{j_{t_2}^{(\alpha)}} \cdots d_{j_{t_k}^{(\alpha)}}$.

The concept of estimable functions of $\boldsymbol{\theta}$ will be stated in the following definitions:

DEFINITION 1.1. *A $p \times 1$ vector $\boldsymbol{\psi}$ is called a parametric function of $\boldsymbol{\theta}$ if each element of $\boldsymbol{\psi}$ is a linear function of unknown effects $\theta_{t_1t_2\cdots t_k}$ $(k \leqq l)$ with* *known constant coefficients, in other words, if* $\boldsymbol{\psi}$ *is such that*

$$\boldsymbol{\psi} = C\boldsymbol{\theta}, \tag{1.10}$$

where C *is a* $p \times v_l$ *matrix with known constant elements.*

DEFINITION 1.2. *A parametric function* $\boldsymbol{\psi}$ *of* $\boldsymbol{\theta}$ *is called an estimable function (or, simply, estimable) if each element of* $\boldsymbol{\psi}$ *has an unbiased linear estimate under the model* (1.9), *in other words, if there exists a* $p \times N$ *matrix* A *of constant elements such that*

$$\mathrm{Exp}\,[A\boldsymbol{y}_T] = \boldsymbol{\psi},$$

identically in $\boldsymbol{\theta}$. *Also* $A\boldsymbol{y}_T$ *is called an unbiased estimate of* $\boldsymbol{\psi}$.

DEFINITION 1.3. *For any given fraction* T *and estimable function* $\boldsymbol{\psi}$, *its unbiased estimate* $\hat{\boldsymbol{\psi}}$ *is called the best linear unbiased estimate (BLUE) of* $\boldsymbol{\psi}$ *if the* α*-th element of* $\hat{\boldsymbol{\psi}}$ *has a minimum variance in the class of all unbiased linear estimates of the* α*-th element of* $\boldsymbol{\psi}$ *for each* $\alpha = 1, 2, \ldots, p$.

For the observation vector $\boldsymbol{y}_T$ and design matrix E_T, consider the following equations for a $v_l \times 1$ vector $\boldsymbol{\theta}^*$:

$$M_T\boldsymbol{\theta}^* = E_T'\boldsymbol{y}_T, \tag{1.11}$$

where $M_T = E_T'E_T$ called the information matrix. These are so called the normal equations.

THEOREM 1.1 (Gauss-Markov Theorem). *For any estimable function* $\boldsymbol{\psi} = C\boldsymbol{\theta}$, *its BLUE* $\hat{\boldsymbol{\psi}}$ *is unique and given by*

$$\hat{\boldsymbol{\psi}} = C\boldsymbol{\theta}^*,$$

where $\boldsymbol{\theta}^*$ *is a solution of the normal equations* (1.11).

Of course, the BLUE $\hat{\boldsymbol{\psi}}$ depends on a fraction T. By matrix theory, there exists always a solution $\boldsymbol{\theta}^*$ of the normal equations (1.11) and it is in general not unique for a given T. However Theorem 1.1 shows that for any two solutions $\boldsymbol{\theta}_1^*$ and $\boldsymbol{\theta}_2^*$ of the normal equations (1.11), $\hat{\boldsymbol{\psi}} = C\boldsymbol{\theta}_1^* = C\boldsymbol{\theta}_2^*$ holds.

As a means of classifying fractions, Box and Hunter [4] introduced the term "resolution." First we shall define a fractional 2^m factorial (briefly, 2^m-FF) design of odd resolution.

DEFINITION 1.4. *A fraction* T *is called a* 2^m*-FF design of resolution* $2l+1$ *if* $\boldsymbol{\theta}$ *itself is estimable, i.e., if* $\boldsymbol{\psi} = C\boldsymbol{\theta}$, *where* $C = I_{v_l}$, *is an estimable function of* $\boldsymbol{\theta}$.

From the model (1.9) and Definition 1.2, it is easy to see that T is a 2^m-FF design of resolution $2l+1$ if and only if its information matrix is nonsingular. From Theorem 1.1, furthermore, it follows that for a 2^m-FF design T of resolution $2l+1$, the BLUE $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is given by

$$\hat{\boldsymbol{\theta}} = V_T E_T' \boldsymbol{y}_T, \tag{1.12}$$

where $V_T = M_T^{-1}$. Note that $\hat{\boldsymbol{\theta}}$ is a unique solution of (1.11). In addition it can be easily shown that its covariance matrix $\mathrm{Var}[\hat{\boldsymbol{\theta}}]$ is given by

$$\mathrm{Var}[\hat{\boldsymbol{\theta}}] = V_T \sigma^2. \tag{1.13}$$

From the nonsingularity of M_T and the model (1.9), we can easily prove the following

THEOREM 1.2. *Let T be a 2^m-FF design of resolution $2l+1$. Then the number of distinct assemblies in T must be at least v_l.*

Next we shall define a 2^m-FF design of even resolution.

DEFINITION 1.5. *A fraction T is called a 2^m-FF design of resolution $2l$ if $\boldsymbol{\theta}_0$ given in* (1.7) *is estimable.*

In a 2^m-FF design of resolution $2l$, in general, the general mean θ_ϕ and l-factor interactions themselves are not estimable, but some linear functions of these effects are estimable. These functions determine alias structures of θ_ϕ and $\theta_{t_1 t_2 \cdots t_l}$. In 2^m-FF designs of even resolution, it is very important to investigate such alias structures (see Sections 11–13). It is well known (see, e.g., Scheffé [21]) that T is a 2^m-FF design of resolution $2l$ if and only if there exists a matrix X of size $p \times N$ such that $XE_T = [O_{p\times 1}, I_p, O_{p\times q}]$, where $p = v_{l-1} - 1$ and $q = \binom{m}{l}$. The symbol $O_{p\times q}$ denotes the $p \times q$ matrix whose elements are all 0. In this case, by considering $C = XE_T$ in Theorem 1.1, we obtain the BLUE $\hat{\boldsymbol{\theta}}_0$ of $\boldsymbol{\theta}_0$,

$$\hat{\boldsymbol{\theta}}_0 = XE_T \boldsymbol{\theta}^*.$$

For general fractional experiments (i.e., fractional s^m or $s_1 \times s_2 \times \cdots \times s_m$ factorial designs), the concept of the term "resolution $2l$ or $2l+1$" can be similarly defined but we shall not consider it here. As compared with designs of odd resolution, in general, it is very difficult to obtain those of even resolution. For earlier work on designs of resolution IV, see, e.g., Anderson and Srivastava [1], Margolin [16, 17], Shirakura [24], Srivastava and Anderson [30, 33], Webb [39].

2. 2^m-BFF designs and B-arrays of strength $2l$

First consider a 2^m-FF design T of resolution $2l+1$ and the covariance matrix $\mathrm{Var}[\hat{\boldsymbol{\theta}}]$ for the design T.

Definition 2.1. *T is called a balanced fractional 2^m factorial (2^m-BFF) design of resolution $2l+1$ if the covariance matrix* $\mathrm{Var}[\hat{\boldsymbol{\theta}}]$ *is invariant under any permutation of m factors.*

Remark. It has been observed in [41] that Definition 2.1 is equivalent to one of the following three statements: (i) For a design $T(P)$ obtained from T by letting $T(P)=PT$, where P is any permutation matrix of order m, $M_T^{-1}=M_{T(P)}^{-1}$ holds, (ii) for any two estimates $\hat{\theta}_{t_1\cdots t_u}$ and $\hat{\theta}_{t'_1\cdots t'_v}$ in the BLUE $\hat{\boldsymbol{\theta}}$,

$$\mathrm{Var}[\hat{\theta}_{t_1\cdots t_u}] = \mathrm{Var}[\hat{\theta}_{\tau(t_1\cdots t_u)}],$$

$$\mathrm{Cov}[\hat{\theta}_{t_1\cdots t_u}, \hat{\theta}_{t'_1\cdots t'_v}] = \mathrm{Cov}[\hat{\theta}_{\tau(t_1\cdots t_u)}, \hat{\theta}_{\tau(t'_1\cdots t'_v)}],$$

where τ is any element of the permutation group $\left\{\tau;\ \tau=\begin{pmatrix} 1 & 2 \cdots m \\ \tau(1) & \tau(2)\cdots\tau(m)\end{pmatrix}\right\}$, and (iii) $\mathrm{Cov}[\hat{\theta}_{t_1\cdots t_u}, \hat{\theta}_{t'_1\cdots t'_v}]$ is a function of u, v and $|\{t_1,\dots, t_u\}\ominus\{t'_1,\dots, t'_v\}|$ (or $|\{t_1,\dots, t_u\}\cap\{t'_1,\dots, t'_v\}|$), and $\mathrm{Var}[\hat{\theta}_{t_1\cdots t_u}]$ is only of u, where the symbols $|S|$ and $S_1\ominus S_2$ denote respectively the cardinality of the set S and the symmetric difference of the sets S_1 and S_2, i.e., $S_1\ominus S_2=S_1\cup S_2-S_1\cap S_2$.

Now we define a balanced array ("partially balanced" array, in the terminology of Chakravarti [5]) of strength t (with 2 symbols), which has a close relationship with a balanced design considered in this paper.

Definition 2.2. *A (0, 1) matrix T of size $m\times N$ is called a balanced array (B-array) of strength t, size N, m constraints and index set $\{\mu_0, \mu_1,\dots, \mu_t\}$ (or indices μ_i ($i=0, 1,\dots, t$)) if for every t-rowed submatrix T^t of T, every vector with weight (or number of nonzero elements) j occurs exactly μ_j times ($j=0, 1,\dots, t$) as a column of T^t.*

For the B-array defined above, it is easily shown that $N=\sum_{j=0}^{t}\binom{t}{j}\mu_j$. Thus the term "size" will be omitted if not necessary.

Let $\varepsilon(t_1\cdots t_u;\ t'_1\cdots t'_v)$ be the element of an information matrix $M_T=E_TE'_T$ in the cell corresponding to $(t_1\cdots t_u;\ t'_1\cdots t'_v)$ for $\theta_{t_1\cdots t_u}$ and $\theta_{t'_1\cdots t'_v}$ in $\boldsymbol{\theta}$. Then the following two theorems have been established by Yamamoto, Shirakura and Kuwada [41]:

Theorem 2.1. *Let T be a 2^m-FF design of resolution $2l+1$. Then a neces-*

sary and sufficient condition for T to be balanced is that the information matrix M_T *has at most* $2l+1$ *distinct elements* γ_i $(i=0, 1,\dots, 2l)$ *such that*

$$\gamma_i = \varepsilon(t_1\cdots t_u;\ t'_1\cdots t'_v) \quad \text{if} \quad |\{t_1,\dots, t_u\} \ominus \{t'_1,\dots, t'_v\}| = i.$$

THEOREM 2.2. *A necessary and sufficient condition for* M_T *to be expressible by such elements* γ_i *is that T is a B-array of strength* $2l$, *m constraints and index set* $\{\mu_0, \mu_1,\dots, \mu_{2l}\}$. *A connection between the elements* γ_i *of* M_T *and the indices* μ_j *of a B-array T is given by*

$$\gamma_i = \sum_{j=0}^{2l} \sum_{p=0}^{i} (-1)^p \binom{i}{p} \binom{2l-i}{j-i+p} \mu_j, \tag{2.1}$$

$$\mu_i = \frac{1}{2^{2l}} \sum_{j=0}^{2l} \sum_{p=0}^{j} (-1)^p \binom{i}{j-p} \binom{2l-i}{p} \gamma_j \tag{2.2}$$

for all $i=0, 1,\dots, 2l$.

Throughout this paper we assume $\binom{a}{b}=0$ if and only if $b>a\geqq 0$ or $b<0$. Next we shall make the definition of a 2^m-BFF design of even resolution.

DEFINITION 2.3. *A* 2^m*-FF design T of resolution* $2l$ *is said to be balanced if the covariance matrix* $\mathrm{Var}[\hat{\boldsymbol{\theta}}_0]$ *for T is invariant under any permutation of m factors.*

In Part III, a 2^m-BFF design of even resolution will be discussed in detail.

A 2^m-FF design of resolution $2l+1$ (or $2l$) is said to be orthogonal if the covariance matrix $\mathrm{Var}[\hat{\boldsymbol{\theta}}]$ (or $\mathrm{Var}[\hat{\boldsymbol{\theta}}_0]$) is diagonal in this design. A B-array of strength t, size N, m constraints and index set $\{\mu_0, \mu_1,\dots, \mu_t\}$ reduces to an orthogonal array with parameters $(N, m, 2, t)$ of index μ when $\mu_0=\mu_1=\cdots=\mu_t$ $(=\mu$, say) (see Raghavarao [19]). It is well known (see, e.g., [41]) that an orthogonal array with parameters $(N, m, 2, 2l)$ (or parameters $(N, m, 2, 2l-1)$) of index μ is equivalent to an orthogonal fractional 2^m factorial design of resolution $2l+1$ (or $2l$). However orthogonal arrays with parameters $(N, m, 2, t)$ of index μ are available only for the special numbers $N=2^t\mu$ and the possibility of the existence of such arrays is in general very small. In such a sense, the class of balanced designs arises naturally as the next wide class to be looked into.

3. TMDPB association schemes and TMDPB association algebras

As a generalization of partially balanced association schemes, multidimensional partially balanced association schemes have been first introduced by Bose and Srivastava [3]. Subsequently the theory has been developed in Srivastava and Anderson [31, 32], Yamamoto, Shirakura and Kuwada [41], Yamamoto and Tamari [43].

Consider p mutually disjoint non-null finite sets of objects $S_1, S_2, \dots, S_p$ with $|S_i| = n_i$, each. Suppose that a relation of association is defined for each ordered pair of objects $x_{ia} \in S_i$ and $x_{jb} \in S_j$, and that x_{jb} is called the α-th associate of x_{ia} for some α belonging to a set of association indices $\Pi^{(i,j)}$. As in the case of partially balanced association schemes, every object is called the zeroth associate of itself and $0 \notin \Pi^{(i,i)}$ is assumed. The following definition is due to [41]:

DEFINITION 3.1. *The relation of association defined among the sets S_1, $S_2, \dots, S_p$ is called a p sets multidimensional partially balanced (MDPB) association scheme if the following conditions are satisfied:*

(i) *The relation of association is symmetrical, i.e., if x_{jb} is the α-th associate of x_{ia}, then the x_{ia} is also the α-th associate of x_{jb}.*

(ii) *With respect to any $x_{ia} \in S_i$, the objects of S_j, distinct from x_{ia}, can be divided into $n^{(i,j)}$ distinct classes and the number of objects in the α-th associate class $S_j(\alpha; x_{ia})$ is $n_\alpha^{(i,j)}$. The numbers $n^{(i,j)}$ and $n_\alpha^{(i,j)}$ are independent of the particular object x_{ia} chosen out of S_i.*

(iii) *Let S_i, S_j and S_k be any three sets where they are not necessarily distinct. Consider the sets $S_k(\beta; x_{ia})$ and $S_k(\gamma; x_{jb})$ where $x_{ia} \in S_i$ and $x_{jb} \in S_j$ are the α-th associates. Then the number of objects common to $S_k(\beta; x_{ia})$ and $S_k(\gamma; x_{jb})$ is $p(i, j, \alpha; k, \beta, \gamma)$ which depends on the pair (x_{ia}, x_{jb}) and S_k only through i, j, α, k, β and γ.*

Note that the condition (i) implies $n^{(i,j)} = n^{(j,i)}$ and $p(i, j, \alpha; k, \beta, \gamma) = p(j, i, \alpha; k, \gamma, \beta)$, and that the number $n_0^{(i,i)} = 1$ can be consistently defined for all i.

Now let $S_0, S_1, S_2, \dots$, and S_l be $l+1$ sets of effects $\{\theta_\phi\}$, $\{\theta_{t_1}\}$, $\{\theta_{t_1 t_2}\}, \dots$, and $\{\theta_{t_1 t_2 \cdots t_l}\}$, the cardinalities of these sets being $1, \binom{m}{1}, \binom{m}{2}, \dots$, and $\binom{m}{l}$, respectively. Suppose a relation of association is defined among these sets in a way such that $\theta_{t_1 \cdots t_u} \in S_u$ and $\theta_{t'_1 \cdots t'_v} \in S_v$ are the α-th associates if

$$|\{t_1, \dots, t_u\} \cap \{t'_1, \dots, t'_v\}| = \min(u, v) - \alpha, \tag{3.1}$$

where $\min(u, v)$ denotes the minimum of the integers u and v. Then the following theorem has been established by Yamamoto, Shirakura and Kuwada [41]:

THEOREM 3.1. *Among the $l+1$ sets of effects $\{\theta_\phi\}, \{\theta_{t_1}\}, \{\theta_{t_1 t_2}\}, \dots, \{\theta_{t_1 \cdots t_l}\}$, the relation of association defined by* (3.1) *is an $l+1$ sets MDPB association scheme with parameters*

$$\Pi^{(u,v)} = \begin{cases} \{0, 1, \dots, \min(u, v)\} & \text{if} \quad u \neq v, \\ \{1, 2, \dots, u\} & \text{if} \quad u = v, \end{cases}$$

$$n^{(u,v)}=\begin{cases}\min(u,v)+1 & \text{if } u\neq v,\\ u & \text{if } u=v,\end{cases}$$

$$n_\alpha^{(u,v)}=\binom{u}{\min(u,v)-\alpha}\binom{m-u}{v-\min(u,v)+\alpha},$$

$$p(u,v,\alpha;w,\beta,\gamma)=\sum_{k=0}^{\min(u,v)-\alpha}\binom{\min(u,v)-\alpha}{k}\binom{v-\min(u,v)+\alpha}{\min(u,w)-\beta-k}$$

$$\cdot\binom{v-\min(u,v)+\alpha}{\min(v,w)-\gamma-k}\binom{m-u-v+\min(u,v)-\alpha}{w-\min(u,w)+\beta-\min(v,w)+\gamma+k}.$$

The association thus defined is called an $l+1$ sets triangular type MDPB (TMDPB) association scheme. As seen from Yamamoto, Fujii and Hamada [40], it can be regarded as a generalization of triangular series of association schemes. To investigate the algebraic structure of an $l+1$ sets TMDPB association scheme, first consider the $\binom{m}{u}\times\binom{m}{v}$ matrices $A_\alpha^{(u,v)}=\|a_{t_1\cdots t_u;\alpha}^{t_1'\cdots t_v'}\|$, ($\alpha=0, 1,\dots,\min(u,v)$; $u,v=0,1,\dots,l$), called the local association matrices. Each matrix $A_\alpha^{(u,v)}$ is defined as follows:

$$a_{t_1\cdots t_u;\alpha}^{t_1'\cdots t_v'}=\begin{cases}1 & \text{if } \theta_{t_1'\cdots t_v'} \text{ is the } \alpha\text{-th associate of } \theta_{t_1\cdots t_u},\\ 0 & \text{otherwise.}\end{cases}\tag{3.2}$$

From (3.1) and Theorem 3.1, we have

$$\begin{aligned}&A_0^{(u,u)}=I_{\binom{m}{u}},\\ &A_\alpha^{(v,u)}=(A_\alpha^{(u,v)})',\\ &\sum_{\alpha=0}^{\min(u,v)}A_\alpha^{(u,v)}=G_{\binom{m}{u}\times\binom{m}{v}},\\ &A_\alpha^{(u,v)}j_{\binom{m}{v}}=n_\alpha^{(u,v)}j_{\binom{m}{u}},\\ &A_\beta^{(u,w)}A_\gamma^{(w,v)}=\sum_{\alpha=0}^{\min(u,v)}p(u,v,\alpha;w,\beta,\gamma)A_\alpha^{(u,v)},\end{aligned}\tag{3.3}$$

where $G_{p\times q}$ denotes the $p\times q$ matrix whose elements are all 1 and, particularly, $j_p=G_{p\times 1}$. Next consider the ordered association matrices $D_\alpha^{(u,v)}$ of size $v_l\times v_l$ obtained in a way such that every matrix has $(l+1)^2$ submatrices $M^{(w,s)}$ of size $\binom{m}{w}\times\binom{m}{s}$ in the w-th row block and s-th column block for $w,s=0,1,\dots,l$, and that all but $M^{(u,v)}=A_\alpha^{(u,v)}$ are zero matrices, i.e., $M^{(w,s)}=O_{\binom{m}{w}\times\binom{m}{s}}$ for $(w,s)\neq(u,v)$. Here $O_{p\times q}$ denotes the $p\times q$ matrix whose elements are all 0. Then, from (3.3) we have

(3.4)
$$D_\alpha^{(v,u)} = (D_\alpha^{(u,v)})',$$
$$\sum_{u=0}^{l} D_0^{(u,u)} = I_{v_l},$$
$$\sum_{u=0}^{l} \sum_{v=0}^{l} \sum_{\alpha=0}^{\min(u,v)} D_\alpha^{(u,v)} = G_{v_l \times v_l},$$
$$D_\beta^{(u,w)} S_\gamma^{(s,v)} = \delta_{ws} \sum_{\alpha=0}^{\min(u,v)} p(u, v, \alpha; w, \beta, \gamma) D_\alpha^{(u,v)},$$

where $\delta_{ws}=1$ or 0 according as $w=s$ or not. The association matrices $B_\alpha^{(u,v)}$ which represent the relation of association of an $l+1$ sets TMDPB association scheme can be defined as follows:

$$B_\alpha^{(u,v)} = \begin{cases} D_\alpha^{(v,u)} + D_\alpha^{(u,v)} & \text{if } u \neq v, \\ D_\alpha^{(u,v)} & \text{if } u = v. \end{cases} \tag{3.5}$$

The algebra $\mathfrak{A} = \{B_\alpha^{(u,v)} | \alpha = 0, 1, \ldots, \min(u, v); 0 \leqq u \leqq v \leqq l\}$ generated by $\binom{l+3}{3}$ symmetric matrices $B_\alpha^{(u,v)}$ is called an $l+1$ sets TMDPB association algebra. The following theorem is due to [41]:

THEOREM 3.2. *The $l+1$ sets TMDPB association algebra $\mathfrak{A}$ is a semi-simple, completely reducible matrix algebra. It can be also represented by the linear closure $[D_\alpha^{(u,v)} | \alpha=0, 1, \ldots, \min(u, v); u, v=0, 1, \ldots, l]$ of all $(l+1)(l+2)(2l+3)/6$ ordered association matrices $D_\alpha^{(u,v)}$.*

Now consider the $\binom{m}{u} \times \binom{m}{v}$ matrices $A_\beta^{(u,v)\#}$, ($\beta=0, 1, \ldots, \min(u, v)$; u, $v=0, 1, \ldots, l$), which are linearly linked with the association matrices $A_\alpha^{(u,v)}$ by the following (see [27], [42]):

$$A_\alpha^{(u,v)} = \sum_{u=0}^{\beta} z_{\beta\alpha}^{(u,v)} A_\beta^{(u,v)\#} \quad \text{for } 0 \leqq \alpha \leqq u \leqq v, \tag{3.6}$$

$$A_\beta^{(u,v)\#} = \sum_{\alpha=0}^{u} z_{(u,v)}^{\beta\alpha} A_\alpha^{(u,v)} \quad \text{for } 0 \leqq \beta \leqq u \leqq v, \tag{3.7}$$

$$A_\beta^{(u,v)\#} = (A_\beta^{(v,u)\#})' \quad \text{for } u > v, \tag{3.8}$$

where

$$z_{\beta\alpha}^{(u,v)} = \sum_{b=0}^{\alpha} (-1)^{\alpha-b} \frac{\binom{u-\beta}{b}\binom{u-b}{u-\alpha}\binom{m-u-\beta+b}{b}\left\{\binom{m-u-\beta}{v-u}\binom{v-\beta}{v-u}\right\}^{\frac{1}{2}}}{\binom{v-u+b}{b}}, \tag{3.9}$$

$$z^{\beta\alpha}_{(u,v)} = \frac{\phi^{\beta} z^{(u,v)}_{\beta\alpha}}{\binom{m}{u}\binom{u}{\alpha}\binom{m-u}{v-u+\alpha}}. \tag{3.10}$$

Here $\phi_\beta = \binom{m}{\beta} - \binom{m}{\beta-1}$. Then the matrices $A^{(u,v)\#}_{\beta}$ have the following properties:

$$\begin{aligned} &\sum_{\beta=0}^{u} A^{(u,u)\#}_{\beta} = I_{\binom{m}{u}}, \\ &A^{(u,v)\#}_{0} = \left\{\binom{m}{u}\binom{m}{v}\right\}^{-1/2} G_{\binom{m}{u}\times\binom{m}{v}}, \\ &A^{(u,w)\#}_{\alpha} A^{(w,v)\#}_{\beta} = \delta_{\alpha\beta} A^{(u,v)\#}_{\beta}, \\ &\text{rank}\,(A^{(u,v)\#}_{\beta}) = \phi_\beta, \end{aligned} \tag{3.11}$$

$$A^{(u,v)\#}_{\beta} = c^{(u,v)}_{\beta} A^{(u,u)\#}_{\beta} A^{(u,v)}_{0} \qquad \text{for} \quad u \leqq v, \tag{3.12}$$

where

$$c^{(u,v)}_{\beta} = \left\{\binom{m-u-\beta}{v-u}\binom{v-\beta}{v-u}\right\}^{-1/2}.$$

Let $D^{(u,v)\#}_{\beta}$ be the matrices obtained by replacing the only nonzero submatrix $A^{(u,v)}_{\beta}$ of $D^{(u,v)}_{\beta}$ by $A^{(u,v)\#}_{\beta}$. From (3.6)–(3.11), we have

$$D^{(u,v)}_{\alpha} = \sum_{\beta=0}^{u} z^{(u,v)}_{\beta\alpha} D^{(u,v)\#}_{\beta} \qquad \text{for} \quad 0 \leqq \alpha \leqq u \leqq v, \tag{3.13}$$

$$D^{(u,v)\#}_{\beta} = \sum_{\alpha=0}^{u} z^{\beta\alpha}_{(u,v)} D^{(u,v)}_{\alpha} \qquad \text{for} \quad 0 \leqq \beta \leqq u \leqq v, \tag{3.14}$$

$$\sum_{u=0}^{l-k} \sum_{\beta=0}^{u} D^{(u,u)\#}_{\beta} = \begin{cases} I_{\nu_l} & \text{if} \quad k = 0, \\ \text{diag}\,[I_{\nu_{l-k}}, O_{p_k \times p_k}] & \text{if} \quad 1 \leqq k \leqq l, \end{cases} \tag{3.15}$$

where $p_k = \sum_{i=0}^{k-1} \binom{m}{l-i}$, and

$$\begin{aligned} &D^{(v,u)\#}_{\beta} = (D^{(u,v)\#}_{\beta})', \\ &D^{(u,w)\#}_{\alpha} D^{(s,v)\#}_{\beta} = \delta_{ws}\delta_{\alpha\beta} D^{(u,v)\#}_{\beta}, \\ &\text{rank}\,(D^{(u,v)\#}_{\beta}) = \phi_\beta. \end{aligned} \tag{3.16}$$

From Theorem 3.2 and (3.13)–(3.16), the following theorem can be established (cf. [42]):

THEOREM 3.3.

