CONTRIBUTIONS TO THE PROBLEM OF APPROXIMATION OF EQUIDISTANT DATA BY ANALYTIC FUNCTIONS*

PART B—ON THE PROBLEM OF OSCULATORY INTERPOLATION. A SECOND CLASS OF ANALYTIC APPROXIMATION FORMULAE

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Introduction. The present second part of the paper has two objectives. Firstly, we wish to carry further the important actuarial work on the subject of osculatory interpolation (Chapters I and II). Secondly, we construct even analytic functions $L(x)$, of extremely fast damping rate, such that the interpolation formula of cardinal type

$$F(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L(x - \nu) \tag{1}$$

reproduces polynomials of a certain degree and reduces to a smoothing formula for integral values of the variable x (Chapter III). This second problem is found to be intimately connected with the subject of osculatory interpolation.

A preliminary remark concerning our notation is necessary. In Part A, Section 2.21 we described various characteristic properties (or "type characteristics") of a polynomial interpolation formula of the form (1), such as: (i) The degree m of the composite polynomial function $L(x)$; (ii) its class C^μ, i.e., order of contact is μ; (iii) the highest degree k of polynomials for which the formula (1) is exact; (iv) the span s of the even polynomial function $L(x)$. For convenience we propose to summarize all these statements by saying that (1) is a formula of type[1]

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[1] The connection of these types characteristics with the notation as used by Greville in his paper *The general theory of osculatory interpolation*, Trans. Actuar. Soc. Amer., **45**, pp. 202–265 (1944), especially pp. 210–211, is as follows: The first three symbols D^m, C^μ, E^k, require no further comment since they are identical respectively with the characteristics 4, 1, and 6 of Greville's classification, pp. 210–211. There remain three further characteristics to be discussed: (i) Whether the formula (1) is an "end-point" or "mid-point" formula. This point is of importance if (1) is written in terms of certral differences, since it, then reduces to either the Everett or else the Steffensen form. The following statement is obvious: *The formula* (1) *is an "end point" or "mid-point" formula depending on whether the span s is even or odd.*

(ii) Greville's adjectives "ordinary" and "modified" agree respectively with our "ordinary" and "smoothing."

(iii) The highest order d of differences involved (explicitly or implicitly). We start with the following question: *Let x be given. How many ordinates y_n enter into the computation of $F(x)$ by* (1)? Assuming that $L(x)$ is continuous, hence $=0$, at the end point $x=s/2$ of its span, we have $L(x-n)\neq 0$, as long as n in such that

$$|x-n| < s/2.$$

This inequality is found to be equivalent to

$$-\frac{s}{2} + x < n < \frac{s}{2} + x. \tag{*}$$

Let $s=2\sigma$ be even (end-point formula) and let now x be anywhere within $0\leqq x\leqq 1$. By (*) $F(x)$ then re-

$$D^m, \quad C^\mu, \quad E^k, \quad s. \tag{2}$$

As an instance we may describe the k-point central interpolation formula of Part A, Section 2.121 as a formula of type[2]

$$D^{k-1}, \quad \begin{Bmatrix} C^0 & \text{if } k \text{ is even} \\ C^{-1} & \text{if } k \text{ is odd} \end{Bmatrix}, \quad E^{k-1}, \quad s = k.$$

In Chapter II we construct a class of ordinary interpolation formulae and two classes of smoothing interpolation formulae. These classes by no means describe all possible osculatory interpolation formulae. Furthermore, a number of interesting problems concerning remainder terms and orders of approximations await solution. No attempt has been made to see which of the numerous formulae tabulated by Greville, loc. cit., are contained in the three classes of formulae developed in Chapter II. The essential progress made in this direction may perhaps be briefly described as follows. The construction of an interpolation formula usually requires the solution of a more or less complicated system of linear equations, unless, as in Lagrange's formula, the basic interpolating functions are obvious from the start. These systems of equations are especially troublesome if one wishes to construct an osculatory interpolation formula of any *general* class. As Greville correctly points out, loc. cit., pp. 255–256, the mere agreement between the number of unknowns with the number of equations which they should satisfy, will, by itself, never prove the existence of a solution. Basically, our parametric representation of spline curves of order k (Part A, Section 3.15, Theorem 5) circumvents this difficulty.

An example which illustrates the operation of this representation is as follows. Let $F(x)$ be defined as equal to 0 for $x \leqq 0$, as well as for $x \geqq 4$. We propose to complete the definition of $F(x)$ in the range $0 \leqq x \leqq 4$ by four cubic arcs joining at $x=1, 2, 3$, in such a way that $F(x)$ be of class C'' for all real x. Of course, we are not interested in the obvious but trivial solution $F(x) \equiv 0$. Let us now count the available parameters and the number of conditions. The 4 cubic arcs furnish $4 \cdot 4 = 16$ parameters. The second order contact requirements at $x=0, 1, 2, 3, 4$ lead to a system of $3 \cdot 5 = 15$ homogeneous equations. The solution of a homogeneous system of 15 equations in 16 unknowns depends on anything from 1 to 16 arbitrary parameters, depending on the rank of the system. Our Theorem 5, for $k=4$, furnishes immediately the one-parameter solution

$$F(x) = c \cdot M_4(x-2) \tag{3}$$

the graph of which is given in Part A, Section 3.13. Again Theorem 5 will easily show that this is the most general solution of the problem. We see how this complicated system of 15 equations in 16 unknowns is explicitly solved by (3). As a variation of the problem, let us now define $F(x)$ to be equal to 0 for $x \leqq 0$, as well as for $x \geqq 3$, and let us propose now to bridge this gap by 3 cubic arcs giving a $F(x)$ of class C''. Now we find that the problem amounts to a system of 12 homogeneous equations in 12 unknowns. This tells us precisely nothing. Again by Theorem 5 we can readily show that

quires all y_n such that $-\sigma < n < \sigma+1$ that is $s = 2\sigma$ consecutive ordinates. Let $s = 2\sigma+1$ be odd (mid-point formula) and let x be anywhere within $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$. Again by (*) $F(x)$ now requires all y_n such that $-\sigma-1 < n < \sigma+1$, hence again $s = 2\sigma+1$ consecutive ordinates. We have therefore proved the following: *The highest order d of differences involved is always related with the span s by the relation $s = d+1$.*

[2] The symbol C^{-1} is to indicate the class of piecewise continuous functions.

the trivial solution $F(x)\equiv 0$ is the only solution. These considerations generalize and allow to characterize our basic functions

$$M_k(x) = \frac{1}{(k-1)!}\,\delta^k x_+^{k-1},$$

up to a multiplicative constant and a shift along the x-axis, as follows:[3] *Let* $F(x)$ *be* $=0$ *for* $x\leqq 0$, *as well as for* $x\geqq n$, *where* n *is a positive integer. We wish to complete the definition of* $F(x)$ *by a succession of* n *arcs, of degree* $k-1$, *joining at* $x=1, 2, \cdots, n-1$ *such as to furnish a* $F(x)$ *of* C^{k-2}. *Then* $n=k$ *is the smallest value of* n *for which this can be done in a non-trivial way and for this minimal value* $n=k$ *the gap is bridged by*

$$F(x) = c\cdot M_k(x - k/2)$$

and in no other way.

The reader who is mainly interested in the numerical applications may pass directly from here to the Appendix where the use of the tables is fully explained and one example is worked out.

1. THE COSINE POLYNOMIALS $\phi_k(u)$ AND CERTAIN RELATED SETS OF POLYNOMIALS

In the present chapter we propose to study further properties of the cosine polynomials

$$\phi_k(u) = \sum_{n=-\infty}^{\infty} M_k(n)\cos nu \tag{1}$$

which were mentioned in Part A, sections 3.14 and 4.1 (for $t=0$). By Part A, section 4.1, formula (6) (for $t=0$) we may also write

$$\phi_k(u) = \sum_{\nu=-\infty}^{\infty} \psi_k(u+2\pi\nu)$$

and therefore

$$\phi_k(u) = (2\sin u/2)^k\cdot\sum_{\nu=-\infty}^{\infty}\frac{(-1)^{\nu k}}{(u+2\pi\nu)^k}. \tag{2}$$

1.1. **Expression of $\phi_k(u)$ in terms of rational polynomials.** We introduce two new sets of periodic functions defined by

$$\rho_k(u) = (2\sin u/2)^k\cdot\sum_{\nu=-\infty}^{\infty}\frac{1}{(u+2\pi\nu)^k}, \tag{3}$$

$$\sigma_k(u) = (2\sin u/2)^k\cdot\sum_{\nu=-\infty}^{\infty}\frac{(-1)^{\nu}}{(u+2\pi\nu)^k}. \tag{3'}$$

A comparison with (2) shows that

$$\phi_k(u) = \begin{cases}\rho_k(u) & \text{if } k \text{ is even,}\\ \sigma_k(u) & \text{if } k \text{ is odd.}\end{cases} \tag{4}$$

[3] Problems of this kind concerning polygonal lines of a certain degree and class are of importance for the theory of formulae of mechanical quadratures. The author expects to discuss this connection elsewhere.

By differentiation of (3) and also (3′) we readily find the recurrence relations

$$\rho_{k+1}(u) = \cos\frac{u}{2}\,\rho_k(u) - \frac{2}{k}\sin\frac{u}{2}\,\rho_k'(u), \tag{5}$$

$$\sigma_{k+1}(u) = \cos\frac{u}{2}\,\sigma_k(u) - \frac{2}{k}\sin\frac{u}{2}\,\sigma_k'(u). \tag{5′}$$

These may be used in a progressive computation of ρ_k and σ_k if we start with

$$\rho_2(u) = 1, \qquad \sigma_1(u) = 1. \tag{6}$$

We prefer, however, to express $\rho_k(u)$ and $\sigma_k(u)$ as polynomials in the variable

$$x = \cos(u/2) \tag{7}$$

by means of

$$\rho_k(u) = U_{k-2}(\cos u/2), \qquad \sigma_k(u) = V_{k-1}(\cos u/2). \tag{8}$$

Substituting into (5), and (5′) respectively we find that the two sequences of polynomials $U_n(x)$, $V_n(x)$, both of exact degree n, satisfy the recurrence relations

$$U_{k+1}(x) = xU_k(x) + \frac{1}{k+2}(1 - x^2)U_k'(x), \tag{9}$$

$$V_{k+1}(x) = xV_k(x) + \frac{1}{k+1}(1 - x^2)V_k'(x), \tag{9′}$$

with initial values which by (6) and (8) are

$$U_0(x) = 1, \qquad V_0(x) = 1. \tag{10}$$

A simple calculation now gives

$$\left.\begin{aligned}
U_1(x) &= x, & V_1(x) &= x,\\
U_2(x) &= \frac{1}{3}(1 + 2x^2), & V_2(x) &= \frac{1}{2}(1 + x^2),\\
U_3(x) &= \frac{1}{3}(2x + x^3), & V_3(x) &= \frac{1}{6}(5x + x^3)\\
U_4(x) &= \frac{1}{15}(2 + 11x^2 + 2x^4), & V_4(x) &= \frac{1}{24}(5 + 18x^2 + x^4)\\
U_5(x) &= \frac{1}{45}(17x + 26x^3 + 2x^5), & V_5(x) &= \frac{1}{120}(61x + 58x^3 + x^5).
\end{aligned}\right\} \tag{11}$$

We record as a lemma the following properties which are readily established by induction.

