The Annals of Mathematical Statistics
1968, Vol. 39, No. 4, 1158–1175

CONTRIBUTIONS TO CENTRAL LIMIT THEORY FOR DEPENDENT VARIABLES[1]

BY R. J. SERFLING[2]

Research Triangle Institute

1. Introduction and summary. Considerations on stochastic models frequently involve sums of dependent random variables (rv's). In many such cases, it is worthwhile to know if asymptotic normality holds. If so, inference might be put on a nonparametric basis, or the asymptotic properties of a test might become more easily evaluated for certain alternatives.

Of particular interest, for example, is the question of when a *weakly stationary* sequence of rv's possesses the central limit property, by which is meant that the sum $\sum_1^n X_i$, suitably normed, is asymptotically normal in distribution. The feeling of many experimenters that the normal approximation is valid in situations "where a stationary process has been observed during a time interval long compared to time lags for which correlation is appreciable" has been discussed by Grenander and Rosenblatt ([10]; 181). (See Section 5 for definitions of stationarity.)

The general class of sequences $\{X_i\}_{-\infty}^{\infty}$ considered in this paper is that whose members satisfy the variance condition

$$\text{(1.1)} \quad \text{Var}\,(\textstyle\sum_{a+1}^{a+n} X_i) \sim nA^2 \quad \text{uniformly in } a \quad (n \to \infty) \quad (A^2 > 0).$$

Included in this class are the weakly stationary sequences for which the covariances r_j have convergent sum $\sum_1^\infty r_j$. A familiar example is a sequence of mutually orthogonal rv's having common mean and common variance.

As a mathematical convenience, it shall be assumed (without loss of generality) that the sequences $\{X_i\}$ under consideration satisfy $E(X_i) \equiv 0$, for the sequences $\{X_i\}$ and $\{X_i - E(X_i)\}$ are interchangeable as far as concerns the question of asymptotic normality under the assumption (1.1). As a practical convenience, it shall be assumed for each sequence $\{X_i\}$ that the absolute central moments $E|X_i - E(X_i)|^\nu$ are bounded uniformly in i for some $\nu > 2$ (ν may depend upon the sequence). When (1.1) holds, this is a mild additional restriction and a typical criterion for verifying a Lindeberg restriction ([15]; 295).

We shall therefore confine attention to sequences $\{X_i\}$ which satisfy the following *basic assumptions* (A):

$$\text{(A1)} \qquad E(X_i) \equiv 0,$$

$$\text{(A2)} \qquad E(T_a^2) \sim A^2 \quad \text{uniformly in } a \quad (n \to \infty) \quad (A^2 > 0),$$

Received 25 May 1967; revised 10 January 1968.

[1] This research was supported in part by the United States Air Force Office of Scientific Research under Contract No. AF 49(638)-1544 and in part by the United States Army Research Office-Durham under Contract No. DA-31-124-ARO-D-323.

[2] Now at Florida State University.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.* www.jstor.org

(A3) $\qquad E|X_i|^{2+\delta} \leqq M \quad (\text{for some } \delta > 0 \text{ and } M < \infty),$

where T_a denotes the normed sum $n^{-\frac{1}{2}} \sum_{a+1}^{a+n} X_i$. Note that the formulations of (A2) and (A3) presuppose (A1).

We shall say, under assumptions (A), that a sequence $\{X_i\}$ *has the central limit property* (clp), or that T_1 is *asymptotically normal* (with mean zero and variance A^2), if

(1.2) $$P\{(nA^2)^{-\frac{1}{2}} \sum_1^n X_i \leqq z\} \to (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2}\, dt \qquad (n \to \infty).$$

The assumptions (A) do not in general suffice for (1.2) to hold. (The reader is referred to Grenander and Rosenblatt ([10]; 180) for examples in which (1.2) does not hold under assumptions (A), one case being a certain strictly stationary sequence of uncorrelated rv's, another case being a certain bounded sequence of uncorrelated rv's.) It is well known, however, that in the case of independent X_i's the assumptions (A) suffice for (1.2) to hold. It is desirable to know in what ways the assumption of independence may be relaxed, retaining assumptions (A), without sacrificing (1.2). Investigators have weakened considerably the moment requirements (A2) and (A3) while retaining strong restrictions on the dependence. However, in many situations of practical interest, assumptions (A) hold but neither strong dependence restrictions nor strong stationarity restrictions seem to apply. Thus it is important to have theorems which take advantage of assumptions (A) when they hold, in order to utilize conclusion (1.2) without recourse to severe additional assumptions. A basic theorem in this regard is offered in Section 4. It is unfortunate that the additional assumptions required, while relatively mild, are not particularly amenable to verification, with present theory. This difficulty is alleviated somewhat by the strong intuitive appeal of the conditions. The variety of ways in which the assumption of independence may be relaxed in itself poses a problem. It is difficult to compare the results of sundry investigations in central limit theory because of the *ad hoc* nature of the suppositions made in each instance. In Section 2 we explore the relationships among certain alternative dependence restrictions, some introduced in the present paper and some already in the literature. Conditions involving the moments of sums $\sum_{a+1}^{a+n} X_i$ are treated in detail in Section 3.

The central limit theorems available for sums of dependent rv's embrace diverse areas of application. The results of Bernstein [2] and Loève [14], [15] have limited applicability within the class of sequences satisfying assumptions (A). A result that is apropos is one of Hoeffding and Robbins [11] for m-dependent sequences (defined in Section 2). In addition to assumptions (A1) and (A3), their theorem requires that, defining $A_a^2 = E(X_{a+m}^2) + 2\sum_1^m E(X_{a+m-j}X_{a+m})$,

(H) $\quad \lim_{n\to\infty} n^{-1} \sum_{i=1}^n A_{a+i}^2 \doteq A^2$ exists uniformly in a $(n \to \infty)$.

Now it can be shown easily that conditions (A2) and (H) are equivalent in the case of an m-dependent sequence satisfying (A1) and (A3). Therefore, a formulation relevant to assumptions (A) is

THEOREM 1.1 (Hoeffding-Robbins). *If $\{X_i\}$ is an m-dependent sequence satisfying assumptions* (A), *then it has the central limit property.*

In the case of a weakly stationary (with mean zero, say) m-dependent sequence, the assumptions of the theorem are satisfied except for (A3), which then is a mild additional restriction. For applications in which the existence of moments is not presupposed, e.g., strictly stationary sequences, Theorem 1.1 has been extended by Diananda [6], [7], [8] and Orey [16] in a series of results reducing the moment requirements while retaining the assumption of m-dependence. In the present paper the interest is in extensions relaxing the m-dependence assumption. A result of Ibragimov [12] in this regard implies

THEOREM 1.2 (Ibragimov). *If* $\{X_i\}$ *is a strictly stationary sequence satisfying assumptions* (A) *and regularity condition* (I), *then it has the central limit property.*

(Condition (I) is defined in Section 2.) Other extensions under condition (I) but not involving stationarity assumptions are Corollary 4.1.3 and Theorem 7.2 below. See also Rosenblatt [17]. Other extensions for strictly stationary sequences, further reducing the dependence restrictions, appear in [12] and [13] and Sections 5 and 6 below.

Section 2 is devoted to dependence restrictions. The restrictions (2.1), (2.2) and (2.3), later utilized in Theorem 4.1, are introduced and shown to be closely related to assumptions (A). Although conditional expectations are involved in (2.2) and (2.3), the restrictions are easily interpreted. It is found, under assumptions (A), that if (2.3) is sufficiently stringent, then (2.1) holds in a stringent form (Theorem 2.1). A link between regularity assumptions formulated in terms of joint probability distributions and those involving conditional expectations is established by Theorem 2.2 and corollaries. Implications of condition (I) are given in Theorem 2.3.

Section 3 is devoted to the particular dependence restriction (2.1). Theorem 3.1 gives, under assumptions (A), a condition necessary and sufficient for (2.1) to hold in the most stringent form, (3.1).

The remaining sections deal largely with central limit theorems. Section 4 obtains the basic result and its general implications. Sections 5, 6 and 7 exhibit particular results for weakly stationary sequences, sequences of martingale differences and bounded sequences.

NOTATION AND CONVENTIONS. We shall denote by $\{X_i\}_{-\infty}^{\infty}$ a sequence of rv's defined on a probability space. Let $\mathfrak{M}_a^b$ denote the σ-algebra generated by events of the form $\{(X_{i_1}, \cdots, X_{i_k}) \varepsilon E\}$, where $a - 1 < i_1 < \cdots < i_k < b + 1$ and E is a k-dimensional Borel set. We shall denote by $\mathcal{P}_a$ the σ-algebra $\mathfrak{M}_{-\infty}^a$ of "past" events, i.e., generated by the rv's $\{X_a, X_{a-1}, \cdots\}$. Conditional expectation given a subfield $\mathcal{B}$ will be represented by $E(\cdot \mid \mathcal{B})$, which is to be regarded as a function measurable $(\mathcal{B})$. All expectations will be assumed finite whenever expressed.