(i) *The* $l+1$ *sets TMDPB association algebra* $\mathfrak{A}$ *is represented by the linear closure of all* $(l+1)(l+2)(2l+3)/6$ *matrices* $D_\beta^{(u,v)\#}$, *i.e.,*

$$\mathfrak{A} = [D_\beta^{(u,v)\#} | \beta = 0, 1, \ldots, \min(u, v);\ u, v = 0, 1, \ldots, l].$$

(ii) *Let* $\mathfrak{A}_\beta$ *be the matrix algebra generated by* $(l-\beta+1)^2$ *matrices* $D_\beta^{(u,v)\#}$ *for each* $\beta=0, 1, \ldots, l$, *i.e.,*

$$\mathfrak{A}_\beta = [D_\beta^{(u,v)\#} | u, v = \beta, \beta+1, \ldots, l],$$

then $\mathfrak{A}_\beta$ *is the minimal two-sided ideal of* $\mathfrak{A}$ *and*

$$\mathfrak{A}_\alpha \mathfrak{A}_\beta = \mathfrak{A}_\beta \mathfrak{A}_\alpha = \delta_{\alpha\beta} \mathfrak{A}_\beta.$$

(iii) *The algebra* $\mathfrak{A}$ *is decomposed into the direct sum of* $l+1$ *ideals* $\mathfrak{A}_\beta$, *i.e.,*

$$\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_l.$$

(iv) *Each ideal* $\mathfrak{A}_\beta$ *has* $D_\beta^{(u,v)\#}$ $(u, v=\beta, \beta+1, \ldots, l)$ *as its basis and it is isomorphic to the complete* $(l-\beta+1)\times(l-\beta+1)$ *matrix algebra with multiplicity* $\phi_\beta = \binom{m}{\beta} - \binom{m}{\beta-1}$.

This theorem implies that for any matrix $B\ (=\sum_{\beta=0}^{l} \sum_{i=0}^{l-\beta} \sum_{j=0}^{l-\beta} \lambda_\beta^{i,j} D_\beta^{(u,v)\#}$, say) belonging to $\mathfrak{A}$, there exists a $v_l \times v_l$ orthogonal matrix P such that

$$P'BP = \operatorname{diag}[\Lambda_0;\ \underbrace{\Lambda_1, \ldots, \Lambda_1}_{\phi_1};\ \cdots;\ \underbrace{\Lambda_l, \ldots, \Lambda_l}_{\phi_\beta}], \tag{3.17}$$

where Λ_β are the $(l-\beta+1)\times(l-\beta+1)$ matrix with (i, j) elements $\lambda_\beta^{i,j}$. The matrix Λ_β is called the irreducible representation of B with respect to each ideal $\mathfrak{A}_\beta$, for which we shall use the following notation:

$$\mathfrak{A}_\beta\colon B \longrightarrow \Lambda_\beta.$$

4. The irreducible representations of the information matrices for B-arrays of strength $2l$

Now consider a B-array T of strength $2l$, m constraints and index set $\{\mu_0, \mu_1, \ldots, \mu_{2l}\}$. Further consider the information matrix M_T for the B-array T as a design. In this section we shall obtain the irreducible representations of M_T with respect to ideals $\mathfrak{A}_\beta$. They will occur in later discussions frequently.

From Theorem 2.1 and (3.1), it is easy to see that if two effects $\theta_{t_1 \cdots t_u}$ and $\theta_{t'_1 \cdots t'_v}$ are the α-th associates, then

$$\varepsilon(t_1\cdots t_u;\ t'_1\cdots t'_v) = \gamma_\omega,$$

where $\omega=|u-v|+2\alpha$, γ_i are given in (2.1) and $\varepsilon(t_1\cdots t_u;\ t'_1\cdots t'_v)$ is the element of M_T corresponding to $\theta_{t_1\cdots t_u}$ and $\theta_{t'_1\cdots t'_v}$. From the definition of association matrices $D_\alpha^{(u,v)}$, therefore, M_T can be expressed as

$$M_T = \sum_{u=0}^{l}\sum_{v=0}^{l}\sum_{\alpha=0}^{\min(u,v)} \gamma_\omega D_\alpha^{(u,v)}.$$

Hence it follows from Theorem 3.2 that the information matrix M_T belongs to the $l+1$ sets TMDPB association algebra $\mathfrak{A}$. From (3.13) M_T can be also expressed as

$$M_T = \sum_{\beta=0}^{l}\sum_{i=0}^{l-\beta}\sum_{j=0}^{l-\beta} \kappa_\beta^{i,j} D_\beta^{(\beta+i,\beta+j)\#}. \tag{4.1}$$

Here

$$\kappa_\beta^{i,j} = \kappa_\beta^{j,i} = \sum_{\alpha=0}^{\beta+i} \gamma_{j-i+2\alpha} z_{\beta\alpha}^{(\beta+i,\beta+j)} \qquad \text{for } 0 \leqq i \leqq j \leqq l-\beta;\ 0 \leqq \beta \leqq l, \tag{4.2}$$

where $z_{\beta\alpha}^{(u,v)}$ are given in (3.9). From Theorem 3.3, therefore, we can obtain the $(l-\beta+1)\times(l-\beta+1)$ symmetric matrices K_β ($\beta=0, 1,\ldots, l$) such that for the B-array T,

$$\mathfrak{A}_\beta\colon M_T \longrightarrow K_\beta,$$

where

$$K_\beta = \begin{bmatrix} \kappa_\beta^{0,0} & \kappa_\beta^{0,1} & \cdots & \kappa_\beta^{0,l-\beta} \\ \vdots & \vdots & & \vdots \\ \kappa_\beta^{l-\beta,0} & \kappa_\beta^{l-\beta,1} & \cdots & \kappa_\beta^{l-\beta,l-\beta} \end{bmatrix}. \tag{4.3}$$

In particular the matrices K_β for the cases $l=2$, 3 are important. Therefore explicit expressions of K_β for $l=2$, 3 are presented in the following example:

EXAMPLE 4.1.

(i) The case $l=2$.

$$\underset{(3\times 3)}{K_0} = \begin{bmatrix} \gamma_0 & m^{1/2}\gamma_1 & \left(\dfrac{m}{2}\right)^{1/2}\gamma_2 \\ & \gamma_0+(m-1)\gamma_2 & \left(\dfrac{m-1}{2}\right)^{1/2}\{2\gamma_1+(m-2)\gamma_3\} \\ (\text{Sym.}) & & \gamma_0+2(m-2)\gamma_2+\dbinom{m-2}{2}\gamma_4 \end{bmatrix},$$

$$\underset{(2\times 2)}{K_1} = \begin{bmatrix} \gamma_0-\gamma_2 & (m-2)^{1/2}(\gamma_1-\gamma_3) \\ \text{(Sym.)} & \gamma_0+(m-4)\gamma_2-(m-3)\gamma_4 \end{bmatrix},$$

$$K_2 = \gamma_0-2\gamma_2+\gamma_4 = 2^4\mu_2,$$

where

$$\gamma_0 = N = \mu_4+\mu_0+4(\mu_3+\mu_1)+6\mu_2, \qquad \gamma_1 = \mu_4-\mu_0+2(\mu_3-\mu_1),$$

$$\gamma_2 = \mu_4+\mu_0-2\mu_2, \qquad \gamma_3 = \mu_4-\mu_0-2(\mu_3-\mu_1),$$

$$\gamma_4 = \mu_4+\mu_0-4(\mu_3+\mu_1)+6\mu_2.$$

(ii) The case $l=3$.

$$\underset{(4\times 4)}{K_0} = \begin{bmatrix} \gamma_0 & m^{1/2}\gamma_1 & \binom{m}{2}^{1/2}\gamma_2 & \binom{m}{3}^{1/2}\gamma_3 \\ & \gamma_0+(m-1)\gamma_2 & \left(\frac{m-1}{2}\right)^{1/2}\{2\gamma_1+(m-2)\gamma_3\} & \left\{\binom{m-1}{2}/3\right\}^{1/2}\{3\gamma_2+(m-3)\gamma_4\} \\ & & \gamma_0+2(m-2)\gamma_2+\binom{m-2}{2}\gamma_4 & \left(\frac{m-2}{3}\right)^{1/2}\left\{3\gamma_1+3(m-3)\gamma_3+\binom{m-3}{2}\gamma_5\right\} \\ \text{(Sym.)} & & & \gamma_0+3(m-3)\gamma_2+3\binom{m-3}{2}\gamma_4+\binom{m-3}{3}\gamma_6 \end{bmatrix},$$

$$\underset{(3\times 3)}{K_1} = \begin{bmatrix} \gamma_0-\gamma_2 & (m-2)^{1/2}(\gamma_1-\gamma_3) \\ & \gamma_0+(m-4)\gamma_2-(m-3)\gamma_4 \\ \text{(Sym.)} & \end{bmatrix}$$

$$\left.\begin{matrix} \left(\frac{m-2}{2}\right)^{1/2}(\gamma_2-\gamma_4) \\ \left(\frac{m-3}{2}\right)^{1/2}\{2\gamma_1+(m-6)\gamma_3-(m-4)\gamma_5\} \\ \gamma_0+(2m-9)\gamma_2+\frac{(m-4)(m-9)}{2}\gamma_4-\binom{m-4}{2}\gamma_6 \end{matrix}\right],$$

$$\underset{(2\times 2)}{K_2} = \begin{bmatrix} \gamma_0-2\gamma_2+\gamma_4 & (m-4)^{1/2}(\gamma_1-2\gamma_3+\gamma_5) \\ \text{(Sym.)} & \gamma_0+(m-7)\gamma_2-(2m-11)\gamma_4+(m-5)\gamma_6 \end{bmatrix},$$

$$K_3 = \gamma_0-3\gamma_2+3\gamma_4-\gamma_6,$$

where

$$\gamma_0 = \mu_6+\mu_0+6(\mu_5+\mu_1)+15(\mu_4+\mu_2)+20\mu_3,$$

$$\gamma_1 = \mu_6-\mu_0+4(\mu_5-\mu_1)+5(\mu_4-\mu_2),$$

$$\gamma_2 = \mu_6+\mu_0+2(\mu_5+\mu_1)-(\mu_4+\mu_2)-4\mu_3,$$

$$\gamma_3 = \mu_6-\mu_0-3(\mu_4-\mu_2), \qquad \gamma_4 = \mu_6+\mu_0-2(\mu_5+\mu_1)-(\mu_4+\mu_2)+4\mu_3,$$

$$\gamma_5 = \mu_6-\mu_0-4(\mu_5-\mu_1)+5(\mu_4-\mu_2), \qquad \gamma_6 = \mu_6+\mu_0-6(\mu_5+\mu_1)+$$
$$+15(\mu_4+\mu_2)-20\mu_3.$$

Part II. 2^m-BFF designs of odd resolution and their optimalities

5. Various properties derived from irreducible representations of the information matrices of 2^m-BFF designs of resolution $2l+1$

For a B-array T of strength $2l$, m constraints and index set $\{\mu_0, \mu_1, \dots, \mu_{2l}\}$, we have observed in Section 2 that T is a 2^m-BFF design of resolution $2l+1$ if and only if its information matrix M_T is nonsingular. We now proceed to consider the characteristic polynomial of M_T of a 2^m-BFF design of resolution $2l+1$ which will make it possible to investigate the balanced designs of higher resolution.

Since $I_{v_l} \in \mathfrak{A}$, if follows that

$$\mathfrak{A}_\beta\colon M_T-\lambda I_{v_l} \longrightarrow K_\beta-\lambda I_{l-\beta+1}.$$

From Theorem 3.3, we have the following theorem (cf. [42]):

THEOREM 5.1. *The characteristic polynomial $\Psi(\lambda)$ of the information matrix M_T of a 2^m-BFF design T of resolution $2l+1$ is given by*

$$\Psi(\lambda) = \det(M_T - \lambda I_{\nu_l}) = \prod_{\beta=0}^{l} \{\det(K_\beta - \lambda I_{l-\beta+1})\}^{\phi_\beta}, \tag{5.1}$$

where $\det(\cdot)$ *stands for the determinant of a matrix.*

From this theorem, we can easily establish the following:

THEOREM 5.2. *Let T be the design of Theorem 5.1. Then*

$$\mathrm{tr}(V_T) = \mathrm{tr}(M_T^{-1}) = \sum_{\beta=0}^{l} \phi_\beta \mathrm{tr}(K_\beta^{-1}), \tag{5.2}$$

$$\det(V_T) = \det(M_T^{-1}) = \prod_{\beta=0}^{l} \{\det(K_\beta^{-1})\}^{\phi_\beta}, \tag{5.3}$$

where $\mathrm{tr}(\cdot)$ *stands for the trace of a matrix.*

From (1.13) we may note that for any 2^m-FF design T of resolution $2l+1$, $\mathrm{tr}(V_T)$ is proportional to the average of the variances of all normalized linear functions of the effects $\theta_{t_1 t_2 \cdots t_k}$ $(k \leqq l)$. On the other hand, $\det(V_T)$ is proportional to the volume of the ellipsoid of concentration (see Cramér [11]). That is, it corresponds to the volume of the region within which the true parametric point may lie with a certain probability. In such a sense, a design T is said to be optimal with respect to the trace or determinant criterion if it minimizes $\mathrm{tr}(V_T)$ or $\det(V_T)$, respectively. It is well known that in the class of all 2^m-FF designs of resolution $2l+1$ with N assemblies, an orthogonal design is optimal with respect to the above two criteria. For studies on optimal designs using various criteria, see, e.g., Hedayat, Raktoe and Federer [13], Kiefer [14, 15], Raktoe and Federer [20], Shirakura [25], Srivastava and Anderson [30, 33].

Let $\bar{T}$ be the matrix obtained from T by interchanging symbols 0 and 1. $\bar{T}$ is called the complement of T. It is easy to see that if T is a B-array of strength $2l$ with indices μ_i, then $\bar{T}$ is that of strength $2l$ with indices $\bar{\mu}_i = \mu_{2l-i}$ $(i=0, 1, \ldots, 2l)$. Furthermore if T is a 2^m-BFF design of resolution $2l+1$, then $\bar{T}$ is also so. Therefore $\bar{T}$ is called the complementary balanced design of T.

THEOREM 5.3. *For a 2^m-BFF design T of resolution $2l+1$ and its complementary design $\bar{T}$,*

$$\begin{aligned} \mathrm{tr}(V_T) &= \mathrm{tr}(V_{\bar{T}}), \\ \det(V_T) &= \det(V_{\bar{T}}). \end{aligned} \tag{5.4}$$

PROOF. This follows immediately from Theorem 3.2 in Shirakura and Kuwada [26].

As will be seen later, this theorem is useful for finding optimal 2^m-BFF designs of resolution VII with respect to the trace and determinant criteria. It may be remarked that (5.4) holds for more general fractional designs (see Srivastava, Raktoe and Pesotan [38]).

From the definition of balanced designs, it follows that T is a 2^m-BFF design of resolution $2l+1$ if and only if $V_T \in \mathfrak{A}$. Thus it is clear that the covariance matrix $\mathrm{Var}[\hat{\boldsymbol{\theta}}]=\sigma^2 V_T$ has at most $\binom{l+3}{3}$ distinct elements. Also we have

$$\mathfrak{A}_\beta:\ \mathrm{Var}[\hat{\boldsymbol{\theta}}] \longrightarrow \sigma^2 K_\beta^{-1}.$$

Using the inverse matrices K_β^{-1}, Shirakura and Kuwada [27] have obtained explicit expressions for all the distinct elements of V_T. That is, let $\kappa_{i,j}^\beta$ be (i, j) elements of K_β^{-1} and let $V_\alpha^{(u,v)}$ be the element of V_T corresponding to $\theta_{t_1\cdots t_u}$ and $\theta_{t'_1\cdots t'_v}$ which are the α-th associates. Then we have

THEOREM 5.4. *For a 2^m-BFF design of resolution $2l+1$,*

$$V_\alpha^{(u,v)} = \sum_{\beta=0}^{u} \kappa_{u-\beta,v-\beta}^{\beta} z_{(u,v)}^{\beta\alpha} \qquad \text{for} \quad 0 \leqq \alpha \leqq u \leqq v \leqq l, \tag{5.5}$$

where $z_{(u,v)}^{\beta\alpha}$ are given in (3.10).

Following a usual procedure in the calculation of $\mathrm{Var}[\hat{\boldsymbol{\theta}}]$, $\mathrm{tr}(\mathrm{Var}[\hat{\boldsymbol{\theta}}])$ and $\det(\mathrm{Var}[\hat{\boldsymbol{\theta}}])$, we have to calculate the inverse of a large $\nu_l \times \nu_l$ $\left(\nu_l = 1+\binom{m}{1}+\cdots+\binom{m}{l}\right)$ matrix M_T. However the expressions of (5.2), (5.3) and (5.5) imply that we have only to calculate the inverse of at most $(l+1)\times(l+1)$ matrix, i.e., K_0. Note that the sizes of matrices K_β do not depend on the number of m factors. For more explicit expressions of $V_\alpha^{(n,v)}$ for the cases $l=2, 3$, see [27].

In the following discussion we shall investigate some combinatorial properties which are useful for obtaining 2^m-BFF designs of resolution $2l+1$. Further deep investigations will be discussed in Sections 6, 7 and 8.

The matrices K_β are obviously dependent on the constraints m and indices μ_i $(i=0, 1,\dots, 2l)$ of a B-array T. The information matrix M_T is in general positive semidefinite. From (5.1), we can establish the following theorems:

THEOREM 5.5. *Let T be a B-array of strength $2l$, m constraints and index set $\{\mu_0, \mu_1,\dots, \mu_{2l}\}$. Then a necessary condition for the existence of T is that every matrix K_β $(\beta=0, 1,\dots, l)$ is positive semidefinite.*

THEOREM 5.6. *Consider the B-array T of Theorem 5.5. Then a necessary and sufficient condition for T to be a 2^m-BFF design of resolution $2l+1$ is that every matrix K_β is positive definite.*

From (2.1), (2.2), (3.9) and (4.2), after some calculations, we can express the elements of K_β in terms of the m and μ_i ($i=0, 1,\ldots, 2l$). For example

(5.6) $$K_l = \kappa_l^{0,0} = 2^{2l}\mu_l,$$

(5.7a) $$\kappa_{l-1}^{0,0} = 2^{2l-2}(\mu_{l+1}+\mu_{l-1}+2\mu_l),$$

(5.7b) $$\kappa_{l-1}^{0,1} = \kappa_{l-1}^{1,0} = 2^{2l-2}(m-2l+2)^{1/2}(\mu_{l+1}-\mu_{l-1}),$$

(5.7c) $$\kappa_{l-1}^{1,1} = 2^{2l-2}\{(m-2l+2)(\mu_{l+1}+\mu_{l-1})-2(m-2l)\mu_l\},$$

(5.8a) $$\kappa_{l-2}^{0,0} = 2^{2l-4}\{\mu_{l+2}+\mu_{l-2}+4(\mu_{l+1}+\mu_{l-1})+6\mu_l\},$$

(5.8b) $$\kappa_{l-2}^{0,1} = \kappa_{l-2}^{1,0} = 2^{2l-4}(m-2l+4)^{1/2}\{\mu_{l+2}-\mu_{l-2}+2(\mu_{l+1}-\mu_{l-1})\},$$

(5.8c) $$\kappa_{l-2}^{0,2} = \kappa_{l-2}^{2,0} = 2^{2l-4}\binom{m-2l+4}{2}^{1/2}(\mu_{l+2}+\mu_{l-2}-2\mu_l),$$

(5.8d) $$\kappa_{l-2}^{1,1} = 2^{2l-4}\{(m-2l+4)(\mu_{l+2}+\mu_{l-2})+4(\mu_{l+1}+\mu_{l-1})-2(m-2l)\mu_l\},$$

(5.8e) $$\kappa_{l-2}^{1,2} = \kappa_{l-2}^{2,1} = 2^{2l-4}\left(\frac{m-2l+3}{2}\right)^{1/2}\{(m-2l+4)(\mu_{l+2}-\mu_{l-2}) \\ -2(m-2l)(\mu_{l+1}-\mu_{l-1})\},$$

(5.8f) $$\kappa_{l-2}^{2,2} = 2^{2l-4}\left[\binom{m-2l+4}{2}(\mu_{l+2}+\mu_{l-2}) \\ -2(m-2l)(m-2l+3)(\mu_{l+1}+\mu_{l-1})+\{3(m-2l)^2+5(m-2l)+4\}\mu_l\right].$$

From (5.6)–(5.8), we thus have as immediate corollaries of Theorem 5.5 and 5.6 the following:

COROLLARY 5.7. *A set of necessary conditions for the existence of the B-array T of Theorem 5.5 is that the following inequalities hold*:

(5.9) $$\mu_l \geqq 0,$$

(5.10a) $$(m-2l+2)(\mu_{l+1}+\mu_{l-1}) \geqq 2(m-2l)\mu_l,$$

(5.10b) $$(m-2l+2)\mu_{l+1}\mu_{l-1}+(\mu_{l+1}\mu_l+\mu_l\mu_{l-1}) \geqq (m-2l)\mu_l^2,$$

(5.11a) $$(m-2l+4)(\mu_{l+2}+\mu_{l-2})+4(\mu_{l+1}+\mu_{l-1}) \geqq 2(m-2l)\mu_l \quad \textit{for} \quad l \geqq 2,$$

(5.11b) $$\binom{m-2l+4}{2}(\mu_{l+2}+\mu_{l-2})+\{3(m-2l)^2+5(m-2l)+4\}\mu_l \\ \geqq 2(m-2l)(m-2l+3)(\mu_{l+1}+\mu_{l-1}) \quad \textit{for} \quad l \geqq 2.$$

COROLLARY 5.8. *A set of necessary conditions for the B-array of Theorem*

5.5 to be a 2^m-BFF design of resolution $2l+1$ is that the inequalities (5.9)–(5.11) *hold with strict inequality in each case.*

From the rest of elements of K_β, we can obtain results similar to Corollaries 5.7 and 5.8. However they are very complicated and will make this paper unduly lengthy.

6. Existence conditions for B-arrays of strength t

For a (0, 1) matrix T of size $m \times N$, let $\tau(i_1, i_2, \dots, i_k; T)$, $(1 \leqq k \leqq m)$, denote the number of times the vector $\boldsymbol{v}$ occurs as a column of T where $\boldsymbol{v}$ contains 1 exactly at the i_1-th, i_2-th,..., i_k-th positions and 0 elsewhere. In particular $\tau(\phi; T)$ denotes the number of times the vector of weight 0 occurs as a column of T. Whenever no emphasis on T is needed, we shall simply write $\tau^m(i_1, i_2, \dots, i_k) = \tau(i_1, i_2, \dots, i_k; T)$. The following two theorems are due to Srivastava [29]:

THEOREM 6.1. *A necessary and sufficient condition for the existence of a B-array T of strength t, $m=t+1$ constraints and index set $\{\mu_0, \mu_1, \dots, \mu_t\}$ is that there exists an integer d such that*

$$(6.1)\qquad \begin{aligned} d &\leqq \psi_{11} = \max_{1 \leqq 2r \leqq t+1} \{0, \sum_{q=0}^{2r-1} (-1)^q \mu_q\}, \\ d &\leqq \psi_{12} = \min_{0 \leqq 2r \leqq t} \{\sum_{q=0}^{2r} (-1)^q \mu_q\}. \end{aligned}$$

Also if there exists an integer d which satisfies (6.1), *then*

$$(6.2)\qquad \begin{aligned} &\tau^{t+1}(i_1, i_2, \dots, i_k) = \sum_{q=1}^{k} (-1)^{k+q} \mu_{q-1} + (-1)^k d \qquad \text{for} \quad 1 \leqq k \leqq t+1, \\ &\tau^{t+1}(\phi) = d. \end{aligned}$$

THEOREM 6.2. *A necessary and sufficient condition for the existence of a B-array T of strength t, $m=t+2$ constraints and index set $\{\mu_0, \mu_1, \dots, \mu_t\}$ is that there exist integers d and d_i $(i=1, 2, \dots, t+2)$ such that*

$$(6.3)\qquad \begin{aligned} &\text{(a)} \quad \psi_{12} \geqq d_i \geqq \psi_{11}, \\ &\text{(b)} \quad d \geqq \psi_{21} = \max_{2 \leqq 2r \leqq t+2} \{0, \sum_{q=0}^{2r-1} (-1)^q q \mu_{2r-1-q} + \max_{\{i_1, \dots, i_{2r}\} \in \mathfrak{M}^1_{2r}} (\sum_{\alpha=0}^{2r} d_{i_\alpha})\}, \\ &\qquad d \leqq \psi_{22} = \min_{0 \leqq 2r \leqq t+1} \{\sum_{q=0}^{2r} (-1)^{q+1} q \mu_{2r-q} \end{aligned}$$

$$+ \min_{\{i_1,\ldots,i_{2r+1}\}\in\mathfrak{M}^1_{2r+1}} \left(\sum_{\alpha=0}^{2r+1} d_{i_\alpha}\right)\},$$

where $\mathfrak{M}^1_k$ *denotes the collection of all subsets of* $\{1, 2,\ldots, t+2\}$ *with cardinality* k. *Also if there exist integers* d *and* d_i *which satisfy* (6.3), *then*

$$\tau^{t+2}(i_1, i_2,\ldots, i_k) = \sum_{q=0}^{k-1}(-1)^{q+1}q\mu_{k-1-q}+(-1)^{k+1}\sum_{\alpha=1}^{k} d_{i_\alpha}+(-1)^k d$$

(6.4) $\qquad$ *for* $1 \leqq k \leqq t+2,$

$$\tau^{t+2}(\phi) = d.$$

DEFINITION 6.1. *For two* (0, 1) *matrices* T_1 *and* T_2 *of size* $m\times N$, T_1 *is said to be isomorphic to* T_2 *if there exist the permutation matrices* Q_1 *and* Q_2 *of size* $m\times m$ *and* $N\times N$, *respectively, such that* $Q_1T_1=T_2Q_2$ *holds.*

From (6.2) and (6.4), we can easily prove the following two corollaries:

COROLLARY 6.3. *The number of nonisomorphic B-arrays of strength* t, $m=t+1$ *constraints and index set* $\{\mu_0, \mu_1,\ldots, \mu_t\}$ *is equal to that of integers* d *satisfying* (6.1).