LEMMA 1. *$U_k(x)$ and $V_k(x)$ are polynomials of exact degree k which are even or odd according as k is even or odd. The coefficients of their highest terms are positive. Also*

$$U_k(1) = V_k(1) = 1, \qquad U_k(-1) = V_k(-1) = (-1)^k. \tag{12}$$

In view of (4) and (8) we find the following expression of $\phi_k(u)$ in terms of our new polynomials:

$$\phi_k(u) = \begin{cases} U_{k-2}(x) & \text{if} \quad k \text{ is even,} \\ V_{k-1}(x) & \text{if} \quad k \text{ is odd.} \end{cases} \tag{13}$$

This shows that the even polynomials $U_{2\nu}$, $V_{2\nu}$ are of special interest.

1.2. **The zeros of the polynomials** U_k **and** V_k. We propose to prove the following proposition:

LEMMA 2. *The zeros of the even polynomials* $U_{2\nu}(x)$ *and* $V_{2\nu}(x)$ *are all simple and purely imaginary.*

We carry through the proof for $U_{2\nu}(x)$ only since the proof for $V_{2\nu}(x)$ is entirely similar. In order to deal with real zeros, we define a new sequence of polynomials $u_k(x)$ by

$$u_k(x) = i^{-k}U_k(xi), \qquad (k = 0, 1, \cdots). \tag{14}$$

These new polynomials are also real and satisfy a recurrence relation which in view of (9) is readily found to be

$$u_{k+1}(x) = xu_k(x) - \frac{1}{k+2}(1 + x^2)u_k'(x). \tag{15}$$

From (11) we find

$$u_0(x) = 1, \qquad u_1(x) = x, \qquad u_2(x) = \tfrac{1}{3}(2x^2 - 1), \qquad u_3(x) = \tfrac{1}{3}(x^3 - 2x), \cdots .$$

In view of (14) it obviously suffices to show that the zeros of $u_k(x)$ ar real and simple, while those of $u_{2\nu}(x)$ are also different from zero. This is readily done by induction as follows. Let $k=2\nu$ be even and let us assume that the k zeros of $u_k(x)$ are

$$-\xi_\nu, -\xi_{\nu-1}, \cdots, -\xi_1, \xi_1, \xi_2, \cdots, \xi_\nu \qquad (0 < \xi_1 < \cdots < \xi_\nu), \tag{16}$$

and therefore simple. This, and the fact that $u_k(x)$ has a highest term of positive coefficient (Lemma 1 and (14)), imply that

$$u_k'(\xi_\nu) > 0,$$

and that the sequence of values of $u_k'(x)$, at the k roots (16), alternate in sign. By (15) we therefore find

$$u_{k+1}(\xi_\nu) < 0$$

and that the values of $u_{k+1}(x)$,at the k roots (16), alternate in sign. Since $u_{k+1}(0)=0$, we conclude that $u_{k+1}(x)$ has ν positive and ν negative zeros which must therefore be simple.

Let now $k=2\nu+1$ be odd and let u_k have the simple zeros

$$-\xi_\nu, \cdots, -\xi_1, 0, \xi_1, \cdots, \xi_\nu \qquad (0 < \xi_1 < \cdots < \xi_\nu). \tag{17}$$

Now we conclude as before that $u_{k+1}(\xi_\nu) < 0$ and that the values of $u_{k+1}(x)$, at the k roots (17), alternate in sign. Again the conclusion is that $u_{k+1}(x)$ has simple real roots none of which vanishes. This proves the theorem by complete induction.

1.3. **A few corollaries.** In this last section of the present chapter we prove several auxiliary propositions which will be used in the next chapter in the derivation of interpolation formulae of various kinds. These propositions represent the solutions of the algebraic problems arising by the Fourier integral transformation of the problems of the construction of those interpolation formulae.

LEMMA 3. *Let $k=2\nu$ be even. We can determine uniquely an even polynomial $P_k(x)$, of degree k, and an odd polynomial $P_{k-1}(x)$, of degree $k-1$, satisfying the identity*

$$U_k(x)P_k(x) + U_{k+1}(x)P_{k-1}(x) \equiv 1. \tag{18}$$

Likewise polynomials $Q_k(x)$ and $Q_{k-1}(x)$, even and odd respectively, exist uniquely such as to satisfy

$$V_k(x)Q_k(x) + V_{k+1}(x)Q_{k-1}(x) \equiv 1. \tag{19}$$

We wish to show first that U_k and U_{k+1} have no common zeros. Indeed a common zero x of U_k and U_{k+1} would, by (9), be a zero of

$$(1 - x^2)U_k'(x).$$

Since by (12) $x \neq \pm 1$, x must be a zero of $U_k'(x)$. But this contradicts our Lemma 2 to the effect that $U_k(x)$ has only simple zeros. The polynomials $U_k(x)$, $U_{k+1}(x)$ having no common divisors, the identity (18) is assured by the elementary theory of the greatest common divisor of two polynomials. We now show that $P_k(x)$ is even and $P_{k-1}(x)$ is odd as follows. Replacing x by $-x$ in (18) we find

$$U_k(x)P_k(-x) - U_{k+1}(x)P_{k-1}(-x) \equiv 1.$$

Since our polynomials P_k, P_{k-1} are uniquely defined by (18) we find

$$P_k(x) = P_k(-x), \qquad P_{k-1}(x) = -P_{k-1}(-x),$$

which prove our statement. An identical reasoning proves the existence of the polynomials Q_k and Q_{k+1} satisfying (19).

The polynomials P_k and P_{k+1} are easily determined for low values of k by the method of indeterminate coefficients. Thus for $k=2$ by (11),

$$U_2(x) = (1 + 2x^2)/3, \qquad U_3(x) = (2x + x^3)/3,$$

from which we find

$$P_2(x) = 2x^2 + 3, \qquad P_1(x) = -4x,$$

satisfying the identity

$$U_2(x)P_2(x) + U_3(x)P_1(x) \equiv 1. \tag{20}$$

Likewise for $k=4$ we have by (11)

$$V_4(x) = (5 + 18x^2 + x^4)/24, \qquad V_5(x) = (61x + 58x^3 + x^5)/120.$$

The corresponding polynomials Q_4, Q_3 are found to be

$$Q_4(x) = (3648 + 4789x^2 + 83x^4)/760, \qquad Q_3(x) = -(1469x + 83x^3)/152.$$

They satisfy the identity

$$V_4(x)Q_4(x) + V_5(x)Q_3(x) \equiv 1. \tag{21}$$

The identities (18), (19) will later be used in the following form. Again for an even k, but replacing k by $k-2$, we get by (18) and (8)

$$\rho_k(u)P_{k-2}(x) + \rho_{k+1}(u)P_{k-3}(x) \equiv 1, \qquad (k \text{ even}, x = \cos u/2).$$

Likewise for an odd k, but replacing k by $k-1$, we obtain from (19) and (8) the identity

$$\sigma_k(u)Q_{k-1}(u) + \sigma_{k+1}(u)Q_{k-2}(k) \equiv 1, \qquad (k \text{ odd}, x = \cos u/2).$$

The even polynomials P_{k-2}, $x^{-1}P_{k-3}$, and Q_{k-1}, $x^{-1}Q_{k-2}$, may now be expressed in powers of

$$1 - x^2 = (\sin u/2)^2.$$

We have therefore proved the following:

Lemma 4. *We can find constants a_ν, a_ν', b_ν, b_ν' such as to satisfy the following two identities*:

For an even k

$$\rho_k(u)\{a_0 - a_2(2\sin u/2)^2 + a_4(2\sin u/2)^4 - \cdots \pm a_{k-2}(2\sin u/2)^{k-2}\}$$
$$+ \rho_{k+1}(u)\{a_0' - a_2'(2\sin u/2)^2 + a_4'(2\sin u/2)^4 - \cdots$$
$$\mp a_{k-4}'(2\sin u/2)^{k-4}\}(2\cos u/2) \equiv 1, \tag{22}$$

and for k odd

$$\sigma_k(u)\{b_0 - b_2(2\sin u/2)^2 + b_4(2\sin u/2)^4 - \cdots \pm b_{k-1}(2\sin u/2)^{k-1}\}$$
$$+ \sigma_{k+1}(u)\{b_0' - b_2'(2\sin u/2)^2 + b_4'(2\sin u/2)^4 - \cdots$$
$$\mp b_{k-3}'(2\sin u/2)^{k-3}\}(2\cos u/2) \equiv 1. \tag{22'}$$

As examples we mention that the identities (20) and (21) become on passing to the variable u

$$\rho_4(u)\{5 - \tfrac{1}{2}(2\sin u/2)^2\} + \rho_5(u)\{-2\}(2\cos u/2) \equiv 1 \tag{23}$$

and

$$\sigma_5(u)\left\{\frac{213}{19} - \frac{991}{760}(2\sin u/2)^2 + \frac{83}{12160}(2\sin u/2)^4\right\}$$
$$+ \sigma_6(u)\left\{-\frac{194}{38} + \frac{83}{1216}\left(2\sin\frac{u}{2}\right)^2\right\}\left(2\cos\frac{u}{2}\right) \equiv 1. \tag{24}$$

The last proposition which we wish to derive here concerns the expansion of $1/\phi_k(u)$ in ascending powers of the variable

$$s = \sin^2 u/2 = 1 - \cos^2 u/2 = 1 - x^2. \tag{25}$$

Let us assume for the moment that k is even. Then by (13)

$$\phi_k(u) = U_{k-2}(x), \qquad (k \text{ even}). \tag{26}$$

Now $U_{k-2}(x)$ is an even polynomial which, by Lemma 2, has purely imaginary zeros. Being an even polynomial, $U_{k-2}(x)$ may be expressed as a polynomial $U^*(s)$ in the variable

$$s = 1 - x^2 \tag{25'}$$

of degree $\kappa=(k-2)/2$. This change of variable transforms the purely imaginary zeros of $U_{k-2}(x)$ into the zeros

$$\alpha_1, \alpha_2, \cdots, \alpha_\kappa \qquad (\kappa = (k-2)/2)$$

of $U^*(s)$ which, by (25′), must all be positive and greater than 1. Finally, since $U_{k-2}(1)=U^*(0)=1$, we have the identity

$$\phi_k(u) = U_{k-2}(x) = \left(1-\frac{s}{\alpha_1}\right)\left(1-\frac{s}{\alpha_2}\right)\cdots\left(1-\frac{s}{\alpha_\kappa}\right). \tag{27}$$

An entirely similar identity is derived for an *odd* k by repeating our arguments for

$$\phi_k(u) = V_{k-1}(x),$$

instead of (26).