2. On regularity assumptions. If a sequence $\{X_i\}$ satisfying assumptions (A) is to have the clp, its dependence structuré must satisfy additional regularity restrictions. Various types of restriction come under consideration. One type involves the growth of the moments of the sums $\sum_{a+1}^{a+n} X_i$ as $n \to \infty$. Such conditions, which are of interest for other purposes also, are treated in detail in

Section 3. Other possible restrictions are typically of two kinds, according as conditional expectations or joint probability distributions are involved in their formulations. The object of this section is to explore the *relationships* among the various regularity assumptions. These relationships are useful in the sequel and also may facilitate the comparison of different central limit theorems in the literature.

A condition on the moments of the sums $\sum_{a+1}^{a+n} X_i$ is given by

$$E|T_a|^{2+\epsilon} = O(n^\gamma), \quad \text{uniformly in } a \text{ as} \quad n \to \infty, \tag{2.1}$$

for some $\epsilon \geqq 0$ and $\gamma \geqq 0$. Under assumptions (A), the condition (2.1) holds with $\gamma = 0$ for $\epsilon = 0$ and with $\gamma = 1 + \frac{1}{2}\epsilon$ for $0 < \epsilon \leqq \delta$ (δ as given in assumption (A3)). Thus (2.1) constitutes a further restriction beyond assumptions (A) only in the case that $\epsilon > 0$ and $\gamma < 1 + \frac{1}{2}\epsilon$, with the most stringent case $\gamma = 0$ being of special interest (see Section 3).

Also closely related to assumptions (A) are the requirements that the mean deviations of the first and second conditional moments of T_a, given the "past" $\mathcal{P}_a$, converge to zero uniformly in a as $n \to \infty$. That is,

$$E|E(T_a \mid \mathcal{P}_a)| \leqq B_1(n) \tag{2.2}$$

and

$$E|E(T_a^2 \mid \mathcal{P}_a) - E(T_a^2)| \leqq B_2(n), \tag{2.3}$$

where $B_1(n)$ and $B_2(n)$ each $\to 0$ as $n \to \infty$. Under assumptions (A), these mean deviations are bounded uniformly in a and thus (2.2) and (2.3) are mild additional restrictions. Nevertheless, if a sequence satisfies assumptions (A) and conditions (2.2) and (2.3) with $B_1(n)$ and $B_2(n)$ each equal to $O(n^{-\theta})$ for some $\theta > 0$, it has the clp (see Section 4). Underlying this result is the fact that, under the assumptions (A), condition (2.3) with $B_2(n) = O(n^{-\theta})$ for some $\theta > 0$ implies a moment restriction of form (2.1) with $\gamma = 0$ for some $\epsilon > 0$.

The relationship just mentioned is a consequence of Theorem 3.1 and the following result.

LEMMA 2.1. *Let* $\{X_i\}$ *satisfy assumption* (A3) *and condition* (2.3) *with* $B_2(n) = O(n^{-\theta})$ *for some* $\theta > 0$. *Then, for some* $\beta > 0$ *and* $C < \infty$,

$$E|E(T_a^2 \mid \mathcal{P}_a) - E(T_a^2)|^{1+\beta} \leqq C. \tag{2.4}$$

PROOF. Let $Y_a = E(T_a^2 \mid \mathcal{P}_a) - E(T_a^2)$. By Loève's c_r-inequalities [15] and a Hölder inequality, we have $E|Y_a|^{1+\frac{1}{2}\delta} \leqq 2^{1+\frac{1}{2}\delta}E|T_a|^{2+\delta}$. By assumption (A3) and Minkowski's inequality, we have $E|T_a|^{2+\delta} \leqq Mn^{1+\frac{1}{2}\delta}$. Hence, uniformly in a,

$$E|Y_a|^{1+\frac{1}{2}\delta} = O(n^{1+\frac{1}{2}\delta}). \tag{2.5}$$

Choose β such that $0 < \beta < \theta\delta/(2 + 2\theta + \delta)$. Let A_a denote the event $\{|Y_a|^\beta \leqq [B_2(n)]^{-1}\}$ and A_a' its complement. Taking the expectation of $|Y_a|^{1+\beta}$ over A_a and A_a' separately, it follows that

$$E|Y_a|^{1+\beta} \leqq [B_2(n)]^{-1}E|Y_a| + [B_2(n)]^{(\frac{1}{2}\delta-\beta)/\beta}E|Y_a^1|^{+\frac{1}{2}\delta}. \tag{2.6}$$

Now $[B_2(n)]^{-1}E|Y_a| \leqq 1$ by (2.3). The second term on the right of (2.6) is $O(1)$ because of (2.5) and the choice of β. Hence (2.4) holds.

We therefore have

THEOREM 2.1. *Let* $\{X_i\}$ *satisfy assumptions* (A) *and condition* (2.3) *with* $B_2(n) = O(n^{-\theta})$ *for some* $\theta > 0$. *Then, for some* $\epsilon > 0$, *condition* (2.1) *holds with* $\gamma = 0$.

Let us now consider some regularity restrictions formulated in terms of the joint probability distributions of the X_i's. Three such conditions of particular interest are:

m-dependence: $\{X_b, X_{b+1}, \cdots, X_{b+s}\}$ and $\{X_{a-r}, X_{a-r+1}, \cdots, X_a\}$ are independent sets of variables if $b - a > m$;

Condition (I): For any event $B \in \mathfrak{M}_{a+k}^{\infty}$, with probability 1,

$$|P(B \mid \mathfrak{M}_{-\infty}^{a}) - P(B)| \leqq \phi(k) \downarrow 0 \qquad (k \to \infty); \tag{2.7}$$

Strong mixing: For any events $A \in \mathfrak{M}_{-\infty}^{a}$ and $B \in \mathfrak{M}_{a+k}^{\infty}$,

$$|P(AB) - P(A)P(B)| \leqq \alpha(k) \downarrow 0 \qquad (k \to \infty). \tag{2.8}$$

Clearly m-dependence is a particular case of (I) in which $\phi(k) = 0$ if $k > m$. Ibragimov [12] shows that (I) is equivalent to the condition that for any events $A \in \mathfrak{M}_{-\infty}^{a}$ and $B \in \mathfrak{M}_{a+k}^{\infty}$,

$$|P(AB) - P(A)P(B)| \leqq \phi(k)P(A). \tag{2.9}$$

Thus we have

$$m\text{-dependence} \Rightarrow \text{condition (I)} \Rightarrow \text{strong mixing}. \tag{2.10}$$

The following theorem is a tool for establishing the relationships linking condition (I) with conditions such as (2.1), (2.2), (2.3) and conditions on the covariances of the X_i's.

THEOREM 2.2. *Let* $\{X_i\}$ *satisfy regularity condition* (I). *Let* ξ *be any* rv *measurable* re $\mathfrak{M}_{a+k}^{\infty}$ *such that* $E|\xi|^p < \infty$ *for some* $p > 1$. *If* $1 \leqq \alpha \leqq p$, *then, with* $1/p + 1/q = 1$,

$$E|E(\xi \mid \mathfrak{M}_{-\infty}^{a}) - E(\xi)|^{\alpha} \leqq 2^{\alpha}[\phi(k)]^{\alpha/q}[E|\xi|^p]^{\alpha/p}. \tag{2.11}$$

PROOF. Let $P(\cdot)$ denote the probability measure induced on $\mathfrak{M}_{a+k}^{\infty}$ by the basic probability model and $P(\cdot \mid \mathfrak{M}_{-\infty}^{a})$ denote a regular conditional probability measure on $\mathfrak{M}_{a+k}^{\infty}$, given $\mathfrak{M}_{-\infty}^{a}$. We shall denote by μ the signed measure $P(\cdot \mid \mathfrak{M}_{-\infty}^{a}) - P(\cdot)$. The space Ω corresponding to the random variables $\{X_{a+k}, X_{a+k+1}, \cdots\}$ has a Hahn decomposition $\Omega = \Omega^+ \cup \Omega^-$ with respect to μ, such that for any measurable subset A of Ω, the sets $A \cap \Omega^+$ and $A \cap \Omega^-$ are measurable and $\mu(A \cap \Omega^+) \geqq 0$, $\mu(A \cap \Omega^-) \leqq 0$. In particular, Ω^+ and Ω^- are measurable since Ω is measurable. Hence we may write, since μ and $-\mu$ are measures, respectively, on Ω^+ and Ω^-,

$$\begin{aligned} |E(\xi \mid \mathfrak{M}_{-\infty}^{a}) - E(\xi)| &= |\textstyle\int_\Omega \xi[dP(\omega \mid \mathfrak{M}_{-\infty}^{a}) - dP(\omega)]| \\ &\leqq |\textstyle\int_{\Omega^+} \xi\, d\mu| + |\textstyle\int_{\Omega^-} \xi\, d(-\mu)| \leqq \textstyle\int_{\Omega^+} |\xi|\, d\mu + \int_{\Omega^-} |\xi|\, d(-\mu). \end{aligned}$$