COROLLARY 6.4. *The number of nonisomorphic B-arrays of strength* t, $m=t+2$ *constraints and index set* $\{\mu_0, \mu_1,\ldots, \mu_t\}$ *is equal to that of sets* $\{d, d_1, d_2,\ldots, d_{t+2}\}$ *such that* d *and* d_i *satisfy* (6.3a, b).

In Theorem 6.2, without loss of generality, we can assume $d_1\geqq d_2\geqq\cdots\geqq d_{t+2}$. Thus we have the following

COROLLARY 6.5. *A necessary and sufficient condition for the existence of a B-array* T *of strength* t, $m=t+2$ *constraints and index set* $\{\mu_0, \mu_1,\ldots, \mu_t\}$ *is that there exist integers* d', d'_i $(i=1,\ldots, t+2)$ *such that*

$$\psi_{12} \geqq d'_1 \geqq d'_2 \geqq\cdots\geqq d'_{t+2} \geqq \psi_{11},$$

(6.5) $$d' \geqq \psi'_{21} = \max_{2\leqq 2r\leqq t+2}\{0, \sum_{q=1}^{2r-1}(-1)^q q\mu_{2r-1-q}+\sum_{i=1}^{2r} d'_i\},$$

$$d' \geqq \psi'_{22} = \min_{0\leqq 2r\leqq t+1}\{\sum_{q=2}^{2r}(-1)^{q+1}q\mu_{2r-q}+\sum_{i=0}^{2r} d'_{t+2-i}\}.$$

As a generalization of Theorem 6.2 and 6.3, we now prove the following theorem:

THEOREM 6.6. *Let* $\mathfrak{M}^2_k$ *be the collection of all subsets of* $\{1, 2,\ldots, t+3\}$ *with cardinality* k *and let* $\mathfrak{M}^{(i)}_k$ *be that of* $\{1, 2,\ldots, t+3\}-\{i\}$, $(1\leqq i\leqq t+3)$. *Then a necessary and sufficient condition for the existence of a B-array* T *of* *strength t, $m=t+3$ constraints and index set $\{\mu_0, \mu_1,\dots, \mu_t\}$ is that there exist integers d, d_i and $d_{i,j}$ $(i, j=1, 2,\dots, t+3;\ i<j)$ such that*

$$\text{(6.6)}\qquad \begin{aligned} &\text{(a)}\quad \psi_{11} \leqq d_{i,j} \leqq \psi_{12}, \\ &\text{(b)}\quad \psi_{21}^{(i)} \leqq d_i \leqq \psi_{22}^{(i)} \qquad \text{for}\quad i = 1, 2,\dots, t+3, \\ &\text{(c)}\quad \psi_{31} \leqq d \leqq \psi_{32}, \end{aligned}$$

where

$$\text{(6.7)}\qquad \begin{aligned} \psi_{21}^{(i)} &= \max_{2\leqq 2r\leqq t+2} \{0, \sum_{q=0}^{2r-1}(-1)^q q\mu_{2r-1-q}+\tilde{d}_{2r}^{(i)}\}, \\ \psi_{22}^{(i)} &= \min_{0\leqq 2r\leqq t+1} \{\sum_{q=0}^{2r}(-1)^{q+1} q\mu_{2r-q}+\underset{\sim}{d}{}_{2r+1}^{(i)}\}, \end{aligned}$$

$$\text{(6.8)}\qquad \begin{aligned} \psi_{31} &= \max_{2\leqq 2r\leqq t+3} \{0, (\sum_{q=0}^{2r-1}(-1)^{q+1}\binom{q}{2}\mu_{2r-1-q}+\tilde{d}_{2r})\}, \\ \psi_{32} &= \min_{2\leqq 2r\leqq t+2} \{(\sum_{q=0}^{2r}(-1)^{q}\binom{q}{2}\mu_{2r-q}+\underset{\sim}{d}{}_{2r+1}),\ \min_{\{i_1\}\in\mathfrak{M}_1^2} d_{i_1}\}. \end{aligned}$$

Here

$$\tilde{d}_k^{(i)} = \max_{\{j_1,\dots,j_k\}\in\mathfrak{M}_k^{(i)}} \{\sum_{\alpha=0}^{k} d_{i,j_\alpha}\}, \qquad \underset{\sim}{d}{}_k^{(i)} = \min_{\{j_1,\dots,j_k\}\in\mathfrak{M}_k^{(i)}} \{\sum_{\alpha=0}^{k} d_{i,j_\alpha}\},$$

$$\tilde{d}_k = \max_{\{i_1,\dots,i_k\}\in\mathfrak{M}_k^2} \{\sum_{\alpha=1}^{k} d_{i_\alpha} - \sum_{\alpha,\beta=1}^{k} d_{i_\alpha,i_\beta}\},$$

$$\underset{\sim}{d}{}_k = \min_{\{i_1,\dots,i_k\}\in\mathfrak{M}_k^2} \{\sum_{\alpha=1}^{k} d_{i_\alpha} - \sum_{\alpha,\beta=1}^{k} d_{i_\alpha,i_\beta}\}.$$

Also if there exist integers d, d_i and $d_{i,j}$ satisfying (6.6a, b, c), *then*

$$\text{(6.9)}\qquad \begin{aligned} \tau^{t+3}(i_1, i_2,\dots, i_k) &= \sum_{q=0}^{k-1}(-1)^q\binom{q}{2}\mu_{k-1-q}+(-1)^k\sum_{\substack{\alpha,\beta=1\\ \alpha<\beta}}^{k} d_{i_\alpha,i_\beta} \\ &\quad +(-1)^{k+1}\sum_{\alpha=1}^{k} d_{i_\alpha}+(-1)^k d \qquad \textit{for}\quad 2\leqq k\leqq t+3, \end{aligned}$$

$$\tau^{t+3}(i_1) = d_{i_1}-d,$$

$$\tau^{t+3}(\phi) = d.$$

PROOF. Let $T^{(i)}$ and $T^{(i,j)}$ $(i, j=1, 2,\dots, t+3;\ i<j)$ be $(t+2)\times N$ and $(t+1)\times N$ matrices obtained from T by omitting the i-th row and the i-th and j-th rows, respectively. Let d_i and $d_{i,j}$ be the numbers of column vectors with weight 0 of $T^{(i)}$ and $T^{(i,j)}$, respectively. If T is a B-array of strength t, then

$T^{(i)}$ and $T^{(i,j)}$ are also of strength t. Thus from Theorem 6.1 and 6.2 it follows that for the B-array $T^{(i)}$, the integers $d_{i,j}$ and d_i must satisfy (6.3a) and (6.3b) (or (6.6a) and (6.6b)). For such integers d_i and $d_{i,j}$, therefore, a necessary and sufficient condition for the existence of a B-array T with indicated indices is equivalent to that there exist nonnegative integers $\tau(i_1, i_2,\ldots, i_k)$ such that the following equations hold:

$$\tau(i_1)+d = d_{i_1},$$

$$\tau(i_1, i_2)+\tau(i_1)+\tau(i_2)+d = d_{i_1,i_2},$$

$$\tau(i_1, i_2, i_3)+\tau(i_1, i_2)+\tau(i_1, i_3)+\tau(i_2, i_3)+\tau(i_1)+\tau(i_2),$$
$$+\tau(i_3)+d = \mu_0,$$

in general, for all permissible k,

$$\tau(i_1, i_2, i_3, i_4,\ldots, i_k)+\tau(i_1, i_2, i_4,\ldots, i_k)+\tau(i_1, i_3, i_4,\ldots, i_k)$$
$$+\tau(i_2, i_3, i_4,\ldots, i_k)+\tau(i_1, i_4,\ldots, i_k)+\tau(i_2, i_4,\ldots, i_k)+\tau(i_3, i_4,\ldots, i_k)$$
$$+\tau(i_4,\ldots, i_k) = \mu_{k-3},$$

where $d=\tau^{t+3}(\phi)$ and $\tau(i_1, i_2,\ldots, i_k)=\tau^{t+3}(i_1, i_2,\ldots, i_k)$. From these equations, it can be easily proved by induction on k that (6.9) hold. The condition (6.6c) is equivalent to that $d\geqq 0$ and $\tau(i_1, i_2,\ldots, i_k)\geqq 0$ for all distinct integers $i_1, i_2,\ldots, i_k$ with $1\leqq i_k\leqq t+3$ and $1\leqq k\leqq t+3$. This completes the proof.

From (6.9), we have

COROLLARY 6.7. *The number of nonisomorphic B-arrays of strength t, $m=t+3$ constraints and index set $\{\mu_0, \mu_1,\ldots, \mu_t\}$ is equal to that of sets $\{\{d_{i,j}\}, \{d_i\}, d\}$ such that* (6.6a, b, c) hold.

For a (0, 1) matrix T of size $m\times N$, let z_q^m $(0\leqq q\leqq m)$ be the number of columns in T which are of weight q. Then the following theorem has been given in [29]:

THEOREM 6.8. *Let T be a B-array of strength t, m constraints and index set $\{\mu_0, \mu_1,\ldots, \mu_t\}$. Then the nonnegative integers z_j^m must satisfy the following equations:*

$$(6.10)\quad \sum_{q=0}^{m}\binom{q}{j}\binom{m-q}{t-j}z_q^m=\binom{m}{t}\binom{t}{j}\mu_j \qquad \text{for } j = 0, 1,\ldots, t.$$

DEFINITION 6.2. *A B-array with m constraints is said to be "trim" if $z_0^m=z_m^m=0$.*

DEFINITION 6.3. *A 2^m-BFF design of resolution $2l+1$ is said to be trim if it is a trim B-array of strength $2l$ and m constraints.*

7. Simple arrays with parameters $(m; \lambda_0, \lambda_1, \ldots, \lambda_m)$

Let $\Omega(k; m)$, $(0 \leqq k \leqq m)$, be the (0, 1) matrix of size $m \times \binom{m}{k}$ whose columns are all distinct vectors with weight k.

DEFINITION 7.1. *A matrix obtained by juxtaposing each $\Omega(k; m)$ λ_k $(k=0, 1, \ldots, m)$ times, i.e.,*

$$[\underbrace{\Omega(0; m): \cdots : \Omega(0; m)}_{\lambda_0}: \underbrace{\Omega(1; m): \cdots : \Omega(1; m)}_{\lambda_1}: \cdots : \underbrace{\Omega(m; m): \cdots : \Omega(m; m)}_{\lambda_m}]$$

is called a simple array (S-array). The numbers $(m; \lambda_0, \lambda_1, \ldots, \lambda_m)$ are called the parameters of the S-array.

Each $\Omega(k; m)$, of course, is an S-array with $\lambda_k=1$. Also it can be easily checked that it is a B-array of strength t with indices $\binom{m-t}{k-i}$ $(i=0, 1, \ldots, t)$. Thus we have

THEOREM 7.1. *An S-array with parameters $(m; \lambda_0, \lambda_1, \ldots, \lambda_m)$ is a B-array of strength t, m constraints and indices $\mu_i = \sum_{k=0}^{m} \binom{m-t}{k-i} \lambda_k$ $(i=0, 1, \ldots, t)$.*

Now we shall investigate some conditions for B-arrays to be S-arrays. From the definition of a B-array, we can easily prove the following:

THEOREM 7.2. *A B-array of strength t, $m=t$ constraints and index set $\{\mu_0, \mu_1, \ldots, \mu_t\}$ is an S-array with parameters $(t; \lambda_0=\mu_0, \lambda_1=\mu_1, \ldots, \lambda_t=\mu_t)$.*

We now prove

THEOREM 7.3. *A B-array of strength t, $m=t+1$ constraints and index set $\{\mu_0, \mu_1, \ldots, \mu_t\}$ is an S-array with parameters $(t+1; \lambda_0, \lambda_1, \ldots, \lambda_{t+1})$, where $\lambda_0 = \tau^{t+1}(\phi)$ and $\lambda_k = \tau^{t+1}(i_1, i_2, \ldots, i_k)$ given in (6.2).*

PROOF. The proof follows from the fact that each $\tau^{t+1}(i_1, i_2, \ldots, i_k)$ in (6.2) depends on distinct integers $i_1, i_2, \ldots, i_k$ only through k.

COROLLARY 7.4. *Let T be a B-array of strength t, m constraints and index set $\{\mu_0, \mu_1, \ldots, \mu_t\}$ and let $T^{(i)}$ $(i=1, 2, \ldots, m)$ be matrices obtained from T by omitting i-th rows. If every $T^{(i)}$ is equivalent to an S-array with parameters $(m-1; \lambda'_0, \lambda'_1, \ldots, \lambda'_{m-1})$ such that $\mu_j = \sum_{k=0}^{m-1} \binom{m-1-t}{k-j} \lambda'_k$ hold for $j=0$,* 1,..., t, *then* T *is also an S-array. Its parameters are given by*

$$\lambda_0 = \tau(\phi; T),$$

$$\lambda_k = \sum_{q=1}^{k}(-1)^{k+q}\lambda'_{q-1}+(-1)^k\lambda_0 \quad \textit{for} \quad 1 \leqq k \leqq m.$$

Proof. From assumption, T is of strength $m-1$, m constraints and index set $\{\lambda'_0, \lambda'_1,\ldots, \lambda'_{m-1}\}$. This completes the proof, because of Theorem 7.3.

Theorem 7.5. *Let* T *be a B-array of strength* t, $m=t+2$ *constraints and index set* $\{\mu_0, \mu_1,\ldots, \mu_t\}$. *If*

$$z_k^{t+2} = 0 \quad \textit{for some} \quad k \quad \textit{with} \quad 1 \leqq k \leqq t+1,$$

where z_k^{t+2} *is the number of columns of* T *which are of weight* k, *then* T *is an S-array with parameters* $\lambda_0=\tau^{t+2}(\phi)$, $\lambda_k=0$ *and* $\lambda_r=\tau^{t+2}(i_1, i_2,\ldots, i_r)$, $(1\leqq r \leqq t+2;\ r\neq k)$, *given in* (6.4).

Proof. It is clear that $z_k^{t+2}=0$ implies $\tau^{t+2}(i_1, i_2,\ldots, i_k)=0$ for all distinct elements $i_1, i_2,\ldots, i_k$ of $\{1, 2,\ldots, t+2\}$. From (6.4), therefore, the value of $\sum_{\alpha=1}^{k} d_{i_\alpha}$ depends on k only. This shows that $d_1=d_2=\cdots=d_{t+2}$. Again from (6.4), this implies that $\tau^{t+2}(i_1, i_2,\ldots, i_r)$ depend on $i_1, i_2,\ldots, i_r$ only through r. This completes the proof.

Corollary 7.6. *Consider the B-array* T *of Theorem* 7.5 *with* $t=6$, $m=8$ *and* $\mu_3=1$. *Then* T *is an S-array with* $\lambda_3+\lambda_5=1$ *and* $\lambda_4=0$.

Proof. Without loss of generality, we assume that T is a trim B-array. Therefore, after some calculation of (6.10), we have

$$\text{(7.1)} \qquad \begin{aligned} z_3^8+z_5^8 &= 56(-3+3\rho_2-2\rho_1+\rho_0) \geqq 0, \\ z_4^8 &= 35(4-3\rho_2+2\rho_1-\rho_0) \geqq 0, \end{aligned}$$

where $\rho_0=\mu_0+\mu_6$, $\rho_1=\mu_1+\mu_5$ and $\rho_2=\mu_2+\mu_4$. From (7.1), it is clear that $0\leqq 4-3\rho_2+2\rho_1-\rho_0\leqq 1$ holds. Now assume that $4-3\rho_2+2\rho_1-\rho_0=1$ holds. Then $z_4^8=35$ and $z_3^8+z_5^8=0$. From Theorem 7.5, T is an S-array, so that $z_4^8 =\binom{8}{4}\lambda_4$. This implies a contradiction. Hence we have $4-3\rho_2+2\rho_1+\rho_0=0$, that is, $z_3^8+z_5^8=56$ and $z_4^8=0$. Again from Theorem 7.5, it follows that T is an S-array with $z_3^8+z_5^8=\binom{8}{3}(\lambda_3+\lambda_5)=56$ and $\lambda_4=0$. This completes the proof.

Theorem 7.7. *A B-array* T *of strength* t, m $(\geqq t+2)$ *constraints and index set* $\{\mu_0, \mu_1,\ldots, \mu_t\}$ *with* $\mu_r=0$ $(0\leqq r\leqq t)$ *is an S-array with parameters* $(m; \lambda_0, \lambda_1,\ldots, \lambda_{r-1}, 0,\ldots, 0, \lambda_{m+r-t+1},\ldots, \lambda_m)$ *which satisfy*

$$(7.2)\quad \begin{aligned} \mu_i &= \sum_{k=0}^{r-1}\binom{m-t}{k-i}\lambda_k \qquad \text{for} \quad i=0, 1,\dots, r-1 \quad (r \neq 0), \\ \mu_{r+1+i} &= \sum_{k=0}^{t-r-1}\binom{m-t}{i-k}\lambda_{m+r-t+1+k} \qquad \text{for} \quad i=0, 1,\dots, t-r-1 \quad (r \neq t). \end{aligned}$$

Note that for two cases $\mu_0=0$ and $\mu_t=0$, the parameters of the S-array take the form of $(m; 0,\dots, 0, \lambda_{m-t+1},\dots, \lambda_m)$ and $(m; \lambda_0,\dots, \lambda_{r-1}, 0,\dots, 0)$, respectively. First we shall prove the following two lemmas:

LEMMA 7.8. *Consider the B-array T of Theorem 7.7. Then the weight q of a column of T must satisfy $q<r$ or $q>m+r-t$.*

PROOF. Assume that there exists a column vector of T with weight q satisfying $r \leqq q \leqq m+r-t$. Then we can obtain a t-rowed submatrix T^t of T such that a column vector with weight r occurs in T^t. This implies $\mu_r \neq 0$, a contradiction. This completes the proof.

In view of Lemma 7.8, the B-array T of Theorem 7.7 can be expressed without loss of generality as

$$T=[T_{(0)}\vdots\, T_{(1)}\vdots \cdots \vdots\, T_{(r-1)}\vdots\, T_{(m+r-t+1)}\vdots \cdots \vdots\, T_{(m)}],$$

where $T_{(q)}$ is a submatrix of T whose columns are only of weight q.

LEMMA 7.9. *Consider the B-array T of Theorem 7.7. Then the submatrices $[T_{(0)}\vdots \cdots \vdots\, T_{(r-1)}]$ and $[T_{(m+r-t+1)}\vdots \cdots \vdots\, T_{(m)}]$ are also B-arrays of strength t and m constraints with index set $\{\mu_0,\dots, \mu_{r-1}, 0,\dots, 0\}$ and $\{0,\dots, 0, \mu_{r+1},\dots, \mu_t\}$, respectively.*

PROOF. The number of times any column vector of weight q $(0 \leqq q \leqq r-1)$ occurs in any t-rowed submatrix of T does not depend on $T_{(m+r-t+1)},\dots, T_{(m)}$. Thus $[T_{(0)}\vdots \cdots \vdots\, T_{(r-1)}]$ is a B-array of strength t, m constraints and index set $\{\mu_0, \mu_1,\dots, \mu_{r-1}, 0,\dots, 0\}$. Similarly it can be shown that $[T_{(m+r-t+1)}\vdots \cdots \vdots\, T_{(m)}]$ is a B-array with the indicated index set.

PROOF OF THEOREM 7.7. We prove by induction that every $T_{(q)}$ $(q=0, 1,\dots, r-1)$ is an S-array. From Lemma 7.9, the index set of the B-array $[T_{(0)}\vdots \cdots \vdots\, T_{(r-1)}]$ is given by $\{\mu_0,\dots, \mu_{r-1}, 0,\dots, 0\}$. Furthermore it is found that the number of times a vector with weight $r-1$ occurs as a column of this array depends on $T_{(r-1)}$ only. Let $\boldsymbol{v}$ be the column vector of $T_{(r-1)}$ which contains 1 exactly at i_1-th,..., i_{r-1}-th positions and 0 elsewhere. Then in a t-rowed submatrix of $T_{(r-1)}$ which includes i_1-th,..., i_{r-1}-th rows, the column vector corresponding to $\boldsymbol{v}$ must occur exactly $\tau(i_1,\dots, i_{r-1}; T_{(r-1)})$ times. From the definition of a B-array, it follows that $\tau(i_1,\dots, i_{r-1}; T_{(r-1)})=\mu_{r-1}$, that is, it does not depend on the

i_1-th,..., i_{r-1}-th positions of v. This shows that $T_{(r-1)}$ is an S-array with $\lambda_{r-1} = \mu_{r-1}$. Assume that $[T_{(j+1)}: T_{(j+2)}: \cdots: T_{(r-1)}]$ is an S-array. Then, since it is a B-array of strength t from Theorem 7.1, it is clear that $[T_{(0)}: \cdots: T_{(j)}]$ is also a B-array of strength t and its index set takes the form of $\{\mu'_0,\ldots, \mu'_j, 0,\ldots, 0\}$. From an argument similar to the above, it follows that $T_{(j)}$ is an S-array with $\lambda_j = \mu'_j$. This proves that $[T_{(0)}: \cdots: T_{(r-1)}]$ is an S-array. In the same way, it can be shown that the B-array $[T_{(m+r-t+1)}: \cdots: T_{(m)}]$ is also an S-array. Clearly the relation (7.2) follows from Theorem 7.1. This completes the proof of Theorem 7.7.

Finally we shall prove the following

THEOREM 7.10. *A necessary and sufficient condition for a B-array T of strength t, m constraints and index set* $\{\mu_0, \mu_1,\ldots, \mu_t\}$ *to be an S-array is that there exist intergers* $d^{t+1}, d^{t+2},\ldots, d^m$ *such that for each* $s = t, t+1,\ldots, m-1$,

(7.3)
$$d^{s+1} \geqq \psi_{11}^{(s)} = \max_{1 \leqq 2r \leqq s+1} \{0, \sum_{q=0}^{2r-1} (-1)^q \mu_q^s\},$$
$$d^{s+1} \leqq \psi_{12}^{(s)} = \min_{0 \leqq 2r \leqq s} \{\sum_{q=0}^{2r} (-1)^q \mu_q^s\},$$

where

(7.4)
$$\mu_k^t = \mu_k \quad \text{for} \quad k = 0, 1,\ldots, t,$$
$$\mu_0^{s+1} = d^{s+1},$$
$$\mu_k^{s+1} = \sum_{q=1}^{k} (-1)^{k+q} \mu_{q-1}^s + (-1)^k d^{s+1} \qquad \text{for} \quad k = 1, 2,\ldots, s+1.$$

If there exist integers d^i *satisfying* (7.3), *then the parameters of the S-array are given by* $(m; \lambda_0 = \mu_0^m, \lambda_1 = \mu_1^m,\ldots, \lambda_m = \mu_m^m)$.

PROOF. Let T^j be a j-rowed submatrix of T. If T is an S-array, then for each $s = t, t+1,\ldots, m-1$, T^{s+1} is also an S-array and a B-array of strength s. Denote its parameters and index set by $(s+1; \mu_0^{s+1}, \mu_1^{s+1},\ldots, \mu_{s+1}^{s+1})$ and $\{\mu_0^s, \mu_1^s,\ldots, \mu_s^s\}$, respectively. Particularly $\mu_k^t = \mu_k$ for $k = 0, 1,\ldots, t$. From Theorems 6.1 and 7.3, it is clear that a connection between the parameters μ_i^{s+1} and the indices μ_i^s is given by (7.4). This implies that there exists an integer d^{s+1} satisfying (7.3) for each $s = t, t+1,\ldots, m-1$. Conversely let $d^{t+1}, d^{t+2},\ldots, d^m$ be integers which satisfy (7.3). Then from Theorems 6.1 and 7.3, we can construct S-arrays $T^{t+1}, T^{t+2},\ldots, T^m$ in sequence. Let $T = T^m$, then T is clearly a B-array of strength t and m constraints with the given index set.

As an immediate corollary to the above theorem, we have

COROLLARY 7.11. *The number of nonisomorphic S-arrays which are equivalent to a B-array of strength t, m constraints and index set* $\{\mu_0, \mu_1,\dots, \mu_t\}$, *is equal to that of sets* $\{d^{t+1}, d^{t+2},\dots, d^m\}$ *satisfying* (7.3).

In Theorem 7.10, note that there may be B-arrays of strength t and m constraints with the same index set which are nonsimple, even if there exist integers d^i satisfying (7.3). However it may be seen from [7–10, 23, 34, 37] and Section 8 that the possibility of the existence of such B-arrays is very small within a certain practical range of N for $t=4, 6$. In such a sense, Theorem 7.10 is very useful for constructing 2^m-BFF designs of resolution V or VII.

8. Optimal 2^9-BFF designs of resolution VII with $130 \leqq N \leqq 150$

Now we shall consider 2^9-BFF designs of resolution VII with N assemblies satisfying $v_l\,(=130) \leqq N \leqq 150$. Two criteria, the trace and determinant criteria, will be used for comparing these designs. As mentioned in Section 5, the two criteria are based on the amounts of (5.2) and (5.3), respectively.