This establishes the following

LEMMA 5. *The reciprocal of the cosine polynomial $\phi_k(u)$ admits of an expansion*

$$\frac{1}{\phi_k(u)} = \sum_{n=0}^{\infty} c_{2n}^{(k)}(2\sin u/2)^{2n} \tag{28}$$

which converges for all real values of u and where the coefficients are positive rational numbers

$$c_{2n}^{(k)} > 0, \qquad (n = 0, 1, 2, \cdots). \tag{29}$$

Indeed, in view of (27), the expansion (28) may be obtained as

$$\frac{1}{\phi_k(u)} = \prod_{\nu=1}^{\kappa}\left(1+\frac{s}{\alpha_\nu}+\frac{s^2}{\alpha_\nu^2}+\cdots\right) = \sum_{n=0}^{\infty} c_{2n}^{(k)}(4s)^n$$

which reduces to (28), in view of (25). In conclusion we notice the following consequences of the identity (28). Since

$$\phi_k(u) = \sum_{\nu=-\infty}^{\infty} \psi_k(u+2\pi\nu),$$

where

$$\psi_k(u) = \left(\frac{2\sin u/2}{u}\right)^k,$$

we have in the neighborhood of the origin $u=0$

$$\phi_k(u) = \left(\frac{2\sin u/2}{u}\right)^k + u^k\cdot(\text{regular function of } u). \tag{30}$$

On multiplying (28) by $\phi_k(u)$ we therefore have

$$\left(\frac{2\sin u/2}{u}\right)^k\cdot\sum_{n=0}^{\infty} c_{2n}^{(k)}(2\sin u/2)^{2n} = 1 + u^k\cdot(\text{regular function})$$

and also

$$\left(\frac{2\sin u/2}{u}\right)^k\cdot\sum_{0\leq 2n<k} c_{2n}^{(k)}(2\sin u/2)^{2n} = 1 + u^k\cdot(\text{regular function}). \tag{31}$$

It is of special interest to point out that if

$$g_{k,m}(u) \equiv \left(\frac{2\sin u/2}{u}\right)^k \cdot \sum_{n=0}^{m-1} c_{2n}^{(k)} (2\sin u/2)^{2n}$$

then

$$g_{k,m}(u) = \begin{cases} 1 + u^{2m}\cdot(\text{regular function}) & \text{if} \quad 2m < k \\ 1 + u^{k}\cdot(\text{regular function}) & \text{if} \quad 2m-2 < k \leqq 2m. \end{cases} \tag{32}$$

As an illustration we find for $k=6$ by (13), and (11), and (25)

$$\frac{1}{\phi_6(u)} = \frac{1}{U_4(x)} = \frac{15}{2+11x^2+2x^4} = \frac{30}{30-30s+4s^2} = \frac{1}{1-s+\dfrac{2}{15}s^2}$$

whence

$$\frac{1}{\phi_6(u)} = 1 + s + \frac{13}{15}s^2 + \cdots$$

or

$$\frac{1}{\phi_6(u)} = 1 + \frac{1}{4}(2\sin u/2)^2 + \frac{13}{240}(2\sin u/2)^4 + \cdots . \tag{33}$$

The relations (32) now become (for $k=6$, $m=1, 2, 3$)

$$\left(\frac{2\sin u/2}{u}\right)^6 = 1 + u^2\cdot(\text{regular function}),$$

$$\left(\frac{2\sin u/2}{u}\right)^6 \left\{1 + \frac{1}{4}(2\sin u/2)^2\right\} = 1 + u^4\cdot(\text{regular function}),$$

$$\left(\frac{2\sin u/2}{u}\right)^6 \left\{1 + \frac{1}{4}(2\sin u/2)^2 + \frac{13}{240}(2\sin u/2)^4\right\} = 1 + u^6\cdot(\text{regular function}).$$

In the next chapter we shall need the numerical values of the coefficients

$$c_{2n}^{(k)} \quad \text{for} \quad 2n < k. \tag{34}$$

It is of interest then to point out that these coefficients (34) may also be otherwise computed as coefficients of a simple generating function. Indeed, from (30) it is clear that the coefficients (34) will not change if on the left-hand side of (28) we replace $\phi_k(u)$ by the first term on the right-hand side of (30). That means, if

$$\left(\frac{u}{2\sin u/2}\right)^k = \sum_{n=0}^{\infty} d_{2n}^{(k)} (2\sin u/2)^{2n} \tag{35}$$

then

$$c_{2n}^{(k)} = d_{2n}^{(k)} \qquad (2n < k). \tag{36}$$

However, the coefficients of (35) are readily determined. Indeed, if we set

$$v = 2\sin u/2 \quad \text{or} \quad u = 2\arcsin v/2, \tag{37}$$

then (35) becomes

$$\left(\frac{2 \arcsin v/2}{v}\right)^k = \sum_{n=0}^{\infty} d_{2n}^{(k)} v^{2n}. \tag{38}$$

Since

$$\frac{2 \arcsin v/2}{v} = 1 + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{v^2}{4} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}\cdot\frac{v^4}{16} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7}\cdot\frac{v^6}{64} + \cdots, \tag{39}$$

we find the expansion (38) by raising (39) to the k-th power. Thus

$$d_0^{(k)} = 1, \qquad d_2^{(k)} = \frac{k}{24}, \qquad d_4^{(k)} = \frac{5k^2 + 22k}{5760}, \cdots. \tag{40}$$

Since all coefficients of (39) are positive, it is clear that the coefficients of (38) are likewise positive. This however does not imply the positivity of the coefficients of (28) beyond the kth term. For $k=6$ the values (40) agree with the coefficients of (33).

II. POLYNOMIAL INTERPOLATION FORMULAE

In this chapter we wish to apply our general Theorem 2 of Part A, section 2.23 and our Lemmas 4 and 5 of the last section in deriving three distinct classes of polynomial interpolation formulae for each value of the positive integer k. The formulae of the first class (Theorem 1 below) are of the ordinary kind (see Part A, section 2.21 a and b), and of the type

$$D^k, \quad C^{k-2}, \quad E^{k-1}, \quad s = \begin{cases} 2k-2 & \text{if } k \text{ is even} \\ 2k-1 & \text{if } k \text{ is odd.} \end{cases}$$

The existence of ordinary interpolation formulae of degree k and class $k-2$ was previously conjectured by Mr. Greville who verified their existence up to and including $k=6$. (See Greville, loc. cit., pp. 212–213.) The formulae of the second class are smoothing interpolation formulae (Theorem 2 below). For a given integral k and each integral m, such that $0 \leqq 2m-2 < k$, a formula is derived which is of the type

$$D^{k-1}, \qquad C^{k-2}, \qquad E^{\min(2m-1,k-1)}, \qquad s = k + 2m - 2.$$

These formulae are derived from an *ordinary* interpolation formula of type

$$D^{k-1}, \qquad C^{k-2}, \qquad E^{k-1}, \qquad s = \infty,$$

discussed in Part A.

The formulae of the third and last class are again *smoothing* interpolation formulae (Theorem 3 below). While in the second class the degree D^{k-1} and the "order of contact" C^{k-2} were fixed, while the degree of exactness E^{2m-1} and the span $s=k+2m-2$ increased apace, in the present class the span $s=k$ is constant. More precisely, a formula is derived for each m such that $0 \leqq 2m \leqq k-1$ which is of type

$$D^{k-1}, \qquad C^{k-2m}, \qquad E^{2m-1}, \qquad s = k.$$

These formulae are derived from the formula of ordinary k-point central interpolation in a manner somewhat reminiscent of Mr. Jenkins' original procedure.

2.1. **Ordinary polynomial interpolation formulae of the Jenkins-Greville type.** We are returning to our basic functions

$$M_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2 \sin u/2}{u}\right)^k e^{iux}du \tag{1}$$

and wish to show that the osculatory interpolation formulae of the type investigated by Jenkins and Greville may be readily derived in terms of these functions. We shall use the operational symbol σ to mean

$$\sigma f(x) = f(x + \tfrac{1}{2}) + f(x - \tfrac{1}{2}).$$

THEOREM 1. *We define the basic polynomial function $L(x)$ by the following two formulae according to the parity of k:*

$$L(x) = a_0 M_k(x) + a_2\delta^2 M_k(x) + \cdots + a_{k-2}\delta^{k-2}M_k(x)$$
$$+ a_0'\sigma M_{k+1}(x) + a_2'\sigma\delta^2 M_{k+1}(x) + \cdots + a_{k-4}'\sigma\delta^{k-4}M_{k+1}(x) \quad (k \text{ even}) \tag{2}$$

$$L(x) = b_0 M_k(x) + b_2\delta^2 M_k(x) + \cdots + b_{k-1}\delta^{k-1}M_k(x)$$
$$+ b_0'\sigma M_{k+1}(x) + b_2'\sigma\delta^2 M_{k+1}(x) + \cdots + b_{k-3}'\sigma\delta^{k-3}M_{k+1}(x), \quad (k \text{ odd}), \tag{3}$$

where the numerical constants a_ν, a_ν', b_ν, b_ν' are those defined in Lemma 4. Then

$$F(x) = \sum_{n=-\infty}^{\infty} y_n L(x - n) \tag{4}$$

is an ordinary polynomial interpolation formula of type

$$D^k, \quad C^{k-2}, \quad E^{k-1}, \quad s = \begin{cases} 2k-2 & \text{if } k \text{ is even,} \\ 2k-1 & \text{if } k \text{ is odd.} \end{cases} \tag{5}$$

Indeed, we notice that

$$\delta e^{iux} = (e^{iu/2} - e^{-iu/2})e^{iux} = 2i \sin u/2 e^{iux},$$
$$\sigma e^{iux} = (e^{iu/2} + e^{-iu/2})e^{iux} = 2 \cos u/2 e^{iux}. \tag{6}$$

Let k now be even, hence $L(x)$ defined by (2), and let

$$L(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)e^{iux}du.$$

We evidently obtain this integral representation by performing the operation

$$a_0 + a_2\delta^2 + \cdots + a_{k-2}\delta^{k-2}$$

on the relation (1) and add to it the result of performing the operation

$$a_0'\sigma + a_2'\sigma\delta^2 + \cdots + a_{k-4}'\sigma\delta^{k-4}$$

on (1), with k replaced by $k+1$. In view of (6) we have

$$\delta^\nu e^{iux} = (2i \sin u/2)^\nu e^{iux}$$
$$\sigma\delta^\nu e^{iux} = (2i \sin u/2)^\nu (2 \cos u/2)e^{iux},$$

and therefore

$$g(u) = \left(\frac{2 \sin u/2}{u}\right)^k \left\{a_0 - a_2(2 \sin u/2)^2 + \cdots \pm a_{k-2}(2 \sin u/2)^{k-2}\right\}$$
$$+ \left(\frac{2 \sin u/2}{u}\right)^{k+1} \left\{a_0' - a_2'(2 \sin u/2)^2 + \cdots \right.$$
$$\left. \mp a_{k-4}'(2 \sin u/2)^{k-4}\right\}(2 \cos u/2). \tag{7}$$

We now turn to Theorem 2 of Part A, section 2.23, which states that (4) is an *ordinary* interpolation formula if and only if the following identity holds:

$$\sum_{\nu=-\infty}^{\infty} g(u + 2\pi\nu) = 1. \tag{8}$$

It should be noticed now that both expressions in (7) contained within braces are periodic functions of period 2π. Since k is even we find that $\sum g(u+2\pi\nu)$ is identical with the left-hand side of our relation I (22). This proves (8) for even k. A precisely similar reasoning for odd k will show that $\sum g(u+2\pi\nu)$ is identical with the left-hand side of I (22′).