Then, by Loève's c_r-inequalities [15],

$$(2.12)\quad |E(\xi \mid \mathfrak{M}_{-\infty}^{a}) - E(\xi)|^p \leqq 2^{p-1}[\textstyle\int_{\Omega^+} |\xi|\, d\mu]^p + 2^{p-1}[\int_{\Omega^-} |\xi|\, d(-\mu)]^p.$$

Now, by regularity condition (I),

$$\int_{\Omega^+} d\mu = P(\Omega^+ \mid \mathfrak{M}_{-\infty}^{a}) - P(\Omega^+) \leqq \phi(k).$$

Since, by Hölder's inequality, we have

$$\int_{\Omega^+} |\xi|\, d\mu \leqq (\textstyle\int_{\Omega^+} |\xi|^p\, d\mu)^{1/p} (\int_{\Omega^+} d\mu)^{1/q},$$

it follows that

$$(2.13)\quad [\textstyle\int_{\Omega^+} |\xi|\, d\mu]^p \leqq [\phi(k)]^{p/q} \int_{\Omega^+} |\xi|^p\, d\mu.$$

By (2.13) and the analogous result for Ω^- and $-\mu$, (2.12) yields

$$\begin{aligned}|E(\xi \mid \mathfrak{M}_{-\infty}^{a}) - E(\xi)|^p &\leqq 2^{p-1}[\phi(k)]^{p/q}[\textstyle\int_{\Omega^+} |\xi|^p\, d\mu + \int_{\Omega^-} |\xi|^p\, d(-\mu)] \\ &\leqq 2^{p-1}[\phi(k)]^{p/q}[E(|\xi|^p \mid \mathfrak{M}_{-\infty}^{a}) + E|\xi|^p],\end{aligned}$$

whence

$$(2.14)\quad E|E(\xi \mid \mathfrak{M}_{-\infty}^{a}) - E(\xi)|^p \leqq 2^p[\phi(k)]^{p/q} E|\xi|^p,$$

from which (2.11) follows, completing the proof.

An easy consequence of the theorem is that the correlations of a sequence $\{X_i\}$ satisfy, under condition (I),

$$(2.15)\quad |\mathrm{corr}\,[X_a, X_{a+k}]| \leqq 2[\phi(k)]^{\frac{1}{2}} \qquad (k \geqq 0).$$

This is obtained by putting $\xi = X_{a+k} - E(X_{a+k})$ and $\alpha = p = 2$ in relation (2.11).

In order to utilize Theorem 2.2 to relate condition (I) to conditions (2.1), (2.2) and (2.3), we prove that (2.2) and (2.3), which involve conditioning with respect to the immediate past, are equivalent, under assumptions (A), to similar restrictions in which the conditioning is with respect to the distant past, namely

$$(2.16)\quad E|E(T_a \mid \mathcal{P}_{a-m})| \leqq B_1(n, m)$$

and

$$(2.17)\quad E|E(T_a^2 \mid \mathcal{P}_{a-m}) - E(T_a^2)| \leqq B_2(n, m),$$

where $B_1(n, m)$ and $B_2(n, m)$ each $\to 0$ as m and n both $\to \infty$. This equivalence also provides a useful simplification in the derivation of a central limit theorem (Section 4). The author is indebted to Professor W. L. Smith for suggesting the possibility and advantage of replacing conditions (2.16) and (2.17) by conditions of type (2.2) and (2.3).

LEMMA 2.2. *Under assumptions* (A), *the conditions* (2.2) *and* (2.3) *are equivalent, respectively, to conditions* (2.16) *and* (2.17). *Furthermore,* $B_i(n)$ *may be chosen to satisfy*

$$(2.18)\quad B_i(n) = B_i(n, m) + O(m^{\frac{1}{2}} n^{-\frac{1}{2}})$$

for each $i = 1, 2$.

PROOF. Trivially (2.2) and (2.3) imply (2.16) and (2.17), respectively. To obtain the converse implication, as well as (2.18), it suffices to show that (2.2) and (2.3) are satisfied for some $B_1(n)$ and $B_2(n)$ which satisfy (2.18). The requirement that $B_1(n)$ and $B_2(n)$ each $\to 0$ as $n \to \infty$ then follows from (2.18) by letting $m \to \infty$ such that $m = O(n)$ as $n \to \infty$. Now, by assumptions (A), we have easily that $E|T_{a+m} - T_a|^2 = O(mn^{-1})$ uniformly in a as m and n both $\to \infty$. It follows that

$$E|T_{a+m} - T_a| = O(m^{\frac{1}{2}}n^{-\frac{1}{2}}) \tag{2.19}$$

and, writing $T_{a+m}^2 - T_a^2 = (T_{a+m} - T_a)(T_{a+m} + T_a)$,

$$E|T_{a+m}^2 - T_a^2| = O(m^{\frac{1}{2}}n^{-\frac{1}{2}}) \tag{2.20}$$

uniformly in a as m and n both $\to \infty$. Applying the elementary relation

$$E|E(Y \mid \mathcal{P}_a) - E(Y)| \leqq E|E(Z \mid \mathcal{P}_a) - E(Z)| + 2E|Y - Z| \tag{2.21}$$

in turn with $Y = T_a$, $Z = T_{a+m}$ and with $Y = T_a^2$, $Z = T_{a+m}^2$, it follows by (2.19) and (2.20) that (2.2) and (2.3) hold with functions $B_1(n)$ and $B_2(n)$ that satisfy (2.18).

THEOREM 2.3. *Let* $\{X_i\}$ *satisfy assumptions* (A) *and regularity condition* (I).

(i) *For any* λ *and* β *such that* $0 < \lambda < 1$ *and* $0 < \beta \leqq \frac{1}{2}\delta$,

$$E|E(T_a \mid \mathcal{P}_a)| \leqq K_1[\phi(n^\lambda)]^{\frac{1}{2}} + K_2 n^{-\frac{1}{2}(1-\lambda)} \tag{2.22}$$

and

$$E|E(T_a^2 \mid \mathcal{P}_a) - E(T_a^2)| \leqq K_3[\phi(n^\lambda)]^{\beta/(1+\beta)}[E|T_a|^{2+2\beta}]^{1/(1+\beta)} + K_4 n^{-\frac{1}{2}(1-\lambda)}, \tag{2.23}$$

where the K_i *are constants not depending upon* λ *or* β.

Suppose, further, that $\phi(n) = O(n^{-\theta})$ *for some* $\theta > 0$.

(ii) *Condition* (2.2) *holds with* $B_1(n) = O(n^{-\frac{1}{2}\theta/(1+\theta)})$.

(iii) *If condition* (2.1) *holds with* $\gamma < \frac{1}{2}\theta\epsilon$ *for some* $\epsilon > 0$, *then condition* (2.3) *holds with* $B_2(n) = O(n^{-(\theta\epsilon - 2\gamma)/(2+\epsilon+2\theta\epsilon)})$.

(iv) *If* $\theta > 1 + 2/\delta$, *then condition* (2.1) *holds with* $\gamma = 0$ *for an* $\epsilon > 0$.

PROOF. (i) follows routinely from Theorem 2.2 with the use of (2.18). (ii) and (iii) follow easily from (i), with $\beta = \frac{1}{2}\epsilon$. Now assume that $\theta > 1 + 2/\delta$. Under assumptions (A), condition (2.1) holds with $\gamma = 1 + \frac{1}{2}\epsilon$ for any $0 < \epsilon < \delta$. In this case $\gamma < \frac{1}{2}\theta\epsilon$, whence, by (ii), condition (2.3) holds with $B_2(n) = O(n^{-\lambda})$ for some $\lambda > 0$. Then Theorem 2.1 asserts that (2.1) must hold with $\gamma = 0$ for some $\epsilon' > 0$, proving (iv).