First we proceed to consider trim B-arrays (or trim designs) T^* (see Definitions 6.2 and 6.3) of strength $t=6$, $m=9$ constraints, size N and index set $\{\mu_0, \mu_1,\dots, \mu_{2l}\}$. To avoid repetition, suppose that such trim B-arrays T^* are considered throughout this section. Further suppose that simply $z_q = z_q^9$ for $q=1, 2,\dots, 8$. Then it follows from Theorem 6.8 that for a trim B-array T^*,

$$
(8.1)\quad
\begin{aligned}
&\text{(a)}\quad 28z_1+7z_2+z_3 = 84\mu_0,\\
&\text{(b)}\quad 28z_1+21z_2+9z_3+2z_4 = 252\mu_1,\\
&\text{(c)}\quad 7z_2+9z_3+6z_4+2z_5 = 252\mu_2,\\
&\text{(d)}\quad z_3+2z_4+2z_5+z_6 = 84\mu_3,\\
&\text{(e)}\quad 2z_4+6z_5+9z_6+7z_7 = 252\mu_4,\\
&\text{(f)}\quad 2z_5+9z_6+21z_7+28z_8 = 252\mu_5,\\
&\text{(g)}\quad z_6+7z_7+28z_8 = 84\mu_6,
\end{aligned}
$$

As in Section 7, define $\rho_0=\mu_0+\mu_6$, $\rho_1=\mu_1+\mu_5$ and $\rho_2=\mu_2+\mu_4$. From (8.1), after some calculations, we obtain

THEOREM 8.1. *For a trim B-array* T^*, *the following hold*:

$$
(8.2)\quad
\begin{aligned}
&\text{(a)}\quad y_1 = -16\mu_3+15\rho_2-12\rho_1+7\rho_0 \geqq 0,\\
&\text{(b)}\quad y_2 = 4(23\mu_3-21\rho_2+15\rho_1-5\rho_0) \geqq 0,
\end{aligned}
$$

(c) $y_3 = 28(-7\mu_3+6\rho_2-3\rho_1+\rho_0) \geqq 0,$

(d) $y_4 = 14(10\mu_3-6\rho_2+3\rho_1-\rho_0) \geqq 0,$

where $y_1=z_1+z_8$, $y_2=z_2+z_7$, $y_3=z_3+z_6$ *and* $y_4=z_4+z_5$.

THEOREM 8.2. *For a trim B-array* T^*,

(8.3)

(a) $N \geqq 42\mu_3,$

(b) $N \geqq \frac{42}{5}(3\rho_2+\mu_3),$

(c) $N \geqq 9\rho_1+39\mu_3,$

(d) $\rho_1 \geqq \frac{1}{3}\mu_3.$

PROOF. It follows from (8.2a, b, c) that $\rho_0+6\rho_1 \geqq (21-9\beta)\rho_2+(12\beta-26)\mu_3$ holds for $\beta \geqq 6/5$. Since $N=\rho_0+6\rho_1+15\rho_2+20\mu_3$, we have $N \geqq 9(4-\beta)\rho_2+6(2\beta-1)\mu_3$ for $\beta \geqq 6/5$. The inequalities (8.3a, b) can be obtained by taking $\beta=4$ and $\beta=6/5$, respectively. From (8.2b, c), also $\rho_0+15\rho_2 \geqq 3\rho_1+19\mu_3$. Similarly we have (8.3c). The inequality (8.3d) can be easily obtained from (8.2a, b, c).

THEOREM 8.3. *For a trim B-array* T^*, $\mu_3 \geqq 4$ *implies* $N \geqq 168$.

PROOF. This follows immediately from (8.3a).

THEOREM 8.4. *Let* T^* *be a trim* 2^9*-BFF design of resolution VII. Then* $\mu_3 \geqq 1$ *and* $\rho_2 > 6/5\mu_3$ *hold.*

PROOF. This follows immediately from (5.9), (5.10a) and Corollary 5.8.

Now we are interested in the designs with $N \leqq 150$. In view of Theorem 8.3 and 8.4, we can restrict only to B-arrays with $1 \leqq \mu_3 \leqq 3$. In the following discussions, we shall make further investigations on trim B-arrays (or trim designs) for each case of $\mu_3=1, 2, 3$. In each case $T^{(i)}$ and $z_k^{(i)}$ ($i=1, 2,\dots, 9$; $k=0, 1,\dots, 8$) denote a B-array obtained from T^* by omitting i-th row and the number of columns of weight k in $T^{(i)}$, respectively.

(a) The case $\mu_3=1$.

THEOREM 8.5. *Let* T^* *be a trim* 2^9*-BFF design of resolution VII with* $\mu_3=1$ *and* $N \leqq 150$, *then* $5 \geqq \rho_2 \geqq 2$, $12 \geqq \rho_1 \geqq 1$ *and* $6\rho_2-3\rho_1+\rho_0=10$ (*i.e.*, $y_4=0$) *hold.*

PROOF. The first two inequalities follow from Theorems 8.2 and 8.4. Clearly $T^{(i)}$ is of strength 6 and 8 constraints with $\mu_3=1$. From Corollary 7.6, therefore, $T^{(i)}$ is also an S-array with a parameter $\lambda_4^{(i)}=0$ for each $i=1, 2,\ldots, 9$. Since $\lambda_k^{(i)}$ is the number of times $\Omega(k; 8)$ occurs as submatrices of $T^{(i)}$, it is found that $z_4=z_5=0$. This completes the proof.

THEOREM 8.6. *Consider the B-array T^* of Theorem 8.5. Then T^* is an S-array with* $(\lambda_0=\lambda_4=\lambda_5=\lambda_6=\lambda_9=0,\ \lambda_3=1)$ *or* $(\lambda_0=\lambda_3=\lambda_4=\lambda_5=\lambda_9=0,\ \lambda_6=1)$.

PROOF. From Theorem 8.5, $y_4=0$ holds. Hence it follows from (8.2c, d) that $z_3+z_6=84$ holds. Again consider a B-array $T^{(i)}$. By Corollary 7.6, it is shown that $T^{(i)}$ is an S-array with $\lambda_4^{(i)}=0$ and $\lambda_3^{(i)}+\lambda_5^{(i)}=1$ for each $i=1, 2,\ldots, 9$. Since $\lambda_k^{(i)}$ are nonnegative integers, $\lambda_3^{(i)}=1$ or 0 according as $\lambda_5^{(i)}=0$ or 1. If $\lambda_3^{(i)}=1$ and $\lambda_5^{(i)}=0$ for some i, then we shall show that $\lambda_3^{(j)}=1$ and $\lambda_5^{(j)}=0$ for all $j=1, 2,\ldots, 9$. It is easy to see that $z_4=0$ and $\lambda_3^{(i)}=1$ imply $z_3 \geqq z_3^{(i)}=56$. Now suppose there exists an integer j such that $\lambda_3^{(j)}=0$ and $\lambda_5^{(j)}=1$. Then $z_5=0$ and $\lambda_5^{(j)}=1$ imply $z_6 \geqq z_5^{(j)}=56$. Thus $z_3+z_6 \geqq 112$ must hold. It contradicts $z_3+z_6=84$. This shows that if $\lambda_3^{(i)}=1$ and $\lambda_5^{(i)}=0$, then $z_3=84$ and $z_6=0$ hold. As in Section 7, therefore, T^* can be expressed without loss of generality as

$$T^* = [T_{(1)}\colon T_{(2)}\colon T_{(3)}\colon T_{(7)}\colon T_{(8)}].$$

It is clear that the number of times a column vector of weight 3 occurs in any 6-rowed submatrix of T^* depends on $T_{(3)}$ only. This implies that $T_{(3)}$ itself must be an S-array with $\lambda_3=1$. Since it is also a B-array of strength 6, the submatrix $[T_{(1)}\colon T_{(2)}\colon T_{(7)}\colon T_{(8)}]$ must be of strength 6. Its index set takes the form of $\{\mu'_0, \mu'_1, \mu'_2, \mu'_3=0, \mu'_4, \mu'_5, \mu'_6\}$. From Theorem 7.7, it follows that this submatrix is an S-array. Hence T^* is an S-array with $\lambda_0=\lambda_4=\lambda_5=\lambda_6=\lambda_9=0$, $\lambda_3=1$. In the same way, we can show that T^* is an S-array with $\lambda_0=\lambda_3=\lambda_4=\lambda_5=\lambda_9=0$, $\lambda_6=1$ in the case when $\lambda_3^{(i)}=0$ and $\lambda_5^{(i)}=1$.

(b) The case $\mu_3=2$.

THEOREM 8.7. *Let T^* be a trim 2^9-BFF design of resolution VII with $\mu_3=2$ and $N \leqq 150$. Then $5 \geqq \rho_2 \geqq 3$ and $8 \geqq \rho_1 \geqq 1$ hold.*

PROOF. This follows from Theorems 8.2 and 8.4.

THEOREM 8.8. *There does not exist any trim B-array T^* with $\mu_3=2$, $\rho_2=5$ and $N \leqq 150$.*

PROOF. In this case $\rho_1 \geqq 6$ and $\rho_1 \leqq 3$ imply $N>150$ and $y_2 \leqq -4(14+5\rho_0) <0$, respectively. Thus the cases (i) $\rho_1=5$ and (ii) $\rho_1=4$ are considered. In the case (i), (8.2a, b) reduce to

$$y_1 = 7\rho_0 - 17 \geqq 0, \qquad y_2 = 4(16-5\rho_0) \geqq 0.$$

This shows that $\rho_0=3$ must hold. For a trim B-array T^* with $\rho_2=5$, $\rho_1=5$ and $\rho_0=3$, consider $T^{(i)}$ and its trim B-array $T^{(i)*}$ for $i=1, 2,\dots, 9$. Then the index set of $T^{(i)*}$ takes the form of $\{\mu_0^{(i)}, \mu_1,\dots, \mu_5, \mu_6^{(i)}\}$, where $0\leqq\mu_0^{(i)}+\mu_6^{(i)}$ $(=\rho_0^{(i)}$, say) $\leqq 3$. From Theorem 6.8,

$$z_1^{(i)}+z_7^{(i)} = 8(\rho_0^{(i)}-2) \geqq 0, \quad z_2^{(i)}+z_6^{(i)} = 28(4-\rho_0^{(i)}) \geqq 0,$$

$$z_3^{(i)}+z_5^{(i)} = 56(\rho_0^{(i)}-1) \geqq 0, \quad z_4^{(i)} = 35(3-\rho_0^{(i)}) \geqq 0$$

hold for $i=1, 2,\dots, 9$. If $\rho_0^{(i)}=2$, then $z_1^{(i)}=z_7^{(i)}=0$ and $z_4^{(i)}=35$. From Theorem 7.5, however, $z_4^{(i)}$ must be a multiple of $\binom{8}{4}=70$. This implies a contradiction. On the other hand, $\rho_0^{(i)}\leqq 1$ implies $z_1^{(i)}+z_7^{(i)}<0$. After all $\rho_0^{(i)}=3$ (i.e., $z_4^{(i)}=0$) for all $i=1, 2,\dots, 9$. Hence $y_4=0$ holds. However it contradicts $y_4=28$ in (8.2d). Next consider the case (ii). Then similarly (8.2a, b) reduce to

$$y_1 = 7\rho_0-5 \geqq 0, \qquad y_2 = 4(1-5\rho_0) \geqq 0.$$

Clearly there does not exist any nonnegative integer ρ_0 satisfying the above inequalities. This completes the proof.

THEOREM 8.9. *There does not exist any trim B-array T^* with $\mu_3=2$, $\rho_2=4$ and $128\leqq N\leqq 150$.*

In view of Theorem 1.1, note that a 2^9-BFF design of resolution VII can not be obtained from a trim B-array with $N<128$ (or a general B-array with $N<130$). To prove the theorem, we need the following three lemmas:

LEMMA 8.10. *If there does not exist a B-array of strength 6 and m constraints with index set $\{\mu_0+\alpha_0+\alpha_1(m-6), \mu_1+\alpha_1, \mu_2, \mu_3, \mu_4, \mu_5+\alpha_2, \mu_6+\alpha_3+\alpha_2(m-6)\}$, where α_i $(i=0, 1, 2, 3)$ are nonnegative integers, then there does not exist any B-array of strength 6 and m constraints with $\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$.*

PROOF. Suppose that there exists a B-array T of strength 6 and m constraints with index set $\{\mu_0, \mu_1,\dots, \mu_6\}$. Further consider a matrix obtained by juxtaposing the array T and an S-array with parameters $(m; \lambda_0=\alpha_0, \lambda_1=\alpha_1, 0,\dots, 0, \lambda_{m-1}=\alpha_2, \lambda_m=\alpha_3)$. From Theorem 7.1, it is clear that this matrix is a B-array with the indicated index set. This implies a contradiction.

LEMMA 8.11. *There does not exist a B-array of strength 6 and 9 constraints with index set $\{10, 4, 2, 2, 2, 3, 8\}$.*

PROOF. This follows immediately from Theorem 6.6.

LEMMA 8.12. *There does not exist an S-array corresponding to a B-array of strength* 6 *and* 9 *constraints with* $\mu_3=2$ *and* $\rho_2=4$.

PROOF. Consider an S-array with parameters $(9; \lambda_0, \lambda_1, \ldots, \lambda_9)$ such that

$$\lambda_2+3\lambda_3+3\lambda_4+\lambda_5 = \mu_2,$$

$$\lambda_3+3\lambda_4+3\lambda_5+\lambda_6 = 2,$$

$$\lambda_4+3\lambda_5+3\lambda_6+\lambda_7 = \mu_4,$$

where $\mu_2+\mu_4=4$. It is easy to see that there do not exist nonnegative integers λ_i satisfying the above equations. This completes the proof, because of Theorem 7.1.

PROOF OF THEOREM 8.9. $\rho_1 \geqq 7$ and $\rho_1 \leqq 3$ imply $N>150$ and $N<128$ respectively. For $4 \leqq \rho_1 \leqq 6$, by using Corollary 6.5, we can construct B-arrays of strength 6 and 8 constraints. Furthermore, in view of Corollary 7.4 and Lemma 8.12, among these B-arrays we can select ones which will be of strength 6 and 9 constraints. The following is a list of index sets of such B-arrays: (i) When $\mu_2=\mu_4=2$ and $\rho_1=6$, $(\mu_0, \mu_1, \mu_5, \mu_6)=(9, 4, 2, 1)$, (8, 4, 2, 2), (7, 4, 2, 3), (6, 4, 2, 4), (5, 4, 2, 5), (7, 3, 3, 3), (6, 3, 3, 4), (5, 3, 3, 5), (8, 4, 2, 1), (7, 4, 2, 2), (6, 4, 2, 3), (5, 4, 2, 4), (6, 3, 3, 3), (5, 3, 3, 4), (7, 4, 2, 1), (6, 4, 2, 2), (5, 4, 2, 3), (5, 3, 3, 3), (4, 3, 3, 4), (ii) when $\mu_2=\mu_4=2$ and $\rho_1=5$, $(\mu_0, \mu_1, \mu_5, \mu_6)=(6, 3, 2, 1)$, (5, 3, 2, 2), (4, 3, 2, 3), (3, 3, 2, 4), (5, 3, 2, 1), (4, 3, 2, 2), (3, 3, 2, 3), (4, 3, 2, 1), (3, 3, 2, 2), and (iii) when $\mu_2=\mu_4=2$ and $\rho_1=4$, $(\mu_0, \mu_1, \mu_5, \mu_6)=(3, 2, 2, 1)$, (2, 2, 2, 2). From Lemmas 8.10 and 8.11, however, it is found that there do not exist B-arrays of strength 6 and 9 constraints with the above index sets. For example, we shall show that there does not exist any B-array with {9, 4, 2, 2, 2, 2, 1}. In Lemma 8.10 consider $\alpha_0=1$, $\alpha_1=0$, $\alpha_2=1$ and $\alpha_3=4$. Then it follows from Lemma 8.11 that this array does not exist. This completes the proof.

THEOREM 8.13. *There does not exist any trim B-array* T^* *with* $\mu_3=2$, $\rho_2=3$ *and* $128 \leqq N \leqq 150$.

PROOF. Clearly $\rho_1 \geqq 8$ and $\rho_1 \leqq 5$ imply $N>150$ and $N<128$, respectively. If $\rho_1=7$, then (8.2b, c) reduce to

$$y_2 = 4(88-5\rho_0) \geqq 0, \qquad y_3 = 28(\rho_0-17) \geqq 0.$$

Thus $\rho_0=17$ (i.e., $y_3=0$) holds. As in Theorem 8.8, consider a trim B-array $T^{(i)*}$. Then from Theorem 6.8,

$$z_2^{(i)}+z_6^{(i)} = 28(12-\rho_0^{(i)}) \geqq 0,$$

$$z_3^{(i)}+z_5^{(i)} = 56(\rho_0^{(i)}-11) \geqq 0, \quad z_4^{(i)} = 35(13-\rho_0^{(i)}) \geqq 0$$

hold for $i=1, 2,\ldots, 9$. From Theorem 7.5, therefore, $\rho_0^{(i)}=11$ (i.e., $z_3^{(i)}=z_5^{(i)}=0$) must hold for all i. Furthermore this implies $y_4=0$. It contradicts $y_4=14(23-\rho_0)\neq 0$ in (8.2d). In the same way, it can be shown that there does not exist T^* with $\rho_1=6$. This completes the proof.

In consequence of Theorems 8.7–8.13, it is found that there does not exist any trim B-array with $\mu_3=2$ and $128\leqq N\leqq 150$.

(c) The case $\mu_3=3$.

THEOREM 8.14. *Let T^* be a trim 2^9-BFF design of resolution VII with $\mu_3=3$ and $128\leqq N\leqq 150$. Then $\rho_2=4$, $3\geqq\rho_1\geqq 2$ and $3\rho_1=\rho_0+3$ (i.e., $y_4=126$, $y_2=y_3=0$, $y_1=9(\rho_1-1)$) hold.*

PROOF. From Theorems 8.2 and 8.4, we have $\rho_2=4$ and $3\geqq\rho_1\geqq 1$. The remaining equalities follow from (8.2b, c, d). Now assume $\rho_1=1$. Then $\rho_0=0$ and $N=126$. It gives a contradiction.

THEOREM 8.15. *The B-array T^* of Theorem 8.14 is an S-array with parameters $\lambda_0=\lambda_2=\lambda_3=\lambda_6=\lambda_7=\lambda_9=0$ and $\lambda_4+\lambda_5=1$.*

PROOF. As in Theorem 8.6, from Theorem 8.14 we can consider T^* as the following form:

$$T^* = [T_{(1)}\colon T_{(4)}\colon T_{(5)}\colon T_{(8)}].$$

The number of times a column vector with weight 1 occurs in any 6-rowed submatrix of T^* depends on $T_{(1)}$ only. This shows that $T_{(1)}$ itself is an S-array with $\lambda_1=\mu_0/3$. Therefore the submatrix $[T_{(4)}\colon T_{(5)}\colon T_{(8)}]$ must be a B-array of strength 6 and its index set takes the form of $\{\mu'_0=0, \mu'_1, \mu_2, \mu_3=3, \mu_4, \mu_5, \mu_6\}$. From Theorem 7.7, this submatrix is an S-array. Since $y_4=126=\binom{9}{4}(\lambda_4+\lambda_5)$, T^* is an S-array with the indicated parameters.

COROLLARY 8.16. *There does not exist any trim B-array with $\mu_2=\mu_4=2$, $\mu_3=3$ and $128\leqq N\leqq 150$.*

PROOF. This follows immediately from Theorems 7.1 and 8.15.

From the above results, we can easily construct trim B-arrays with $128\leqq N\leqq 150$. Furthermore it is found that all the B-arrays obtained are fortunately S-arrays. General B-arrays can be easily obtained from trim B-arrays by adding column vectors, each being of weight 0 or 9. Among all the B-arrays for each

TABLE 8.1 Optimal 2^9-BFF designs of resolution VII with respect to the trace criterion

N	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	tr (V_T)	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9
*130	4	4	3	1	1	3	4	1.60156	0	1	0	1	0	0	0	1	0	1
*131	4	4	3	1	1	3	5	1.59277	0	1	0	1	0	0	0	1	0	2
*132	4	4	3	1	1	3	6	1.58984	0	1	0	1	0	0	0	1	0	3
133	4	4	3	1	1	3	7	1.58838	0	1	0	1	0	0	0	1	0	4
*134	5	4	3	1	1	3	7	1.58690	1	1	0	1	0	0	0	1	0	4
135	5	4	3	1	1	3	8	1.58599	1	1	0	1	0	0	0	1	0	5
*136	6	4	3	1	1	3	8	1.58521	2	1	0	1	0	0	0	1	0	5
137	6	4	3	1	1	3	9	1.58458	2	1	0	1	0	0	0	1	0	6
138	7	4	3	1	1	3	9	1.58410	3	1	0	1	0	0	0	1	0	6
*139	7	5	3	1	1	3	4	1.52246	0	2	0	1	0	0	0	1	0	1
*140	7	5	3	1	1	3	5	1.51367	0	2	0	1	0	0	0	1	0	2
*141	7	5	3	1	1	3	6	1.51074	0	2	0	1	0	0	0	1	0	3
*142	7	5	3	1	1	3	7	1.50928	0	2	0	1	0	0	0	1	0	4
143	7	5	3	1	1	3	8	1.50840	0	2	0	1	0	0	0	1	0	5
144	7	5	3	1	1	3	9	1.50781	0	2	0	1	0	0	0	1	0	6
145	8	5	3	1	1	3	9	1.50732	1	2	0	1	0	0	0	1	0	6
146	8	5	3	1	1	3	10	1.50690	1	2	0	1	0	0	0	1	0	7
147	9	5	3	1	1	3	10	1.50657	2	2	0	1	0	0	0	1	0	7
148	10	6	3	1	1	3	4	1.49609	0	3	0	1	0	0	0	1	0	1
*149	10	6	3	1	1	3	5	1.48730	0	3	0	1	0	0	0	1	0	2
*150	10	6	3	1	1	3	6	1.48437	0	3	0	1	0	0	0	1	0	3

* This design is also optimal with respect to the determinant criterion.