There remains the problem of showing that (4) is of the type as stated in the Theorem. Since $L(x)$ is by (2), (3), a linear combination of functions of the form

$$M_k(x + n), \qquad M_{k+1}(x + \tfrac{1}{2} + n),$$

it is clear that $L(x)$ is a polynomial line of degree k, of class C^{k-2}, with discontinuities at $x=n$, or $x=n+1/2$, according to whether k is even or odd. Finally (4) is exact for the degree $k-1$, again by Theorem 2 of Part A, section 2.23. There remains the discussion of the span s of $L(x)$. Now the span of $M_k(x)$ is $=k$ (see Part A, section 3.13) and therefore the span of $\delta^\nu M_k(x)$ is equal to $k+\nu$, while the span of $\sigma\delta^\nu M_k(x)$ is equal to $k+\nu+1$. Now it is immediately verified that the two terms of (2) involving δ^{k-2} and $\sigma\delta^{k-4}$ are both of span $2k-2$, while the similar two terms of (3) are both of span $2k-1$. This completes a proof of the Theorem.

As illustrations we mention that the identities I(23) and I(24) corresponding to the cases $k=4$ and $k=5$ give rise to the basic functions

$$L(x) = 5M_4(x) + \tfrac{1}{2}\delta^2 M_4(x) - 2\sigma M_5(x) \tag{9}$$

and

$$L(x) = \frac{213}{19} M_5(x) + \frac{991}{760}\delta^2 M_5(x) + \frac{83}{12160}\delta^4 M_5(x) - \frac{194}{38}\sigma M_6(x) - \frac{83}{1216}\sigma\delta^2 M_6(x). \tag{10}$$

These two basic functions give rise to ordinary interpolation formulae (4) which are of the types

$$D^4,\ C^2,\ E^3,\ s = 6 \quad \text{and} \quad D^5,\ C^3,\ E^4,\ s = 9, \tag{11}$$

respectively.

Incidentally, the characteristic function of (9) is, by I(23),

$$g(u) = \left(\frac{2 \sin u/2}{u}\right)^4 \left(5 - 2\sin^2\frac{u}{2}\right) - 4\left(\frac{2 \sin u/2}{u}\right)^5 \cos u/2$$

or

$$g(u) = \left(\frac{2 \sin u/2}{u}\right)^4 \left(4 + \cos u - 4\frac{\sin u}{u}\right).$$

This agrees with our formula (11″) of Part A, section 2.122, as in fact the basic function (9) is identical with Jenkins' function there described by formula (11).

In concluding we wish to mention the numerical results for $k=6$. In this case we need the identity

$$U_4(x)P_4(x) + U_5(x)P_3(x) \equiv 1. \tag{12}$$

By (11) we have

$$U_4(x) = \frac{1}{15}(2 + 11x^2 + 2x^4), \qquad U_5(x) = \frac{1}{45}(17x + 26x^3 + 2x^5).$$

By indeterminate coefficients we find

$$P_4(x) = \frac{15}{2} + \frac{115}{7}x^2 + \frac{27}{21}x^4, \qquad P_3(x) = -\frac{285}{14}x - \frac{27}{7}x^3,$$

from which, on passing to the variable I(25), the identity (12) becomes

$$\rho_6(u)\left\{\frac{353}{14} - \frac{133}{28}(2\sin u/2)^2 + \frac{9}{448}(2\sin u/2)^4\right\}$$
$$+ \rho_7(u)\left\{-\frac{339}{28} + \frac{27}{56}(2\sin u/2)^2\right\}(2\cos u/2) \equiv 1.$$

The basic function corresponding to $k=6$ is therefore

$$L(x) = \frac{353}{14}M_6(x) + \frac{133}{28}\delta^2 M_6(x) + \frac{9}{448}\delta^4 M_6(x)$$
$$- \frac{339}{28}\sigma M_7(x) - \frac{27}{56}\sigma\delta^2 M_7(x), \tag{13}$$

giving rise to an ordinary interpolation formula of type

$$D^6, \ C^4, \ E^5 \quad \text{and} \quad s = 10.$$

2.2. **A first class of smoothing interpolation formulae derived from an ordinary interpolation formula of type D^{k-1}, C^{k-2}.** We start by recalling an ordinary polynomial interpolation formula derived in Part A, section 4.2. Indeed, the formula (9) of that section furnishes, for $t=0$, the following polynomial basic function

$$L_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\psi_k(u)}{\phi_k(u)} e^{iux} du. \tag{14}$$

The corresponding formula

$$F(x) = \sum_{n=-\infty}^{\infty} y_n L_k(x-n) \tag{15}$$

is, as we know, an ordinary polynomial interpolation formula of type

$$D^{k-1}, \qquad C^{k-2}, \qquad E^{k-1}, \qquad s = \begin{cases} \infty & \text{if } k \geqq 3, \\ k & \text{if } k = 1, 2. \end{cases} \tag{16}$$

We turn now to the expansion I(28) of Lemma 5, substituting I(28) into the integral (14) and integrating term-wise we obtain the expansion

$$L_k(x) = M_k(x) - c_2^{(k)}\delta^2 M_k(x) + c_4^{(k)}\delta^4 M_k(x) - \cdots, \qquad (k \geqq 3). \tag{17}$$

On comparing the present interpolation formula (15) with the formula (4) of the Jenkins-Greville type, we notice, by (5) and (16), that they are both of class C^{k-2} and that they are both exact for the degree $k-1$. The degree of (15) is lower by 1 than the degree of (4). This reduction of the degree to the lowest possible value $k-1$, for a formula of class C^{k-2}, was achieved at the price of having an infinite span. The infinite span of (15) clearly disqualifies this interpolation formula as far as numerical purposes are concerned.

We now turn to the *partial sums* of the series (17). They will yield smoothing interpolation formulae of considerable practical importance. Indeed, let

$$L_{k,m}(x) = M_k(x) - c_2^{(k)}\delta^2 M_k(x) + \cdots + (-1)^{m-1} c_{2m-2}^{(k)}\delta^{2m-2} M_k(x), \quad (2m - 2 < k). \tag{18}$$

The characteristic function of this basic function is identical with the left-hand side of the identity I(32). In view of our Theorem 2 of Part A, this identity I(32) proves that (18) is the basic function of a smoothing interpolation formula which is exact for the degree equal to $\min(2m-1, k-1)$. It is, moreover, visibly of degree $k-1$, of class C^{k-2}, and of span $s=k+2m-2$. One further important point is in need of proof, namely that the formulae based on (18) actually do smooth any given sequence (see Definition b of Part A, section 2.2). This will readily follow from Lemma 5. Indeed the characteristic function $\phi_{m,k}(u)$ of the formula

$$F(n) = \sum_\nu y_\nu L_{k,m}(n - \nu) \tag{19}$$

is, by Theorem 2, Part A, given by

$$\phi_{k,m}(u) = \sum_{\nu=-\infty}^{\infty} g_{k,m}(u + 2\pi\nu).$$

By I(32) and I(28) we now have

$$\phi_{k,m}(u) = \sum_{\nu=-\infty}^{\infty} g_{k,m}(u + 2\pi\nu) = \phi_k(u)\sum_{n=0}^{m-1} c_{2n}^{(k)}(2 \sin u/2)^{2n}$$
$$< \phi_k(u)\sum_{n=0}^{\infty} c_{2n}^{(k)}(2 \sin u/2)^{2n} = 1, \qquad (0 < u < 2\pi). \tag{20}$$

Since obviously $\phi_{m,k}(u) > 0$, for all u, we see that (19) is indeed a smoothing formula according to our definition. Recalling the relations I(36), I(38), we may therefore state the following Theorem:

THEOREM 2. *Let k be a positive integer and m an integer such that $0<2m<k+2$. Let the positive rational numbers $d_{2n}^{(k)}$ be defined by the expansion*

$$\left(\frac{2 \arcsin v/2}{v}\right)^k = \sum_{n=0}^{\infty} d_{2n}^{(k)} v^{2n}. \tag{21}$$

Then

$$L_{k,m}(x) = M_k(x) - d_2^{(k)}\delta^2 M_k(x) + d_4^{(k)}\delta^4 M_k(x) - \cdots + (-1)^{m-1} d_{2m-2}^{(k)}\delta^{2m-2} M_k(x) \tag{22}$$

gives rise to a smoothing interpolation formula

$$F(x) = \sum_{n=-\infty}^{\infty} y_n L_{k,m}(x-n) \tag{23}$$

of type

$$D^{k-1}, \qquad C^{k-2}, \qquad E^{\min(2m-1,k-1)}, \qquad s = k + 2m - 2. \tag{24}$$

Moreover, the formula (23) *preserves the degree* $k-1$. (*See Part* A, *section* 2.21, *Definition* d.)

If k is fixed and m increases then the smoothing power of our formula (23) *decreases according to our definition of Part* A, *section* 1.12, *Definition* 2.

The last statement concerning the decreasing smoothing power of (23) follows from (20), since $\phi_{k,m}(u)$ increases strictly as m increases while u is constant $(0<u<2\pi)$.

Notice, by (24), how on increasing m by one unit both the degree of exactness, as well as the span, increase by two units.

As illustration we find from I(40) that

$$L_{k,2}(x) = M_k(x) - \frac{k}{24}\delta^2 M_k(x), \qquad (k \geqq 4), \tag{25}$$

yields a smoothing formula of type

$$D^{k-1}, \qquad C^{k-2}, \qquad E^3, \qquad s = k + 2. \tag{26}$$

The characteristic function of (25) is

$$g_{k,2}(u) = \left(\frac{2\sin u/2}{u}\right)^k \left(1 + \frac{k}{6}\sin^2\frac{u}{2}\right) \tag{27}$$

or

$$g_{k,2}(u) = \left(\frac{2\sin u/2}{u}\right)^k \left\{1 + \frac{k}{12} - \frac{k}{12}\cos u\right\}.$$

For $k=4$ this function $g_{4,2}(u)$ agrees with the integrand of our formula (12″) of Part A, section 2.123. Also Mr. Jenkins' basic function, as given by formula (12) of Part A, section 2.123, may be derived by working out the various polynomial expressions of

$$L_{4,2}(x) = M_4(x) - \tfrac{1}{6}\delta^2 M_4(x) \tag{28}$$

from the explicit expressions of $M_4(x)$ (see Part A, section 3.13, (14)).