3. On moments of sums. A condition restricting the growth of the moments of the partial sums $\sum_{a+1}^{a+n} X_i$ (as $n \to \infty$) was given in Section 2 by

$$E|T_a|^{2+\epsilon} = O(n^\gamma), \quad \text{uniformly in } a \text{ as } n \to \infty, \tag{2.1}$$

for some $\epsilon \geqq 0$ and $\gamma \geqq 0$. Without loss of generality, it may be assumed that $\gamma \leqq 1 + \frac{1}{2}\epsilon$, since (2.1) implies that the absolute moments $E|X_i|^{2+\epsilon}$ are uniformly bounded, which in turn implies by Minkowski's inequality that (2.1) holds with

$\gamma = 1 + \frac{1}{2}\epsilon$. Moreover, it is seen easily that for sequences satisfying assumptions (A) we must have $\gamma \geqq 0$ in condition (2.1). For such sequences, as noted in the previous section, (2.1) is not a further restriction if $\epsilon = 0$ or if $\gamma = 1 + \frac{1}{2}\epsilon$. Therefore, given assumptions (A), it is of interest to know when (2.1) holds with $0 \leqq \gamma < 1 + \frac{1}{2}\epsilon$ for an $\epsilon > 0$.

The relationship of (2.1) to other types of regularity assumption was considered in Section 2. In the present section we seek information, under assumptions (A), about the *existence* of (non-trivial) conditions of type (2.1) and about the *stringency* of such conditions. The most stringent possible case, $\gamma = 0$, has received primary attention in the literature, but also the cases in which $0 < \gamma < 1 + \frac{1}{2}\epsilon$ merit investigation (see Section 4).

Conditions of type (2.1) are relevant not only to the central limit question for a sequence $\{X_i\}$ but also to the rates of certain convergences, such as the convergence of the moments of T_a to those of the limiting distribution of T_a, the convergence of the sample mean $\bar{X}_n = n^{-\frac{1}{2}}T_1$ to a value μ, and the convergence to zero of the probability that $|\bar{X}_n - \mu|$ exceeds a given fixed number.

The possibility of condition (2.1) with $\gamma = 0$ for an $\epsilon > 0$, that is, for an $\epsilon > 0$ and a finite M_0,

$$E|T_a|^{2+\epsilon} \leqq M_0 \quad (\text{all } a, \text{ all } n), \tag{3.1}$$

has been studied for various special classes of sequence $\{X_i\}$. The condition (3.1) has been shown by Brillinger [4] to hold for any sequence of independent, identically distributed rv's and by von Bahr [1] to hold for certain sequences of independent, but not identically distributed, rv's. The interest of the present paper includes situations when considerable dependence may exist in the sequence $\{X_i\}$. Doob ([9]; 225) has proved (3.1) for a class of Markov chains satisfying Doeblin's condition. Ibragimov [12] has adapted Doob's argument to obtain (3.1) for any strictly stationary sequence satisfying regularity condition (I), a result including that of Brillinger.

Below we shall show that under the basic assumptions (A), a mild additional dependence restriction is necessary and sufficient for (3.1) to hold. This result (Theorem 3.1) was utilized in the previous section to reach conclusions to the effect that, under assumptions (A), condition (3.1) is implied by either of the regularity assumptions (2.3) or (I), provided that the relevant function $B_2(\cdot)$ or $\phi(\cdot)$ converges to zero sufficiently fast. Thus (3.1) holds for any m-dependent sequence satisfying assumptions (A) and in particular for any sequence of independent rv's having common (zero) mean, common variance and uniformly bounded absolute moments $E|X_i|^{2+\delta}$ for some $\delta > 0$. It further follows from Theorem 3.1 that (3.1) holds for any sequence of martingale differences (see Section 6), under assumptions (A).

We also obtain some results which apply under assumptions (A) without a further dependence restriction, but which yield condition (2.1) in forms less stringent than (3.1), for example with $\gamma = \frac{1}{2}\epsilon$ for any *bounded* sequence.

THEOREM 3.1. *Let* $\{X_i\}$ *satisfy assumptions* (A). *A necessary and sufficient* *condition for* (3.1) *to hold for some* $\epsilon > 0$ *is that*

$$E|E(T_a^2 \mid \mathcal{P}_a) - E(T_a^2)|^{1+\beta} \leqq M_1 \tag{3.2}$$

hold for some $\beta > 0$.

PROOF. The necessity follows easily, by choosing $\beta = \frac{1}{2}\epsilon$ and applying Loève's c_r-inequalities [15] and a Hölder inequality.

For the sufficiency proof, it shall be assumed (without loss of generality) that $\beta \leqq \min [\frac{1}{2}, \frac{1}{2}\delta]$. Then, choosing $\epsilon = 2\beta$, we have

$$0 < \epsilon \leqq \min [1, \delta]. \tag{3.3}$$

Let $m = [\frac{1}{2}n]$, the greatest integer $\leqq \frac{1}{2}n$, and define

$$R_a = \textstyle\sum_{a+1}^{a+m} X_i, \qquad S_a = \sum_{a+m+1}^{a+2m} X_i.$$

Since $\epsilon \leqq 1$, we have

$$\begin{aligned} E|R_a + S_a|^{2+\epsilon} &\leqq E[(R_a + S_a)^2(|R_a|^\epsilon + |S_a|^\epsilon)] \\ &\leqq E|R_a|^{2+\epsilon} + E|S_a|^{2+\epsilon} + 2E|R_a|\,|S_a|^{1+\epsilon} \\ &\quad + 2E|R_a|^{1+\epsilon}|S_a| + E|R_a|^2|S_a|^\epsilon + E|R_a|^\epsilon|S_a|^2. \end{aligned} \tag{3.4}$$

Now, letting $\Delta = E(S_a^2 \mid \mathcal{P}_{a+m}) - E(S_a^2)$, we have, for $0 < s \leqq 2$,

$$E(|S_a|^s \mid \mathcal{P}_{a+m}) \leqq [E(S_a^2)]^{s/2} + |\Delta|^{s/2}. \tag{3.5}$$

Hence, for $r + s = 2 + \epsilon$ and $0 < s \leqq 2$,

$$\begin{aligned} E|R_a|^r|S_a|^s &= E[|R_a|^r E(|S_a|^s \mid \mathcal{P}_{a+m})] \leqq [E|R_a|^r][E(S_a^2)]^{s/2} + E(|R_a|^r|\Delta|^{s/2}) \\ &\leqq [E|R_a|^{2+\epsilon}]^{r/(2+\epsilon)}\{[E(S_a^2)]^{s/2} + [E|\Delta|^{1+\frac{1}{2}\epsilon}]^{s/(2+\epsilon)}\}. \end{aligned} \tag{3.6}$$

By assumptions (A), $E(S_a^2) \leqq C_0 m$ for a finite C_0 not depending on a, and, by (3.2), $E|\Delta|^{1+\frac{1}{2}\epsilon} \leqq M_1 m^{1+\frac{1}{2}\epsilon}$. Therefore, by (3.6), for r and s satisfying $r + s = 2 + \epsilon$ and $0 \leqq s \leqq 2$, we have

$$E|R_a|^r|S_a|^s \leqq Cm^{s/2}[E|R_a|^{2+\epsilon}]^{r/(2+\epsilon)}, \tag{3.7}$$

for a finite constant C not depending upon r and s. (We may take C to be $\max_{0 \leqq t \leqq 2} [C_0^{t/2} + M_1^{t/(2+\epsilon)}]$.)

Define, for positive integers h,

$$A_h = h^{-(1+\frac{1}{2}\epsilon)} \sup_a E|\textstyle\sum_{a+1}^{a+h} X_i|^{2+\epsilon},$$

which is finite by assumption (A3) since $\epsilon \leqq \delta$. Then, by (3.4) and (3.7), we have

$$E|R_a + S_a|^{2+\epsilon} \leqq m^{1+\frac{1}{2}\epsilon} A_m[2 + g(A_m)], \tag{3.8}$$

where $g(z) = C[2z^{-(1+\epsilon)/(2+\epsilon)} + 2z^{-1/(2+\epsilon)} + z^{-\epsilon/(2+\epsilon)} + z^{-2/(2+\epsilon)}]$.

By assumption (A3), $E|X_i|^{2+\epsilon}$ is uniformly bounded, say by $K < \infty$. Since

$g(z) \to 0$ as $z \to \infty$, there exists $z_0 > 1$ such that

$$[2 + g(z)]^{1/(2+\epsilon)} \leqq 2^{\frac{1}{2}} - z_0^{-1} \tag{3.9}$$

for $z \geqq z_0$.