TABLE 8.2 Optimal 2^9-BFF designs of resolution VII with respect to the determinant criterion

N	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	tr (V_T)	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9
133	5	4	3	1	1	3	6	1.58842	1	1	0	1	0	0	0	1	0	3
135	6	4	3	1	1	3	7	1.58614	2	1	0	1	0	0	0	1	0	4
137	7	4	3	1	1	3	8	1.58473	3	1	0	1	0	0	0	1	0	5
138	4	4	3	1	1	4	6	1.58630	0	1	0	1	0	0	0	1	1	0
143	8	5	3	1	1	3	7	1.50881	1	2	0	1	0	0	0	1	0	4
144	8	5	3	1	1	3	8	1.50792	1	2	0	1	0	0	0	1	0	5
145	9	5	3	1	1	3	8	1.50760	2	2	0	1	0	0	0	1	0	5
146	9	5	3	1	1	3	9	1.50700	2	2	0	1	0	0	0	1	0	6
147	7	5	3	1	1	4	6	1.51719	0	2	0	1	0	0	0	1	1	0
148	7	5	3	1	1	4	7	1.50596	0	2	0	1	0	0	0	1	1	1

TABLE 8.3 Covariance matrices for optimal 2^9-BFF designs of resolution VII

N	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$V_0^{(0,0)}$ $V_0^{(0,1)}$	$V_0^{(0,2)}$ $V_0^{(0,3)}$	$V_0^{(1,1)}$ $V_1^{(1,1)}$	$V_0^{(1,2)}$ $V_1^{(1,2)}$
130	4	4	3	1	1	3	4	0.017578	−0.001519	0.017578	0.001519
								0.001519	−0.000651	−0.001519	−0.000651
131	4	4	3	1	1	3	5	0.014526	−0.001010	0.017565	0.001553
								0.001316	−0.000346	−0.001533	−0.000617
132	4	4	3	1	1	3	6	0.013509	−0.000841	0.017560	0.001564
								0.001248	−0.000244	−0.001537	−0.000606
133	4	4	3	1	1	3	7	0.013000	−0.000756	0.017558	0.001570
								0.001214	−0.000193	−0.001539	−0.000600
133	5	4	3	1	1	3	6	0.013471	−0.000811	0.017545	0.001583
								0.001224	−0.000258	−0.001552	−0.000587
134	5	4	3	1	1	3	7	0.012947	−0.000720	0.017542	0.001589
								0.001185	−0.000209	−0.001555	−0.000581
135	5	4	3	1	1	3	8	0.012630	−0.000664	0.017541	0.001594
								0.001162	−0.000180	−0.001557	−0.000577
135	6	4	3	1	1	3	7	0.012919	−0.000701	0.017534	0.001600
								0.001170	−0.000217	−0.001563	−0.000571
136	6	4	3	1	1	3	8	0.012597	−0.000643	0.017532	0.001604
								0.001145	−0.000188	−0.001565	−0.000566

$V_0^{(1,3)}$ $V_1^{(1,3)}$	$V_0^{(2,2)}$ $V_1^{(2,2)}$	$V_2^{(2,2)}$ $V_0^{(2,3)}$	$V_1^{(2,3)}$ $V_2^{(2,3)}$	$V_0^{(3,3)}$ $V_1^{(3,3)}$	$V_2^{(3,3)}$ $V_3^{(3,3)}$
−0.001519	0.011882	0.000380	−0.000380	0.011882	0.000380
0.000651	−0.000705	0.000705	0.000488	−0.000705	−0.000488
−0.001499	0.011797	0.000295	−0.000431	0.011851	0.000349
0.000671	−0.000790	0.000654	0.000437	−0.000736	−0.000519
−0.001492	0.011768	0.000267	−0.000448	0.011841	0.000339
0.000678	−0.000818	0.000637	0.000420	−0.000746	−0.000529
−0.001489	0.011754	0.000253	−0.000456	0.011836	0.000334
0.000682	−0.00832	0.000629	0.000412	−0.000751	−0.000534
−0.001501	0.011746	0.000244	−0.000437	0.011836	0.000334
0.000669	−0.000841	0.000648	0.000431	−0.000751	−0.000534
−0.001497	0.011730	0.000228	−0.000445	0.011831	0.000329
0.000673	−0.000857	0.000640	0.000423	−0.000756	−0.000539
−0.001495	0.011720	0.000218	−0.000450	0.011828	0.000327
0.000675	−0.000867	0.000635	0.000418	−0.000759	−0.000541
−0.001501	0.011717	0.000215	−0.000440	0.011829	0.000327
0.000669	−0.000870	0.000645	0.000428	−0.000758	−0.000541
−0.001499	0.011706	0.000205	−0.000445	0.011826	0.000324
0.000671	−0.000880	0.000640	0.000423	−0.000761	−0.000544

TABLE 8.3 (continued)

N	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$V_0^{(0,0)}$	$V_0^{(0,1)}$	$V_0^{(0,2)}$	$V_0^{(0,3)}$	$V_0^{(1,1)}$	$V_1^{(1,1)}$	$V_0^{(1,2)}$	$V_1^{(1,2)}$
137	6	4	3	1	1	3	9	0.012382	0.001128	−0.000604	−0.000169	0.017531	−0.001566	0.001607	−0.000563
137	7	4	3	1	1	3	8	0.012577	0.001135	−0.000630	−0.000194	0.017528	−0.001570	0.001610	−0.000560
138	7	4	3	1	1	3	9	0.012359	0.001117	−0.000590	−0.000174	0.017526	−0.001571	0.001614	−0.000556
138	4	4	3	1	1	4	6	0.020745	0.002094	−0.001732	−0.000878	0.017342	−0.001419	0.001485	−0.000737
139	7	5	3	1	1	3	4	0.017456	0.001424	−0.001444	−0.000671	0.014418	−0.001207	0.000902	−0.000400
140	7	5	3	1	1	3	5	0.014404	0.001221	−0.000936	−0.000366	0.014404	−0.001221	0.000936	−0.000366
141	7	5	3	1	1	3	6	0.013387	0.001153	−0.000766	−0.000264	0.014400	−0.001225	0.000947	−0.000355
142	7	5	3	1	1	3	7	0.012878	0.001119	−0.000682	−0.000214	0.014398	−0.001227	0.000953	−0.000349
143	7	5	3	1	1	3	8	0.012573	0.001099	−0.000631	−0.000183	0.014396	−0.001229	0.000956	−0.000346
143	8	5	3	1	1	3	7	0.012867	0.001117	−0.000673	−0.000219	0.014397	−0.001228	0.000955	−0.000347
144	7	5	3	1	1	3	9	0.012370	0.001085	−0.000597	−0.000163	0.014395	−0.001230	0.000958	−0.000344

$V_0^{(1,3)}$	$V_1^{(1,3)}$	$V_0^{(2,2)}$	$V_1^{(2,2)}$	$V_2^{(2,2)}$	$V_0^{(2,3)}$	$V_1^{(2,3)}$	$V_2^{(2,3)}$	$V_0^{(3,3)}$	$V_1^{(3,3)}$	$V_2^{(3,3)}$	$V_3^{(3,3)}$
−0.001498	0.000672	0.011699	−0.000887	0.000198	0.000637	−0.000448	0.000420	0.011824	−0.000763	0.000323	−0.000546
−0.001502	0.000668	0.011698	−0.000889	0.000196	0.000643	−0.000442	0.000426	0.011825	−0.000762	0.000323	−0.000545
−0.001500	0.000670	0.011691	−0.000896	0.000189	0.000640	−0.000445	0.000423	0.011823	−0.000764	0.000321	−0.000547
−0.001450	0.000539	0.011852	−0.000727	0.000366	0.000674	−0.000383	0.000513	0.011700	−0.000789	0.000393	−0.000377
−0.000956	0.000346	0.011498	−0.000871	0.000431	0.001007	−0.000295	0.000356	0.011444	−0.000926	0.000376	−0.000275
−0.000936	0.000366	0.011414	−0.000956	0.000346	0.000956	−0.000346	0.000305	0.011414	−0.000956	0.000346	−0.000305
−0.000929	0.000373	0.011385	−0.000984	0.000318	0.000939	−0.000363	0.000288	0.011403	−0.000966	0.000336	−0.000315
−0.000926	0.000376	0.011371	−0.000999	0.000303	0.000931	−0.000371	0.000280	0.011398	−0.000971	0.000331	−0.000320
−0.000924	0.000378	0.011363	−0.001007	0.000295	0.000926	−0.000376	0.000275	0.011395	−0.000975	0.000328	−0.000323
−0.000927	0.000375	0.011364	−0.001005	0.000297	0.000935	−0.000367	0.000284	0.011396	−0.000974	0.000328	−0.000323
−0.000922	0.000380	0.011357	−0.001013	0.000289	0.000922	−0.000380	0.000271	0.011393	−0.000977	0.000326	−0.000326

TABLE 8.3 (continued)

N	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$V_0^{(0,0)}$	$V_0^{(0,1)}$	$V_0^{(0,2)}$	$V_0^{(0,3)}$	$V_0^{(1,1)}$	$V_1^{(1,1)}$	$V_0^{(1,2)}$	$V_1^{(1,2)}$
144	8	5	3	1	1	3	8	0.012556	0.001096	−0.000619	−0.000189	0.014396	−0.001229	0.000958	−0.000344
145	8	5	3	1	1	3	9	0.012348	0.001081	−0.000584	−0.000170	0.014395	−0.001230	0.000961	−0.000341
145	9	5	3	1	1	3	8	0.012545	0.001094	−0.000612	−0.000193	0.014395	−0.001230	0.000960	−0.000343
146	8	5	3	1	1	3	10	0.012199	0.001071	−0.000558	−0.000155	0.014394	−0.001231	0.000962	−0.000340
146	9	5	3	1	1	3	9	0.012334	0.001079	−0.000575	−0.000174	0.014394	−0.001231	0.000962	−0.000340
147	9	5	3	1	1	3	10	0.012183	0.001069	−0.000549	−0.000160	0.014394	−0.001231	0.000964	−0.000338
147	7	5	3	1	1	4	6	0.020674	0.002023	−0.001679	−0.000896	0.014467	−0.001139	0.000844	−0.000486
148	10	6	3	1	1	3	4	0.017415	0.001393	−0.001420	−0.000678	0.013364	−0.001103	0.000696	−0.000316
148	7	5	3	1	1	4	7	0.015872	0.001459	−0.001070	−0.000477	0.014400	−0.001206	0.000916	−0.000415
149	10	6	3	1	1	3	5	0.014364	0.001189	−0.000911	−0.000373	0.013351	−0.001117	0.000730	−0.000283
150	10	6	3	1	1	3	6	0.013346	0.001121	−0.000741	−0.000271	0.013346	−0.001121	0.000741	−0.000271

$V_0^{(1,3)}$	$V_1^{(1,3)}$	$V_0^{(2,2)}$	$V_1^{(2,2)}$	$V_2^{(2,2)}$	$V_0^{(2,3)}$	$V_1^{(2,3)}$	$V_2^{(2,3)}$	$V_0^{(3,3)}$	$V_1^{(3,3)}$	$V_2^{(3,3)}$	$V_3^{(3,3)}$
−0.00925	0.000377	0.011355	−0.001015	0.000288	0.000930	−0.000372	0.000279	0.011393	−0.000977	0.000325	−0.000326
−0.000923	0.000379	0.011349	−0.001021	0.000281	0.000926	−0.000376	0.000275	0.011391	−0.000979	0.000323	−0.000328
−0.000926	0.000377	0.011350	−0.001019	0.000283	0.000933	−0.000370	0.000281	0.011392	−0.000978	0.000324	−0.000327
−0.000922	0.000380	0.011345	−0.001025	0.000277	0.000924	−0.000378	0.000273	0.011390	−0.000980	0.000322	−0.000329
−0.000924	0.000378	0.011344	−0.001026	0.000276	0.000929	−0.000373	0.000278	0.011390	−0.000980	0.000322	−0.000329
−0.000923	0.000379	0.011339	−0.001031	0.000272	0.000927	−0.000375	0.000276	0.011388	−0.000981	0.000321	−0.000330
−0.000991	0.000283	0.011420	−0.000907	0.000438	0.000957	−0.000302	0.000392	0.011371	−0.000956	0.000389	−0.000219
−0.000769	0.000244	0.011371	−0.000927	0.000448	0.001108	−0.000267	0.000312	0.011298	−0.000999	0.000375	−0.000203
−0.000941	0.000332	0.011343	−0.000984	0.000361	0.000904	−0.000355	0.000339	0.011335	−0.000992	0.000353	−0.000256
−0.000748	0.000264	0.011286	−0.001012	0.000363	0.001057	−0.000318	0.000261	0.011268	−0.001030	0.000345	−0.000234
−0.000741	0.000271	0.011258	−0.001040	0.000335	0.001040	−0.000335	0.000244	0.011258	−0.001040	0.000335	−0.000244

number $N=\mu_0+\mu_6+6(\mu_1+\mu_5)+15(\mu_2+\mu_4)+20\mu_3$ with $130 \leqq N \leqq 150$, we can find the required optimal designs with respect to the trace and determinant criteria. In view of Theorem 5.3, however, note that we may restrict our attention to B-arrays such that (i) $\mu_2>\mu_4$ if $\mu_2 \neq \mu_4$, (ii) $\mu_1>\mu_5$ if $\mu_2=\mu_4$ and $\mu_1 \neq \mu_5$, or (iii) $\mu_0 \geqq \mu_6$ if $\mu_2=\mu_4$ and $\mu_1=\mu_5$. In Table 8.1, the optimal 2^9-BFF designs T of resolution VII with respect to the trace criterion are given with the values of $\mathrm{tr}(V_T)$ and the parameters λ_i ($i=0, 1,\dots, 9$) of the corresponding S-arrays. Note that the optimal designs are completely determined by knowing the values λ_i. Next let us consider the optimal designs with respect to the determinant criterion. In this case it is interesting that for $N=130$–132, 134, 136, 139–142 and 149–150, these designs are identical with the designs of Table 8.1, and moreover that for the remaining values of N but $N=138$ and 147, these designs are the second-best designs with respect to the trace criterion. These are given in Table 8.2 with the values of $\mathrm{tr}(V_T)$ and λ_i. By Theorem 5.4, we can easily obtain the distinct elements $V_\alpha^{(u,v)}$ of V_T for each optimal design of Tables 8.1 and 8.2. These are given in Table 8.3.

Part III. 2^m-BFF designs of even resolution derived from B-arrays of strength $2l$ and their optimalities

9. S_l type 2^m-BFF designs and their optimality

Consider a B-array T of strength $2l$, m constraints and index set $\{\mu_0, \mu_1,\dots, \mu_{2l}\}$ such that the following condition is satisfied:

$$(9.1) \qquad \begin{aligned} &\det(K_\beta) \neq 0 \qquad \text{for all} \quad \beta = 0, 1,\dots, l-1, \\ &K_l = 0, \end{aligned}$$

where K_β are the $(l-\beta+1)\times(l-\beta+1)$ matrices given in (4.3). Note that a 2^m-BFF design of resolution $2l+1$ can be no longer obtained from such an array T, since its information matrix M_T is singular. The following theorem has been established by Shirakura [24]:

THEOREM 9.1. *Let T be the above B-array. Then T is a fractional design in which*

(a) $\boldsymbol{\theta}_1$ *and* $\boldsymbol{\psi}_\beta=A_\beta^{(l,l)\#}\boldsymbol{\theta}_2$ ($\beta=0, 1,\dots, l-1$) *are estimable where* $\boldsymbol{\theta}_1$ *and* $\boldsymbol{\theta}_2$ *are given in* (1.7),

(b) *the BLUE* $\hat{\boldsymbol{\psi}}_{1\beta}=(\hat{\boldsymbol{\theta}}'_1, \hat{\boldsymbol{\psi}}'_\beta)'$ *of* $(\boldsymbol{\theta}'_1, \boldsymbol{\psi}'_\beta)'$ *is given by*

$$(9.2) \qquad \hat{\boldsymbol{\psi}}_{1\beta} = X_{1\beta}E'_T\boldsymbol{y}_T \qquad \textit{for} \quad \beta = 0, 1,\dots, l-1,$$

where

(9.3) $$X_{1\beta} = \sum_{\substack{\alpha=0\\ \alpha\neq\beta}}^{l-1} \sum_{i=0}^{l-\beta-1} \sum_{j=0}^{l-\beta} \kappa_{i,j}^{\alpha} D_{\alpha}^{(\alpha+i,\alpha+j)\#} + \sum_{i=0}^{l-\beta} \sum_{j=0}^{l-\beta} \kappa_{i,j}^{\beta} D_{\beta}^{(\beta+i,\beta+j)\#}$$

($\kappa_{i,j}^{\beta}$ are (i, j) elements of K_{β}^{-1}),

(c) *the covariance matrix* $\mathrm{Var}[\hat{\boldsymbol{\theta}}_1]$ *is invariant under any permutation of m factors.*

From Definition 2.3, the designs obtained in this theorem are a subclass of 2^m-BFF designs of resolution $2l$.

DEFINITION 9.1. *A B-array T of strength 2l, m constraints and index set $\{\mu_0, \mu_1, \ldots, \mu_{2l}\}$ is called an S_l type 2^m-BFF design if T satisfies Condition* (9.1).

It is easy to see that the covariance matrix $\mathrm{Var}[\hat{\boldsymbol{\theta}}_1]$ has at most $\binom{l+2}{3}$ distinct elements. By using the method similar to Theorem 5.4, we can obtain the following

THEOREM 9.2. *Let T be an S_l type 2^m-BFF design and consider the elements $V_{\alpha}^{(u,v)}\sigma^2$ of* $\mathrm{Var}[\hat{\boldsymbol{\theta}}_1]$ *corresponding to $\theta_{t_1\cdots t_u}$ and $\theta_{t'_1\cdots t'_v}$ which are α-th associates. Then*

(9.4) $$V_{\alpha}^{(u,v)} = \sum_{\beta=0}^{u} \kappa_{u-\beta,v-\beta}^{\beta} z_{(u,v)}^{\beta\alpha} \qquad \text{for} \quad 0 \leqq \alpha \leqq u \leqq v \leqq l-1,$$

where $z_{(u,v)}^{\beta\alpha}$ are given in (3.10).

Now we shall state some combinatorial properties of S_l type 2^m-BFF designs. From (5.6), $K_l=0$ is equivalent to $\mu_l=0$. To construct S_l type 2^m-BFF designs, first of all, we must investigate B-arrays of strength $2l$ with $\mu_l=0$. From Theorem 7.7, we can establish

THEOREM 9.3. *T is a B-array of strength 2l, m constraints and index set $\{\mu_0, \mu_1, \ldots, \mu_{2l}\}$ with $\mu_l=0$ if and only if T is an S-array with parameters (m; $\lambda_0, \ldots, \lambda_{l-1}, 0, \ldots, 0, \lambda_{m-l+1}, \ldots, \lambda_m$), where*

(9.5) $$\mu_i = \sum_{k=0}^{l-1} \binom{m-2l}{k-i} \lambda_k, \qquad \mu_{l+1+i} = \sum_{k=0}^{l-1} \binom{m-2l}{i-k} \lambda_{m-l+1+k}$$

for $i=0, 1, \ldots, l-1$.

COROLLARY 9.4. *A necessary and sufficient condition for the existence of the B-array of Theorem* 9.3 *is that the following inequalities hold:*

$$\sum_{i=0}^{l-1}(-1)^{i+k}\binom{m-2l-1+i-k}{i-k}\mu_i \geqq 0,$$

$$\sum_{i=0}^{l-1}(-1)^{i+k}\binom{m-2l-1+k-i}{k-i}\mu_{l+1+i} \geqq 0$$

for all $k=0, 1,\dots, l-1$.

PROOF. See Shirakura [22].

The following two theorems are due to [24].

THEOREM 9.5. *Let T be an S_l type 2^m-BFF design. Then the number of distinct assemblies in T must be at least* $v_l^1=v_l-\phi_l=1+m+\binom{m}{2}+\cdots+\binom{m}{l-2}+2\binom{m}{l-1}$.

THEOREM 9.6. *If there exists an S_l type 2^m-BFF design T with N_0 ($\geqq v_l^1$) assemblies, then for $N>N_0$, $(N-N_0+1)$ nonisomorphic S_l type 2^m-BFF designs with N assemblies can be obtained from T.*

THEOREM 9.7. *A necessary condition for the existence of an S_l type 2^m-BFF design is that the following strict inequalities hold*:

$$\mu_{l-1}\mu_{l+1} > 0,$$

$$(m-2l+4)(\mu_{l-2}+\mu_{l+2}) > 4(m-2l)(\mu_{l-1}+\mu_{l+1}) \qquad \textit{for} \quad l \geqq 2.$$

PROOF. This follows from (5.7), (5.8f) and Condition (9.1).

THEOREM 9.8. *Consider the case $l=2$. Then there exist always S_2 type 2^m-BFF designs for any N ($\geqq v_2^1=2m+1$) assemblies.*

PROOF. Consider an S-array T with parameters $(m;\ \lambda_0=1,\ \lambda_2=1,\ 0,\dots,0,\ \lambda_{m-1}=1,\ \lambda_m=0)$. From Theorem 9.3, then T is equivalent to a B-array of strength 4, size $N=2m+1$, m constraints and index set $\{\mu_0=(m-3),\ \mu_1=1,\ \mu_2=0,\ \mu_3=1,\ \mu_4=(m-4)\}$. It is easy to check that the matrices K_0 and K_1 in Example 4.1, (i) are nonsingular for the B-array T. This implies that T is just an S_2 type 2^m-BFF design with the smallest number $v_2^1=2m+1$ of assemblies. Because of Theorem 9.6, the proof of this theorem is completed.

Now consider the case $l=3$. In this case the smallest number is $v_3^1=1+m+2\binom{m}{2}$. Consider an S-array T with parameters $(m;\ \lambda_0=0,\ \lambda_1=1,\ \lambda_2=1,\ 0,\dots,0,\ \lambda_{m-2}=1,\ \lambda_{m-1}=0,\ \lambda_m=1)$, which is identical with a B-array of strength 6, size $N=v_3^1$, m constraints and index set $\{\mu_0=\binom{m-5}{2},\ \mu_1=m-5,\ \mu_2=1,\ \mu_3=0,$

$\mu_4=1,\ \mu_5=m-6,\ \mu_6=\binom{m-6}{2}+1\}$. Unlike the case $l=2$, it is very complicated to show that for a general number m, the array T satisfies Condition (9.1). However for each value of m within a practical range, we shall be able to show that T satisfies Condition (9.1).

From Theorems 9.6, 9.8 and the above statements, we may say that for any given N, there are in general a large number of possible S_l type 2^m-BFF designs. Among these, we must choose one which maximizes information in some sense. For this purpose, Shirakura [24] has introduced the following amount for an S_l type 2^m-BFF design T:

$$S_T = \sum_{\beta=0}^{l-1} \phi_\beta \operatorname{tr}(K_\beta^{-1}). \tag{9.6}$$

Let ψ_β^* be $\phi_\beta\times 1$ vector whose elements are composed of ϕ_β independent linear functions in $\left\{\phi_\beta/\binom{m}{l}\right\}^{-1/2}\psi_\beta$. Then S_T can be rewritten as

$$S_T = \operatorname{tr}(\operatorname{Var}[\hat{\theta}_1])/\sigma^2 + \sum_{\beta=0}^{l-1} \operatorname{tr}(\operatorname{Var}[\hat{\psi}_\beta^*])/\sigma^2,$$

where $\hat{\psi}_\beta^*$ is the BLUE of ψ_β^*. From (3.17), (4.1) and Condition (9.1), it is also found that S_T denotes the trace of a generalized inverse matrix of M_T.

DEFINITION 9.2. *For given N assemblies, an S_l type 2^m-BFF design T is said to be optimal with respect to the generalized trace (GT) criterion if T minimizes S_T.*

10. Optimal S_3 type 2^m-BFF designs with $m=6, 7$

In view of the previous section, we are interested in optimal S_l type 2^m-BFF designs with respect to the GT criterion for desirable numbers m and $N\geqq v_l^1$. In this section, for the special case $l=3$, the optimal designs will be obtained for $m=6, 7$ and for every N with $v_3^1\left(=1+m+2\binom{m}{2}\right)\leqq N<v_3\left(=1+m+\binom{m}{2}+\binom{m}{3}\right)$. In this case, note that since there exist always 2^m-BFF designs of resolution VII with $N\geqq v_3$ assemblies (see [23]), we need not consider S_3 type 2^m-BFF designs for larger N. For the optimal designs for $m=8$, see [24].

From Condition (9.1) and Theorem 9.3, first consider a B-array of strength 6, m constraints and index set $\{\mu_0, \mu_1, \mu_2, \mu_3=0, \mu_4, \mu_5, \mu_6\}$, and the corresponding S-array with parameters $(m;\ \lambda_0, \lambda_1, \lambda_2, \lambda_3=0,\dots, \lambda_{m-3}=0, \lambda_{m-2}, \lambda_{m-1}, \lambda_m)$. From (9.5), we have

$$\mu_0 = \lambda_0+(m-6)\lambda_1+\binom{m-6}{2}\lambda_2,\quad \mu_1 = \lambda_1+(m-6)\lambda_2,$$

$$\mu_2 = \lambda_2, \quad \mu_4 = \lambda_{m-2}, \quad \mu_5 = \lambda_{m-1}+(m-6)\lambda_{m-2}, \tag{10.1}$$

$$\mu_6 = \lambda_m+(m-6)\lambda_{m-1}+\binom{m-6}{2}\lambda_{m-2}.$$

From Corollary 9.4 and Theorem 9.7, we can obtain the following

THEOREM 10.1. *A necessary condition for the existence of an S_3 type 2^m-BFF design is that the following inequalities hold*:

$$\text{(a)} \quad \mu_2 \geqq 1, \quad \mu_4 \geqq 1,$$

$$\text{(b)} \quad \mu_0+\binom{m-5}{2}\mu_2 \geqq (m-6)\mu_1, \quad \mu_1 \geqq (m-6)\mu_2, \tag{10.2}$$

$$\mu_6+\binom{m-5}{2}\mu_4 \geqq (m-6)\mu_5, \quad \mu_5 \geqq (m-6)\mu_4.$$

Now we shall prove

THEOREM 10.2. *A necessary condition for the existence of an S_3 type 2^m-BFF design with $N<\nu_3$ is that the following inequalities hold*:

$$\text{(a)} \quad \frac{m+1}{3} \geqq (\mu_2+\mu_4) \qquad \text{for} \quad m \neq 7,$$

$$\text{(b)} \quad 3 \geqq (\mu_2+\mu_4) \qquad \text{for} \quad m = 7. \tag{10.3}$$

PROOF. From (10.2b), it is easy to verify that $\mu_0+\mu_6+6(\mu_1+\mu_5)\geqq(m^2-m-30)(\mu_2+\mu_4)/2$ holds. Since $N=\mu_0+\mu_6+6(\mu_1+\mu_5)+15(\mu_2+\mu_4)<\nu_3$, we have $\nu_3>\binom{m}{2}(\mu_2+\mu_4)$. This shows that

$$\frac{m+1}{3}+\frac{2(m+1)}{m(m-1)} > (\mu_2+\mu_4). \tag{10.4}$$

Let $m+1=3t+r$ where $0\leqq r\leqq 2$. Since we are assuming $m\geqq 6$, we have $t\geqq 2$. Now we shall show that (10.3a, b) hold for each case $r=0, 1, 2$. For $r=0$, the left hand side of (10.4) reduces to $t+6t/(9t^2-9t+2)$. Clearly $r=0$ implies $m\geqq 8$, so that $t\geqq 3$. It is easy to see that $0<6t/(9t^2-9t+2)<1$ holds for $t\geqq 3$. Hence we have $t\geqq(\mu_2+\mu_4)$. For $r=1$, the left hand side of (10.4) reduces to $t+(3t^2+5t+2)/(9t^2-3t)$. Since $0<(3t^2+5t+2)/(9t^2-3t)<1$ holds for $t\geqq 2$, we have $t\geqq(\mu_2+\mu_4)$. Finally consider the case $r=2$. Then the left hand side of (10.4) reduces to $t+(3t^2+4t+2)/(9t^2+3t)$. Similarly it can be shown that $0<(3t^2+4t+2)/(9t^2+3t)<1$ holds for $t\geqq 3$. Thus $t>(\mu_2+\mu_4)$ for $m\geqq 11$. When $m=7$, from (10.4) it is clear that (10.3b) holds. This completes the proof.