Likewise, by I(40)

$$L_{k,3}(x) = M_k(x) - \frac{k}{24}\delta^2 M_k(x) + \frac{k(5k+22)}{5760}\delta^4 M_k(x), \qquad (k \geqq 6), \tag{29}$$

yields a smoothing formula of type

$$D^{k-1}, \qquad C^{k-2}, \qquad E^5, \qquad s = k + 4. \tag{30}$$

2.3. **A second class of smoothing interpolation formulae derived from the ordinary k-point central interpolation formula.** Among the smoothing interpolation formulae (23) described by Theorem 2 the one of most interest is obtained by letting m

assume its largest value. If k is even, m is maximal if $2m-2=k-2$ or $m=k/2$. If k is odd, m is maximal if $2m-2=k-1$ or $m=(k+1)/2$. In either case

$$\max m = \mu = \left[\frac{k+1}{2}\right], \tag{31}$$

where $[x]$ represents the largest integer not exceeding x. The corresponding basic function (22) is

$$L_{k,\mu}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin u/2}{u}\right)^k\{1 + d_2^{(k)}(2\sin u/2)^2 + \cdots + d_{2\mu-2}^{(k)}(2\sin u/2)^{2\mu-2}\}e^{iux}du. \tag{32}$$

We recall that the smoothing interpolation formula based on this function is by (24) of the type

$$D^{k-1}, \qquad C^{k-2}, \qquad E^{k-1}, \qquad s = k + 2\mu - 2. \tag{33}$$

Indeed, the formula is exact for the degree $k-1$ because of

$$\left(\frac{2\sin u/2}{u}\right)^k\{1 + d_2^{(k)}(2\sin u/2)^2 + \cdots + d_{2\mu-2}^{(k)}(2\sin u/2)^{2\mu-2}\} = 1 + u^k\cdot(\text{regular function}). \tag{34}$$

An interesting counterpart to (32) is obtained as follows. An identity of the type (34) may also be obtained if in the expression within braces we replace $2\sin u/2$ by u. Indeed, rational constants $\gamma_{2\nu}^{(k)}$ may be determined such that

$$\left(\frac{2\sin u/2}{u}\right)^k\{1 + \gamma_2^{(k)}u^2 + \gamma_4^{(k)}u^4 + \cdots + \gamma_{2\mu-2}^{(k)}u^{2\mu-2}\} = 1 + u^k\cdot(\text{regular function}). \tag{35}$$

Lemma 6. *The basic function*

$$\Gamma_{k,\mu}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin u/2}{u}\right)^k\{1 + \gamma_2^{(k)}u^2 + \cdots + \gamma_{2\mu-2}^{(k)}u^{2\mu-2}\}e^{iux}du \tag{36}$$

is identical, for all real values of x, with the basic function $C_k(x)$ of the k-point central interpolation method (see Part A, *section* 2.121).

Notice first that by differentiation of

$$M_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin u/2}{u}\right)^k e^{iux}du$$

we obtain

$$M_k^{(2\nu)}(x) = (-1)^\nu\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin u/2}{u}\right)^k u^{2\nu}e^{iux}du, \qquad (2\nu < k). \tag{37}$$

Therefore the integral (36) may also be written as[4]

[4] Assuming (41) already established, we see by (38) and the relations

$$M_k^{(\nu)}(x) = \delta^\nu M_{k-\nu}(x), \qquad (0 \leqq \nu \leqq k-1), \tag{*}$$

(see Part A, section 3.15, formula (23)) that we may express $C_k(x)$ as follows

$$\Gamma_{k,\mu}(x) = M_k(x) - \gamma_2^{(k)} M_k''(x) + \gamma_4^{(k)} M_k^{(4)}(x) - \cdots + (-1)^{\mu-1} \gamma_{2\mu-2}^{(k)} M_k^{(2\mu-2)}(x). \quad (38)$$

Now $M_k(x)$ and all its derivatives are functions of span $s=k$. Therefore also $\Gamma_{k,\mu}(x)$ has the span $s=k$. Furthermore by (35), and Theorem 2 of Part A, we conclude that

$$F(x) = \sum_{n=-\infty}^{\infty} y_n \Gamma_{k,\mu}(x-n) \quad (39)$$

is an interpolation formula of the following type characteristics:

$$E^{k-1}, \qquad s = k. \quad (40)$$

These last two properties (40) allow us to show readily that

$$\Gamma_{k,\mu}(x) = C_k(x) \quad \text{for all real } x. \quad (41)$$

Indeed, let k be even, $k=2\kappa$. Let $P_0(x)$ be the polynomial of degree $k-1$ defined by the following k conditions

$$P_0(-\kappa+1) = P_0(-\kappa+2) = \cdots = P_0(-1) = 0, \qquad P_0(0) = 1,$$
$$P_0(1) = P_0(2) = \cdots = P_0(\kappa) = 0. \quad (42)$$

Since (39) is exact for the degree $k-1$ we have the identity

$$P_0(x) = \sum_{n=-\infty}^{\infty} P_0(n)\Gamma_{k,\mu}(x-n), \quad \text{for all real } x. \quad (43)$$

We now restrict x to the range

$$0 \leqq x \leqq 1. \quad (44)$$

Then we may write (43) as

$$P_0(x) = \sum_{n=-\kappa+1}^{n=\kappa} P_0(n)\Gamma_{k,\mu}(x-n)$$

since $\Gamma_{k,\mu}(x-n)=0$ if $|x-n| \geqq \kappa$. In view of (42) this identity reduces to the single term, for $n=0$:

$$P_0(x) = \Gamma_{k,\mu}(x), \qquad (0 \leqq x \leqq 1),$$

and therefore (41) holds for the range (44). Likewise, applying the formula (39) to the polynomial $P_1(x)$, of degree k, defined by

$$P_1(-\kappa+2) = \cdots = P_1(-1) = 0, \qquad P_1(0) = 1, \qquad P_1(1) = \cdots = P_1(\kappa+1) = 0,$$

we find that (41) holds in the range $1 \leqq x \leqq 2$, and so forth. Similar arguments obviously apply, with obvious modifications, to the case of an odd k.

The coefficients $\gamma_{2\nu}^{(k)}$ are the expansion coefficients of

$$\left(\frac{u}{2\sin u/2}\right)^k = \sum_{\nu=0}^{\infty} \gamma_{2\nu}^{(k)} u^{2\nu}. \quad (45)$$

$$C_k(x) = M_k(x) - \gamma_2^{(k)} \delta^2 M_{k-2}(x) + \gamma_4^{(k)} \delta^4 M_{k-4}(x) + \cdots + (-1)^{\mu-1} \gamma_{2\mu-2}^{(k)} \delta^{2\mu-2} M_{k-2\mu+2}(x). \quad (^{**})$$

This formula reveals at a glance the following fact: *If k is even, then $C_k^{(2\nu)}(x)$ ($\nu=0, 1, 2, \cdots$) are continuous. If k is odd then $C_k^{(2\nu+1)}(x)$ ($\nu=0, 1, 2, \cdots$) are continuous.* The author learned this property from Kingsland Camp, *Notes on Interpolation*, Trans. Actuar. Soc. Amer., **38**, p. 22 (1937).

In N. E. Nörlund's *Differenzenrechnung*, page 143, we find the expansion

$$\left(\frac{t}{\sin t}\right)^k = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{t^{2\nu}}{(2\nu)!} D_{2\nu}^{(k)}.$$

The coefficient $D_{2\nu}^{(k)}$ is a polynomial in k of degree ν. Nörlund's Table 6 on page 460, loc. cit., lists these polynomials for $\nu=0, 1, \cdots, 6$. We therefore have

$$\left(\frac{u}{2 \sin u/2}\right)^k = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{u^{2\nu}}{(2\nu)!} \cdot \frac{D_{2\nu}^{(k)}}{2^{2\nu}} \tag{46}$$

whence

$$\gamma_{2\nu}^{(k)} = (-1)^\nu \frac{D_{2\nu}^{(k)}}{(2\nu)! 2^{2\nu}} \cdot \tag{47}$$

The first few values are

$$\gamma_0^{(k)} = 1, \qquad \gamma_2^{(k)} = \frac{k}{24}, \qquad \gamma_4^{(k)} = \frac{k(5k+2)}{5760} \cdot \tag{48}$$

The expansion coefficients of $t/\sin t$ are positive (see Nörlund, loc. cit., Chapter II, sections 2, 3). Therefore *the coefficients* $\gamma_{2\nu}^{(k)}$ *are all positive.* We shall use this fact later.

In view of the results of our last section it seems natural to consider the partial sums

$$\Gamma_{k,m}(x) = M_k(x) - \gamma_2^{(k)} M_k''(x) + \cdots + (-1)^{m-1} \gamma_{2m-2}^{(k)} M_k^{(2m-2)}(x),$$
$$\left(1 \leqq m < \mu = \left[\frac{k+1}{2}\right]\right), \tag{49}$$

of the sum (38). The properties of the interpolation formulae based on these functions are described by the following theorem:

THEOREM 3. *The formula*

$$F(x) = \sum y_\nu \Gamma_{k,m}(x-\nu), \qquad \left(1 \leqq m < \mu = \left[\frac{k+1}{2}\right]\right), \tag{50}$$

is a smoothing interpolation formula of the type

$$D^{k-1}, \qquad C^{k-2m}, \qquad E^{2m-1}, \qquad s = k. \tag{51}$$

The smoothing power of (50) *decreases, as* m *increases from* $m=1$ *to* $m=\mu-1$, *until for* $m=\mu$ (50) *reduces to the* (*ordinary*) k-*point central interpolation formula.*

Indeed, the characteristic function of (49) is

$$g(u) = \left(\frac{2 \sin u/2}{u}\right)^k \left\{1 + \gamma_2^{(k)} u^2 + \cdots + \gamma_{2m-2}^{(k)} u^{2m-2}\right\}. \tag{52}$$

From (45) we conclude that

$$g(u) = 1 + u^{2m} \cdot (\text{regular function})$$

and therefore (50) is exact for the degree $2m-1$, by Theorem 2 of Part A. The remaining three characteristics

$$D^{k-1}, \qquad C^{k-2m}, \qquad s = k,$$

are evident on inspection of (49). There remains the investigation of the characteristic function of the corresponding smoothing formula

$$F(n) = \sum_{\nu} y_\nu \Gamma_{k,m}(n-\nu). \tag{53}$$

By Theorem 2, Part A, this characteristic function is

$$\chi_m(u) = \sum_{\nu=-\infty}^{\infty} g(u+2\pi\nu).$$

From (52) and I(3), we obtain

$$\chi_m(u) = \phi_k(u) + \gamma_2^{(k)}(2\sin u/2)^2\phi_{k-2}(u) + \cdots + \gamma_{2m-2}^{(k)}(2\sin u/2)^{2m-2}\phi_{k-2m+2}(u). \tag{54}$$

This expression is obviously positive for all values of u. For $m=\mu$, however, we obtain the identity

$$\chi_\mu(u) \equiv 1 \tag{55}$$

since, by Lemma 6, we have before us an *ordinary* interpolation formula. Since (54) is a partial sum of the left-hand side of (55) we have therefore proved the inequalities

$$0 < \chi_m(u) < 1 \qquad (1 \leqq m < \mu,\ 0 < u < 2\pi). \tag{56}$$

Then (53) is indeed a smoothing formula. The final statement of the Theorem is evident from (54), since $\chi_m(u)$ increases with m.