Define $a_h = \max(A_h, z_0, K)$. Since $z[2 + g(z)]$ is a nondecreasing function of z, we have $A_m[2 + g(A_m)] \leqq a_m[2 + g(a_m)]$. Hence by (3.8) and (3.9) it follows that

$$E|R_a + S_a|^{2+\epsilon} \leqq m^{1+\frac{1}{2}\epsilon}a_m(2^{\frac{1}{2}} - z_0^{-1})^{2+\epsilon}. \tag{3.10}$$

Since $n - 1 \leqq 2m \leqq n$, we obtain

$$\begin{aligned}[E|\textstyle\sum_{a+1}^{a+n} X_i|^{2+\epsilon}]^{1/(2+\epsilon)} &\leqq [E|R_a + S_a|^{2+\epsilon}]^{1/(2+\epsilon)} + [E|X_{a+n}|^{2+\epsilon}]^{1/(2+\epsilon)} \\ &\leqq m^{\frac{1}{2}}a_m^{1/(2+\epsilon)}[2^{\frac{1}{2}} - z_0^{-1} + m^{-\frac{1}{2}}],\end{aligned} \tag{3.11}$$

by Minkowski's inequality and (3.10).

Let N_0 be an integer such that $m^{-\frac{1}{2}} < z_0^{-1}$ if $m > N_0$. Then, for $n > N_0$,

$$E|\textstyle\sum_{a+1}^{a+n} X_i|^{2+\epsilon} \leqq (2m)^{1+\frac{1}{2}\epsilon}a_m \leqq n^{1+\frac{1}{2}\epsilon}a_m, \tag{3.12}$$

by (3.11). It follows from (3.12) and the definition of A_n that $A_n \leqq a_m$ whenever $n > N_0$. Hence $a_n \leqq a_m$ for $n > N_0$. It follows easily that, for $n > N_0$, $a_n \leqq M_0 = \max(a_1, \cdots, a_{N_0})$. Therefore, by (3.12),

$$E|\textstyle\sum_{a+1}^{a+n} X_i|^{2+\epsilon} \leqq n^{1+\frac{1}{2}\epsilon}M_0 \tag{3.13}$$

for $n > N_0$. But also (3.13) holds when $n \leqq N_0$, by the definitions of the a_i's and M_0. Hence (3.1) holds. This completes the proof.

It should be noted that under assumptions (A), a condition of form (3.2) with $\beta = 0$ holds automatically, so that (3.2) is a slight strengthening of an implication of these assumptions. Despite this close attachment, (3.2) is unfortunately not so amenable to practical verification as the assumptions (A). The following results are possibly more readily applicable, although the conclusions are not as strong.

LEMMA 3.1. *Let $0 < \alpha < \beta$. If $E|X_i|^\beta < M_0$ (all i) and $E|T_a|^\alpha \leqq M_0 n^\gamma$, where $M_0 < \infty$, then*

$$E|T_a|^\theta \leqq M_0 n^{\frac{1}{2}\theta - (\frac{1}{2}\alpha - \gamma)(\beta - \theta)/(\beta - \alpha)}, \tag{3.14}$$

for $\alpha < \theta < \beta$.

PROOF. Put $p = (\beta - \alpha)/(\beta - \theta)$ and $q = (\beta - \alpha)/(\theta - \alpha)$. Then $1/p + 1/q = 1$ and $\alpha/p + \beta/q = \theta$. Hence, applying Hölder's inequality,

$$E|T_a|^\theta \leqq (E|T_a|^\alpha)^{1/p}(E|T_a|^\beta)^{1/q} \leqq M_0 n^{\gamma/p + \beta/2q},$$

which reduces to (3.14) since $\gamma/p + \beta/2q = \frac{1}{2}\theta + (\gamma - \frac{1}{2}\alpha)/p$.

Putting $\beta = 2 + \delta$, $\alpha = 2$, $\gamma = 0$ and $\theta = 2 + \epsilon$ in the lemma, we obtain

THEOREM 3.2. *Let $\{X_i\}$ satisfy assumptions (A). Then, for $0 \leqq \epsilon \leqq \delta$, condition (2.1) holds with $\gamma = \frac{1}{2}\epsilon + \epsilon/\delta$.*

For $\epsilon < \delta$, the result is sharper than that implied by Minkowski's inequality, i.e., condition (2.1) with $\gamma = 1 + \frac{1}{2}\epsilon$. A still sharper relation holds for a bounded sequence.

THEOREM 3.3. *Let* $\{X_i\}$ *be a bounded sequence satisfying assumptions* (A1) *and* (A2). *Then, for any* $\epsilon > 0$, *condition* (2.1) *holds with* $\gamma = \frac{1}{2}\epsilon$.

PROOF. Let $\{X_i\}$ be bounded by $M_1 < \infty$ and put $Y_i = X_i/M_1$. The sequence $\{Y_i\}$ satisfies assumptions (A1) and (A2) since $\{X_i\}$ does and also we have $E|Y_i|^{2+\delta} \leqq 1$ for every $\delta > 0$. Hence, for every $\delta > 0$, $\{Y_i\}$ satisfies the assumptions of Lemma 3.1 with $\beta = 2 + \delta$, $\alpha = 2$, $\gamma = 0$ and $\theta = 2 + \epsilon$, for a constant M_0, $1 < M_0 < \infty$, which does not depend upon δ. Therefore, for the normed sums $T_a' = n^{-\frac{1}{2}} \sum_{a+1}^{a+n} Y_i$, (3.14) implies

$$E|T_a'|^{2+\epsilon} \leqq M_0 n^{\frac{1}{2}\epsilon + \epsilon/\delta}. \tag{3.15}$$

Since M_0 does not depend upon δ and δ may be taken arbitrarily large, we may let $\delta \to \infty$ in (3.15), yielding condition (2.1) with $\gamma = \frac{1}{2}\epsilon$ for the $\{Y_i\}$ sequence and hence a similar condition for the $\{X_i\}$ sequence, for every $\epsilon > 0$.

4. A basic central limit theorem. Here we shall derive a central limit theorem under the basic assumptions (A) and dependence restrictions of type (2.1), (2.2) and (2.3). These conditions have been introduced and discussed in previous sections. Let us recall the following aspects. The assumptions (A) are commonly satisfied and amenable to verification. Given these assumptions, the conditions (2.1), (2.2) and (2.3) are not severe additional restrictions but are not, with present theory, very amenable to verification, although they have some intuitive appeal. Further, under assumptions (A), if (2.3) is sufficiently stringent, then (2.1) holds automatically in the most stringent form (3.1). Finally, under assumptions (A), the conditions (2.2) and (2.3) are equivalent to seemingly weaker restrictions (2.16) and (2.17), respectively.

To have that the normed sum $S = (nA^2)^{-\frac{1}{2}} \sum_1^n X_i$ is asymptotically standard normal, it is equivalent to have that its characteristic function, $f_n(t) = E(e^{itS})$, satisfies

$$f_n(t) \to e^{-\frac{1}{2}t^2} \quad \text{as} \quad n \to \infty, \tag{4.1}$$

for every t. To obtain (4.1) we shall break the sum S into partial sums, and correspondingly the quantity $f_n(t) - \exp(-\frac{1}{2}t^2)$ into parts, and introduce Taylor expansions for exponential quantities. These preliminary steps will not entail any assumptions about the sequence $\{X_i\}$, except that certain expectations exist. We shall make use of (2.1) to neglect "remainder" terms of the Taylor expansions and shall employ the other assumptions to establish the asymptotic behavior of linear and quadratic terms. By recombination of the parts, (4.1) will follow.

Take $0 < \alpha < 1$ and let $v = [n^{1-\alpha}]$ and $k = [n/v]$, where $[a]$ denotes the greatest integer $\leqq a$. Then $n = kv + r$ $(0 \leqq r < v)$ and $k \sim n^\alpha$ as $n \to \infty$. Defining

$$U_i = (nA^2)^{-\frac{1}{2}}[X_{(i-1)k+1} + \cdots + X_{ik}] \quad (i = 1, \cdots, v),$$

$$Z_m = \sum_{i=1}^{m} U_i \qquad (m = 1, \cdots, v),\ Z_0 = 0,$$

$$\phi_m(t) = e^{-(1-m/v)\frac{1}{2}t^2} \qquad (m = 0, 1, \cdots, v),$$

and

$$\psi_m(t) = \phi_m(t)E(e^{itZ_m}) \qquad (m = 0, 1, \cdots, v),$$

we have $S - Z_v = o_p(1)$ by (A1), (A2) and Chebyshev's inequality, so that

$$(4.2) \qquad f_n(t) - e^{-\frac{1}{2}t^2} \sim \psi_v(t) - \psi_0(t) = \sum_{m=0}^{v-1} [\psi_{m+1}(t) - \psi_m(t)].$$

The representation (4.2) is an adaptation of one used, for example, by Ibragimov [13].