From the above results, we can easily construct S_3 type 2^m-BFF designs for

TABLE 10.1 Optimal S_3 type 2^m-BFF designs

$m=6$

N	μ_0	μ_1	μ_2	μ_4	μ_5	μ_6	S_T	λ_0	λ_1	λ_2	λ_4	λ_5	λ_6
37	0	1	1	1	0	1	1.20979	0	1	1	1	0	1
38	1	1	1	1	0	1	1.16667	1	1	1	1	0	1
39	1	1	1	1	0	2	1.15368	1	1	1	1	0	2
40	2	1	1	1	0	2	1.14619	2	1	1	1	0	2
41	2	1	1	1	0	3	1.14179	2	1	1	1	0	3

$m=7$

N	μ_0	μ_1	μ_2	μ_4	μ_5	μ_6	S_T	λ_0	λ_1	λ_2	λ_5	λ_6	λ_7
50	1	2	1	1	1	1	1.43426	0	1	1	1	0	1
51	2	2	1	1	1	1	1.41425	1	1	1	1	0	1
52	2	2	1	1	1	2	1.40466	1	1	1	1	0	2
53	3	2	1	1	1	2	1.39952	2	1	1	1	0	2
54	3	2	1	1	1	3	1.39624	2	1	1	1	0	3
55	4	2	1	1	1	3	1.39388	3	1	1	1	0	3
56	1	2	1	1	2	1	1.15878	0	1	1	1	1	0
57	2	2	1	1	2	1	1.13936	1	1	1	1	1	0
58	2	2	1	1	2	2	1.12012	1	1	1	1	1	1
59	3	2	1	1	2	2	1.11531	2	1	1	1	1	1
60	3	2	1	1	2	3	1.11032	2	1	1	1	1	2
61	4	2	1	1	2	3	1.10804	3	1	1	1	1	2
62	4	2	1	1	2	4	1.10571	3	1	1	1	1	3
63	2	3	1	1	2	1	1.09260	0	2	1	1	1	0

TABLE 10.2 Covariance matrices for optimal S_3 type 2^m-BFF designs

$m=6$

N	μ_0	μ_1	μ_2	μ_4	μ_5	μ_6	$V_0^{(0,0)}$	$V_0^{(0,1)}$	$V_0^{(0,2)}$	$V_0^{(1,1)}$
37	0	1	1	1	0	1	0.02833	0.00187	−0.00083	0.03042
38	1	1	1	1	0	1	0.02832	0.00195	−0.00098	0.02995
39	1	1	1	1	0	2	0.02800	0.00150	−0.00135	0.02933
40	2	1	1	1	0	2	0.02799	0.00152	−0.00139	0.02926
41	2	1	1	1	0	3	0.02788	0.00137	−0.00152	0.02905

$V_1^{(1,1)}$	$V_0^{(1,2)}$	$V_1^{(1,2)}$	$V_0^{(2,2)}$	$V_1^{(2,2)}$	$V_2^{(2,2)}$
−0.00083	−0.00021	−0.00021	0.03250	0.00125	0.00125
−0.00130	0.00065	0.00065	0.03092	−0.00033	−0.00033
−0.00192	0.00013	0.00013	0.03049	−0.00076	−0.00076
−0.00199	0.00027	0.00027	0.03020	−0.00105	−0.00105
−0.00220	0.00010	0.00010	0.03005	−0.00120	−0.00120

$m=7$	N	μ_0	μ_1	μ_2	μ_4	μ_5	μ_6	$V_0^{(0,0)}$	$V_0^{(0,1)}$	$V_0^{(0,2)}$	$V_0^{(1,1)}$
	50	1	2	1	1	1	1	0.02980	0.00058	−0.00376	0.05237
	51	2	2	1	1	1	1	0.02742	0.00019	−0.00266	0.05231
	52	2	2	1	1	1	2	0.02708	0.00045	−0.00235	0.05211
	53	3	2	1	1	1	2	0.02645	0.00036	−0.00206	0.05210
	54	3	2	1	1	1	3	0.02632	0.00045	−0.00195	0.05203
	55	4	2	1	1	1	3	0.02604	0.00041	−0.00181	0.05202
	56	1	2	1	1	2	1	0.03125	0.00000	−0.00446	0.04167
	57	2	2	1	1	2	1	0.02979	−0.00049	−0.00384	0.04150
	58	2	2	1	1	2	2	0.02734	0.00000	−0.00279	0.04141
	59	3	2	1	1	2	2	0.02673	−0.00012	−0.00253	0.04138
	60	3	2	1	1	2	3	0.02604	0.00000	−0.00223	0.04136
	61	4	2	1	1	2	3	0.02573	−0.00006	−0.00210	0.04135
	62	4	2	1	1	2	4	0.02539	0.00000	−0.00195	0.04134
	63	2	3	1	1	2	1	0.03097	−0.00028	−0.00419	0.03939

$V_1^{(1,1)}$	$V_0^{(1,2)}$	$V_1^{(1,2)}$	$V_0^{(2,2)}$	$V_1^{(2,2)}$	$V_2^{(2,2)}$
−0.00666	−0.00231	0.00116	0.02633	−0.00145	0.00203
−0.00672	−0.00213	0.00134	0.02582	−0.00196	0.00151
−0.00692	−0.00237	0.00111	0.02555	−0.00223	0.00124
−0.00693	−0.00233	0.00115	0.02541	−0.00237	0.00110
−0.00700	−0.00241	0.00107	0.02531	−0.00247	0.00101
−0.00700	−0.00239	0.00109	0.02525	−0.00253	0.00094
−0.00521	0.00000	0.00000	0.02487	−0.00191	0.00255
−0.00537	0.00021	0.00021	0.02460	−0.00218	0.00228
−0.00547	0.00000	0.00000	0.02415	−0.00263	0.00183
−0.00549	0.00005	0.00005	0.02404	−0.00274	0.00172
−0.00551	0.00000	0.00000	0.02392	−0.00287	0.00159
−0.00552	0.00002	0.00002	0.02386	−0.00293	0.00154
−0.00553	0.00000	0.00000	0.02380	−0.00299	0.00147
−0.00515	−0.00020	0.00047	0.02432	−0.00227	0.00238

each $m \geqq 6$ and each N with $v_3^1 \leqq N < v_3$. Among these, we can obtain the required optimal design T such that S_T in (9.6) is a minimum. In Table 10.1, the optimal designs for $m=6$, 7 are given with the values of λ_0, λ_1, λ_2, λ_{m-2}, λ_{m-1}, λ_m in (10.1). The distinct 20 elements $V_\alpha^{(u,v)}$ in (9.4) for the designs are also given in Table 10.2. As in Theorem 5.3, for an S_t type 2^m-BFF design T and its complementary design $\bar{T}$, we have $S_T = S_{\bar{T}}$. Thus it may be remarked that for the designs in Table 10.1, their complementary designs are also optimal with respect to the GT criterion.

11. Alias structures of l-factor interactions in S_l type 2^m-BFF designs and their estimability

In this section we shall make certain investigations on alising of l-factor interactions in S_l type 2^m-BFF designs. It has been observed in Section 9 that $\boldsymbol{\psi}_\beta = A_\beta^{(l,l)\#}\boldsymbol{\theta}_2$ ($\beta=0, 1,\dots, l-1$) are estimable in an S_l type 2^m-BFF design T. From (3.7) and (3.11), $\boldsymbol{\psi}_\beta$ are such that

(i) every element of $\boldsymbol{\psi}_0$ represents the mean of effects of l-factor interactions, i.e.,

$$\boldsymbol{\psi}_0 = \frac{1}{\binom{m}{l}} \sum_{\{t_1,t_2,\dots,t_l\}\in\mathfrak{M}_l} \theta_{t_1t_2\cdots t_l}\boldsymbol{j}_{\binom{m}{l}},$$

(ii) the elements of $\boldsymbol{\psi}_\beta$ ($\beta\neq 0$) represent contrasts between effects of l-factor interactions, i.e.,

$$\boldsymbol{j}'_{\binom{m}{l}}\boldsymbol{\psi}_\beta = 0 \qquad \text{for} \quad \beta\neq 0,$$

(iii) any two contrasts, one belonging to $\boldsymbol{\psi}_\alpha$ and other to $\boldsymbol{\psi}_\beta$ ($\alpha\neq\beta$), are orthogonal, i.e.,

$$\boldsymbol{\psi}'_\alpha\boldsymbol{\psi}_\beta = 0 \qquad \text{for} \quad \alpha\neq\beta, \quad \text{and}$$

(iv) there are ϕ_β independent contrasts in each $\boldsymbol{\psi}_\beta$ ($\beta\neq 0$).

From the above statements, it is found that in all $\binom{m}{l-1}$ $(=\phi_0+\phi_1+\cdots+\phi_{l-1})$ independent linear functions of $\theta_{t_1t_2\cdots t_l}$ are estimable in the design T. However to observe the pattern of aliasing, a more simple expression for alias structures of l-factor interactions is needed. We establish the following

THEOREM 11.1. *In an S_l type 2^m-BFF design,*

$$\boldsymbol{\psi} = A_0^{(l-1,l)}\boldsymbol{\theta}_2 \tag{11.1}$$

is an estimable function of $\boldsymbol{\theta}_2$ where $A_0^{(l-1,l)}$ is the local association matrix of size $\binom{m}{l-1}\times\binom{m}{l}$ defined in (3.2). There are just $\binom{m}{l-1}$ independent linear functions of $\boldsymbol{\theta}_2$ in $\boldsymbol{\psi}$.

PROOF. From (3.11) and (3.12), we have $A_\beta^{(l-1,l)\#}A_\beta^{(l,l)\#} = A_\beta^{(l-1,l)\#} = c_\beta^{(l-1,l)}A_\beta^{(l-1,l-1)\#}A_0^{(l-1,l)}$ for all $\beta=0, 1,\dots, l-1$. Hence the estimability of $\boldsymbol{\psi}_\beta$ ($\beta=0, 1,\dots, l-1$) implies that $\sum_{\beta=0}^{l-1}A_\beta^{(l-1,l-1)\#}A_0^{(l-1,l)}\boldsymbol{\theta}_2$ is estimable. Since $\sum_{\beta=0}^{l-1}A_\beta^{(l-1,l-1)\#}=I_p$, where $p=\binom{m}{l-1}$, it is clear that $\boldsymbol{\psi}$ is estimable. From (3.6) and (3.11), $A_0^{(l-1,l)}A_0^{(l,l-1)} = \sum_{\beta=0}^{l-1}(z_{\beta 0}^{(l-1,l)})^2A_\beta^{(l-1,l-1)\#}$. From (3.9),

$(z_{\beta 0}^{(l-1,l)})^2 = (m-l+1-\beta)(l-\beta) \neq 0$ for all $\beta=0, 1,\dots, l-1$, so that rank $(A_0^{(l-1,l)}) = \text{rank}(A_0^{(l-1,l)}A_0^{(l,l-1)}) = \binom{m}{l-1}$. This completes the proof.

EXAMPLE 11.1.

(i) Consider an S_2 type 2^m-BFF design ($l=2$). Then $\boldsymbol{\theta}_2' = (\theta_{12}, \theta_{13},\dots, \theta_{1m}, \theta_{23},\dots, \theta_{m-1m})$, $\left(1\times\binom{m}{2}\right)$, and $\text{rank}(A_0^{(1,2)}) = m$. $\boldsymbol{\psi}$ reduces to

$$\boldsymbol{\psi} = \begin{bmatrix} \theta_{12}+\theta_{13}+\theta_{14}+\cdots+\theta_{1m} \\ \theta_{12}+\theta_{23}+\theta_{24}+\cdots+\theta_{2m} \\ \theta_{13}+\theta_{23}+\theta_{34}+\cdots+\theta_{3m} \\ \vdots \\ \theta_{1m}+\theta_{2m}+\theta_{3m}+\cdots+\theta_{m-1m} \end{bmatrix}.$$

(ii) Consider an S_3 type 2^m-BFF design ($l=3$). Then $\boldsymbol{\theta}_2' = (\theta_{123}, \theta_{124},\dots, \theta_{12m}, \theta_{134},\dots, \theta_{m-2m-1m})$, $\left(1\times\binom{m}{3}\right)$, and $\text{rank}(A_0^{(2,3)}) = \binom{m}{2}$. $\boldsymbol{\psi}$ reduces to

$$\boldsymbol{\psi} = \begin{bmatrix} \theta_{123}+\theta_{124}+\theta_{125}+\cdots+\theta_{12m} \\ \theta_{123}+\theta_{134}+\theta_{135}+\cdots+\theta_{13m} \\ \theta_{124}+\theta_{134}+\theta_{145}+\cdots+\theta_{14m} \\ \vdots \\ \theta_{1m-1m}+\theta_{2m-1m}+\cdots+\theta_{m-2m-1m} \end{bmatrix}.$$

COROLLARY 11.2. *For an S_l type 2^m-BFF design T, the BLUE $\hat{\boldsymbol{\psi}}$ of $\boldsymbol{\psi}$ is given by*

$$\hat{\boldsymbol{\psi}} = X_1 E_T' \boldsymbol{y}_T, \tag{11.2}$$

where X_1 is the $p\times v_l$ matrix such that

$$X_1 = \sum_{\beta=0}^{l-1} (c_\beta^{(l-1,l)})^{-1} [O_{p\times q_\beta} : \kappa_{l-\beta,0}^{\beta} A_\beta^{(l-1,\beta)\#} : \kappa_{l-\beta,1}^{\beta} A_\beta^{(l-1,\beta+1)\#} : \cdots : \kappa_{l-\beta,l-\beta}^{\beta} A_\beta^{(l-1,l)\#}].$$

$\left(p=\binom{m}{l-1}, q_\beta = \sum_{i=0}^{\beta}\binom{m}{i-1}\right.$ *and, particularly,* $\left.[O_{p\times 0}: A] = A\right)$.

PROOF. This follows immediately from (9.2), (9.3) and Theorem 11.1.

REMARK. From Theorem 9.5, the rank of the information matrix M_T of an S_l type 2^m-BFF design T is $\nu_l^1 = 1 + \binom{m}{1} + \cdots + \binom{m}{l-2} + 2\binom{m}{l-1}$. Since $\nu_l^1 - \nu_{l-1} = \binom{m}{l-1}$, from the design T we can not obtain more than $\binom{m}{l-1}$ independent linear functions of $\theta_{t_1 t_2 \cdots t_l}$ which are estimable. Therefore it follows from Theorem 11.1 that any estimable function $\boldsymbol{\psi}^*$ of $\boldsymbol{\theta}_2$ is completely determined by $\boldsymbol{\psi}^* = C^* \boldsymbol{\psi}$, where C^* is a matrix of appropriate size.

THEOREM 11.3. *In an S_l type 2^m-BFF design, no l-factor interaction itself is estimable.*

PROOF. Assume that some l-factor interaction $\theta_{t_1 t_2 \cdots t_l}$ is estimable in this design. Let $\boldsymbol{t}$ be the $\binom{m}{l} \times 1$ vector obtained from $\boldsymbol{\theta}_2$ by replacing $\theta_{t_1 \cdots t_l}$ with 1 and the remaining effects with 0. Now we shall show that $\operatorname{rank}(A) > \binom{m}{l-1}$, where $A = [A_0^{(l,l-1)} : \boldsymbol{t}]$. Since $A_0^{(l-1,l)} A_0^{(l,l-1)}$ is nonsingular, $\det(A'A) = \det(A_0^{(l-1,l)} A_0^{(l,l-1)})(1-s)$, where $s = \boldsymbol{t}' A_0^{(l,l-1)} (A_0^{(l-1,l)} A_0^{(l,l-1)})^{-1} A_0^{(l-1,l)} \boldsymbol{t}$. From (3.6), (3.9), (3.11) and (3.12), we have $(A_0^{(l-1,l)} A_0^{(l,l-1)})^{-1} = \sum_{\beta=0}^{l-1} (z_{\beta 0}^{(l-1,l)})^{-2} A_\beta^{(l-1,l-1)\#}$ and $A_0^{(l,l-1)} A_\beta^{(l-1,l-1)\#} A_0^{(l-1,l)} = (c_\beta^{(l-1,l)})^{-2} A_\beta^{(l,l)\#}$ for $\beta = 0, 1, \ldots, l-1$. Since $z_{\beta 0}^{(l-1,l)} = (c_\beta^{(l-1,l)})^{-1}$ and $\sum_{\beta=0}^{l-1} A_\beta^{(l,l)\#} = I_{\binom{m}{l}} - A_l^{(l,l)\#}$, it is clear that $1 - s = \operatorname{tr}(\boldsymbol{t}\boldsymbol{t}' A_l^{(l,l)\#})$. From (3.7), therefore, $(1-s) \neq 0$, so that $\det(A'A) \neq 0$. From matrix theory, it is found that there does not exist any $\binom{m}{l-1} \times 1$ vector $\boldsymbol{x}$ satisfying

$$A_0^{(l,l-1)} \boldsymbol{x} = \boldsymbol{t}.$$

This contradicts that $\theta_{t_1 \cdots t_l}$ is estimable.

In view of this theorem, consider a situation where some of l-factor interactions can be assumed negligible. By Theorem 11.1, we can easily prove the following lemma:

LEMMA 11.4. *In an S_l type 2^m-BFF design, r ($\leqq \binom{m}{l-1}$) l-factor interactions themselves are estimable if the column vectors of $A_0^{(l-1,l)}$ corresponding to these effects are independent, and if the remaining l-factor interactions can be neglected.*

Now let us consider an experiment with the special factor f_{t_1} such that every l-factor interaction involving it can not be ignored. From properties of the matrix $A_0^{(l-1,l)}$, then we may suppose without loss of generality that it is the first factor f_1. Thus we denote the vector composed of all $\binom{m-1}{l-1}$ l-factor interactions involving the factor f_1 by

$$\boldsymbol{\theta}_2^1 = (\{\theta_{1t_2t_3\cdots t_l}\})', \qquad \left(\binom{m-1}{l-1}\times 1\right).$$

THEOREM 11.5. *In an S_l type 2^m-BFF design, $\boldsymbol{\theta}_2^1$ is estimable under the assumption that the remaining l-factor interactions are negligible.*

PROOF. From the definition of association matrices, $A_0^{(l-1,l)}$ can be written in the form of

$$(11.3) \qquad A_0^{(l-1,l)} = \left[\begin{array}{c:c} \tilde{A}_0^{(l-2,l-1)} & O_{\binom{m-1}{l-2}\times\binom{m-1}{l}} \\ \hdashline I_{\binom{m-1}{l-1}} & \tilde{A}_0^{(l-1,l)} \end{array}\right],$$

where $\tilde{A}_0^{(u,v)}$ are the local association matrices, defined by (3.2), for $(m-1)$ factors $f_2, f_3, \ldots, f_m$. The first $\binom{m-1}{l-1}$ columns of $A_0^{(l-1,l)}$ are clearly independent. This completes the proof, because of Lemma 11.4.

Note that since $\mathrm{rank}(A_0^{(l-1,l)}) = \binom{m}{l-1}$, among the remaining l-factor interactions we can recover $\binom{m-1}{l-2} = \binom{m}{l-1} - \binom{m-1}{l-1}$ $(=z$, say) those. Consider the following matrix:

$$(11.4) \qquad \left[\begin{array}{c:c} \tilde{A}_0^{(l-2,l-1)} & O_{z\times z} \\ \hdashline I_{\binom{m-1}{l-1}} & F^1_{j_1j_2\cdots j_z} \end{array}\right] \qquad (= A_{j_1j_2\cdots j_z}, \text{ say}),$$

where $F^1_{j_1j_2\cdots j_z}$ is the $\binom{m-1}{l-1}\times z$ matrix composed of j_1-th, j_2-th,..., j_z-th columns of $\tilde{A}_0^{(l-1,l)}$. Then it is easy to see that $A_{j_1\cdots j_z}$ is nonsingular if and only if $(\tilde{A}_0^{(l-2,l-1)}F^1_{j_1\cdots j_z})$ is so. However it is in general difficult to observe whether $(\tilde{A}_0^{(l-2,l-1)}F^1_{j_1\cdots j_z})$ is nonsingular or not. The following lemma is very useful:

LEMMA 11.6. *Let $F^2_{j_1j_2\cdots j_z}$ be the $z\times z$ matrix composed of j_1-th, j_2-th,..., j_z-th columns of $\tilde{A}_0^{(l-2,l)}$. Then $F^2_{j_1\cdots j_z}$ is nonsingular if and only if $(\tilde{A}_0^{(l-2,l-1)}F^1_{j_1\cdots j_z})$ is so.*

PROOF. From (3.3), we have $\tilde{A}_0^{(l-2,l-1)}\tilde{A}_0^{(l-1,l)} = 2\tilde{A}_0^{(l-2,l)}$. Hence $\tilde{A}_0^{(l-2,l-1)} \cdot F^1_{j_1\cdots j_z} = 2F^2_{j_1\cdots j_z}$ holds. This completes the proof.

Let $\boldsymbol{\theta}^2_{2(j_1j_2\cdots j_z)}$ be the $z\times 1$ vector composed of z effects which are obtained from $\boldsymbol{\theta}_2$ corresponding to j_1-th, j_2-th,..., j_z-th columns of $\tilde{A}_0^{(l-1,l)}$ in (11.3). Then we establish the following

THEOREM 11.7. *If the matrix $F^2_{j_1j_2\cdots j_z}$ of Lemma 11.6 is nonsingular, then*

$\boldsymbol{\theta}_2^1$ and $\boldsymbol{\theta}_{2(j_1 j_2 \cdots j_z)}^2$ are estimable in an S_l type 2^m-BFF design under the assumption that the remaining l-factor interactions are negligible. Furthermore their BLUEs $\hat{\boldsymbol{\theta}}_2^1$ and $\hat{\boldsymbol{\theta}}_{2(j_1 \cdots j_z)}^2$ are given as follows:

$$\text{(11.5)} \qquad \begin{aligned} \hat{\boldsymbol{\theta}}_2^1 &= \boldsymbol{y}_2 + F_{j_1\cdots j_z}^1 (F_{j_1\cdots j_z}^2)^{-1} (\boldsymbol{y}_1 - \tilde{A}_0^{(l-2,l-1)})/2, \\ \hat{\boldsymbol{\theta}}_{2(j_1\cdots j_z)}^2 &= (F_{j_1\cdots j_z}^2)^{-1} (\tilde{A}_0^{(l-2,l-1)} \boldsymbol{y}_2 - \boldsymbol{y}_1)/2, \end{aligned}$$

where $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are the $z\times 1$ and $\binom{m-1}{l-1}\times 1$ vectors, respectively such that $(\boldsymbol{y}_1', \boldsymbol{y}_2') = \hat{\boldsymbol{\psi}}'$ in (11.2).

PROOF. The proof of the first part of the theorem follows immediately from (11.4), Lemmas 11.4 and 11.6. Now we shall show that (11.5) holds. From (11.1), (11.2), (11.4), (11.5) and the assumption of this theorem, we have

$$\mathrm{Exp}\,[\hat{\boldsymbol{\psi}}] = \boldsymbol{\psi} = A_{j_1\cdots j_z} \begin{bmatrix} \boldsymbol{\theta}_2^1 \\ \boldsymbol{\theta}_{2(j_1\cdots j_z)}^2 \end{bmatrix}.$$

It is easily shown that the inverse matrix of $A_{j_1\cdots j_z}$ is given by

$$(A_{j_1\cdots j_z})^{-1} = \left[\begin{array}{c|c} \frac{1}{2} F_{j_1\cdots j_z}^1 (F_{j_1\cdots j_z}^2)^{-1} & I_{\binom{m-1}{l-1}} - \frac{1}{2} F_{j_1\cdots j_z}^1 (F_{j_1\cdots j_z}^2)^{-1} \tilde{A}_0^{(l-2,l-1)} \\ \hline -(F_{j_1\cdots j_z}^2)^{-1} & (F_{j_1\cdots j_z}^2)^{-1} \tilde{A}_0^{(l-2,l-1)} \end{array}\right].$$

This completes the proof.

Designs of resolution less than or equal to VII are thus far very important. For the cases $l=2$, 3, therefore, we shall make further investigations on recovering l-factor interactions. First consider the case $l=2$ $(4 \leqq m)$. In this case

$$\boldsymbol{\theta}_1^2 = (\theta_{12}, \theta_{13}, \ldots, \theta_{1m})', \quad ((m-1)\times 1),$$
$$\mathrm{rank}\,(A_0^{(1,2)}) = m, \quad z = 1 \quad \text{and}$$
$$\tilde{A}_0^{(0,2)} = \boldsymbol{j}'_{\binom{m-1}{2}} = (1, 1, \ldots, 1).$$

Therefore the matrix $F_{j_1}^2$ of Lemma 11.6 is nonsingular for every $j_1 = 1, 2, \ldots, \binom{m-1}{2}$. From Theorem 11.7, we can easily obtain

THEOREM 11.8. *In an S_2 type 2^m-BFF design, the two-factor interactions θ_{1i} $(i=2, 3, \ldots, m)$ and any two-factor interaction θ_{jk} in $\{\theta_{t_1 t_2}\}$, $(t_1 \geqq 2)$, are estimable ignoring the remaining two-factor interactions.*

Next consider the case $l=3$ $(6 \leqq m)$. Then

$$\boldsymbol{\theta}_2^1 = (\theta_{123}, \theta_{124}, \ldots, \theta_{12m}, \theta_{134}, \ldots, \theta_{1m-1m})', \quad \left(\binom{m-1}{2} \times 1\right),$$

$$\operatorname{rank}(\tilde{A}_0^{(2,3)}) = \binom{m}{2} \quad \text{and} \quad z = m-1.$$

In this case, besides the special factor f_1, further consider the special two factors f_{t_2} and f_{t_3} such that every three-factor interaction involving these two factors can not be ignored. As before, we may suppose without loss of generality that they are the second and third factors f_2 and f_3. Therefore we can obtain the $(m-3) \times 1$ vector

$$\boldsymbol{\theta}_{2(1\,2 \cdots m-3)}^2 = (\theta_{234}, \theta_{235}, \ldots, \theta_{23m})'.$$

Since $z-(m-3)=2$, two effects $\theta_{t_1t_2t_3}$ and $\theta_{t'_1t'_2t'_3}$ can be further recovered from the rest. Now suppose that at least one of the two effects involves the factor f_2 or f_3, and therefore suppose without loss of generality that the effect involves the factor f_2, i.e., $\theta_{t_1t_2t_3} = \theta_{2t_2t_3}$. Consequently the following theorem can be established:

THEOREM 11.9. *In an S_3 type 2^mBFF design, $\boldsymbol{\theta}_2^1$, $\boldsymbol{\theta}_{2(1\,2 \cdots m-3)}^2$ and the above two effects $\theta_{2t_2t_3}$ and $\theta_{t'_1t'_2t'_3}$ $(4 \leqq t_2 < t_3 \leqq m,\ 3 \leqq t'_1 < t'_2 < t'_3 \leqq m)$ are estimable ignoring the remaining three-factor interactions.*

PROOF. First consider the case where the other effect involves the factor f_3, i.e., $\theta_{t'_1t'_2t'_3} = \theta_{3t'_2t'_3}$. Further suppose that the j_{z-1}-th and j_z-th columns of $\tilde{A}_0^{(1,3)}$ correspond to the effects $\theta_{2t_2t_3}$ and $\theta_{3t'_2t'_3}$, respectively. Of course, the i-th column of $\tilde{A}_0^{(1,3)}$ corresponds to the i-th effect in $\boldsymbol{\theta}_{2(1\,2 \cdots m-3)}^2$ for each $i = 1, 2, \ldots, m-3$. Then the $z \times z$ submatrix $F_{1\,2 \cdots m-3\,j_{z-1}j_z}^2$ of $\tilde{A}_0^{(1,3)}$, defined in Lemma 11.6, can be explicitly written in the form

$$F_{1\,2 \cdots m-3\,j_{z-1}j_z}^2 = \left[\begin{array}{cc|cc} 1 & 1 \cdots 1 & 1 & 0 \\ 1 & 1 \cdots 1 & 0 & 1 \\ \hline \multicolumn{2}{c|}{I_{m-3}} & \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{array}\right], \tag{11.6}$$

where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are $(m-3) \times 1$ (0, 1) vectors with weight 2. In this case, it is easy to verify that the matrix of (11.6) is nonsingular. Next consider the case $4 \leqq t'_1 < t'_2 < t'_3 \leqq m$. Then the submatrix composed of the last two columns of (11.6) is exchanged for

$$\left[\begin{array}{cc|c} 0 & 0 & \boldsymbol{a}'_2 \\ 1 & 0 & \boldsymbol{a}'_1 \end{array}\right]',$$

where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are vectors with weight 2 and 3, respectively. Similarly it can be easily shown that the new matrix is also nonsingular. This completes the proof, because of Theorem 11.7.