As illustrations we mention the following four special cases, two from each end of the range of values of m.

(i) $m=2$. The formula based on

$$\Gamma_{k,2}(x) = M_k(x) - \frac{k}{24}M_k''(x) \qquad (k \geqq 4) \tag{57}$$

has the type

$$D^{k-1}, \qquad C^{k-4}, \qquad E^3, \qquad s = k. \tag{57'}$$

(ii) $m=3$. The formula based on

$$\Gamma_{k,3}(x) = M_k(x) - \frac{k}{24}M_k''(x) + \frac{k(5k+2)}{5760}M_k^{(4)}(x), \qquad (k \geqq 6), \tag{58}$$

has the type

$$D^{k-1}, \qquad C^{k-6}, \qquad E^5, \qquad s = k. \tag{58'}$$

The values (48) were used.

(iii) $m=\mu-1$. The formula based on[5]

[5] The formula (59) is especially instructive because we can observe very clearly how the addition to $C_k(x)$ of the extra term removes the crudest discontinuities of $C_k(x)$. Indeed, by the formula (*) of our preceding footnote we may write (59) as

$$\Gamma_{k,\mu-1}(x) = C_k(x) + (-1)^{\mu} \overset{(k)}{\gamma}_{2\mu-2} M_k^{(2\mu-2)}(x) \qquad (k \geqq 4) \tag{59}$$

has the type

$$D^{k-1}, \qquad \begin{Bmatrix} C^2 & \text{if} & k \text{ is even} \\ C^1 & \text{if} & k \text{ is odd} \end{Bmatrix}, \qquad E^{k-3}, \qquad s = k. \tag{59'}$$

(iv) $m=\mu-2$. The formula based on

$$\Gamma_{k,\mu-2}(x) = C_k(x) + (-1)^{\mu} \overset{(k)}{\gamma}_{2\mu-2} M_k^{(2\mu-2)}(x) + (-1)^{\mu-1} \overset{(k)}{\gamma}_{2\mu-4} M_k^{(2\mu-4)}(x) \quad (k \geqq 6) \tag{60}$$

has the type

$$D^{k-1}, \qquad \begin{Bmatrix} C^4 & \text{if} & k \text{ is even} \\ C^3 & \text{if} & k \text{ is odd} \end{Bmatrix}, \qquad E^{k-5}, \qquad s = k. \tag{60'}$$

These formulae show clearly how an increase in "order of contact" is compensated by a corresponding loss in "reproductive power" and vice versa.

III. A SECOND CLASS OF ANALYTIC INTERPOLATION FORMULAE

In Part A, section 4.2, we described a class of *ordinary* analytic interpolation formulae of basic function

$$L_k(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\psi_k(u, t)}{\phi_k(u, t)} e^{iux} du \qquad (k = 1, 2, \cdots ; t > 0). \tag{1}$$

These interpolation formulae are exact for the degree $k-1$. The basic function (1) as well as those of the smoothing interpolation formulae derived from it in Part A, section 4.3, dampen out like a descending exponential function. In the present last chapter we wish to construct smoothing analytic interpolation formulae of basic functions dampening out like

$$\exp(-c^2x^2),$$

hence much more rapidly. In view of the development of section 2.2 it would seem fairly obvious how such formulae may be derived. We clearly need an analogue of Lemma 5 which we state as a **conjecture**: *The reciprocal of $\phi_k(u,t)$ admits of an expansion*

$$\Gamma_{k,\mu-1}(x) = C_k(x) + (-1)^{\mu} \overset{(k)}{\gamma}_{2\mu-2} \delta^{2\mu-2} M_{k-2\mu+2}(x). \tag{59'}$$

Let k be *odd*, hence $2\mu-2=k-1$ and therefore

$$\Gamma_{k,\mu-1}(x) = C_k(x) + (-1)^{\mu} \overset{(k)}{\gamma}_{k-1} \delta^{k-1} M_1(x). \tag{59''}$$

As seen from the graph of $M_1(x)$, the corrective term is a step-function with discontinuities at $x=n+1/2$ whose values are proportional with the binomial coefficients of order $k-1$: $\binom{k-1}{\nu}$. Their addition to $C_k(x)$ offsets the discontinuities of $C_k(x)$ and turn it into a function (59″) of class C^1. If k is *even*, hence $2\mu-2=k-2$, we have

$$\Gamma_{k,\mu-1}(x) = C_k(x) + (-1)^{\mu} \overset{(k)}{\gamma}_{k-2} \delta^{k-2} M_2(x). \tag{59'''}$$

As seen from the graph of $M_2(x)$, the corrective term is now an ordinary polygonal line with vertices at $x=n$, whose ordinates (at these vertices) are proportional to the binomial coefficients of order $k-2$: $\binom{k-2}{\nu}$. Again, the superposition of this polygonal line on $C_k(x)$ offsets the corners of $C_k(x)$ and turns it into a function (59‴) of class C^2. The formulae (59‴), (59″) are especially convenient for constructing tables of these functions from existing tables of $C_k(x)$, i.e., tables of Lagrange interpolation coefficients.

$$\frac{1}{\phi_k(u, t)} = \sum_{n=0}^{\infty} c_{2n}^{(k)}(t)(2 \sin u/2)^{2n}, \tag{2}$$

which converges for all real values of u and where the coefficients are all positive

$$c_{2n}^{(k)}(t) > 0, \qquad (n = 0, 1, 2, \cdots). \tag{3}$$

A proof of this conjecture would require a closer function-theoretic study of the entire periodic function $\phi_k(u, t)$ which has not been carried through as yet.

Since

$$\phi_k(u, t) = \sum_{\nu=-\infty}^{\infty} \psi_k(u + 2\pi\nu, t)$$

(see Part A, section 4.1, formula (6)) we have

$$\phi_k(u, t) = \psi_k(u, t) + u^k \cdot (\text{regular function}).$$

Therefore the expansion (2) agrees in its terms of order less than k with the similar terms of the expansion

$$\frac{1}{\psi_k(u, t)} = e^{t(u/2)^2}\left(\frac{u}{2 \sin u/2}\right)^k = \sum_{n=0}^{\infty} d_{2n}^{(k)}(t)(2 \sin u/2)^{2n}, \qquad (-\pi \leqq u \leqq \pi). \tag{4}$$

Hence

$$c_{2n}^{(k)}(t) = d_{2n}^{(k)}(t), \qquad (0 \leqq 2n < k). \tag{5}$$

The expansion (4) is readily determined and its coefficients are found to be positive as follows. We turn back to section 1.3 where in terms of the variable

$$v = 2 \sin u/2 \tag{6}$$

we have by I(35)'

$$\left(\frac{u}{2 \sin u/2}\right)^k = \sum_{n=0}^{\infty} d_{2n}^{(k)} v^{2n} \qquad (-\pi \leqq u \leqq \pi). \tag{7}$$

Also by I(39)

$$u/2 = \arcsin v/2 = \frac{v}{2} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{v^3}{8} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}\cdot\frac{v^5}{32} + \cdots \qquad (-2 \leqq v \leqq 2). \tag{8}$$

On substituting (8) into the exponential series we find the expansion

$$e^{t(u/2)^2} = \sum_{n=0}^{\infty} e_{2n}(t) v^{2n} \qquad (-2 \leqq v \leqq 2) \tag{9}$$

with positive coefficients, the first three of which are found to be

$$e_0(t) = 1, \qquad e_2(t) = \frac{t}{4}, \qquad e_4(t) = \frac{t^2}{32} + \frac{t}{48}. \tag{10}$$

On multiplying the series (7) and (9) we obtain the expansion (4). From the values I(40) and (10) we readily find

$$d_0^{(k)}(t) = 1, \qquad d_2^{(k)}(t) = \frac{k}{24} + \frac{t}{4}, \qquad d_4^{(k)}(t) = \frac{5k^2 + 22k}{5760} + \frac{tk}{96} + \frac{t}{48} + \frac{t^2}{32}. \tag{11}$$

Our arguments of section 2.2 may now be repeated leading to the following theorem:

THEOREM 4. *Let k be a positive integer and m an integer such that* $0 < 2m < k+2$. *Then*

$$L_{k,m}(x, t) = M_k(x, t) - d_2^{(k)}(t)\delta^2 M_k(x, t) + \cdots + (-1)^{m-1} d_{2m-2}^{(k)}(t)\delta^{2m-2} M_k(x) \tag{12}$$

gives rise to a "smoothing" analytic interpolation formula

$$F(x) = \sum_{n=-\infty}^{\infty} y_n L_{k,m}(x - n, t) \tag{13}$$

which is exact for the degree $\min(2m-1, k-1)$. *Moreover* (13) *always preserves the degree* $k-1$.

The adjective "smoothing" was purposely written in quotation marks in order to indicate that there is no general proof as yet that (13) always reduces, for integral values of the variable x, to a smoothing formula in the sense of our Definition 1 of Part A, section 1.1. For indeed, (2) and (3), which imply such a proof, were only conjectured. In the Appendix we give 8-place tables of the three basic functions

$$\left.\begin{aligned} L_1(x) &= L_{4,2}(x, 1/8), \\ L_2(x) &= L_{4,2}(x, 1/2), \\ L_3(x) &= L_{6,3}(x, 1/2), \end{aligned}\right\} \tag{14}$$

as well as 7-place tables of their first and second derivatives. For these three sets of values of the parameters k, t, and m, the interpolation formula (13) is indeed a smoothing formula. This point is verified by an inspection of the corresponding characteristic functions

$$\phi_i(u) = L_i(0) + 2L_i(1)\cos u + 2L_i(2)\cos 2u + \cdots, \qquad (i = 1, 2, 3,). \tag{15}$$

From the values of $L_i(n)$, as given by our tables, we computed the following table for these characteristic functions:

u	$\phi_1(u)$	$\phi_2(u)$	$\phi_3(u)$
0°	1.00000	1.00000	1.00000
30°	.99734	.99519	.99952
60°	.96332	.93655	.97760
90°	.85492	.76500	.84693
120°	.67727	.51297	.56702
150°	.50474	.29296	.27879
180°	.43283	.20728	.16123

Since $0 < \phi_i(u) < 1$ for $0 < u \leqq 180°$, all three formulae (13) are smoothing formulae according to Part A, section 1.1. Also $\phi_2(u) < \phi_1(u)$ implies that L_2 gives a stronger smoothing formula as compared to L_1.

Our set of tables is intended mainly for the purpose of illustrating the method. A more complete set of tables would be needed in order to furnish smoothing of a desired strength, as required by the needs of the numerical data at hand.