Let us write

$$(4.3) \qquad \begin{aligned} \psi_{m+1}(t) - \psi_m(t) &= \phi_{m+1}(t)E(e^{itZ_{m+1}}) - \phi_m(t)E(e^{itZ_m}) \\ &= \phi_{m+1}(t)E[e^{itZ_m}(e^{itU_{m+1}} - e^{-t^2/2v})]. \end{aligned}$$

For real y, put

$$(4.4) \qquad e^{-y} = 1 - y + Q(y)y^2, \qquad e^{iy} = 1 + iy - \tfrac{1}{2}y^2 + R(y).$$

It is easily seen that $|Q(y)| \leqq \frac{1}{2}$ if $y \geqq 0$, that $|R(y)| \leqq |y|^3$, and that $|R(y)| \leqq y^2$, whence also $|R(y)| \leqq |y|^{2+\epsilon}$ if $0 \leqq \epsilon \leqq 1$. Using the identities (4.4) in (4.3), we obtain

$$\begin{aligned} &\psi_{m+1}(t) - \psi_m(t) \\ &= \phi_{m+1}(t)E\{e^{itZ_m}[itU_{m+1} - \tfrac{1}{2}t^2U_{m+1}^2 + \tfrac{1}{2}t^2v^{-1} + R(tU_{m+1}) - Q(\tfrac{1}{2}t^2v^{-1})\tfrac{1}{4}t^4v^{-2}]\} \\ &= \phi_{m+1}(t)E\{e^{itZ_m}[itE(U_{m+1} \mid \mathcal{P}_{mk}) - \tfrac{1}{2}t^2E(U_{m+1}^2 \mid \mathcal{P}_{mk}) + \tfrac{1}{2}t^2v^{-1} + R(tU_{m+1}) \\ &\quad + Q(\tfrac{1}{2}t^2v^{-1})\tfrac{1}{4}t^4v^{-2}]\} \end{aligned}$$

and thus

$$(4.5) \qquad \begin{aligned} |\psi_{m+1}(t) - \psi_m(t)| \leqq |t|E|E(U_{m+1} \mid \mathcal{P}_{mk})| &+ \tfrac{1}{2}t^2E|E(U_{m+1}^2 \mid \mathcal{P}_{mk}) - v^{-1}| \\ &+ E|R(tU_{m+1})| + \tfrac{1}{8}t^4v^{-2}. \end{aligned}$$

We therefore have

$$(4.6) \qquad |f_n(t) - e^{-\frac{1}{2}t^2}| \leqq H_1 + H_2 + H_3 + H_4 + o(1), \quad \text{as} \quad n \to \infty,$$

where

$$H_1 = |t| \sum_{m=0}^{v-1} E|E(U_{m+1} \mid \mathcal{P}_{mk})|, \quad H_2 = \tfrac{1}{2}t^2 \sum_{m=0}^{v-1} E|E(U_{m+1}^2 \mid \mathcal{P}_{mk}) - E(U_{m+1}^2)|,$$

$$H_3 = \sum_{m=0}^{v-1} E|R(tU_{m+1})|, \quad \text{and} \quad H_4 = \tfrac{1}{2}t^2 \sum_{m=0}^{v-1} |E(U_{m+1}^2) - v^{-1}|.$$

LEMMA 4.1. *Under assumption* (A2), $H_4 \to 0$ *as* $n \to \infty$.

PROOF. By (A2), there is a function $g(h) = o(1)$ $(h \to \infty)$ such that $|k^{-1}nE(U_i^2) - 1| \leqq g(k)$ $(i = 1, 2, \cdots, m)$. Hence, for each i,

$$|E(U_i^2) - v^{-1}| \leqq g(k)kn^{-1} + |kn^{-1} - v^{-1}|$$

and therefore

$$H_4 \leqq \tfrac{1}{2}t^2 g(k) vkn^{-1} + |vkn^{-1} - 1| \leqq \tfrac{1}{2}t^2 g(k) + rn^{-1} = o(1) \quad \text{as} \quad n \to \infty.$$

We shall now let additional assumptions play a role, in order that (4.6) yield (4.1).

LEMMA 4.2. *Let* $\{X_i\}$ *satisfy condition* (2.1) *for an* $\epsilon > 0$. *Then* $H_3 \to 0$ *as* $n \to \infty$ *if the number* γ *in* (2.1) *satisfies*

$$\gamma < (\tfrac{1}{6}\epsilon + \tfrac{1}{3}\beta)(\alpha^{-1} - 1) \tag{4.7}$$

where $\beta = \min(\epsilon, 1)$.

PROOF. By condition (2.1), we have

$$E|U_i|^{2+\epsilon} \leqq M_0 n^{-(1+\frac{1}{2}\epsilon)} k^{1+\frac{1}{2}\epsilon+\gamma}, \tag{4.8}$$

for a finite M_0 not depending upon i, k or n.

Let $\rho = \epsilon - \beta$. Defining

$$l = v^{-1/3-\eta},$$

where $\eta > 0$ may be chosen arbitrarily small, we have $l^3 = o(v^{-1})$ as $n \to \infty$.

Now $|R(y)| \leqq |y|^{2+\beta}$ and $|R(y)| \leqq |y|^3$. Hence, letting B denote the set $\{|U_i| > \rho l\}$, we may write

$$E|R(tU_i)| \leqq \int_B |tU_i|^{2+\beta}\, dP(U_i) + \rho^3 t^3 l^3$$

and obtain easily

$$E|R(tU_i)| \leqq t^{2+\beta}(\rho l)^{-\rho} E|U_i|^{2+\epsilon} + \rho^3 t^3 l^3, \tag{4.9}$$

with the convention that $0^0 = 1$.

Applying (4.8) to (4.9) and noting that $vk \leqq n$, $k \sim n^\alpha$ and $vl^3 = o(1)$, we have $H_3 = O(n^\Delta)$, where

$$\Delta = \alpha\gamma - \tfrac{1}{2}\epsilon(1 - \alpha) + \rho(1 - \alpha)(\tfrac{1}{3} + \eta).$$

It follows that η may be chosen small enough to make $\Delta < 0$ if relation (4.7) holds.

It should be noted that the requirement (4.7) is less stringent when $\epsilon \leqq 1$ than when $\epsilon > 1$. For, if $\epsilon \leqq 1$, (4.7) becomes

$$\gamma < \tfrac{1}{2}\epsilon(\alpha^{-1} - 1), \tag{4.10}$$

whereas if $\epsilon > 1$ the requirement (4.7) is more stringent than (4.10). Possibly a different method of proof would yield a result not involving this seeming inconsistency.

LEMMA 4.3. *If* $\{X_i\}$ *satisfies condition* (2.3), *then* $H_2 \to 0$ *as* $n \to \infty$.

PROOF. By (2.3) we have, for $m = 0, 1, \cdots, v - 1$,

$$A^2 nk^{-1} E|E(U_{m+1}^2 \mid \mathcal{P}_{mk}) - E(U_{m+1}^2)| \leqq B_2(k) \downarrow 0 \qquad (k \to \infty).$$

Hence $H_2 = O(vkn^{-1})B_2(k) = o(1)$ as $n \to \infty$.

A similar argument yields

LEMMA 4.4. *If* $\{X_i\}$ *satisfies condition* (2.2) *with* $B_1(n) = O(n^{-\theta})$ *for some* $\theta > \frac{1}{2}(\alpha^{-1} - 1)$, *then* $H_1 \to 0$ *as* $n \to \infty$.

We may now combine these lemmas to obtain (4.1) via (4.6). Note that Lemmas 4.2 and 4.4 involve α in restrictions which tend to be mutually opposing, in that Lemma 4.2 requires α to be "small enough" while Lemma 4.4 requires it to be "large enough." There exists a choice of α satisfying both restrictions if and only if

$$\gamma < 2\theta[\tfrac{1}{6}\epsilon + \tfrac{1}{3}\min(\epsilon, 1)]. \tag{4.11}$$

We also remark that the four lemmas taken together actually involve all three basic assumptions since (A3) is implied by (2.1) and (A1) is presupposed in the formulations of (A2), (A3), (2.1) and (2.2).