12. Existence of a 2^m-BFF design of resolution IV with the minimum number of assemblies

It has been shown in Webb [39] and Margolin [17] that the minimum number of assemblies must be $2m$ for a general 2^m-FF design of resolution IV. On the other hand, from Theorem 9.5 the corresponding number for S_2 type 2^m-BFF designs must be $\nu_2^{\frac{1}{2}}=2m+1$. This difference follows from the fact that the general mean θ_ϕ itself is estimable in S_2 type 2^m-BFF designs. In this section we shall show that a 2^m-BFF design of resolution IV with $N=2m$ assemblies can be obtained from a B-array of strength 4 and m constraints, that is, there exists a 2^m-BFF design of resolution IV with the minimum number of assemblies. First consider an S-array T with parameters $(m;\ \lambda_0=0,\ \lambda_2=1,\ 0,\dots,\ 0,\ \lambda_{m-1}=1,\ \lambda_m=0)$, which is equivalent to a B-array of strength 4, size $N=2m$, m constraints and index set $\{\mu_0=(m-4),\ \mu_1=1,\ \mu_2=0,\ \mu_3=1,\ \mu_4=(m-4)\}$. Then the matrices K_0 and K_1 given in Example 4.1, (i) reduce to the following

$$(12.1)\qquad K_0=\begin{bmatrix} 2m & 0 & 2(m-4)\binom{m}{2}^{1/2} \\ 0 & 2(m-2)^2 & 0 \\ 2(m-4)\binom{m}{2}^{1/2} & 0 & (m-1)(m-4)^2 \end{bmatrix},$$

$$K_1=\begin{bmatrix} 8 & 0 \\ 0 & (m-2) \end{bmatrix}.$$

These matrices are clearly of rank$(K_0)=2$ and $\det(K_1)\neq 0$. Let $\boldsymbol{o}_p$ be the $p\times 1$ vector whose elements are all 0, i.e., $\boldsymbol{o}_p=O_{p\times 1}$. Let C_0 be a $\nu_2\times\nu_2$ $\left(\nu_2=1+m+\binom{m}{2}\right)$ matrix such that

$$C_0=\left[\begin{array}{c:c:c} 0 & \boldsymbol{o}'_m & \boldsymbol{o}'_{\binom{m}{2}} \\ \hdashline \boldsymbol{o}_m & I_m & O_{m\times\binom{m}{2}} \\ \hdashline A_0^{(2,0)\#} & O_{\binom{m}{2}\times m} & h_1A_0^{(2,2)\#}+h_2A_1^{(2,2)\#} \end{array}\right],$$

where $h_1=(m-4)(m-1)^{1/2}/(2m)^{1/2}$ and h_2 is any real number. Then we shall prove

LEMMA 12.1. *For the B-array T mentioned above and its information matrix M_T, there exists a $v_2 \times v_2$ matrix X_0 such that $X_0 M_T = C_0$.*

PROOF. The matrix C_0 is also expressed as

$$C_0 = (D_0^{(1,1)\#} + D_0^{(2,0)\#} + h_1 D_0^{(2,2)\#}) + (D_1^{(1,1)\#} + h_2 D_1^{(2,2)\#}).$$

From Theorem 3.3, the matrix C_0 belongs to the 3 sets TMDPB association algebra $\mathfrak{A}$. Therefore it follows from (3.17) that the irreducible representations of C_0 with respect to ideals $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are given as follows:

$$\mathfrak{A}_0 : C_0 \longrightarrow \Gamma_0^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & h_1 \end{bmatrix},$$

$$\mathfrak{A}_1 : C_0 \longrightarrow \Gamma_0^1 = \begin{bmatrix} 1 & 0 \\ 0 & h_2 \end{bmatrix}.$$

From (12.1), it is easily shown that there exist 3×3 and 2×2 matrices X_0^0 and X_0^1, respectively, such that $X_0^0 K_0 = \Gamma_0^0$ and $X_0^1 K_1 = \Gamma_0^1$. Let X_0 be a matrix such that $X_0 \in \mathfrak{A}$ and the irreducible representations of X_0 are X_0^0 and X_0^1. Then it is easy to check that $X_0 M_T = C_0$ holds.

THEOREM 12.2. *The B-array T of Lemma 12.1 is a 2^m-BFF design of resolution IV in which a parametric function of $\boldsymbol{\theta}' = (\theta_\phi; \{\theta_i\}; \{\theta_{ij}\})$,*

$$\boldsymbol{\zeta}_0 = C_0 \boldsymbol{\theta} = (0, \boldsymbol{\theta}_0, \theta_\phi A_0^{(0,2)\#} + \boldsymbol{\theta}'_2\{h_1 A_0^{(2,2)\#} + h_2 A_1^{(2,2)\#}\})' \tag{12.2}$$

is estimable, where $\boldsymbol{\theta}'_0 = (\{\theta_i\})$ and $\boldsymbol{\theta}'_2 = (\{\theta_{ij}\})$. Its BLUE is given by

$$\hat{\boldsymbol{\zeta}}_0 = X_0 E'_T \boldsymbol{y}_T, \tag{12.3}$$

where X_0 is given in Lemma 12.1.

PROOF. From (1.9) and Lemma 12.1, it follows that

$$\mathrm{Exp}[\hat{\boldsymbol{\zeta}}_0] = X_0 E'_T \mathrm{Exp}[\boldsymbol{y}_T] = X_0 M_T = C_0 \boldsymbol{\theta} = \boldsymbol{\zeta}_0.$$

Hence $\boldsymbol{\zeta}_0$ is an estimable function of $\boldsymbol{\theta}$. On the other hand, it follows from Gauss-Markov Theorem that the BLUE $\hat{\boldsymbol{\zeta}}_0$ of $\boldsymbol{\zeta}_0$ is uniquely given by $\hat{\boldsymbol{\zeta}}_0 = C_0 \boldsymbol{\theta}^*$ where $\boldsymbol{\theta}^*$ is a solution of the normal equations (1.11). Thus we have $\hat{\boldsymbol{\zeta}}_0 = X_0 M_T \boldsymbol{\theta}^* = X_0 E'_T \boldsymbol{y}_T$. Clearly we also have $\mathrm{Var}[\hat{\boldsymbol{\zeta}}_0] = X_0 M_T X'_0 \sigma^2 = C_0 X'_0 \sigma^2$. Since $C_0 X'_0 \in \mathfrak{A}$, it is found that $\mathrm{Var}[\hat{\boldsymbol{\zeta}}_0]$ is invariant under any permutation of m factors. This completes the proof.

COROLLARY 12.3. *For the design T of Theorem 12.2,*

$$(12.4)\quad \hat{\boldsymbol{\theta}}_0 = \left[\boldsymbol{o}_m : \left\{\left(\frac{x_{11}}{m}+\frac{m-1}{8m}\right)A_0^{(1,1)}+\frac{1}{m}\left(x_{11}-\frac{1}{8}\right)A_1^{(1,1)}\right\} : O_{m\times\binom{m}{2}}\right]E_T'\boldsymbol{y}_T,$$

$$(12.5)\quad \begin{aligned} \mathrm{Var}\,[\hat{\theta}_i] &= \left(\frac{x_{11}}{m}+\frac{m-1}{8m}\right)\sigma^2, \\ \mathrm{Cov}\,[\hat{\theta}_i, \hat{\theta}_j] &= \frac{1}{m}\left(x_{11}-\frac{1}{8}\right)\sigma^2, \end{aligned}$$

$$(12.6)\quad \mathrm{Cov}\,[\hat{\theta}_i, \hat{\zeta}_{0k}] = 0,$$

where $x_{11}=1/2(m-2)^2$ *and* $\hat{\zeta}_{0k}$ *is the BLUE of k-th element of the vector* $\{\theta_\phi A_0^{(2,0)\#} + (h_1 A_0^{(2,2)\#}+h_2 A_1^{(2,2)\#})\boldsymbol{\theta}_2\}$.

PROOF. Let x_{ij} $(i, j=0, 1, 2)$ be (i, j) elements of X_0^0. From Lemma 12.1, we have $x_{00}=x_{01}=x_{10}=x_{02}=x_{12}=0$ and $x_{11}=1/2(m-2)^2$. Furthermore $X_0^1 = \mathrm{diag}\,[1/8, h_2/8(m-2)]$. Therefore the $m\times v_2$ submatrix of X_0 whose rows correspond to the block of main effects θ_i is given by

$$\left[\boldsymbol{o}_m : x_{11}A_0^{(1,1)\#}+\frac{1}{8}A_1^{(1,1)\#} : O_{m\times\binom{m}{2}}\right].$$

From (3.7), (12.2) and (12.3), we thus have (12.4). Since $\mathrm{Var}\,[\hat{\boldsymbol{\zeta}}_0]=C_0 X_0'\sigma^2 \in \mathfrak{A}$, we have the irreducible representations of $C_0 X_0'$, i.e.,

$$\mathfrak{A}_0 : C_0X_0' \longrightarrow \mathrm{diag}\,[0,\ x_{11},\ 1/2m],$$

$$\mathfrak{A}_1 : C_0X_0' \longrightarrow \mathrm{diag}\,[1,\ h_2/(m-2)]/8.$$

Hence

$$\mathrm{Var}\,[\hat{\boldsymbol{\zeta}}_0] = \left[x_{11}D_0^{(1,1)\#}+\frac{1}{8}D_1^{(1,1)\#}+\frac{1}{2m}D_0^{(2,2)\#}+\frac{h_2}{m-2}D_1^{(2,2)\#}\right]\sigma^2$$

and, particularly, from the definition of $D_\beta^{(u,v)\#}$

$$\mathrm{Var}\,[\hat{\boldsymbol{\theta}}_0] = \left((x_{11}A_0^{(1,1)\#}+\frac{1}{8}A_1^{(1,1)\#}\right)\sigma^2.$$

This shows that (12.5) and (12.6) hold.

Next, as in Section 11, we shall investigate the alias structure of θ_ϕ and θ_{ij}. Unlike an S_2 type 2^m-BFF design, note that, in general, θ_ϕ itself is not estimable in the design T.

THEOREM 12.4. *Suppose* $m>4$. *For the design T of Theorem 12.2,*

$$d\theta_\phi \boldsymbol{j}_m + A_0^{(1,2)}\boldsymbol{\theta}_2$$

is estimable, where $d=(m-4)/2$.

PROOF. From (3.11) and (3.12), the estimability of $\boldsymbol{\zeta}_0$ in (12.2) implies that $\theta_\phi/m^{1/2}\cdot\boldsymbol{j}_m+(h_1c_0^{(1,2)}A_0^{(1,1)\#}A_0^{(1,2)}+h_2c_1^{(1,2)}A_1^{(1,1)\#}A_0^{(1,2)})\boldsymbol{\theta}_2$ is estimable. Now recall that $h_1\neq 0$ and h_2 is any real number. By letting $h_2=h_1c_0^{(1,2)}/c_1^{(1,2)}$, from (3.11) it can be easily shown that $d\theta_\phi\boldsymbol{j}_m+(A_0^{(1,1)\#}+A_1^{(1,1)\#})A_0^{(1,2)}\boldsymbol{\theta}_2=d\theta_\phi\boldsymbol{j}_m+A_0^{(1,2)}\boldsymbol{\theta}_2$ is estimable.

COROLLARY 12.5. *In the design T of Theorem* 12.2 ($m>4$), *the general mean and* $(m-1)$ *two-factor interactions involving the special factor are estimable under the assumption that the remaining two-factor interactions are negligible.*

PROOF. Without loss of generality, we can assume that the special factor is the first factor. From the assumption and (11.3), therefore, $d\theta_\phi\boldsymbol{j}_m+A^{(1,2)}\boldsymbol{\theta}_2$ can be written as

$$\left[\begin{array}{c|c} 1 & 1\ 1\cdots 1 \\ \hline 1 & \\ 1 & I_{m-1} \\ \vdots & \\ 1 & \end{array}\right]\begin{bmatrix} d\theta_\phi \\ \theta_{12} \\ \theta_{13} \\ \vdots \\ \theta_{1m}\end{bmatrix}.$$

This shows that θ_ϕ, θ_{12}, $\theta_{13},\dots,\theta_{1m}$ are estimable.

Finally consider the case where $m=4$. Then we establish the following

THEOREM 12.6. *Let T be a B-array of strength* 4, $m=4$ *constraints and index set* $\{0, 1, 0, 1, 0\}$. *Then in this design T the general mean* θ_ϕ *and the differences* $(\theta_{ij}-\theta_{pq})$ *are estimable, where* $\{i, j\}\cap\{p, q\}=\phi$ *and* $\{i, j\}\cup\{p, q\}=\{1, 2, 3, 4\}$.

PROOF. In this case, $\boldsymbol{\theta}=(\theta_{12}, \theta_{13}, \theta_{14}, \theta_{23}, \theta_{34})$, $h_1=0$ and $A_1^{(2,2)\#}=(A_0^{(2,2)}-A_2^{(2,2)})/2$. Also recall that h_2 is any real number. Therefore by considering $h_2=0$, it follows from Theorem 12.2 that the general mean θ_ϕ itself is estimable. On the other hand, when $h_2=2$, from (3.2) it can be shown that $\theta_\phi+(\theta_{ij}-\theta_{pq})$ is estimable. This completes the proof, because of the estimability of θ_ϕ.

As an easy corollary to Theorem 12.6, we have

COROLLARY 12.7. *Consider the B-array T of Theorem* 12.6. *Then in this design T the* $(m-1)$ *two-factor interactions involving the special factor themselves are estimable under the assumption that the remaining two-factor interactions are negligible.*

13. Various types of 2^m-BFF designs of resolution $2l$ and their optimality

It has been observed in Section 9 that B-arrays satisfying Condition (9.1) yield 2^m-BFF designs of resolution $2l$. By further investigations of the properties of matrices K_β in (4.3), other types of 2^m-BFF designs of resolution $2l$ can be similarly obtained from B-arrays of strength $2l$.

Let $K_\beta^{(0)}$ be the $(l-\beta)\times(l-\beta)$ matrices obtained from K_β by cutting the last row and column. Consider the following condition: For r integers β_i with $0\leqq\beta_1<\beta_2<\cdots<\beta_r\leqq l$,

$$\kappa_{\beta_i}^{l-\beta_i,l-\beta_i}=0,$$

$$\det(K_{\beta_i}^{(0)})\neq 0 \qquad (\beta_i\leqq l-1), \tag{13.1}$$

$$\det(K_\alpha)\neq 0 \quad \text{for all} \quad \alpha \text{ with } \alpha\neq\beta_i \quad \text{and} \quad 0\leqq\alpha\leqq l.$$

Note that this condition is equivalent to Condition (9.1) when $r=1$ and $\beta_1=l$.

EXAMPLE 13.1. Let us consider an S-array with parameters $(m=8;\ \lambda_0=1, \lambda_1=1, \lambda_2=0, \lambda_3=0, \lambda_4=1, \lambda_5=0, \lambda_6=0, \lambda_7=1, \lambda_8=0)$. It is equivalent to a B-array of strength 6 ($l=3$), size $N=87$, $m=8$ constraints and index set $\{\mu_0=2, \mu_1=1, \mu_2=1, \mu_3=2, \mu_4=1, \mu_5=1, \mu_6=2\}$. From Example 4.1, (ii), it is easily checked that this array satisfies $\kappa_2^{1,1}=0$, $\det(K_2^{(0)})\neq 0$ and $\det(K_\beta)\neq 0$ ($\beta=0, 1, 3$). Here $r=1$ and $\beta_1=2$.

Using an argument similar to Section 12, we shall show that B-arrays of strength $2l$, m constraints and index set $\{\mu_0, \mu_1,\ldots, \mu_{2l}\}$ satisfying Condition (13.1) yield 2^m-BFF designs of resolution $2l$.

LEMMA 13.1. *The condition* $\kappa_\beta^{l-\beta,l-\beta}=0$ *implies* $\kappa_\beta^{j,l-\beta}=\kappa_\beta^{l-\beta,j}=0$ *for all* $j=0, 1,\ldots, l-\beta-1$.

PROOF. From Theorem 5.5, the matrix K_β is positive semidefinite. Hence it is easy to verify that $\kappa_\beta^{l-\beta,l-\beta}=0$ implies $\kappa_\beta^{j,l-\beta}=\kappa_\beta^{l-\beta,j}=0$ for $j=0, 1,\ldots, l-\beta-1$.

Let

$$C=\operatorname{diag}\,[I_{v_{l-1}},\ \sum_{\alpha=0}^{l} h_\alpha A_\alpha^{(l,l)\#}],$$

where h_α are real numbers such that $h_\beta=0$ for $\beta=\beta_i$ ($i=1, 2,\ldots, r$). Then we shall prove

LEMMA 13.2. *For a B-array T satisfying Condition* (13.1), *there exists a*

$v_l \times v_l$ matrix X such that $XM_T = C$ holds.

PROOF. From (3.15), the $v_l \times v_l$ matrix C is also expressed as

$$C = \sum_{u=0}^{l-1} \sum_{\alpha=0}^{u} D_\alpha^{(u,u)\#} + \sum_{\alpha=0}^{l} h_\alpha D_\alpha^{(l,l)\#}$$
$$= \sum_{\alpha=0}^{l-1} \{ \sum_{u=0}^{l-\beta-1} D_\alpha^{(u+\alpha,u+\alpha)\#} + h_\alpha D_\alpha^{(l,l)\#} \} + h_l D_l^{(l,l)\#}.$$

This implies $C \in \mathfrak{A}$. Thus it follows from (3.17) that

$$\mathfrak{A}_\alpha: C \longrightarrow \Gamma_\alpha = \begin{cases} \operatorname{diag}[I_{l-\alpha}, h_\alpha] & \text{for } \alpha = 0, 1, \dots, l-1, \\ h_l & \text{for } \alpha = l. \end{cases}$$

Let

$$X_\alpha = \begin{cases} \Gamma_\alpha \operatorname{diag}[K_\alpha^{(0)-1}, 0] & \text{for } \alpha = \beta_1, \beta_2, \dots, \beta_r, \\ \Gamma_\alpha K_\alpha^{-1} & \text{otherwise,} \end{cases}$$

and let

$$X = \sum_{\alpha=0}^{l} \sum_{i=0}^{l-\alpha} \sum_{j=0}^{l-\alpha} \chi_\alpha^{i,j} D_\beta^{(\beta+i,\beta+j)}, \tag{13.2}$$

where $\chi_\alpha^{i,j}$ are (i, j) elements of X_α. Since $XM_T \in \mathfrak{A}$ and from Lemma 13.1

$$\mathfrak{A}_\alpha: XM_T \longrightarrow X_\alpha K_\alpha = \Gamma_\alpha \quad \text{for } \alpha = 0, 1, \dots, l,$$

it is easy to see that $XM_T = C$ holds.

THEOREM 13.3. *Let T be the B-array of Lemma 13.2. Then a parametric function,*

$$\boldsymbol{\Psi} = C\boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{\theta}_1 \\ (\sum_{\alpha=0}^{l} h_\alpha A_\alpha^{(l,l)\#})\boldsymbol{\theta}_2 \end{bmatrix} \tag{13.3}$$

is an estimable function of $\boldsymbol{\theta}$. The BLUE $\hat{\boldsymbol{\Psi}}$ of $\boldsymbol{\Psi}$ is given by

$$\hat{\boldsymbol{\Psi}} = X E_T' \boldsymbol{y}_T, \tag{13.4}$$

where X is given in (13.2).

PROOF. From (1.9), Lemma 13.2 and Gauss-Markov Theorem, it is easy to verify that $\boldsymbol{\Psi}$ is an estimable function of $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\Psi}}$ of (13.4) is the BLUE of $\boldsymbol{\Psi}$.

THEOREM 13.4. *For the B-array of Lemma* 13.2, *the covariance matrix*

$\mathrm{Var}[\hat{\boldsymbol{\Psi}}]$ *is given as follows*:

(13.5) $$\mathrm{Var}[\hat{\boldsymbol{\Psi}}] = XC\sigma^2$$

$$= [\sum_{\alpha=0}^{l-1} \sum_{i=0}^{l-\alpha-1} \sum_{j=0}^{l-\alpha-1} \kappa_{i,j}^{\alpha} D_{\alpha}^{(\alpha+i,\alpha+j)\#} + \sum_{\alpha=0}^{l-1} \{ \sum_{i=0}^{l-\alpha-1} h_\alpha \kappa_{i,l-\alpha}^{\alpha}$$

$$\cdot (D_{\alpha}^{(\alpha+i,l)\#} + D_{\alpha}^{(l,\alpha+i)\#}) + h_\alpha^2 \kappa_{l-\alpha,l-\alpha}^{\alpha} D_{\alpha}^{(l,l)\#}\} + h_l^2 \kappa_{0,0}^{l} D_l^{(l,l)\#}]\sigma^2,$$

where $\kappa_{i,j}^{\alpha}$ *are the* (i, j) *elements of* $K_\alpha^{(0)-1}$ *or* K_α^{-1} *according as* $\alpha=\beta_k$ $(k=1, 2,\ldots, r)$ *or not.*

PROOF. From (13.4) and Lemma 13.2, we have

$$\mathrm{Var}[\hat{\boldsymbol{\Psi}}] = XE_T' \mathrm{Var}[\boldsymbol{y}_T]E_T X' = XM_T X'\sigma^2 = XC\sigma^2.$$

Since $XC \in \mathfrak{A}$, it is clear that the irreducible representations of XC with respect to ideals $\mathfrak{A}_\alpha$ $(\alpha=0, 1,\ldots, l)$ are given by

$$X_\alpha \Gamma_\alpha = \begin{bmatrix} \kappa_{0,0}^{\alpha} & \cdots & \kappa_{0,l-\alpha-1}^{\alpha} & h_\alpha \kappa_{0,l-\alpha}^{\alpha} \\ & & \vdots & \vdots \\ & & \kappa_{l-\alpha-1,l-\alpha-1}^{\alpha} & h_\alpha \kappa_{l-\alpha-1,l-\alpha}^{\alpha} \\ (\mathrm{Sym.}) & & & h_\alpha^2 \kappa_{l-\alpha,l-\alpha}^{\alpha} \end{bmatrix}.$$

This leads to (13.5).

Let X_{11} be the $v_{l-1} \times v_{l-1}$ submatrix whose rows and columns are composed of the first v_{l-1} those of X. From (13.2) we have

(13.6) $$\mathrm{diag}[X_{11}, O_{\binom{m}{l}\times\binom{m}{l}}] = \sum_{\alpha=0}^{l-1} \sum_{i=0}^{l-\alpha-1} \sum_{j=0}^{l-\alpha-1} \kappa_{i,j}^{\alpha} D^{(\alpha+i,\alpha+j)\#} = X^{(0)}, \text{ say.}$$

From (13.5), therefore, we have

(3.7) $$\mathrm{Var}[\hat{\boldsymbol{\theta}}_1] = X_{11}\sigma^2,$$

where $\hat{\boldsymbol{\theta}}_1$ is the BLUE of $\boldsymbol{\theta}_1$ given in (13.4). Since $X^{(0)} \in \mathfrak{A}$, it is clear that $\mathrm{Var}[\hat{\boldsymbol{\theta}}_1]$ is invariant under any permutation of m factors. Thus we establish.

THEOREM 13.5. *B-arrays satisfying Condition* (13.1) *are* 2^m*-BFF designs of resolution* $2l$ *such that the vectors* $A_\alpha^{(l,l)\#}\boldsymbol{\theta}_2$ $(\alpha \neq \beta_1, \beta_2,\ldots, \beta_r; 0 \leqq \alpha \leqq l)$ *are estimable.*

DEFINITION 13.1. *A B-array* T *of strength* $2l$, m *constraints and index set* $\{\mu_0, \mu_1,\ldots, \mu_{2l}\}$ *is called an* $S_l(\beta_1, \beta_2,\ldots, \beta_r)$ *type* 2^m*-BFF design if* T *satisfies Condition* (13.1).

Of course, we may say that an $S_l(\beta_1,\dots,\beta_r)$ type 2^m-BFF design is identical with an S_l type 2^m-BFF design if $r=1$ and $\beta_1=l$.

THEOREM 13.6. *For an $S_l(\beta_1,\beta_2,\dots,\beta_r)$ type 2^m-BFF design T, the number of distinct assemblies in T must be at least* $\nu_l^1(\beta_1,\beta_2,\dots,\beta_r)=\nu_l-\sum_{i=0}^{r}\phi_{\beta_i}$.

PROOF. This follows from the fact that from (3.17) and Condition (13.1), $\text{rank}(M_T)=\nu_l^1(\beta_1,\dots,\beta_r)$ holds.