APPENDIX

Description of the tables and their use for the analytic approximation of equidistant data. In the Tables I, II, and III, we have tabulated the following three functions

$$L_1(x) = M_4(x, 1/8) - \frac{19}{96}\delta^2 M_4(x, 1/8) \tag{1}$$

$$L_2(x) = M_4(x, 1/2) - \frac{7}{24}\delta^2 M_4(x, 1/2) \tag{2}$$

$$L_3(x) = M_6(x, 1/2) - \frac{3}{8}\delta^2 M_6(x, 1/2) + \frac{199}{1920}\delta^4 M_6(x, 1/2) \tag{3}$$

and their first two derivatives. The function $M_k(x, t)$ occurring in these definitions may be defined by the integral

$$M_k(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t(u/2)^2}\left(\frac{2 \sin u/2}{u}\right)^k \cos uxdu.$$

A given sequence of equidistant ordinates

$$\{y_n\} \tag{4}$$

is approximated by either one of the three analytic functions

$$F_i(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_i(x - \nu), \qquad (i = 1, 2, 3). \tag{5}$$

The choice among these approximations depends on the amount of smoothing desired. The formula (5), for $i=1$ and $i=2$, is exact for (i.e., reproduces) cubic polynomials. For $i=3$ the formula (5) is exact for quintic polynomials. For the same data (4), the sequence $\{F_2(n)\}$ is always smoother than the sequence $\{F_1(n)\}$. Generally, the sequence $\{F_3(n)\}$ should be smoother than the sequence $\{F_1(n)\}$.

The first and second derivatives of the approximation (5) may be computed by the similar formulae

$$F_i'(x) = \sum y_\nu L_i'(x - \nu), \tag{6}$$

$$F_i''(x) = \sum y_\nu L_i''(x - \nu). \tag{7}$$

The arrangement of our tables is such as to facilitate the computation of $F_i(x)$ by (5), as explained in the Appendix to Part A.

An example of smoothing with subtabulation to tenths. We propose to compute a table of the approximation $F_2(x)$, in the range $31 \leqq x \leqq 34$, for the same ordinates $\{y_n\}$ as were used in our example of Part A (Appendix). The ordinates which we now require are given by the following table:

n	y_n	Δ	Δ^2	Δ^3	Δ^4	Δ^5
26	32840					
27	34790	1950				
28	37260	2470	520			
29	40440	3180	710	190		
30	44750	4310	1130	420	230	
31	51120	6370	2060	930	510	280
32	59390	8270	1900	− 160	−1090	−1600
33	67550	8160	− 110	−2010	−1850	− 760
34	73820	6270	−1890	−1780	230	2080
35	77830	4010	−2260	− 370	1410	1180
36	80240	2410	−1600	660	1030	− 380
37	81660	1420	− 990	610	− 50	−1080
38	82330	670	− 758	240	− 370	− 320
39	82680	350	− 320	430	190	560

From these values and the Table II of $L_2(x)$ and $L_2''(x)$ we obtain the following tables of the approximation $F_2(x)$ and its second derivative $F_2''(x)$ shown with their differences.

x	$F_2(x)$	Δ	Δ^2	Δ^3	Δ^4	$F_2''(x)$	Δ	Δ^2	Δ^3	Δ^4
31.0	51232.76					1901.77				
31.1	51989.86	75710				1767.20	−13457			
31.2	52764.62	77476	1766			1611.60	−15560	−2103		
31.3	53555.48	79086	1610	−156		1436.84	−17476	−1916	187	
31.4	54360.69	80521	1435	−175	−19	1245.42	−19142	−1666	250	63
31.5	55178.34	81765	1244	−191	−16	1040.29	−20513	−1371	295	45
31.6	56006.39	82805	1040	−204	−13	824.64	−21565	−1052	319	24
31.7	56842.68	83629	824	−216	−12	601.66	−22298	− 733	319	0
31.8	57684.98	84230	601	−223	− 7	374.38	−22728	− 430	303	−16
31.9	58531.03	84605	375	−226	− 3	145.59	−22879	− 151	279	−24
32.0	59378.53	84750	145	−230	− 4	− 82.29	−22788	91	242	−37
32.1	60225.21	84668	− 82	−227	3	− 307.15	−22486	302	211	−31
32.2	61068.82	84361	− 307	−225	2	− 527.12	−21997	489	187	−24
32.3	61907.17	83835	− 526	−219	6	− 740.51	−21339	658	169	−18
32.4	62738.12	83095	− 740	−214	5	− 945.74	−20523	816	158	−11
32.5	63559.62	82150	− 945	−205	9	−1141.23	−19549	974	158	0
32.6	64369.71	81009	−1141	−196	9	−1325.45	−18422	1127	153	− 5
32.7	65166.57	79686	−1323	−182	14	−1496.90	−17145	1277	150	− 3
32.8	65948.46	78189	−1497	−174	8	−1654.18	−15728	1417	140	−10
32.9	66713.83	76537	−1652	−155	19	−1796.14	−14196	1532	115	−25
33.0	67461.25	74742	−1795	−143	12	−1921.91	−12577	1619	87	−28
33.1	68189.46	72821	−1921	−126	17	−2030.98	−10907	1670	51	−36
33.2	68897.38	70792	−2029	−108	18	−2123.19	− 9221	1686	16	−35
33.3	69584.08	68670	−2122	− 93	15	−2198.74	− 7555	1666	− 20	−36
33.4	70248.81	66473	−2197	− 75	18	−2258.04	− 5930	1625	− 41	−21
33.5	70890.96	64215	−2258	− 61	14	−2301.69	− 4365	1565	− 60	−19
33.6	71510.12	61916	−2299	− 41	20	−2330.31	− 2862	1503	− 62	− 2
33.7	72105.98	59586	−2330	− 31	10	−2344.57	− 1426	1436	− 67	− 5
33.8	72678.41	57243	−2343	− 13	18	−2345.07	− 50	1376	− 60	7
33.9	73227.40	54899	−2344	− 1	12	−2332.47	1260	1310	− 66	− 6
34.0	73753.07	52567	−2332	12	13	−2307.47	2500	1240	− 70	− 4

A comparison of the approximation $F_2(x)$ with the strictly interpolating function $F(x)$, obtained in the Appendix of Part A, is of interest. The function $F(x)$ was obtained by the formula

$$F(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_k(x - \nu, t), \qquad (k = 4, t = 1/2),$$

where, in view of III(1) and III(2), we may define $L_k(x, t)$ by the expansion

$$L_k(x, t) = M_k(x, t) - c_2^{(k)}(t)\delta^2 M_k(x, t) + c_4^{(k)}(t)\delta^4 M_k(x, t) - \cdots. \tag{8}$$

Our present approximation $F_2(x)$ was computed by the formula (5), for $i=2$, where $L_2(x)$, by (2), III(5) and III(11), happens to be identical with the sum

$$L_2(x) = M_k(x, t) - c_2^{(k)}(t)\delta^2 M_k(x, t)$$

of the first two terms of the series (8). A comparison of the tables of $F(x)$ and $F_2(x)$ shows that their difference in the range $31 \leqq x \leqq 34$ nowhere exceeds 0.23% of the value of $F(x)$.

TABLE I. $L_1(x) = L_{4,2}(x, 1/8)$, $L_1'(x)$, $L_1''(x)$

$L_1(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
3	−.00041123	−.00018550	−.00007621	−.00002830	−.00000943
2	−.03462580	−.02756299	−.02099452	−.01533104	−.01073240
1	.14220425	.08277004	.03516336	−.00066601	−.02556532
0	.78566556	.77475976	.74281594	.69204028	.62577328
−1	.14220425	.21251290	.29166041	.37661030	.46353840
−2	−.03462580	−.04145730	−.04699226	−.04985257	−.04841746
−3	−.00041123	−.00083691	−.00157669	−.00277249	−.00458632
−4			−.00000003	−.00000017	−.00000074

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
3	−.00000280	−.00000074	−.00000017	−.00000003	
2	−.00718751	−.00458632	−.00277249	−.00157669	−.00083691
1	−.04092087	−.04841746	−.04985257	−.04699226	−.04145730
0	.54811118	.46353840	.37661030	.29166041	.21251290
−1	.54811118	.62577328	.69204028	.74281594	.77475976
−2	−.04092087	−.02556532	−.00066601	.03516336	.08277004
−3	−.00718751	−.01073240	−.01533104	−.02099452	−.02756299
−4	−.00000280	−.00000943	−.00002830	−.00007621	−.00018550

$L_1'(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
3	.0030922	.0015611	.0007155	.0002953	.0001089
2	.0709642	.0690870	.0616092	.0514079	.0405956
1	−.6512052	−.5359047	−.4163489	−.3016954	−.1986416
0	.0000000	−.2168063	−.4183406	−.5915396	−.7269119
−1	.6512052	.7515608	.8262912	.8662852	.8650057
−2	−.0709642	−.0638857	−.0445045	−.0099839	.0416488
−3	−.0030922	−.0056121	−.0094213	−.0147669	−.0217952
−4		−.0000001	−.0000006	−.0000026	−.0000103

$L_1'(x)$

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
3	.0000357	.0000103	.0000026	.0000006	.0000001
2	.0304945	.0217952	.0147669	.0094213	.0056121
1	−.1113051	−.0416488	.0099839	.0445045	.0638857
0	−.8188067	−.8650057	−.8662852	−.8262912	−.7515608
−1	.8188067	.7269119	.5915396	.4183406	.2168063
−2	.1113051	.1986416	.3016954	.4163489	.5359047
−3	−.0304945	−.0405956	−.0514079	−.0616092	−.0690870
−4	−.0000357	−.0001089	−.0002953	−.0007155	−.0015611

$L_1''(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
4	−.0000004	−.0000001			
3	−.0197354	−.0114013	−.0059477	−.0027759	−.0011500
2	.0202421	−.0521077	−.0926044	−.1078983	−.1061742
1	1.0966561	1.1912649	1.1844282	1.0974670	.9568292
0	−2.1943249	−2.1159568	−1.8927238	−1.5554043	−1.1427100
−1	1.0966561	.8923803	.5868711	.2020705	−.2337191
−2	.0202421	.1269947	.2654156	.4283134	.6058876
−3	−.0197354	−.0311722	−.0454298	−.0617347	−.0788289
−4	−.0000004	−.0000019	−.0000092	−.0000377	−.0001346

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
3	−.0004201	−.0001346	−.0000377	−.0000092	−.0000019
2	−.0947351	−.0788289	−.0617347	−.0454298	−.0311722
1	.7867260	.6058876	.4283134	.2654156	.1269947
0	−.6915708	−.2337191	.2020705	.5868711	.8923803
−1	−.6915708	−1.1427100	−1.5554043	−1.8927238	−2.1159568
−2	.7867260	.9568292	1.0974670	1.1844282	1.1912649
−3	−.0947351	−.1061742	−.1078983	−.0926044	−.0521077
−4	−.0004201	−.0011500	−.0027759	−.0059477	−.0114013
−5					−.0000001