We have

THEOREM 4.1. *Let* $\{X_i\}$ *satisfy assumptions* (A) *and conditions*

$$E|T_a|^{2+\epsilon} = O(n^{\gamma}) \quad \textit{uniformly in } a \quad (n \to \infty), \tag{2.1}$$

$$E|E(T_a \mid \mathcal{P}_a)| \leqq B_1(n) \downarrow 0 \qquad (n \to \infty), \tag{2.2}$$

and

$$E|E(T_a^2 \mid \mathcal{P}_a) - E(T_a^2)| \leqq B_2(n) \downarrow 0 \qquad (n \to \infty). \tag{2.3}$$

If $B_1(n) = O(n^{-\theta})$, $\theta > 0$, *and*

$$\gamma < 2\theta[\tfrac{1}{6}\epsilon + \tfrac{1}{3}\min(\epsilon, 1)], \tag{4.11}$$

then $\{X_i\}$ *has the central limit property.*

In many applications (2.1) holds with $\gamma = 0$, in which case (4.11) is trivially satisfied. Or, applying Theorem 2.1 to eliminate (2.1) and (4.11) as explicit conditions, we have

COROLLARY 4.1.1. *Let* $\{X_i\}$ *satisfy assumptions* (A) *and conditions* (2.2) *and* (2.3) *with* $B_i(n) = O(n^{-\theta})$ *for some* $\theta > 0$. *Then* $\{X_i\}$ *has the central limit property.*

The dependence restrictions (2.2) and (2.3) have a certain intuitive appeal. A restriction analogous to (2.3), but involving the conditional variance of T_a rather than the conditional second moment, may be of interest in some applications. By the conditional variance of T_a is meant

$$\mathrm{Var}\,(T_a \mid \mathcal{P}_a) = E(T_a^2 \mid \mathcal{P}_a) - E^2(T_a \mid \mathcal{P}_a).$$

It is not difficult to obtain from Theorem 4.1 the following result.

COROLLARY 4.1.2. *Let* $\{X_i\}$ *satisfy assumptions* (A) *and suppose that for some function* $g(n) = O(n^{-\theta})$, $\theta > 0$,

$$E[E^2(T_a \mid \mathcal{P}_a)] \leqq g(n) \tag{4.12}$$

and

$$E|\mathrm{Var}\,(T_a \mid \mathcal{P}_a) - \mathrm{Var}\,(T_a)| \leqq g(n). \tag{4.13}$$

Then $\{X_i\}$ has the central limit property.

COROLLARY 4.1.3. *Let $\{X_i\}$ satisfy assumptions* (A) *and regularity condition* (I). *If $\phi(m) = O(m^{-\lambda})$, $\lambda > 1 + 2/\delta$, then $\{X_i\}$ has the central limit property.*

PROOF. We apply Theorem 2.3. Since $\lambda > 1 + 2/\delta$, (2.1) holds with $\gamma = 0$ for an $\epsilon > 0$. Hence (2.2) and (2.3) hold with $B_i(n) = O(n^{-\theta})$, where θ is the smaller of $\frac{1}{2}\lambda/(1 + \lambda)$ and $\lambda\epsilon/(2 + \epsilon + 2\lambda\epsilon)$. Thus the conditions of Theorem 4.1 are satisfied.

5. Weakly stationary sequences. A sequence $\{X_i\}$ is *weakly stationary* if the X_i's have a common mean and a common variance (r_0) and the covariances satisfy Cov $[X_a, X_{a+k}] = r_k$, a function depending only on k ($k = 1, 2, \cdots$). A sequence is *strictly stationary* if, for every choice of integers $s \geqq 1$ and $k_1, \cdots, k_s$, the joint probability distribution of $(X_{a+k_1}, \cdots, X_{a+k_s})$ does not depend upon a. A strictly stationary sequence for which Var $(X_0) < \infty$ is stationary in the weak sense also. In the case of a Gaussian stochastic process, the two concepts of stationarity coincide.

Taking the common mean to be zero, a weakly stationary sequence satisfies assumptions (A) if $\sum_1^\infty r_j$ converges and $E|X_i|^{2+\delta} \leqq M < \infty$ for some $\delta > 0$. (See Lemma 5.1 below.) These conditions are satisfied in typical practical situations. For example, considerations about moving average schemes involve weakly stationary sequences of uncorrelated rv's, in which case $\sum_1^\infty r_j = 0$.

LEMMA 5.1. *If $\{X_i\}$ is weakly stationary (with mean zero) and $\sum_1^\infty r_j$ converges, then the normed sums T_a satisfy assumption* (A2), *with $A^2 = r_0 + 2\sum_1^\infty r_j$.*

PROOF.

$$E(T_a^2) = n^{-1}\sum_{a+1}^{a+n} E(X_i^2) + 2n^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} E(X_{a+i}X_{a+j})$$
$$= r_0 + 2n^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} r_{j-i} = r_0 + 2n^{-1}\sum_{i=1}^{n-1} R_i,$$

where $R_i = \sum_{j=1}^{i} r_j$. Since $\{R_n\}$ converges to $\sum_1^\infty r_j$, so does $n^{-1}\sum_1^n R_i$, whence $E(T_a^2) \to A^2$ uniformly in a, i.e. (A2) holds.

As a result of Lemma 5.1 and Corollary 4.1.1, we have

THEOREM 5.1. *Let $\{X_i\}$ be weakly stationary (with mean zero) with $\sum_1^\infty r_j$ convergent and have uniformly bounded moments $E|X_i|^{2+\delta}$ for some $\delta > 0$. If $\{X_i\}$ satisfies conditions* (2.2) *and* (2.3) *with $B_i(n) = O(n^{-\theta})$ for some $\theta > 0$, it has the central limit property.*

A similar result involving the conditional variance of T_a follows from Corollary 4.1.2 with Lemma 5.1. It is thus clear from an intuitive standpoint that the central limit property holds for wide classes of weakly stationary and strictly stationary sequences. (See later sections for results apropos to bounded sequences and sequences of martingale differences.)

A consequence of relation (2.15) is

LEMMA 5.2. *If $\{X_i\}$ is weakly stationary and satisfies regularity condition* (I) *with $\sum_1^\infty [\phi(j)]^{\frac{1}{2}} < \infty$, then $\sum_1^\infty |r_j| < \infty$.*

This result, with Lemma 5.1 and Corollary 4.1.3, yields

THEOREM 5.2. *Let $\{X_i\}$ be a weakly stationary sequence having uniformly* *bounded moments* $E|X_i|^{2+\delta}$ *for some* $\delta > 0$ *and satisfying regularity condition* (I) *with* $\phi(m) = O(m^{-\theta})$, $\theta > \max[2, 1 + 2/\delta]$. *Then* $\{X_i\}$ *has the central limit property.*

For a *strictly* stationary sequence, Theorem 5.2 is contained in results of Ibragimov [12], but Theorem 5.1 augments previous results.

6. Sequences of martingale differences. We call $\{X_i\}$ a sequence of *martingale differences* (md) if $E|X_i| < \infty$ and $E(X_i \mid \mathcal{P}_{i-1}) = 0$ (each i). (The sums $S_n = X_1 + \cdots + X_n$ form a martingale.) If correlations exist, the rv's X_i are uncorrelated. An m-dependent sequence is a sequence of md if $E|X_i| < \infty$ (all i). If $\{X_i\}$ is a sequence of md with $E(X_i^2) \equiv \sigma^2 < \infty$, then $\{X_i\}$ is weakly stationary.

THEOREM 6.1. *Let* $\{X_i\}$ *be a sequence of* md *satisfying assumptions* (A). *If*

$$E\left|n^{-1} \textstyle\sum_{a+1}^{a+n} [E(X_i^2 \mid \mathcal{P}_a) - E(X_i^2)]\right| \leqq B_2(n) \downarrow 0 \qquad (n \to \infty), \tag{6.1}$$

then $\{X_i\}$ *has the central limit property.*

PROOF. A sequence of md satisfies (2.2) automatically with $B_1(n) \equiv 0$ and hence $O(n^{-\theta})$ for any $\theta > 0$. Under assumption (A3), condition (2.1) holds with $\epsilon = \delta$ and $\gamma = 1 + \frac{1}{2}\delta$. With θ chosen sufficiently large, the requirement (4.11) of Theorem 4.1 may thus be satisfied. It remains to show that condition (2.3) is satisfied.