As in Theorems 5.4 and 9.2, from (13.7) we can obtain the following

THEOREM 13.7. *For an $S_l(\beta_1,\beta_2,\dots,\beta_r)$ type 2^m-BFF design T, the $\binom{l+2}{3}$ distinct elements $V_\alpha^{(u,v)}$ of the covariance matrix* $\text{Var}[\hat{\boldsymbol{\theta}}_1]$ *are explicitly given by*

$$V_\alpha^{(u,v)} = \sum_{\beta=0}^{u}\kappa_{u-\beta,v-\beta}^{\beta}z_{(u,v)}^{\beta\alpha} \quad \text{for} \quad 0\leqq\alpha\leqq u\leqq v\leqq l-1. \tag{13.8}$$

In general, for given $N\geqq\nu_l^1(\beta_1,\dots,\beta_r)$, there are more than one distinct $S_l(\beta_1,\dots,\beta_r)$ type 2^m-BFF designs. Note that these designs can estimate a common parameter vector $\boldsymbol{\theta}_1$. As a measure for comparing these designs, the amount of $\text{tr}(\text{Var}[\hat{\boldsymbol{\theta}}_1])$ will be used. Let

$$S_T^{(0)} = \text{tr}(\text{Var}[\hat{\boldsymbol{\theta}}_1])/\sigma^2. \tag{13.9}$$

Then we can establish the following theorem:

THEOREM 13.8. *For an $S_l(\beta_1,\beta_2,\dots,\beta_r)$ type 2^m-BFF design T, $S_T^{(0)}$ in (13.9) can be expressed as*

$$S_T^{(0)} = \sum_{\beta=0}^{l-1}\phi_\beta\,\text{tr}(K_\beta^{(0)-1}). \tag{13.10}$$

PROOF. From (13.7) and (13.8),

$$\text{tr}(\text{Var}[\hat{\boldsymbol{\theta}}_1])/\sigma^2 = \text{tr}(X_{11}) = \sum_{u=0}^{l-1}\binom{m}{u}V_0^{(u,u)}$$

$$= \sum_{\beta=0}^{l-1}\phi_\beta(\kappa_{0,0}^{\beta}+\kappa_{1,1}^{\beta}+\cdots+\kappa_{l-\beta-1,l-\beta-1}^{\beta}).$$

This completes the proof.

In view of Definition 9.2, we make

DEFINITION 13.2. *For given N assemblies, an $S_l(\beta_1,\beta_2,\dots,\beta_r)$ type 2^m-BFF design T is said to be optimal with respect to the partial generalized trace (PGT) criterion if $S_T^{(0)}$ is a minimum.*

EXAMPLE 13.2. Consider $m=8$, $l=3$ and $N=87$. Let T_1 be a B-array of strength 6, 8 constraints and index set {8, 4, 1, 0, 1, 3, 7}. Then it is easy to check that T_1 is an $S_3(\beta_1,\dots,\beta_r)$ type 2^8-BFF design with $r=1$ and $\beta_1=3$, i.e., T_1 is of S_3 type. By using the PGT criterion, now let us compare this design T_1 and the design T of Example 13.1. From (13.10), we have $S_T^{(0)}=0.52654$ and $S_{T_1}^{(0)}=1.18125$. Thus the design T is better than T_1 with respect to the PGT criterion. In fact, as will be seen from the next section, T is an optimal $S_3(\beta_1,\dots,\beta_r)$ type 2^8-BFF design with respect to the PGT criterion. However, T_1 is an optimal S_3 type 2^8-BFF design with respect to the GT criterion.

14. Optimal $S_3(\beta_1, \beta_2,\dots, \beta_r)$ type 2^m-BFF designs with $m=6, 7, 8$

In this section, optimal $S_3(\beta_1,\dots,\beta_r)$ type 2^m-BFF designs with respect to the PGT criterion will be presented for $6\leqq m\leqq 8$ and for every number of N with $v_3^1(\beta_1,\dots,\beta_r)\leqq N<v_3$. For 2^m-BFF designs of resolution VI, as pointed out in Section 10, we are usually interested in ones for which the number of assemblies is less than v_3. First we shall begin by investigating combinatorial properties of $S_3(\beta_1,\dots,\beta_r)$ type 2^m-BFF designs which are not of S_3 type (i.e., $r=1$; $\beta_1\neq 3$). For those of S_3 type 2^m-BFF designs, see Section 9. From (2.1), (2.2), (3.9), (4.2), (5.7) and (5.8), we have

(14.1)
$$\kappa_2^{0,1}=2^4(m-4)^{1/2}(\mu_4-\mu_2),$$
$$\kappa_2^{1,1}=2^4\{(m-4)(\mu_4+\mu_2)-2(m-6)\mu_3\};$$

(14.2)
$$\kappa_1^{0,2}=2^2\binom{m-2}{2}^{1/2}(\mu_5+\mu_1-2\mu_3),$$
$$\kappa_1^{1,2}=2^2\left(\frac{m-3}{2}\right)^{1/2}\{(m-2)(\mu_5-\mu_1)-2(m-6)(\mu_4-\mu_2)\},$$
$$\kappa_1^{2,2}=2^2\left\{\binom{m-2}{2}(\mu_5+\mu_1)-2(m-3)(m-6)(\mu_4+\mu_2)+(3m^2-31m+82)\mu_3\right\};$$

(14.3)
$$\kappa_0^{1,3}=\left\{\binom{m-1}{2}/3\right\}^{1/2}\{m(\mu_0+\mu_6)-2(m-6)(\mu_1+\mu_5)-m(\mu_2+\mu_4)+4(m-6)\mu_3\},$$
$$\kappa_0^{3,3}=\binom{m}{3}(\mu_0+\mu_6)-(m-1)(m-2)(m-6)(\mu_1+\mu_5)+\tfrac{1}{2}(m-2)(5m^2-53m+144)(\mu_2+\mu_4)-\tfrac{2}{3}(m-6)(5m^2-39m+82)\mu_3.$$

From (14.1)–(14.3) and Lemma 13.1, we have

LEMMA 14.1. *For a B-array of strength* 6, *m constraints and index set* $\{\mu_0, \mu_1, \dots, \mu_6\}$,

(i) *if* $\kappa_2^{1,1}=0$, *then*

$$\mu_2 = \mu_4, \quad (m-4)\mu_2 = (m-6)\mu_3, \tag{14.4}$$

(ii) *if* $\kappa_1^{2,2}=0$, *then*

$$2\mu_3 = \mu_1+\mu_5, \quad 2(m^2-9m+22)\mu_3 = (m-3)(m-6)(\mu_2+\mu_4), \tag{14.5}$$

(iii) *if* $\kappa_0^{3,3}=0$, *then*

$$\frac{2(m-1)(m-2)(m-6)}{3}(\mu_1+\mu_5) = \frac{8(m-2)(m^2-10m+27)}{3}(\mu_2+\mu_4)$$

$$-4(m-6)(m^2-7m+14)\mu_3, \tag{14.6}$$

$$2\binom{m}{3}(\mu_0+\mu_6) = 3(m-2)(m^2-9m+24)(\mu_2+\mu_4)$$

$$-\frac{16(m-6)(m^2-6m+11)}{3}\mu_3.$$

THEOREM 14.2. *Let* $m=6$ *and consider an* $S_3(\beta_1,\dots,\beta_r)$ *type* 2^6*-BFF design* T *with* $v_3^1(\beta_1,\dots,\beta_r)\leqq N<v_3\ (=42)$ *which is not of* S_3 *type. Then, apart from an interchange of* 0 *and* 1, T *exists only when it is one of B-arrays of strength* 6 *with index set* $\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$ *such that*

(i) $N = 32$, $\{0, 1, 0, 1, 0, 1, 0\}$, $(r=2;\ \beta_1=0,\ \beta_2=2)$,

(ii) $N = 32+\omega_{01}+\omega_{11}$, $\{1+\omega_{01}, 0, 1, 0, 1, 0, 1+\omega_{11}\}$, $(r=2;\ \beta_1=1,$ $\beta_2=3)$,

where ω_{01} *and* ω_{11} *are nonnegative integers with* $\omega_{01}+\omega_{11}\leqq 9$,

(iii) $N = 33+\omega_{02}+\omega_{12}$, $\{1+\omega_{02}, 1, 0, 1, 0, 1, \omega_{12}\}$, $(r=1;\ \beta_1=2)$,

where ω_{02} *and* ω_{12} *are nonnegative integers with* $\omega_{02}+\omega_{12}\leqq 8$,

(iv) $N = 38$, $\{0, 2, 0, 1, 0, 1, 0\}$, $(r=2;\ \beta_1=0,\ \beta_2=2)$,

(v) $N = 38+\omega_{03}+\omega_{13}$, $\{\omega_{03}, 2, 0, 1, 0, 1, \omega_{13}\}$, $(r=1;\ \beta_1=2)$,

where ω_{03} *and* ω_{13} *are nonnegative integers with* $1\leqq\omega_{03}+\omega_{13}\leqq 3$.

PROOF. From Lemma 14.1, $\kappa_2^{1,1}=0$, $\kappa_1^{2,2}=0$ and $\kappa_0^{3,3}=0$ imply $\mu_2=\mu_4=0$, $\mu_1=\mu_3=\mu_5=0$ and $\mu_0=\mu_2=\mu_4=\mu_6=0$, respectively. In Section 9, recall that $\kappa_3^{0,0}=0$ implies $\mu_3=0$. From the definition of a B-array, it follows that for any given index set, there exists always a B-array of strength 6 and 6 constraints. Therefore we can easily construct B-arrays with $v_3^1(\beta_1,\dots,\beta_r)\leqq N<42$ which satisfy Condition (13.1). This completes the proof.

THEOREM 14.3. *There does not exist any $S_3(\beta_1,\dots,\beta_r)$ type 2^7-BFF design with $v_3^1(\beta_1,\dots,\beta_r)\leqq N<v_3$ $(=64)$ which is not of S_3 type.*

PROOF. First consider $\kappa_2^{1,1}=0$. From (14.4), $3\mu_2=\mu_3$ and $\mu_2=\mu_4$ hold. From the nonsingularity of $K_2^{(0)}$, it follows that μ_2, μ_3 and μ_4 must be positive integers. Thus μ_3 is a multiple of 3. This implies $N>64$, a contradiction. For the case $\kappa_1^{2,2}=0$, from (14.5) we have $2\mu_3=\mu_1+\mu_3$ and $4\mu_3=\mu_2+\mu_4$. Since $K_1^{(0)}$ is nonsingular, it is clear that $\mu_1+\mu_3$ and $\mu_2+\mu_4$ must be multiples of 2 and 4, respectively. This implies $N>64$, a contradiction. Finally consider $\kappa_0^{3,3}=0$. Then, from (14.6) we have $5\{15(\mu_2+\mu_4)-7(\mu_0+\mu_6)\}=48\mu_3$. Similarly it can be shown that this contradicts $N>64$ or $\det(K_0^{(0)})\neq 0$. This completes the proof.

THEOREM 14.4. *Let $m=8$ and consider an $S_3(\beta_1,\dots,\beta_r)$ type 2^8-BFF design T with $v_3^1(\beta_1,\dots,\beta_r)\leqq N<v_3$ $(=93)$ which is not of S_3 type. Then, apart from an interchange of 0 and 1, T exists only when it is one of B-arrays of strength 6 with index set $\{\mu_0,\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6\}$ such that*

$$N=86+\omega_0+\omega_1,\quad \{3+\omega_0,1,1,2,1,1,2+\omega_1\},\quad (r=1;\beta_1=2),$$

where ω_0 and ω_1 are nonnegative integers with $\omega_0+\omega_1\leqq 6$. Furthermore T is equivalent to an S-array with parameters ($m=8$; $\lambda_0=1+\omega_0$, $\lambda_1=1$, $\lambda_2=0$, $\lambda_3=0$, $\lambda_4=1$, $\lambda_5=0$, $\lambda_6=0$, $\lambda_7=1$, $\lambda_8=\omega_1$).

PROOF. We shall use the same methods as in Theorems 14.2 and 14.3. For $\kappa_1^{2,2}=0$, (14.5) reduces to $2\mu_3=\mu_1+\mu_5$ and $14\mu_3=5(\mu_2+\mu_4)$. Since $\mu_3=0$ implies $\det(K_1^{(0)})=0$, it is clear that μ_3 and $(\mu_2+\mu_4)$ must be multiples of 5 and 14, respectively. This gives $N>93$, a contradiction. For $\kappa_0^{3,3}=0$, (14.6) reduces to $7(\mu_0+\mu_6)=18(\mu_2+\mu_4-\mu_3)$ and $7(\mu_1+\mu_5)=22(\mu_2+\mu_4-\mu_3)$. If $\mu_3\neq\mu_2+\mu_4$, then $(\mu_0+\mu_6)$ and $(\mu_1+\mu_5)$ must be multiples of 18 and 22, respectively, which leads to the contradiction $N\geqq 93$. Thus the case $\mu_3=\mu_2+\mu_4$, $\mu_0=\mu_1=\mu_5=\mu_6$ is considered. This implies $\kappa_0^{1,1}=\gamma_0+(m-1)\gamma_1=0$ (see Example 4.1, (ii)), so that $\det(K_0^{(0)})=0$. This gives a contradiction. Finally consider $\kappa_2^{1,1}=0$. From (14.4), then $\mu_2=\mu_4$ and $\mu_3=2\mu_2$ hold. Since $\mu_2=0$ or $\mu_2\geqq 2$ implies $\det(K_0^{(0)})=0$ or $N>93$, respectively, the case $\mu_3=2$, $\mu_2=\mu_4=1$ is considered. Therefore we suppose a B-array T with index set $\{\mu_0,\mu_1,\mu_2=1,$

$\mu_3=2$, $\mu_4=1$, μ_5, $\mu_6\}$, where $0<\mu_1+\mu_5\leqq 2$. The inequality $\mu_1+\mu_5\leqq 2$ is due to Shirakura [23], Theorem 4.1. Also $\mu_1+\mu_5=0$ implies that the distinct number of assemblies in T is less than $\nu_3^1(\beta_1,\ldots,\beta_r)=73$ if $r=1$ and $\beta_1=2$. From Corollary 6.5, it is easily shown that apart from an interchange of 0 and 1, the possible index set of T is one of $\{2+\omega'_{01}, 1, 1, 2, 1, 1, 2+\omega'_{11}\}$ and $\{2+\omega'_{02}, 1, 1, 2, 1, 0, \omega'_{12}\}$, where ω'_{01}, ω'_{11}, ω'_{02} and ω'_{12} are nonnegative integers satisfying $\omega'_{01}+\omega'_{11}\leqq 7$ and $\omega'_{02}+\omega'_{12}\leqq 14$. Simultaneously it is found that all B-arrays of strength 6 with these index sets are identical with S-arrays. Among the B-arrays obtained above, particularly, $\{2, 1, 1, 2, 1, 1, 2\}$ and $\{2+\omega'_{02}, 1, 1, 2, 1, 0, \omega'_{12}\}$

TABLE 14.1 Optimal S_3 $(\beta_1, \beta_2,\ldots, \beta_r)$ type 2^m-BFF designs

$m=6$	N	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$S_T^{(0)}$	types
	*32	0	1	0	1	0	1	0	0.68750	$r=2$; $\beta_1=0$, $\beta_2=2$
	*32	1	0	1	0	1	0	1	0.68750	$r=2$; $\beta_1=1$, $\beta_2=3$
	*33	2	0	1	0	1	0	1	0.67676	$r=2$; $\beta_1=1$, $\beta_2=3$
	*34	2	0	1	0	1	0	2	0.66602	$r=2$; $\beta_1=1$, $\beta_2=3$
	*35	3	0	1	0	1	0	2	0.66243	$r=2$; $\beta_1=1$, $\beta_2=3$
	*36	3	0	1	0	1	0	3	0.65885	$r=2$; $\beta_1=1$, $\beta_2=3$
	*37	4	0	1	0	1	0	3	0.65706	$r=2$; $\beta_1=1$, $\beta_2=3$
	*38	0	2	0	1	0	1	0	0.62305	$r=2$; $\beta_1=0$, $\beta_2=2$
	*39	1	2	0	1	0	1	0	0.62305	$r=1$; $\beta_1=2$
	*39	0	2	0	1	0	1	1	0.62305	$r=1$; $\beta_1=2$
	*40	1	2	0	1	0	1	1	0.61111	$r=1$; $\beta_1=2$
	*41	2	2	0	1	0	1	1	0.60917	$r=1$; $\beta_1=2$
	*41	1	2	0	1	0	1	2	0.60917	$r=1$; $\beta_1=2$

$m=7$	N	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$S_T^{(0)}$	types
	50	1	2	1	0	1	1	1	0.94936	$r=1$; $\beta_1=3$
	51	2	2	1	0	1	1	1	0.93579	$r=1$; $\beta_1=3$
	52	2	2	1	0	1	1	2	0.92836	$r=1$; $\beta_1=3$
	53	3	2	1	0	1	1	2	0.92469	$r=1$; $\beta_1=3$
	54	3	2	1	0	1	1	3	0.92209	$r=1$; $\beta_1=3$
	55	4	2	1	0	1	1	3	0.92037	$r=1$; $\beta_1=3$
	56	1	2	1	0	1	2	1	0.84524	$r=1$; $\beta_1=3$
	57	2	2	1	0	1	2	1	0.83698	$r=1$; $\beta_1=3$
	58	2	2	1	0	1	2	2	0.82444	$r=1$; $\beta_1=3$
	59	3	2	1	0	1	2	2	0.82131	$r=1$; $\beta_1=3$
	60	3	2	1	0	1	2	3	0.81780	$r=1$; $\beta_1=3$
	61	4	2	1	0	1	2	3	0.81619	$r=1$; $\beta_1=3$
	62	4	2	1	0	1	2	4	0.81450	$r=1$; $\beta_1=3$
	*63	5	2	1	0	1	2	4	0.81353	$r=1$; $\beta_1=3$

TABLE 14.1 (continued)

$m=8$	N	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$S_T^{(0)}$	types
	65	3	3	1	0	1	2	2	1.60112	$r=1;\ \beta_1=3$
	66	4	3	1	0	1	2	2	1.58894	$r=1;\ \beta_1=3$
	67	4	3	1	0	1	2	3	1.58023	$r=1;\ \beta_1=3$
	68	5	3	1	0	1	2	3	1.57599	$r=1;\ \beta_1=3$
	69	5	3	1	0	1	2	4	1.57291	$r=1;\ \beta_1=3$
	70	6	3	1	0	1	2	4	1.57076	$r=1;\ \beta_1=3$
	71	6	3	1	0	1	2	5	1.56918	$r=1;\ \beta_1=3$
	72	3	3	1	0	1	3	3	1.28477	$r=1;\ \beta_1=3$
	73	4	3	1	0	1	3	3	1.27668	$r=1;\ \beta_1=3$
	74	4	3	1	0	1	3	4	1.26510	$r=1;\ \beta_1=3$
	75	5	3	1	0	1	3	4	1.26150	$r=1;\ \beta_1=3$
	76	5	3	1	0	1	3	5	1.25747	$r=1;\ \beta_1=3$
	77	6	3	1	0	1	3	5	1.25550	$r=1;\ \beta_1=3$
	78	6	3	1	0	1	3	6	1.25342	$r=1;\ \beta_1=3$
	79	7	3	1	0	1	3	6	1.25219	$r=1;\ \beta_1=3$
	80	5	4	1	0	1	3	3	1.20743	$r=1;\ \beta_1=3$
	81	5	4	1	0	1	3	4	1.19826	$r=1;\ \beta_1=3$
	82	6	4	1	0	1	3	4	1.19257	$r=1;\ \beta_1=3$
	83	6	4	1	0	1	3	5	1.18902	$r=1;\ \beta_1=3$
	84	7	4	1	0	1	3	5	1.18614	$r=1;\ \beta_1=3$
	85	7	4	1	0	1	3	6	1.18421	$r=1;\ \beta_1=3$
	86	8	4	1	0	1	3	6	1.18246	$r=1;\ \beta_1=3$
	*87	3	1	1	2	1	1	2	0.52654	$r=1;\ \beta_1=2$
	*88	3	1	1	2	1	1	3	0.50228	$r=1;\ \beta_1=2$
	*89	4	1	1	2	1	1	3	0.49706	$r=1;\ \beta_1=2$
	*90	4	1	1	2	1	1	4	0.49327	$r=1;\ \beta_1=2$
	*91	5	1	1	2	1	1	4	0.49149	$r=1;\ \beta_1=2$
	*92	5	1	1	2	1	1	5	0.48991	$r=1;\ \beta_1=2$

imply $\det(K_0)=0$ and $\det(K_1)=0$, respectively. This completes the proof.

In Table 14.1, optimal $S_3(\beta_1,\ldots,\beta_r)$ type 2^m-BFF designs with respect to the PGT criterion are presented for any given N assemblies, which satisfy (i) $m=6$, $32\leqq N\leqq 41$, (ii) $m=7$, $50\leqq N\leqq 63$ and (iii) $m=8$, $65\leqq N\leqq 92$. As in Tables 8.1 and 10.1, note that for the designs of Table 14.1, their complementary designs are also optimal. From Section 9 and Theorems 14.2–14.4, it is found that for any N with ($m=7$, $42\leqq N\leqq 63$) and ($m=8$, $65\leqq N\leqq 86$), the optimal designs can be chosen in the class of S_3 type 2^m-BFF designs. Furthermore, as seen from Table 10.1 and Shirakura [24], it is interesting that many of the optimal designs are also optimal with respect to the GT criterion. In Table 14.1, the designs

TABLE 14.2 Covariance matrices for optimal S_3 ($\beta_1, \beta_2, \ldots, \beta_r$) type 2^m-BFF designs

	N	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$V_0^{(0,0)}$	$V_0^{(0,1)}$	$V_0^{(0,2)}$	$V_0^{(1,1)}$
$m=6$	32	0	1	0	1	0	1	0	0.03125	0.00000	0.00000	0.03125
	32	1	0	1	0	1	0	1				
	33	2	0	1	0	1	0	1	0.03076	0.00049	−0.00049	0.03076
	34	2	0	1	0	1	0	2	0.03027	0.00000	−0.00098	0.03027
	35	3	0	1	0	1	0	2	0.03011	0.00016	−0.00114	0.03011
	36	3	0	1	0	1	0	3	0.02995	0.00000	−0.00130	0.02995
	37	4	0	1	0	1	0	3	0.02987	0.00008	−0.00138	0.02987
	38	0	2	0	1	0	1	0	0.02832	0.00195	−0.00098	0.02832
	39	1	2	0	1	0	1	0	0.02832	0.00195	−0.00098	0.02832
	39	0	2	0	1	0	1	1				
	40	1	2	0	1	0	1	1	0.02811	0.00177	−0.00136	0.02817
	41	2	2	0	1	0	1	1	0.02808	0.00174	−0.00143	0.02814
	41	1	2	0	1	0	1	2				

	N	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$V_0^{(0,0)}$	$V_0^{(0,1)}$	$V_0^{(0,2)}$	$V_0^{(1,1)}$
$m=7$	63	5	2	1	0	1	2	4	0.02520	−0.00003	−0.00187	0.04134

	N	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$V_0^{(0,0)}$	$V_0^{(0,1)}$	$V_0^{(0,2)}$	$V_0^{(1,1)}$
$m=8$	87	3	1	1	2	1	1	2	0.01173	−0.00100	0.00015	0.01925
	88	3	1	1	2	1	1	3	0.01130	0.00000	0.00000	0.01649
	89	4	1	1	2	1	1	3	0.01129	−0.00022	−0.00003	0.01590
	90	4	1	1	2	1	1	4	0.01116	0.00000	−0.00008	0.01552
	91	5	1	1	2	1	1	4	0.01111	−0.00010	−0.00011	0.01534
	92	5	1	1	2	1	1	5	0.01104	0.00000	−0.00014	0.01519

which are not optimal with respect to the GT criterion will be indicated by the asteric *. In Table 14.2, the distinct elements $V_\alpha^{(u,v)}$ in (13.8) are also given for these designs. For constructions of the designs, see Theorems 14.2 and 14.4.

Finally it may be remarked that for the case $m=6$, the number of assemblies $N=32$ obtained in Theorem 14.2 is the minimum number for designs of resolution VI. Indeed it has been stated in Webb [39] that the minimum number must be $N=m^2-m+2$ in a general 2^m-FF design of resolution VI. For $m=7$ and 8, we have $N=44$ and 58, respectively. However it is unknown whether there exists a 2^m-BFF design ($m \neq 6$) of resolution VI with the minimum number.

$V_1^{(1,1)}$	$V_0^{(1,2)}$	$V_1^{(1,2)}$	$V_0^{(2,2)}$	$V_1^{(2,2)}$	$V_2^{(2,2)}$
0.00000	0.00000	0.00000	0.03125	0.00000	0.00000
−0.00049	0.00049	0.00049	0.03076	−0.00049	−0.00049
−0.00098	0.00000	0.00000	0.03027	−0.00098	−0.00098
−0.00114	0.00016	0.00016	0.03011	−0.00114	−0.00114
−0.00130	0.00000	0.00000	0.02995	−0.00130	−0.00130
−0.00138	0.00008	0.00008	0.02987	−0.00138	−0.00138
−0.00098	0.00195	0.00000	0.02832	−0.00098	0.00098
−0.00098	0.00195	0.00000	0.02832	−0.00098	0.00098
−0.00113	0.00162	−0.00033	0.02760	−0.00170	0.00026
−0.00116	0.00157	−0.00039	0.02748	−0.00181	0.00014

$V_1^{(1,1)}$	$V_0^{(1,2)}$	$V_1^{(1,2)}$	$V_0^{(2,2)}$	$V_1^{(2,2)}$	$V_2^{(2,2)}$
−0.00554	0.00001	0.00001	0.02376	−0.00302	0.00144

$V_1^{(1,1)}$	$V_0^{(1,2)}$	$V_1^{(1,2)}$	$V_0^{(2,2)}$	$V_1^{(2,2)}$	$V_2^{(2,2)}$
0.00710	−0.00042	−0.00042	0.01289	0.00073	−0.00100
0.00434	0.00000	0.00000	0.01282	0.00067	−0.00107
0.00375	−0.00009	−0.00009	0.01281	0.00065	−0.00108
0.00336	0.00000	0.00000	0.01278	0.00063	−0.00110
0.00318	−0.00004	−0.00004	0.01277	0.00062	−0.00111
0.00304	0.00000	0.00000	0.01276	0.00061	−0.00113

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