TABLE II. $L_2(x)=L_{4,2}(x,\ 1/2)$, $L_2'(x)$, $L_2''(x)$

$L_2(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
5	−.00000001				
4	−.00003297	−.00001721	−.00000871	−.00000428	−.00000203
3	−.00453670	−.00313034	−.00210474	−.00137859	−.00087930
2	−.04033977	−.03775949	−.03379714	−.02913417	−.02429574
1	.20271717	.14753253	.09914645	.05829935	.02523450
0	.68438455	.67711287	.65567464	.62117134	.57534514
−1	.20271717	.26343912	.32793367	.39399436	.45907812
−2	−.04033977	−.04070831	−.03791256	−.03092183	−.01869149
−3	−.00453670	−.00640787	−.00882100	−.01183236	−.01545906
−4	−.00003297	−.00006127	−.00011052	−.00019364	−.00032972
−5	−.00000001	−.00000003	−.00000007	−.00000018	−.00000042

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
4	−.00000094	−.00000042	−.00000018	−.00000007	−.00000003
3	−.00054590	−.00032972	−.00019364	−.00011052	−.00006127
2	−.01965692	−.01545906	−.01183236	−.00882100	−.00640787
1	−.00024449	−.01869149	−.03092183	−.03791256	−.04070831
0	.52044824	.45907812	.39399436	.32793367	.26343912
−1	.52044824	.57534514	.62117134	.65567464	.67711287
−2	−.00024449	.02523450	.05829935	.09914645	.14753253
−3	−.01965692	−.02429574	−.02913417	−.03379714	−.03775949
−4	−.00054590	−.00087930	−.00137859	−.00210474	−.00313034
−5	−.00000094	−.00000203	−.00000428	−.00000871	−.00001721

$L_2'(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
5	.0000001				
4	.0002093	.0001145	.0000607	.0000311	.0000154
3	.0162494	.0120195	.0086291	.0060153	.0040721
2	.0162409	.0339777	.0441321	.0482532	.0478931
1	−.5820676	−.5195203	−.4469715	−.3695748	−.2920635
0	.0000000	−.1448007	−.2821139	−.4050264	−.5077160
−1	.5820676	.6294198	.6567760	.6601752	.6369100
−2	−.0162409	.0104651	.0471784	.0943909	.1518602
−3	−.0162494	−.0213049	−.0270545	−.0332049	−.0392596
−4	−.0002093	−.0003705	−.0006358	−.0010581	−.0017083
−5	−.0000001	−.0000002	−.0000006	−.0000015	−.0000034

$L_2'(x)$

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
4	.0000074	.0000034	.0000015	.0000006	.0000002
3	.0026769	.0017083	.0010581	.0006358	.0003705
2	.0444848	.0392596	.0332049	.0270545	.0213049
1	−.1876748	−.1518602	−.0943909	−.0471784	−.0104651
0	−.5858619	−.6369100	−.6601752	−.6567760	−.6294198
−1	.5858619	.5077160	.4050264	.2821139	.1448007
−2	.1876748	.2920635	.3695748	.4469715	.5195203
−3	−.0444848	−.0478931	−.0482532	−.0441321	−.0339777
−4	−.0026769	−.0040721	−.0060153	−.0086291	−.0120195
−5	−.0000074	−.0000154	−.0000311	−.0000607	−.0001145

$L_2''(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
5	−.0000010	−.0000004	−.0000001		
4	−.0012284	−.0007087	−.0003952	−.0002129	−.0001108
3	−.0465343	−.0380479	−.0298698	−.0225871	−.0164844
2	.2201282	.1369510	.0687454	.0162834	−.0210828
1	.5579563	.6842412	.7580806	.7819152	.7615818
0	−1.4606815	−1.4227560	−1.3118991	−1.1365778	− .9099683
−1	.5579763	.3809905	.1594624	−.0960473	.3711575
−2	.2201282	.3157558	.4193623	.5245097	.6230076
−3	−.0465343	−.0543645	−.0601451	−.0620423	−.0578316
−4	−.0012284	−.0020585	−.0033357	−.0052283	−.0079270
−5	−.0000010	−.0000025	−.0000057	−.0000126	−.0000270

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
4	−.0000557	−.0000270	−.0000126	−.0000057	−.0000025
3	−.0116249	−.0079270	−.0052283	−.0033357	−.0020585
2	−.0450247	−.0578316	−.0620423	−.0601451	−.0543645
1	.7053804	.6230076	.5245097	.4193623	.3157558
0	−.6486751	−.3711575	−.0960473	.1594624	.3809905
−1	−.6486751	−.9099683	−1.1365778	−1.3118991	−1.4227560
−2	.7053804	.7615818	.7819152	.7580806	.6842412
−3	−.0450247	−.0210828	.0162834	.0687454	.1369510
−4	−.0116249	−.0164844	−.0225871	−.0298698	−.0380479
−5	−.0000557	−.0001108	−.0002129	−.0003952	−.0007087
−6				−.0000001	−.0000004

TABLE III. $L_3(x)=L_{6,3}(x, 1/2)$, $L_3'(x)$, $L_3''(x)$

$L_3(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
6	.00000024	.00000012	.00000005	.00000002	.00000001
5	.00008286	.00005169	.00003150	.00001874	.00001088
4	.00268003	.00209760	.00160285	.00119677	.00087361
3	.00146844	.00384345	.00530620	.00602933	.00618622
2	−.06657958	−.06134625	−.05394260	−.04532872	−.03631488
1	.20814167	.14737265	.09315196	.04655844	.00818720
0	.70841267	.70114512	.67967926	.64500315	.59869152
−1	.20814167	.27390708	.34268907	.41215975	.47975381
−2	−.06657958	−.06856746	−.06617125	−.05825575	−.04376882
−3	.00146844	−.00197317	−.00659556	−.01244745	−.01948472
−4	.00268003	.00333890	.00404886	.00476644	.00542627
−5	.00008286	.00012978	.00019870	.00029745	.00043550
−6	.00000024	.00000049	.00000096	.00000183	.00000340

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
5	.00000616	.00000340	.00000183	.00000096	.00000049
4	.00062367	.00043550	.00029745	.00019870	.00012978
3	.00593823	.00542627	.00476644	.00404886	.00333890
2	−.02754330	−.01948472	−.01244745	−.00659556	−.00197317
1	−.02182865	−.04376882	−.05825575	−.06617125	−.06856746
0	.54280388	.47975381	.41215975	.34268907	.27390708
−1	.54280388	.59869152	.64500315	.67967926	.70114512
−2	−.02182865	.00818720	.04655844	.09315196	.14737265
−3	−.02754330	−.03631488	−.04532872	−.05394260	−.06134625
−4	.00593823	.00618622	.00602933	.00530620	.00384345
−5	.00062367	.00087361	.00119677	.00160285	.00209760
−6	.00000616	.00001088	.00001874	.00003150	.00005169
−7		.00000001	.00000002	.00000005	.00000012

$L_3'(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
6	−.0000017	−.0000009	−.0000004	−.0000002	−.0000001
5	−.0003813	−.0002500	−.0001598	−.0000996	−.0000605
4	−.0062356	−.0053949	−.0044992	−.0036324	−.0028472
3	.0288514	.0189098	.0106360	.0041176	−.0007095
2	.0379708	.0648953	.0815717	.0893524	.0898535
1	−.6356363	−.5771501	−.5054638	−.4254184	−.3417929
0	.0000000	−.1447858	−.2828738	−.4080067	−.5147714
−1	.6356363	.6763588	.6953825	.6897327	.6576776
−2	−.0379708	.0001221	.0497076	.1103706	.1808527
−3	−.0288514	−.0401754	−.0523583	−.0646105	−.0758656
−4	.0062356	.0069002	.0072273	.0070158	.0060307
−5	.0003813	.0005675	.0008239	.0011670	.0016121
−6	.0000017	.0000033	.0000063	.0000116	.0000206

$L_3'(x)$

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
5	−.0000358	−.0000206	−.0000116	−.0000063	−.0000033
4	−.0021709	−.0016121	−.0011670	−.0008239	−.0005675
3	−.0040148	−.0060307	−.0070158	−.0072273	−.0069002
2	.0847953	.0758656	.0646105	.0523583	.0401754
1	−.2590022	−.1808527	−.1103706	−.0497076	−.0001221
0	−.5989336	−.6576776	−.6897327	−.6953825	−.6763588
−1	.5989336	.5147714	.4080067	.2828738	.1447858
−2	.2590022	.3417929	.4254184	.5054638	.5771501
−3	−.0847953	−.0898535	−.0893524	−.0815717	−.0648953
−4	.0040148	.0007095	−.0041176	−.0106360	−.0189098
−5	.0021709	.0028472	.0036324	.0044992	.0053949
−6	.0000358	.0000605	.0000996	.0001598	.0002500
−7		.0000001	.0000002	.0000004	.0000009

$L_3'(x)$

x	$x+.0$	$x+.1$	$x+.2$	$x+.3$	$x+.4$
6	.0000117	.0000061	.0000031	.0000015	.0000007
5	.0015631	.0010868	.0007350	.0004835	.0003094
4	.0077612	.0088518	.0089242	.0083240	.0073357
3	−.1069900	−.0913730	−.0739648	−.0565175	−.0403140
2	.3239975	.2160536	.1197566	.0385786	−.0256675
1	.5032414	.6588169	.7667843	.8261081	.8390021
0	−1.4591699	−1.4253081	−1.3259553	−1.1676245	−.9605716
−1	.5032414	.3045939	.0708748	−.1867923	−.4548011
−2	.3239975	.4384255	.5526106	.6585611	.7474910
−3	−.1069900	−.1186224	−.1237228	−.1195410	−.1033391
−4	.0077612	.0052612	.0009447	−.0055676	−.0145672
−5	.0015631	.0021860	.0029705	.0039174	.0050049
−6	.0000117	.0000217	.0000392	.0000685	.0001165
−7				.0000001	.0000001

x	$x+.5$	$x+.6$	$x+.7$	$x+.8$	$x+.9$
6	.0000003	.0000001	.0000001		
5	.0001926	.0001165	.0000685	.0000392	.0000217
4	.0061776	.0050049	.0039174	.0029705	.0021860
3	−.0261859	−.0145672	−.0055676	.0009447	.0052612
2	−.0726654	−.1033391	−.1195410	−.1237228	−.1186224
1	.8104398	.7474910	.6585611	.5526106	.4384255
0	−.7179590	−.4548011	−.1867923	−.0708748	.3045939
−1	−.7179590	−.9605716	−1.1676245	−1.3259553	−1.4253081
−2	.8104398	.8390021	.8261081	.7667843	.6588169
−3	−.0726654	−.0256675	.0385786	.1197566	.2160536
−4	−.0261859	−.0403140	−.0565175	−.0739648	−.0913730
−5	.0061776	.0073357	.0083240	.0089242	.0088518
−6	.0001926	.0003094	.0004835	.0007350	.0010868
−7	.0000003	.0000007	.0000015	.0000031	.0000061