Since, for $j > i > 0$,

$$E(X_{a+i}X_{a+j} \mid \mathcal{P}_a) = E[X_{a+i}E(X_{a+j} \mid \mathcal{P}_{a+i}) \mid \mathcal{P}_a] = 0, \tag{6.2}$$

we have, for a sequence of md,

$$E(T_a^2 \mid \mathcal{P}_a) = n^{-1} \textstyle\sum_{a+1}^{a+n} E(X_i^2 \mid \mathcal{P}_a), \tag{6.3}$$

as well as

$$E(T_a^2) = n^{-1} \textstyle\sum_{a+1}^{a+n} E(X_i^2). \tag{6.4}$$

By (6.3) and (6.4), condition (2.3) has the form (6.1).

The requirement (6.1) is a relatively mild dependence restriction. It is satisfied, e.g., if the conditional variance of X_i given the "distant past" converges in the mean (uniformly) to the unconditional variance as the "distance" increases without limit, that is, if

$$E\left|E(X_{a+n}^2 \mid \mathcal{P}_a) - E(X_a^2)\right| \to 0 \quad \text{uniformly in } a \qquad (n \to \infty). \tag{6.5}$$

The following corollary pertains to weakly stationary sequences.

COROLLARY 6.1.1. *If* $\{X_i\}$ *is a sequence of* md *for which* $E(X_i^2) \equiv \sigma^2 < \infty$, $E|X_i|^{2+\delta} \leqq M < \infty$ *for some* $\delta > 0$, *and*

$$E\left|E(X_{a+n}^2 \mid \mathcal{P}_a) - \sigma^2\right| \to 0 \quad \textit{uniformly in } a \qquad (n \to \infty), \tag{6.6}$$

then $\{X_i\}$ *has the central limit property.*

Theorem 6.1 has an interesting comparison with the following result of Ibragimov [13].

THEOREM 6.2. (Ibragimov). *Let* $\{X_i\}$ *be a strictly stationary ergodic sequence of* md *with* $E(X_1^2) = \sigma^2 < \infty$. *Then* $\{X_i\}$ *has the central limit property.*

We shall compare the results under the assumption that $E(X_i^2) = \sigma^2 < \infty$ (all i). In this case, (6.1) becomes

$$E\,|n^{-1} \textstyle\sum_{a+1}^{a+n} E(X_i^2 \mid \mathcal{P}_a) - \sigma^2| \leqq B_2(n) \downarrow 0 \qquad (n \to \infty). \tag{6.7}$$

We see that Theorem 6.1 imposes a mild additional restriction, (A3), but reduces the assumptions {strict stationarity, ergodicity} to {weak stationarity, (6.7)}. In order to show that this is indeed a reduction, it suffices to prove that

$$\{\text{strict stationarity, ergodicity, (A3)}\} \Rightarrow (6.7), \tag{6.8}$$

since in our case strict stationarity $\Rightarrow$ weak stationarity. Define $\xi_i = X_i^2 - E(X_i^2)$. By (A3), there exists a finite M such that

$$E\,|n^{-1} \textstyle\sum_{a+1}^{a+n} \xi_i|^{1+\frac{1}{2}\delta} \leqq M. \tag{6.9}$$

The ergodicity implies that

$$n^{-1} \textstyle\sum_{a+1}^{a+n} \xi_i \to 0 \quad \text{with pr. 1} \qquad (n \to \infty), \tag{6.10}$$

and the strict stationarity implies that the convergence is uniform in a. Since $\delta > 0$, (6.9) and (6.10) together imply convergence in the (first) mean ([15], p. 164), i.e.,

$$E\,|n^{-1} \textstyle\sum_{a+1}^{a+n} \xi_i| \to 0 \quad \text{uniformly in} \quad a \qquad (n \to \infty), \tag{6.11}$$

from which (6.7) follows easily. Thus (6.8) is proved.

Under regularity condition (I), a particularly simple result holds, a consequence of Theorem 2.2 and the fact that (6.5) $\Rightarrow$ (6.1).

COROLLARY 6.1.2. *If* $\{X_i\}$ *is a sequence of* md *satisfying assumptions* (A) *and regularity condition* (I), *then* $\{X_i\}$ *has the central limit property.*

The condition (I) is stronger than the ergodic hypothesis (see Billingsley [3], p. 12).

In conclusion, we prove a statement made in Section 3.

THEOREM 6.3. *Let* $\{X_i\}$ *be a sequence of* md *satisfying assumptions* (A). *Then condition* (3.1) *holds.*

PROOF. Put $\xi_i = X_i^2 - E(X_i^2)$. By (6.3) and (6.9),

$$E\,|E(T_a^2 \mid \mathcal{P}_a) - E(T_a^2)|^{1+\frac{1}{2}\delta} = E\,|n^{-1} \textstyle\sum_{a+1}^{a+n} \xi_i|^{1+\frac{1}{2}\delta} \leqq M.$$

Hence condition (3.2) of Theorem 3.1 is satisfied, yielding (3.1) for some $\epsilon > 0$.

7. Bounded sequences. In Section 3 it was seen that a bounded sequence satisfying assumptions (A1) and (A2) also satisfies condition (2.1) with $\gamma = \frac{1}{2}\epsilon$, for any $\epsilon > 0$. Then, if ϵ is chosen $\leqq 1$, the requirement (4.11) of Theorem 4.1 is met whenever $\theta > \frac{1}{2}$. This gives

THEOREM 7.1. *Let* $\{X_i\}$ *be a bounded sequence satisfying* (A1), (A2) *and conditions* (2.2) *and* (2.3), *with* $B_1(n) = O(n^{-\theta})$, $\theta > \frac{1}{2}$. *Then* $\{X_i\}$ *has the central limit property.*

Consider also the following consequence of Corollary 4.1.3.

THEOREM 7.2. *Let* $\{X_i\}$ *be a bounded sequence satisfying* (A1), (A2) *and regularity condition* (I) *with* $\phi(m) = O(m^{-\theta})$, $\theta > 1$. *Then* $\{X_i\}$ *has the central limit property.*

These results illustrate that boundedness of a sequence is a favorable condition toward its having the central limit property. A comparison of Theorem 7.2 with Ibragimov's theorems indicates that boundedness and strict stationarity, though vastly different in nature, are about equally productive as a further regularity assumption given condition (I).

8. Acknowledgment. The author acknowledges with gratitude the encouragement and helpful suggestions contributed by Professors M. R. Leadbetter, W. L. Smith and Wassily Hoeffding.

REFERENCES

[1] VON BAHR, BENGT (1965). On the convergence of moments in the central limit theorem. *Ann. Math. Statist.* **36** 808–818.

[2] BERNSTEIN, S. (1927). Sur l'extension du théorème du calcul des probabilités aux sommes de quantités dependantes. *Mathematische Annalen* **97** 1–59.

[3] BILLINGSLEY, PATRICK (1965). *Ergodic Theory and Information.* Wiley, New York.

[4] BRILLINGER, D. (1962). A note on the rate of convergence of a mean. *Biometrika* **49** 574–576.

[5] COX, D. R. and LEWIS, P. A. W. (1966). *The Statistical Analysis of Series of Events.* Methuen, London.

[6] DIANANDA, P. H. (1953). Some probability limit theorems with statistical applications. *Proc. Cambridge Philos. Soc.* **49** 239–246.

[7] DIANANDA, P. H. (1954). The central limit theorem for m-dependent variables asymptotically stationary to second order. *Proc. Cambridge Philos. Soc.* **50** 287–292.

[8] DIANANDA, P. H. (1955). The central limit theorem for m-dependent variables. *Proc. Cambridge Philos. Soc.* **51** 92–95.

[9] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[10] GRENANDER, U. and ROSENBLATT, M. (1957). *Statistical Analysis of Stationary Time Series.* Wiley, New York.

[11] HOEFFDING, WASSILY and ROBBINS, HERBERT (1948). The central limit theorem for dependent random variables. *Duke Math. J.* **15** 773–780.

[12] IBRAGIMOV, I. A. (1962). Some limit theorems for stationary processes. *Theor. Prob. Appl.* **7** 349–382.

[13] IBRAGIMOV, I. A. (1963). A central limit theorem for a class of dependent random variables. *Theor. Prob. Appl.* **8** 83–89.

[14] LOÈVE, MICHEL (1945). Etude asymptotique des sommes de variables aléatoires liées. *J. de Math.* **24**.

[15] LOÈVE, MICHEL (1960). *Probability Theory,* (2nd ed.). Van Nostrand, Princeton.

[16] OREY, STEPHEN (1958). A central limit theorem for m-dependent random variables. *Duke Math. J.* **25** 543–546.

[17] ROSENBLATT, M. (1956). A central limit theorem and a strong mixing condition. *Proc. Nat. Acad. Sci.* **42** 43–47.